Géométrie de dimension infinie appliquée à l'analyse des formes

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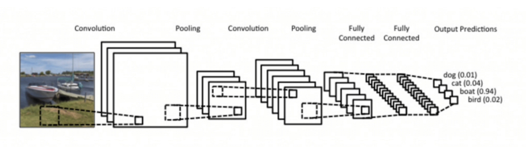
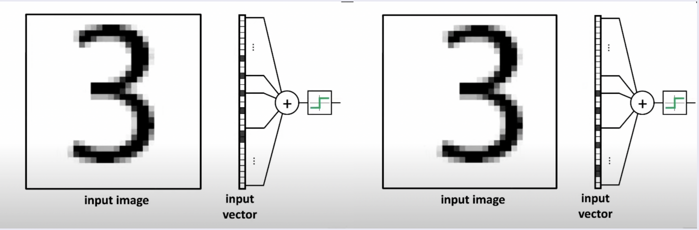
Why Geometry?

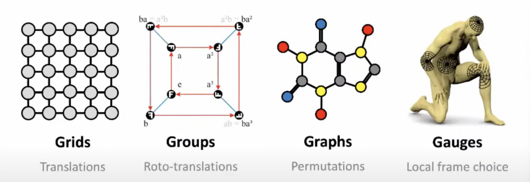
Traditional Deep Learning pipeline





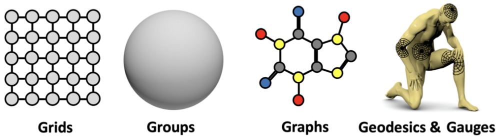
Shift invariance has to be learned...one needs a lot of data for this! see S+SSPR2020 Keynote Talk : Michael Bronstein

Idea: incorporate geometric structures into the Deep Learning pipeline



S+SSPR2020 Keynote Talk : The 4G of Geometric Deep Learning by Michael Bronstein

Idea: incorporate geometric structures into the Deep Learning pipeline



The "5G" of Geometric Deep Learning: Grids, Group (homogeneous spaces with global symmetries), Graphs (and sets as a particular case), and Manifolds, where geometric priors are manifested through global isometry invariance (which can be expressed using Geodesics) and local Gauge symmetries.

The 5G of Geometric Deep Learning by Michael Bronstein

Why Geometry?

- to avoid data augmentation and manage ressources
- to develop better models of data representation
- to make deep learning more green
- because most datasets cluster on non-linear spaces

Why infinite-dimensional Geometry?

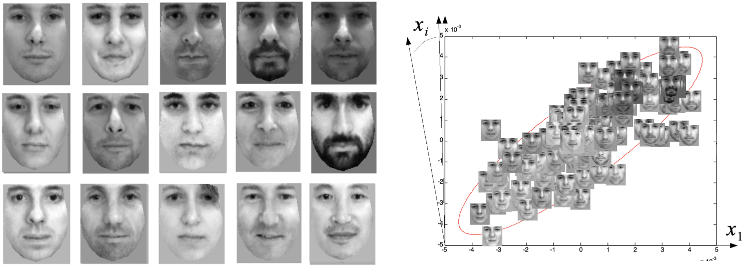


Figure: Representation of the manifold of faces.

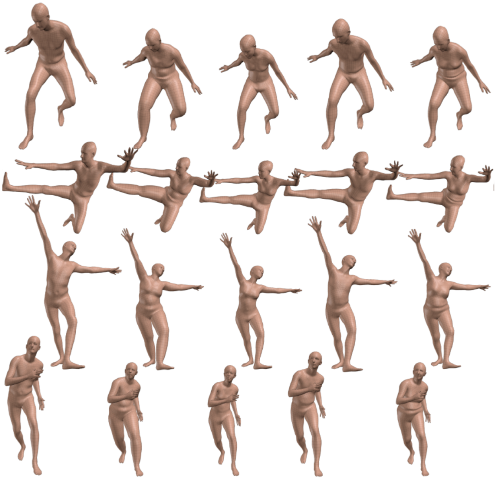


Figure 13: Animating SMPL. *Decomposition of SMPL parameters into pose and shape: Shape parameters, $\vec{\beta}$, vary across different subjects from left to right, while pose parameters, $\vec{\theta}$, vary from top to bottom for each subject.*

Figure: Representation of the manifold of Human bodies and poses, see SMPL: a skinned multi-person linear model

Shape spaces are non-linear manifolds

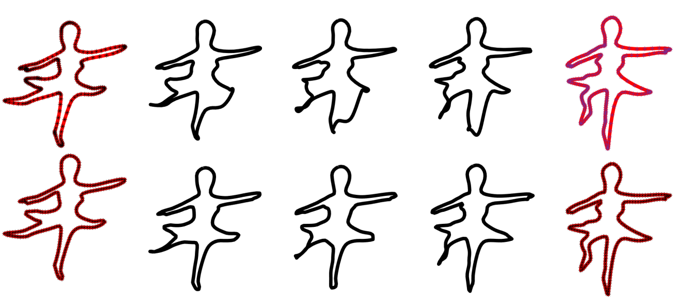


Figure: First line : linear interpolation between some parameterized ballerinas, second line : linear interpolation between arc-length parameterized ballerinas.

Shape spaces are non-linear manifolds

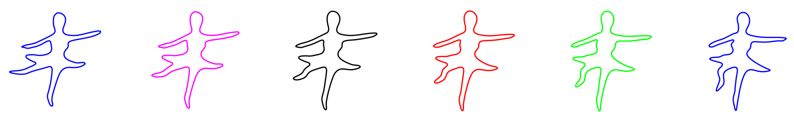


Figure: Geodesic between some parameterized ballerinas generated using the algorithm free to download at http://ssamg.stat.fsu.edu/software. The first and last shapes were taken from http://ed.ted.com/lessons/the-physics-of-the-hardest-move-in-ballet-arleen-sugano.

Why infinite-dimensional geometry?

- because most shape spaces are infinite-dimensional
- natural objects on a finite-dimensional manifold are elements of an infinite-dimensional space (vector fields, Riemannian metrics, mesures...)
- existence of geodesics on a finite-dimensional manifold is an infinite-dimensional phenomenon
  - initial value problem or shooting : geodesic is a solution of a Cauchy problem, i.e. a fixed point of a contraction in an appropriate infinite-dimensional space of curves
  - 2 boundary value problem: geodesic is a curve minimising an energy functional on a infinite-dimesnional space of curves
- Each time one want to vary the geometry of a finite-dimensional manifold, one ends up with a infinite-dimensional manifold (of Riemannian metric, of connexions, of symplectic forms....)

Outline

Part I : Geometric Objects

1. **Manifolds**
2. **Fiber bundles**
3. **Sections of a fiber bundle**
4. Application: **Resampling** using structural invariants of curves

Outline

Part II : Infinite-dimensional Geometry

1. What are the **Model** spaces of infinite-dimensional geometry?
2. What are the **Tools** from Functional Analysis?
3. Which **Geometric structures** can we consider?
4. What are the **Traps** of infinite-dimensional geometry?

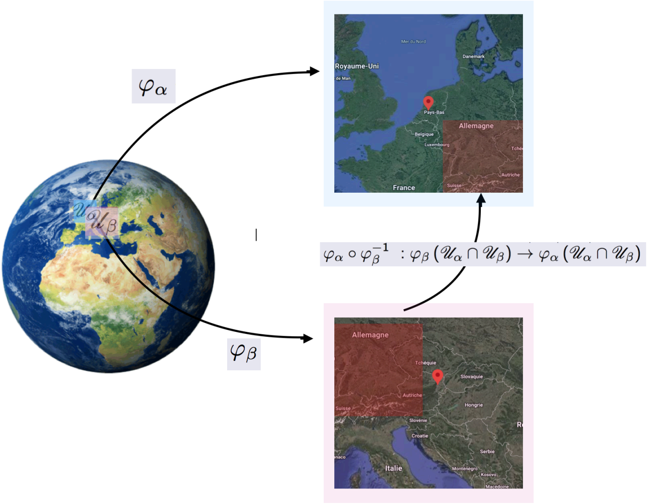
Outline

Part III : Shape Analysis

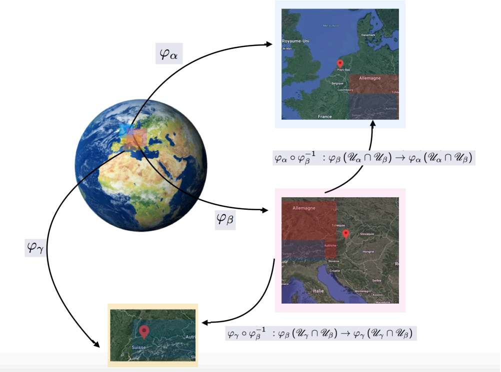
1. Shape spaces as **Quotient versus Sections** of fiber bundles
2. 3 different ways of putting a **intrinsic Riemannian metric** on Shape space

ORGANISATION: Cours 1, Cours 2, Cours 3, TD

Definition of an infinite-dimensional manifold



Definition of an infinite-dimensional manifold



The notion of manifold is build out from the notion of SMOOTH maps (or $\mathscr{C}^k$, or $\mathscr{C}^w$) between MODEL SPACES, the crucial condition on the set of smooth maps is the CHAIN RULE.

Charts and complete Atlas

Definition of a chart

Definition of an atlas

$\mathscr{C}^k$ equivalent atlases

Manifold = Hausdorff topological space with an equivalence class of $\mathscr{C}^k$ atlases

Charts and complete Atlas

Definition of a chart

A **chart** on a topological space $\mathscr{M}$ is a triple $(\mathscr{U}, \varphi, \mathscr{F}_\alpha)$ where $\mathscr{U}$ is an open set in $\mathscr{M}$ and ϕ an homeomorphism from $\mathscr{U}$ to an open set in a model topological vector space $\mathscr{F}_\alpha$.

Definition of an atlas

An **atlas** on a topological space $\mathscr{M}$ is a collection of charts $(\mathscr{U}_\alpha, \varphi_\alpha, \mathscr{F}_\alpha)_{\alpha \in \mathscr{I}}$ such that $\cup_{\alpha \in \mathscr{I}} \mathscr{U}_\alpha = \mathscr{M}$

Atlas of class $\mathscr{C}^k$

An **atlas** $\mathscr{A} = (\mathscr{U}_\alpha, \varphi_\alpha, \mathscr{F}_\alpha)_{\alpha \in \mathscr{I}}$ on $\mathscr{M}$ is of class $\mathscr{C}^k$ if all transition maps are $\mathscr{C}^k$-maps between the model topological vector spaces :
$\forall (\mathscr{U}_\alpha, \varphi_\alpha, \mathscr{F}_\alpha) \in \mathscr{A}$ and $(\mathscr{U}_\beta, \varphi_\beta, \mathscr{F}_\beta) \in \mathscr{A}$ such that $\mathscr{U}_\alpha \cap \mathscr{U}_\beta \neq \emptyset$
$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(\mathscr{U}_\alpha \cap \mathscr{U}_\beta) \to \varphi_\alpha(\mathscr{U}_\alpha \cap \mathscr{U}_\beta)$ is of class $\mathscr{C}^k$

Charts and complete Atlas

$\mathscr{C}^k$ equivalent atlases

Two atlases on a topological space $\mathscr{M}$ are said to be $\mathscr{C}^k$ **equivalent** if their union is of class $\mathscr{C}^k$

Manifolds of class $\mathscr{C}^k$

A **manifold of class** $\mathscr{C}^k$ $(k \geq 0)$ is an Hausdorff topological space endowed with an equivalence class of $\mathscr{C}^k$-atlases.

Hausdorff space

A topological space $\mathscr{M}$ is said to be **Hausdorff** if for any pair of distinct points $f_0 \neq f_1$ in $\mathscr{M}$ one can find two disjoints open sets $\mathscr{U}_0 \cap \mathscr{U}_1 = \emptyset$ in $\mathscr{M}$ such that $f_0 \in \mathscr{U}_0$ and $f_1 \in \mathscr{U}_1$

Manifolds of class $\mathscr{C}^k$, but which k?

Figure: Left : Maserati, Right : Aston Martin.

Manifolds of class $\mathscr{C}^k$, but which k?

Figure: Left : La Géode, Paris, Right : Cloud Gate, Chicago.

What is a fiber bundle?

What is a fiber bundle?

What is a fiber bundle?

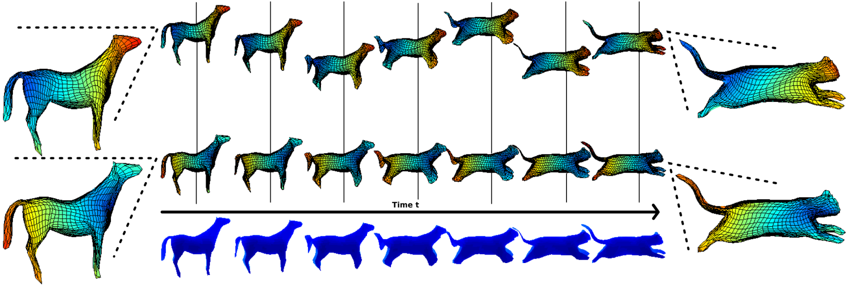
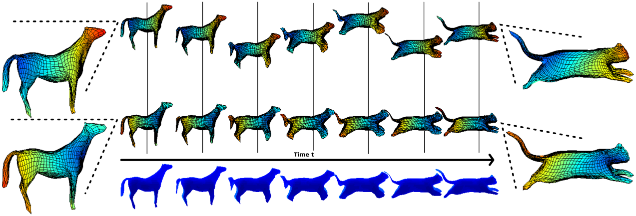


Figure: Picture from *Gauge invariant framework for shape analysis of surfaces*, A. B. Tumpach, H. Drira, M. Daoudi, and A. Srivastava, IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 38, no. 1, 2016.

Fiber Bundles



Definition of a fiber bundle

A fiber bundle over a manifold $\mathscr{B}$ with fiber $\mathscr{F}$ is a manifold $\mathscr{M}$ together with a projection $p : \mathscr{M} \to \mathscr{B}$ such that

- $\forall x \in \mathscr{B}, p^{-1}(x)$ is isomorphic to $\mathscr{F}$
- $\forall x \in \mathscr{B}$, there exists a neighborhood $\mathscr{U}$ of x and an isomorphism $\Phi : \mathscr{U} \times \mathscr{F} \to p^{-1}(\mathscr{U})$ that trivialises the manifold $\mathscr{M}$ locally.

Tangent Spaces and Tangent bundle

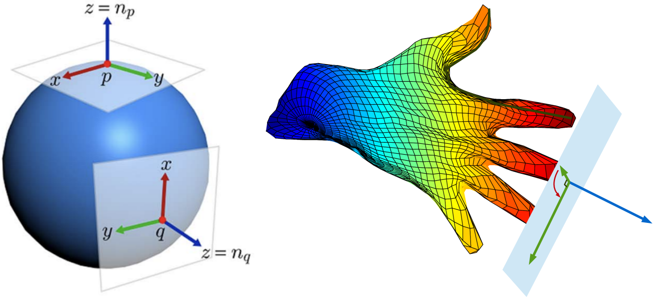
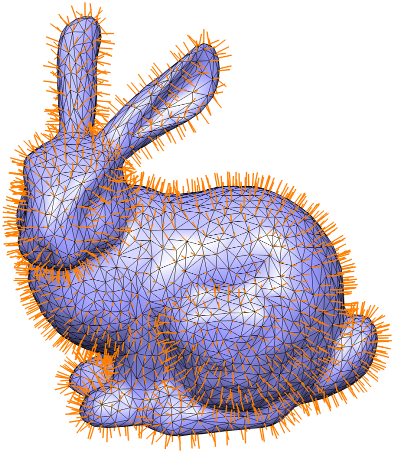


Figure: Picture taken from https://wiki.tizen.org/images/0/0c/Chapter_11 and from *Gauge invariant framework for shape analysis of surfaces*, A. B. Tumpach, H. Drira, M. Daoudi, and A. Srivastava, IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 38, no. 1, 2016.

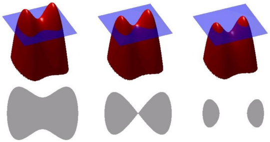
Normal Spaces and Normal bundle



Submersion Theorem

Definition of a submersion

An application $f : \mathscr{N} \to \mathscr{M}$ between two manifolds $\mathscr{N}$ and $\mathscr{M}$ is a submersion at $x \in \mathscr{N}$ if the derivative $df_x : T_x\mathscr{N} \to T_{f(x)}\mathscr{M}$ is surjective.



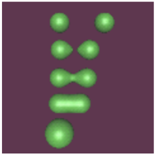


Figure: Pictures by Eduard Gröller, Thomas Theussl, Peter Rautek to illustrate implicit modeling.

Theorem

If $f : \mathscr{N} \to \mathscr{M}$ is a submersion at any point of $f^{-1}(\{c\})$ where $c \in \mathscr{M}$, then $f^{-1}(\{c\})$ is a submanifold of $\mathscr{N}$.

Definition of an immersion

A morphism $f : \mathscr{N} \to \mathscr{M}$ between two manifolds $\mathscr{N}$ and $\mathscr{M}$ is an immersion if for any $x \in \mathscr{N}$, the derivative $df_x : T_x\mathscr{N} \to T_{f(x)}\mathscr{M}$ is injective.

Definition of an embedding

A morphism $f : \mathscr{N} \to \mathscr{M}$ between two manifolds $\mathscr{N}$ and $\mathscr{M}$ is an embedding if f is an immersion and an homeomorphism from $\mathscr{N}$ on $f(\mathscr{N})$.

Embedding Theorem

Theorem

If $f : \mathscr{N} \to \mathscr{M}$ is an embedding, then $f(\mathscr{N})$ is a submanifold of $\mathscr{M}$.

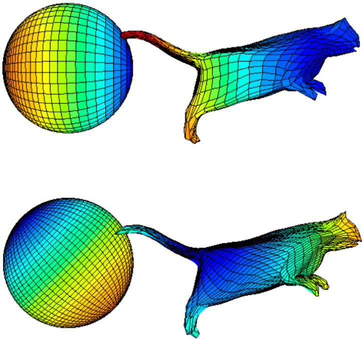


Figure: Two cats parameterized by a sphere.

Parameterization

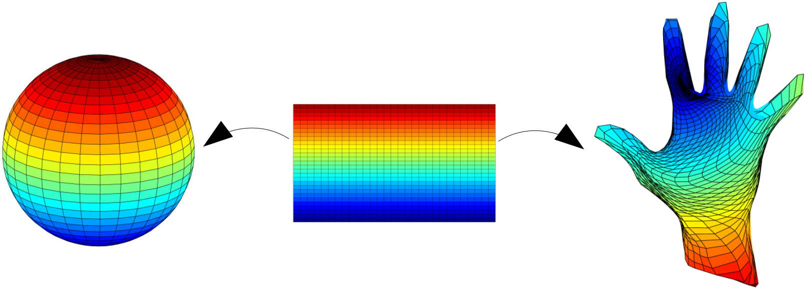


Figure: Parameterization of a sphere and a hand by a square.

Parameterization

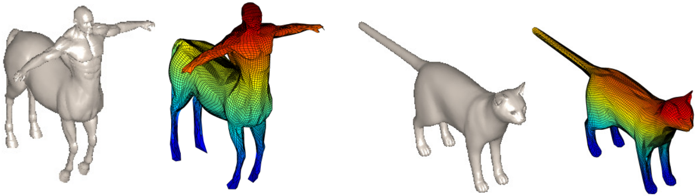


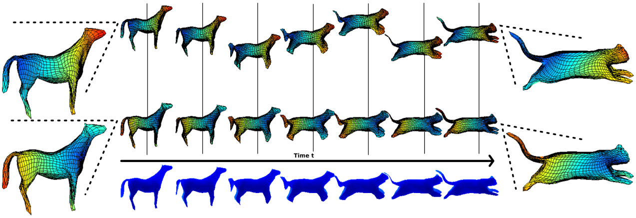
Figure: Divers surfaces from the dataset Tosca without and with parameterization.

Whitney's Theorem

Any smooth manifold of dimension n can be embedded smoothly into $\mathbb{R}^{2m}$.

Figure: The Klein bottle is a smooth 2-dimensional manifold that cannot be embedded in the 3-dimensional space, but can be embedded in $\mathbb{R}^4$.

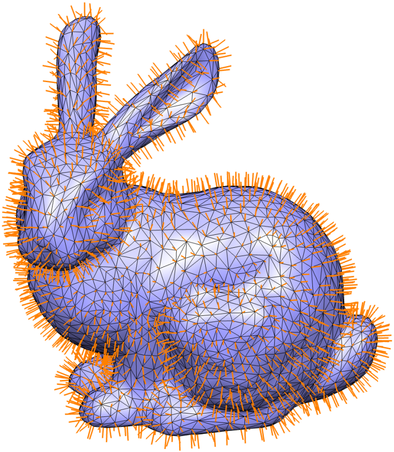
Sections of a fiber Bundles



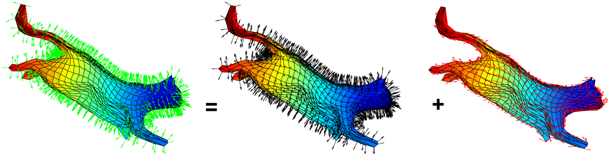
Definition of a section

A smooth section of a fiber bundle $\mathscr{M}$ over a manifold $\mathscr{B}$ with fiber $\mathscr{F}$ is a smooth map s from the base $\mathscr{B}$ into the manifold $\mathscr{M}$ that projects to the identity: $p \circ s(x) = x, \forall x \in \mathscr{B}$.

Sections of a fiber Bundles



Sections of a fiber Bundles



Sections of a fiber Bundles

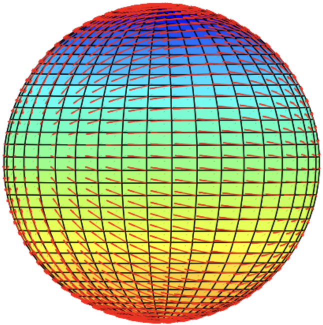


Figure: Vector field on a sphere

Sections of a fiber Bundles

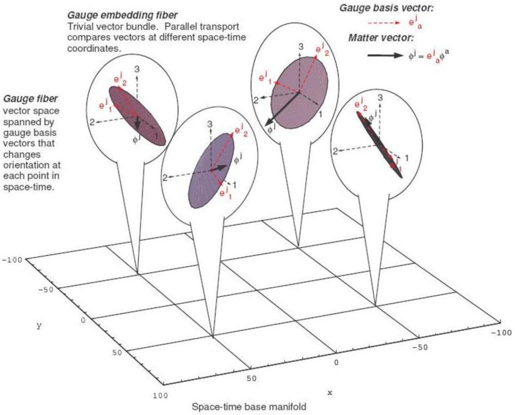
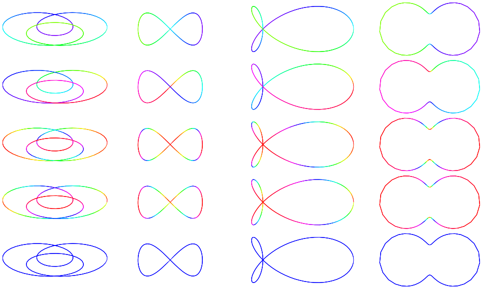


Figure: For more informations on the Geometry of Gauge Theories, see the homepage of Mario A Serna Jr

Sections of a fiber Bundles



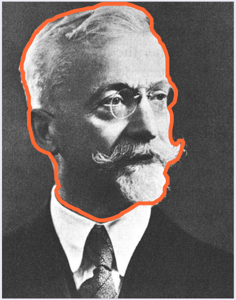
Application in Computer graphics

Figure 1: *"Icons in PAD": a collage of shapes generated with our scale- and rotation-invariant shape descriptor, namely pyramid of arclength descriptor (PAD); this novel descriptor improves the shape-matching efficiency, thus facilitating the generation of complex results.*

Figure: See Pyramid of Arclength Descriptor for Generating Collage of Shapes, Kwan, Sinn, Han, Wong and Fu, ACM Transactions on Graphics (SIGGRAPH Asia 2016 issue), Vol. 35, No. 6, November 2016

Curvature of a 2D-Curve

Consider a 2D simple closed curve. After the choice of a starting point and a direction, there is a unique way to travel the curve at unit speed : this is the arc-length parameterization. The rate of turning angle of the velocity vector is called the signed curvature of the curve. For instance, when moving along the external outline of the glasses, this curvature equals the inverse of the radius of the glasses.



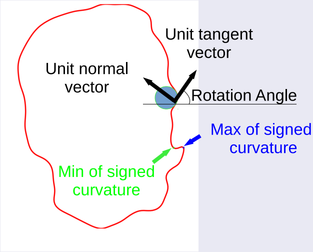


Figure: Elie Cartan and the moving frame associated to the contour of his head.

Curvature of a 2D-Curve

The curvature function of Elie Cartan's head looks like this in different parameterizations :

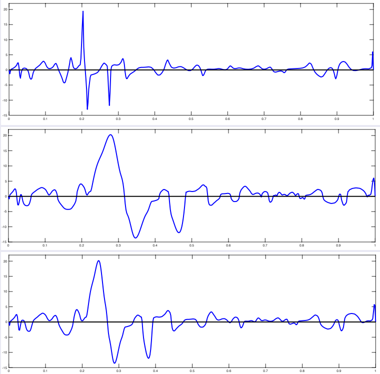


Figure: Signed curvature of Elie Cartan's head for the parameterization proportional to arc-length (first line), proportional to the curvature-length (second line), and proportional to the curvarc length (third line).

Arc-length Parameterization

It is easy to resample a curve using the arc-length parameterization : one computes the distances between points and resample uniformly using interpolation with splines.

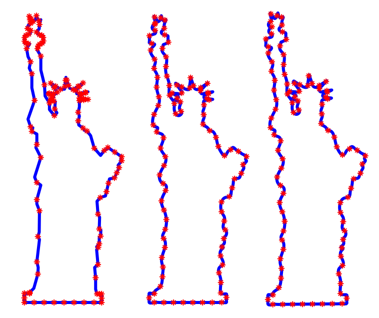


Figure:
The statue of Liberty (left), a uniform resampling using Matlab function spline (middle), a reconstruction of the statue using its discrete curvature (right).

Curvature-length Parameterization

Arc-length parameterization is used to compute the curvature function. Than one can integrate the absolute value of the curvature along the curve and renormalized to have a total integral equal to 1. The resulting function is used to define the curvature-length parameterization of the curve and resample the curve accordingly.

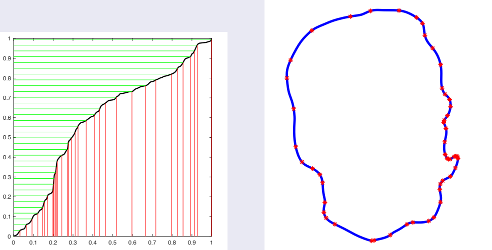


Figure: Integral of the (renormalized) absolute value of the curvature (left), and corresponding resampling of Elie's Cartan head (right).

Curvature-length Parameterization

The draw-back of curvature-length parameterization is that it does not put points at all on flat pieces of the curve. In order to fix this, instead of integrating the curvature, one can integrate λ + curvature, where λ is a parameter.

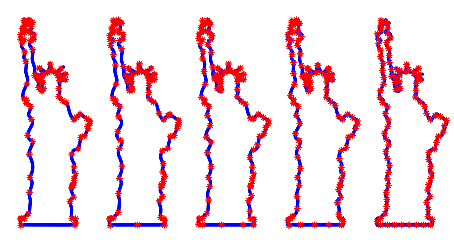


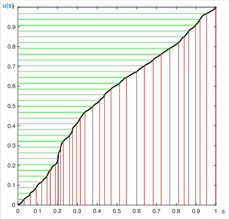
Figure:
Resampling of the statue of Liberty proportional to the intergral of λ + curvature, for (from left to right) $\lambda = 0; \lambda = 0.3; \lambda = 1; \lambda = 2; \lambda = 100$.

Curvarc-length Parameterization

The Curvarc-length parameterization is defined by

$$u(s) = \frac{\int_0^s (L + |\kappa(s)|)ds}{\int_0^1 (L + |\kappa(s)|)ds}$$

where L is the length of the curve, and κ the curvature function.



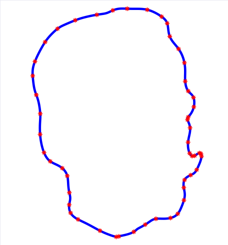


Figure: Integral of the (renormalized) curvarc length (left), and corresponding resampling of Elie's Cartan head (right).

Application: Reseach on Rheumatoid Arthritis by Georg Langs at Computational Imaging Research Lab, Vienna.

Rheumatoid Arthritis

Rheumatoid Arthritis is a long-term autoimmune disorder that primarily affects joints leading to pain and stiffness. The causes of this disease are not clear. The quantification of disease progression is crucial in order to adapt the treatment to the patient.

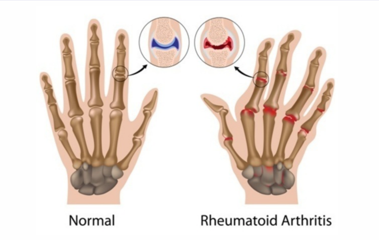


Figure: credit: Georg Langs, CIR, Vienna.

Rheumatoid Arthritis

X-ray modality is used to detect and quantify Rheumatoid Arthritis.
X-Ray projective radiography creates black and white images of bones.

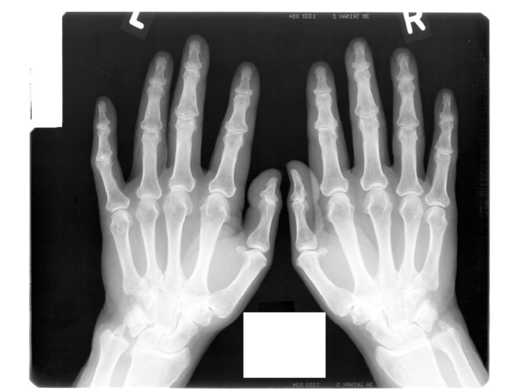


Figure: credit: Georg Langs, CIR, Vienna.

Rheumatoid Arthritis

Key features to detect are Erosion and Joint Space shrinking.

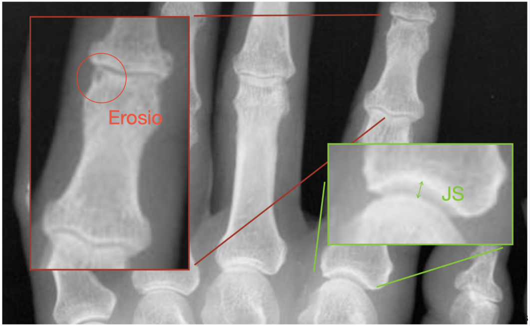


Figure: credit: Georg Langs, CIR, Vienna.

Rheumatoid Arthritis

To measure Joint Space, one uses landmarks along the contours of bones.
Difficulty : landmarks have to be placed at the same anatonical positions.

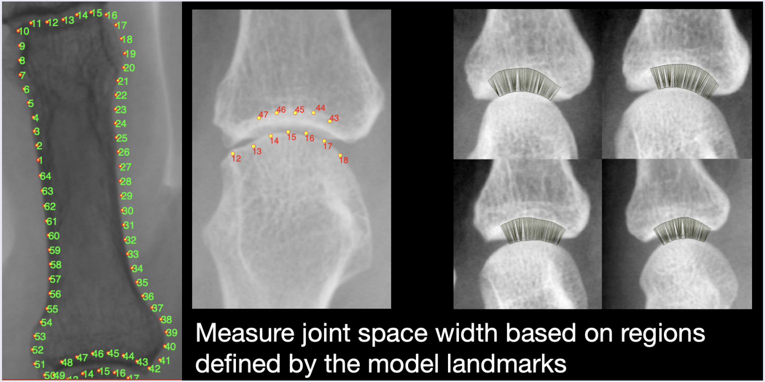
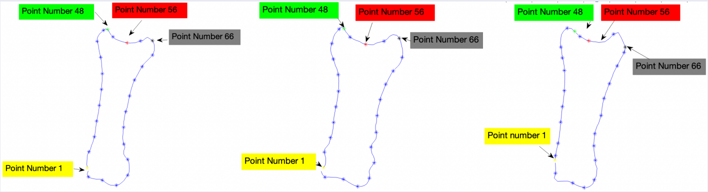


Figure: credit: Georg Langs, CIR, Vienna.

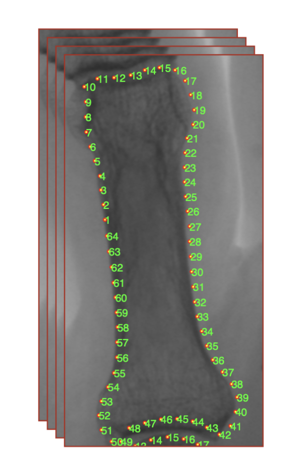
Rheumatoid Arthritis

Difficulty : landmarks have to be placed at the same anatonical positions.
These positions have geometric characteristics depending on curvature.
Task : automatically setup landmarks at correct anatonical positions.

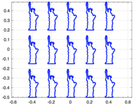
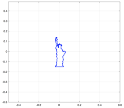
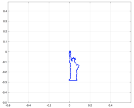
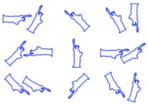


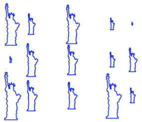
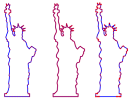
Problem to solve

Find the best canonical parameterization for the bones...



Group Actions on Images

Group G	Some elements of one orbit under the group G	a preferred element in the orbit	another choice of preferred element in the orbit
$\mathbb{R}^3$ acting by translation		 centered curve : $\int_0^1 \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} \|f'(s)\| ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.	 curve starting at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
SO(3) acting by rotation		 axes of approximating ellipse aligned	 tangent vector at starting point horizontal

Group G	Some elements of one orbit under the group G	a preferred element in the orbit	another choice of preferred element in the orbit
$\mathbb{R}^+$ acting by scaling		 length = 1	 enclosed area =1
$\text{Diff}^+([0,1])$ acting by reparameterization		 arc-length parameterization	 curvature proportional parameterization

Outline

Part II : Infinite-dimensional Geometry

1. What are the **Model** spaces of infinite-dimensional geometry?
2. What are the **Tools** from Functional Analysis?
3. Which **Geometric structures** can we consider?
4. What are the **Traps** of infinite-dimensional geometry?

Complete metric spaces

Metric space

A metric space is a space $\mathscr{M}$ endowed with a **distance function** $d \ : \mathscr{M} \times \mathscr{M} \to \mathbb{R}$

- $d(f_0, f_1) = 0 \Leftrightarrow f_0 = f_1$
- $d(f_0, f_1) = d(f_1, f_0)$
- $d(f_0, f_1) \leq d(f_0, f_2) + d(f_2, f_0)$

Cauchy sequence

A sequence $\{f_k\}_{k\in\mathbb{N}}$ in a metric space $\mathscr{M}$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists $N > 0$ suth that $d(f_n, f_m) < \varepsilon$ for all $n, m > N$

Complete metric space

A metric space $\mathscr{M}$ is said to be **complete** if any Cauchy sequence of elements in $\mathscr{M}$ converges

What are the Model spaces of infinite-dim. geometry?

Hilbert $\subset$ Banach $\subset$ Fréchet $\subset$ Locally Convex Spaces

Hilbert space H = **complete** vector space for the distance given by an inner product = $\langle \cdot, \cdot \rangle \ : H \times H \rightarrow \mathbb{R}$

- symmetric :$\langle x, y \rangle = \langle y, x \rangle$
- bilinear : $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$
- non-negative : $\langle x, x \rangle \geq 0$
- definite : $\langle x, x \rangle = 0 \Rightarrow x = 0$

$H^* = H$ (Riesz Theorem).

What are the Model spaces of infinite-dim. geometry?

Hilbert $\subset$ **Banach** $\subset$ Fréchet $\subset$ Locally Convex spaces

Banach space B = complete vector space for the distance given by a norm = $\| \cdot \| \ : B \rightarrow \mathbb{R}^+$

- triangle inequality : $\|x + y\| \leq \|x\| + \|y\|$
- absolute homogeneity : $\|\lambda x\| = |\lambda| \|x\|$.
- non-negative : $\|x\| \geq 0$
- definite : $\|x\| = 0 \Rightarrow x = 0$.

B^* = Banach space.

What are the Model spaces of infinite-dim. geometry?

Hilbert $\subset$ Banach $\subset$ **Fréchet** $\subset$ Locally Convex spaces

Fréchet space F = complete Hausdorff vector space for the distance $d\ : F \times F \rightarrow \mathbb{R}^+$ given by a countable family of semi-norms $\|\cdot\|_n$:

$$d(x, y) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

$F^* \neq$ Fréchet space if F not Banach, but locally convex
$F^{**} =$ Fréchet space.

What are the Model spaces of infinite-dim. geometry?

Hilbert $\subset$ Banach $\subset$ Fréchet $\subset$ **Locally Convex spaces**

Locally Convex spaces = Hausdorff topological vector space whose topology is given by a (possibly not countable) family of semi-norms.

References :

- **Analysis on locally convex spaces**, K.-H. Neeb.
- **The convenient setting of global analysis**, Kriegl, Michor
- **Diffeological spaces**, Souriau
- **Bastiani calculus on locally convex spaces**, Bastiani
- **Frölicher spaces**, Frölicher
- **Ringed spaces**, Egeileh, Michel, and Wurzbacher
- **Comparative smootheologies**, Stacey

What are the smooth maps between the model spaces?

Differentiable function on $\mathbb{R}$

For a function $f : \mathscr{U} \subset \mathbb{R} \to \mathbb{R}$, there are 3 equivalent notions of been **differentiable** at $x \in \mathscr{U}$

- $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists and is finite
- $\exists L$ such that $\lim_{h\to 0} \frac{f(x+h)-f(x)-Lh}{h} = 0$
- there exists a function $g : \mathscr{U} \subset \mathbb{R} \to \mathbb{R}$, such that $f(x+h) = f(x) + f'(x)h + g(h)$ and $\lim_{h\to 0} \frac{g(h)}{h} = 0$

Remark

On $\mathbb{R}$, a differentiable function is automatically continuous

In the Banach context

Fréchet differentiability in the Banach context

Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be two Banach spaces. A map $P : \mathscr{U} \subset \mathscr{B}_1 \to \mathscr{B}_2$ is **Fréchet differentiable** at $f_0 \in \mathscr{B}_1$ if there exists a **continuous linear operator** $DP(f_0) : \mathscr{B}_1 \to \mathscr{B}_2$ such that

$$P(f_0 + h) = P(f_0) + DP(f_0)(h) + \|h\|_1 \cdot \varepsilon(h) \quad \text{with} \quad \lim_{h \to 0} \|\varepsilon(h)\|_2 = 0$$

Remark

No continuity is assumed in the definition of Fréchet differentiability, but **Fréchet differentiable at** $f_0 \Rightarrow$ **continuous at** f_0

In the Banach context

$\mathscr{C}^1$ in the Banach context

A map $P : \mathscr{U} \subset \mathscr{B}_1 \rightarrow \mathscr{B}_2$ between Banach spaces is $\mathscr{C}^1$ if it is Fréchet differentiable on $\mathscr{U}$ and the derivative DP is continuous as a map from $\mathscr{U}$ into the Banach space $L_c(\mathscr{B}_1, \mathscr{B}_2)$ of continuous linear operators from $\mathscr{B}_1$ to $\mathscr{B}_2$

Smooth maps between Banach spaces

By induction one defines the notion of smooth maps on Banach spaces.

In the Fréchet context

Directional derivative

Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two Fréchet spaces and $P : \mathscr{U} \subset \mathscr{F}_1 \to \mathscr{F}_2$ a **continuous** non-linear map. P admits a **derivative at** f_0 **in the direction** of $h \in \mathscr{F}_1$ if the following limit exists

$$DP(f_0)(h) = \lim_{t \to 0} \frac{P(f_0 + th) - P(f_0)}{t}$$

One says that P is differentiable at f_0 if it admits directional derivatives in every direction $h \in \mathscr{F}_1$

$\mathscr{C}^1$ in the Fréchet context

A map $P : \mathscr{U} \subset \mathscr{F}_1 \to \mathscr{F}_2$ between Fréchet spaces is $\mathscr{C}^1$ if it is differentiable in $\mathscr{U}$ and the derivative DP is continuous as a map from $\mathscr{U} \times \mathscr{F}_1$ into $\mathscr{F}_2$

Comparaison of the two notions of $\mathscr{C}^1$-maps

Remarks

no linearity assumed but if P is $\mathscr{C}^1$ then $DP(f)h$ is always linear in h

On a Banach space

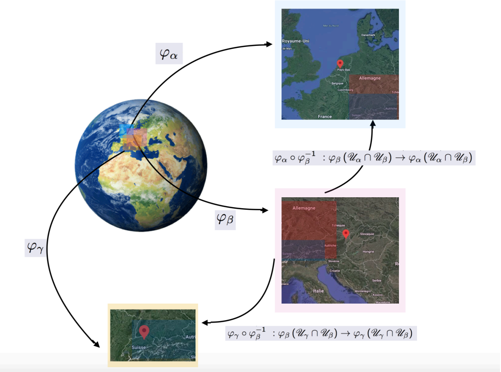
- $\mathscr{C}^1$ in the Banach context $\Leftrightarrow$ $\mathscr{C}^1$ in the Fréchet context
- $\mathscr{C}^2$ in the Fréchet context $\Leftrightarrow$ $\mathscr{C}^1$ in the Banach context [Keller]

Taylor formula

If $P\ : \mathscr{U} \subseteq \mathscr{F} \rightarrow G$ is $\mathscr{C}^2$ and if the path connecting f and $f+h$ lies in $\mathscr{U}$ then

$$P(f+h) = P(f) + DP(f)(h) + \int_0^1 (1-t) D^2 P(f+th)(h,h) dt$$

Definition of an infinite-dimensional manifold



The notion of manifold is build out from the notion of SMOOTH maps (or $\mathscr{C}^k$, or $\mathscr{C}^w$) between MODEL SPACES, the crucial condition on the set of smooth maps is the CHAIN RULE.

Examples from Geometry

Spheres

The sphere in a Hilbert space is a smooth Hilbert manifold

Remark : The sphere in a Banach space is not smooth unless the Banach space is an Hilbert space

Linear Grassmannians

The projective space of an Hilbert space is a Hilbert manifold
The Grassmannian of p-dimensional subspaces in an Hilbert space is Hilbert manifold (p finite)
The Grassmannian of subspaces in an Hilbert space with infinite dimension and codimension is a Banach manifold

Examples from Geometry

Manifolds of maps

The space of smooth maps from a compact manifold into a finite-dimensional manifold is a Fréchet manifold

Space of sections

The space of smooth section of a finite-dimensional vector bundle over a compact manifold is a Fréchet manifold

Examples from Geometry

Non-linear Grassmannians and non-linear Flags

The space of embeddings $N \hookrightarrow M$ from a compact manifold N into a finite-dimensional manifold M is a Fréchet manifold. More generally the space of flags $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_k \subset M$ is a Fréchet manifold

Reference

- M. Bauer, M. Bruveris, P.W. Michor, *Overview of the Geometries of Shape Spaces and Diffeomorphism Groups*
- F. Gay-Balmaz, C. Vizman, *Vortex sheets in ideal 3D fluids, coadjoint orbits, and characters*
- S. Haller, C. Vizman, *nonlinear Flag manifolds as coadjoint orbits*

Examples from Geometry

Manifolds of curves

The space of H^1 curves from $[0, 1]$ into a finite-dimensional Riemannian manifold M is a Hilbert manifold, and the critical points of the energy functional are the geodesics of M

Arc-length parameterized curves

The space of arc-length parameterized smooth curves $c \ : [0, 1] \to \mathbb{R}^n$ is a Fréchet submanifold of the space of all parameterized curves.

What are the Tools from Functional Analysis?

Banach-Picard fixed point Theorem or Contraction Theorem

(E, d) complete metric space
$f \ : E \to E$ contraction of E : $d(f(x), f(y)) \leq kd(x, y)$ where $k \in (0, 1)$

$$\Rightarrow \begin{cases} \exists\,!\ x \in E, f(x) = x \\ \forall x_0 \in E,\ \textit{the sequence } x_{n+1} = f(x_n) \textit{ converges to } x \end{cases}$$

What are the Key Tools from Functional Analysis?

Hahn-Banach Theorem

E locally convex space
$A \subset E$ a convex
$x \in E$, $x \notin \overline{A}$

$$\Rightarrow \exists \textit{ continuous functional } \ell \ : E \to \mathbb{R} \textit{ with } \ell(x) \notin \overline{\ell(A)}$$

What are the Tools from Functional Analysis?

Open mapping Theorem

$$\begin{cases} F \text{ Fréchet} \\ G \text{ Fréchet} \end{cases} \text{ or } \begin{cases} F \text{ webbed locally convex} \\ G \text{ inductive limit of Baire locally convex spaces} \end{cases}$$

$L : F \to G$ *continuous, linear, and surjective*

$\Rightarrow L$ *is open*

What are the Tools from Functional Analysis?

Cauchy-Lipschitz Theorem in the Banach case

- $\mathscr{I} \subset \mathbb{R}$ be an interval containing 0
- $\mathscr{U}$ open set of a Banach space $\mathscr{B}$
- $P \; : \mathscr{I} \times \mathscr{U} \rightarrow \mathscr{B}$

such that

- $\|P(t,f)\| \leq C \quad \forall (t,f) \in \mathscr{I} \times \mathscr{U}$
- $\|P(t,f_1) - P(t,f_0)\| \leq C' \|f_1 - f_0\| \quad \forall t \in \mathscr{I}, \forall f_0, f_1 \in \mathscr{U}$

For any $f_0 \in \mathscr{U}$ we can find a neighboorhood $\tilde{\mathscr{U}}$ of f_0 and an $\varepsilon > 0$ such that for any $f \in \tilde{\mathscr{U}}$ the Cauchy problem

$$\frac{d}{dt}\phi(t,f) = P(t,\phi(t,f))$$

has a unique solution with initial condition $\phi(0,f) = f$ on $[-\varepsilon,\varepsilon]$. Moreover if P is $\mathscr{C}^p$, $t \rightarrow \phi(t,x)$ is $\mathscr{C}^p$ for any $f \in \tilde{\mathscr{U}}$

What are the Tools from Functional Analysis?

Cauchy Theorem in the Fréchet case

Let $P : \mathscr{U} \subset \mathscr{F} \to \mathscr{F}$ be a **smooth Banach map**. Then $\forall f_0 \in \mathscr{U}$, $\exists \tilde{\mathscr{U}} \ni f_0$ and $\varepsilon > 0$ s.t. $\forall f \in \tilde{\mathscr{U}}$

$$\frac{d}{dt} f = P(f)$$

has a unique solution with initial condition $f(0) = f$ on $0 \leq t \leq \varepsilon$ depending smoothly on t and f.

What are the Tools from Functional Analysis?

Inverse function Theorem

Theorem

*Let $f : \mathscr{U} \subset B_1 \to B_2$ be a $\mathscr{C}^1$-map between **Banach** spaces. If $Df(a)$ is invertible at $a \in \mathscr{U}$, then there exists an open neighborhood $\mathscr{V}_a$ of $a \in \mathscr{U}$ and an open neighborhood $\mathscr{V}_{f(a)} \subset B_2$ such that $f : \mathscr{V}_a \to \mathscr{V}_{f(a)}$ is a $\mathscr{C}^1$-diffeomorphism.*

Counterexample : $\exp : \mathrm{Lie}(\mathrm{Diff}(\mathbb{S}^1)) \to \mathrm{Diff}(\mathbb{S}^1)$ not locally onto.

What are the Tools from Functional Analysis?

Inverse function Theorem

Theorem (Nash-Moser)

Let $f : \mathscr{U} \subset F_1 \to F_2$ be a smooth ***tame*** *map between* ***Fréchet*** *spaces. Suppose that the equation for the derivative $Df(x)(h) = k$ has a unique solution $h = L(x)k$ for all $x \in \mathscr{U}$ and $\forall k \in F_2$ and that the family of inverses $L : \mathscr{U} \times F_2 \to F_1$ is a smooth tame map. Then f is locally invertible and each local inverse is a smooth tame map.*

For the ideas of the proof, see the video
https://www.youtube.com/watch?v=2Jaw64CsoLc

What are the Tools from Functional Analysis?

Theorems :	Hilbert	Banach	Fréchet	Locally Convex
Banach-Picard	$\surd$	$\surd$	$\surd$	X
Open Mapping	$\surd$	$\surd$	$\surd$	F webbed G limit of Baire
Hahn-Banach	$\surd$	$\surd$	$\surd$	$\surd$
Cauchy Theorem	$\surd$	$\surd$	Hamilton	X
Inverse function	$\surd$	$\surd$	Nash-Moser	X

What are the Toys we can play with?

Riemannian $\subset$ Symplectic $\subset$ Poisson Geometry

Riemannian metric = smoothly varying inner product on a manifold M

$$g_x \; : \; \begin{array}{ccc} T_xM \times T_xM & \to & \mathbb{R} \\ (U, V) & \mapsto & g_x(U, V) \end{array}$$

strong Riemannian metric = for every $x \in M$, $g_x \; : T_xM \to (T_xM)^*$ is an isomorphism

weak Riemannian metric = for every $x \in M$, $g_x \; : T_xM \to (T_xM)^*$ is just injective

Levi-Cevita connection may not exist for a weak Riemannian metric.

What are the Toys we can play with?

Riemannian $\subset$ **Symplectic** $\subset$ Poisson Geometry

Symplectic form = smoothly varying skew-symmetric bilinear form

$$\begin{array}{cccc} \omega_x & : & T_xM \times T_xM & \rightarrow & \mathbb{R} \\ & & (U, V) & \mapsto & \omega_x(U, V) \end{array}$$

with $dw = 0$ and $(T_xM)^{\perp_w} = \{0\}$

strong symplectic form = for every $x \in M$, $\omega_x \; : T_xM \rightarrow (T_xM)^*$ is an isomorphism

weak symplectic form = for every $x \in M$, $\omega_x \; : T_xM \rightarrow (T_xM)^*$ is just injective

Darboux Theorem does not hold for a weak symplectic form

What are the Toys we can play with?

Riemannian $\subset$ **Symplectic** $\subset$ Poisson Geometry

Hamiltonian Mechanics

(M, g) **strong Riemannian manifold**

- $$\begin{array}{cccc} \flat : & T_xM & \simeq & T_x^*M \\ & U & \mapsto & g_x(U, \cdot) \end{array} \qquad \flat^{-1} = \sharp$$
- **Kinetic energy = Hamiltonian**
  $$\begin{array}{cccc} H : & T^*M & \to & \mathbb{R} \\ & \eta_x & \mapsto & g_x(\eta_x^\sharp, \eta_x^\sharp) \end{array}$$

(T^*M, ω) **strong symplectic manifold**

- $\pi : T^*M \to M$
- $\omega = d\theta$
- $$\begin{array}{cccc} \theta_{(x,\eta)} : & T_{x,\eta}T^*M & \to & \mathbb{R} \\ & X & \mapsto & \eta(\pi_*(X)) \end{array}$$ **Liouville 1-form**

geodesic flow = flow of Hamiltonian vector field $X_H : dH = \omega(X_H, \cdot)$

What are the Toys we can play with?

Riemannian $\subset$ **Symplectic** $\subset$ **Poisson Geometry**

Poisson bracket = family of bilinear maps
$\{\cdot,\cdot\}_U \ : \mathscr{C}^\infty(U) \times \mathscr{C}^\infty(U) \to \mathscr{C}^\infty(U)$, U open in M with

- skew-symmetry $\{f,g\}_U = -\{g,f\}_U$
- Jacobi identity $\{f,\{g,h\}_U\}_U + \{g,\{h,f\}_U\}_U + \{h,\{f,g\}_U\}_U = 0$
- Leibniz rule $\{f,gh\}_U = \{f,g\}_U h + g\{f,h\}_U$

A strong symplectic form defines a Poisson bracket by
$\{f,g\} = \omega(X_f, X_g)$ where $df = \omega(X_f,\cdot)$ and $dg = \omega(X_g,\cdot)$

A Poisson bracket may not be given by a bivector field

What are the Toys we can play with?

$$\left.\begin{array}{l}\text{Riemannian}\\ \text{Symplectic}\\ \text{Complex}\end{array}\right\} \subset \text{ Kähler} \subset \text{hyperkähler Geometry}$$

Complex structure = smoothly varying endomorphism J of the tangent space s.t. $J^2 = -1$.

Integrable complex structure : s. t. there exists an holomorphic atlas
Formally integrable complex structure : with Nijenhuis tensor = 0

Newlander-Nirenberg Theorem is not true in general :
formal integrability does not imply integrability.

What are the traps of infinite-dimensional geometry?

In infinite-dimensional geometry, the golden rule is :
"Never believe anything you have not proved yourself!"

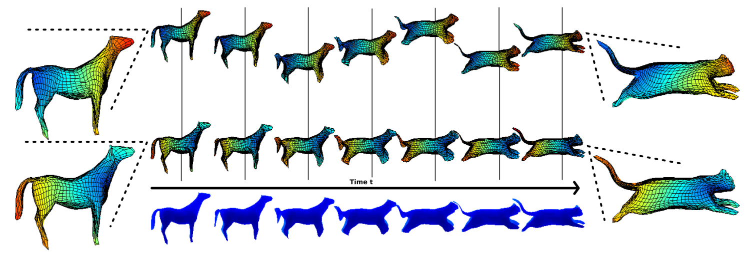
- The distance function associated to a Riemannian metric may by the zero function (Example by Michor and Mumford).
- Levi-Cevita connection may not exist for weak Riemannian metrics
- Hopf-Rinow Theorem does not hold in general : geodesic completeness $\neq$ metric completeness
- Darboux Theorem does not apply to weak symplectic forms
- A formally integrable complex structure does not imply the existence of a holomorphic atlas
- the tangent space differs from the space of derivations, even on a Hilbert space
- a Poisson bracket may not be given by a bivector field, even on a Hilbert space, (example by Beltita, Golinski and Tumpach)

Outline

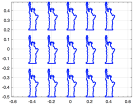
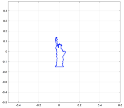
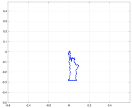
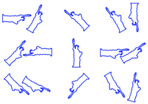
Part III : Shape Analysis

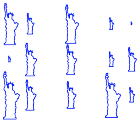
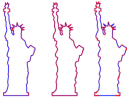
1. Shape spaces as **Quotient versus Sections** of fiber bundles
2. 3 different ways of putting a **intrinsic Riemannian metric** on Shape space
   1. Quotient metric
   2. Gauge invariant metric
   3. induced metric on a section

Shape spaces



Pre-shape space $\mathscr{F} := \{f \text{ embedding } : \mathbb{S}^2 \to \mathbb{R}^3\} \subset \mathscr{C}^\infty(\mathbb{S}^2, \mathbb{R}^3)$
Shape space $\mathscr{S} :=$ 2-dimensional submanifolds of $\mathbb{R}^3$

Group G	Some elements of one orbit under the group G	a preferred element in the orbit	another choice of preferred element in the orbit
$\mathbb{R}^3$ acting by translation		 centered curve : $\int_0^1 \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} \|f'(s)\| ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.	 curve starting at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
SO(3) acting by rotation		 axes of approximating ellipse aligned	 tangent vector at starting point horizontal

Group G	Some elements of one orbit under the group G	a preferred element in the orbit	another choice of preferred element in the orbit
$\mathbb{R}^+$ acting by scaling		 length = 1	 enclosed area =1
$\mathrm{Diff}^+([0,1])$ acting by reparameterization		 arc-length parameterization	 curvature proportional parameterization

Let $I = [0,1]$ and consider the set of plane curves of class $\mathscr{C}^k$ modulo translation, denoted $\mathscr{C}^k(I,\mathbb{R}^2)/\mathbb{R}^2$ endowed with the norm

$$\|\gamma\|_{\mathscr{C}^k} := \sum_{j=1}^{k} \max_{s\in I} |\gamma^{(j)}(s)|, \tag{1}$$

For $I = [0,1]$, the space of smooth immersions

$$\mathcal{C}(I) = \bigcap_{k=1}^{\infty} \mathcal{C}^k(I) = \{\gamma \in \mathscr{C}^\infty(I,\mathbb{R}^2)/\mathbb{R}^2, \gamma'(s) \neq 0, \forall s \in I\}.$$

is an open set of $\mathscr{C}^\infty(I,\mathbb{R}^2)/\mathbb{R}^2$ for the topology induced by the family of norms $\|\cdot\|_{\mathscr{C}^k}$, hence a Fréchet manifold.

$$\mathcal{C}_1(I) = \{\gamma \in \mathcal{C}(I) \ : \int_0^1 |\gamma'(s)| ds = 1\}.$$

$$\mathscr{A}_1(I) = \{\gamma \in \mathcal{C}(I) \ : |\gamma'(s)| = 1, \ \forall s \in I\} \subset \mathcal{C}_1(I).$$

Theorem (A.B.T, S.Preston)

$\mathscr{A}_1([0,1])$ *is diffeomorphic to the quotient Fréchet manifold* $\mathcal{C}_1([0,1])/\operatorname{Diff}^+([0,1])$.

1.Quotient metric

We will consider the 2-parameter family of elastic metrics on $\mathcal{C}_1(I)$ introduced by Mio et al. :

$$G^{a,b}(w,w) = \int_0^1 \left(a\,(D_s w\cdot \mathrm{v})^2 + b\,(D_s w\cdot \mathrm{n})^2\right)|\gamma'(t)|\,dt, \qquad (2)$$

where a and b are positive constants, γ is any parameterized curve in $\mathcal{C}_1(I)$, w is any element of the tangent space $T_\gamma\mathcal{C}_1(I)$, with $D_s w = \frac{w'}{|\gamma'|}$ denoting the arc-length derivative of w, $\mathrm{v} = \gamma'/|\gamma'|$ and $\mathrm{n} = \mathrm{v}^\perp$.

Since the reparameterization group preserves the elastic metric $G^{a,b}$, it defines a quotient elastic metric on the quotient space $\mathcal{C}_1([0,1])/\operatorname{Diff}^+([0,1])$, which we will denote by $\overline{G}^{a,b}$.

$$\overline{G}^{a,b}([w],[w]) = \inf_{u\in T_\gamma\mathcal{O}} G^{a,b}(w+u, w+u)$$

Pull-back of the quotient metric on arc-length parameterized curves

Since $\mathscr{A}_1([0,1])$ is diffeomorphic to the quotient Fréchet manifold $\mathcal{C}_1([0,1])/\operatorname{Diff}^+([0,1])$, we can pull-back the quotient elastic metric $\overline{G}^{a,b}$ to the space of arc-length parameterized curves $\mathscr{A}_1([0,1])$ and define

$$\widetilde{G}^{a,b}(w,w) = G^{a,b}([w],[w]) = \inf_{u \in T_\gamma \mathscr{O}} G^{a,b}(w+u, w+u)$$

where w is tangent to $\mathscr{A}_1([0,1])$.
This minimum is achieved by the unique vector $P_h(w) \in [w]$ belonging to the horizontal space $\mathrm{Hor}_\gamma := T_\gamma \mathscr{O}^\perp$ at γ. In other words:

$$\widetilde{G}^{a,b}(w,w) = G^{a,b}(P_h(w), P_h(w)), \tag{3}$$

where $P_h(w) \in T_\gamma \mathcal{C}_1([0,1])$ is the projection of w onto the horizontal space.

Theorem (A.B.T- S. Preston)

Let w *be a tangent vector to the manifold* $\mathscr{A}_1([0,1])$ *at* γ *and write* $w' = \Phi\,\mathrm{n}$, *where* Φ *is a real function in* $\mathscr{C}^\infty([0,1],\mathbb{R})$. *Then the projection* $P_h(w)$ *of* $w \in T_\gamma\mathscr{A}_1([0,1])$ *onto the horizontal space* Hor_γ *reads* $P_h(w) = w - m\,\mathrm{v}$ *where* $m \in \mathscr{C}^\infty([0,1],\mathbb{R})$ *is the unique solution of*

$$-\frac{a}{b}m'' + \kappa^2 m = \kappa\Phi, \qquad m(0) = 0, \quad m(1) = 0 \tag{4}$$

where κ *is the curvature function of* γ.

A.B.T., S. Preston, *Quotient elastic metrics on the manifold of arc-length parameterized plane curves*, Journal of Geometric Mechanics.

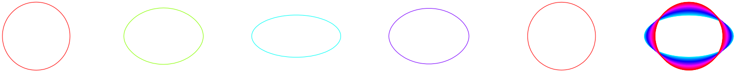


Figure: Toy example: initial path joining a circle to the same circle via an ellipse. The 5 first shapes at the left correspond to the path at time $t = 0$, $t = 0.25$, $t = 0.5$, $t = 0.75$ and $t = 1$. The right picture shows the entire path, with color varying from red ($t = 0$) to blue ($t = 0.5$) to red again ($t = 1$).

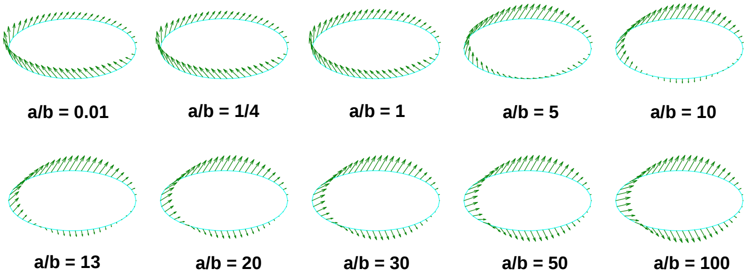


Figure: Gradient of the energy functional at the middle of the path depicted in Fig. **??** for $b = 1$ and different values of the parameter a/b.

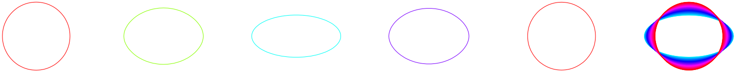


Figure: Toy example: initial path joining a circle to the same circle via an ellipse. The 5 first shapes at the left correspond to the path at time $t = 0$, $t = 0.25$, $t = 0.5$, $t = 0.75$ and $t = 1$.

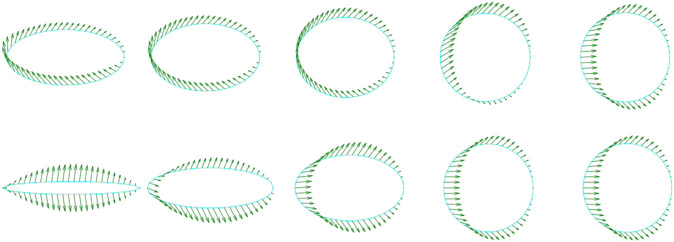


Figure: Gradient of the energy functional at the middle of the path connecting a circle to the same circle via an ellipse for different values of the eccentricity of the middle ellipse. The first line corresponds to the values of parameters $a = 0.01$ and $b = 1$. The second line corresponds to $a = 100$ and $b = 1$.

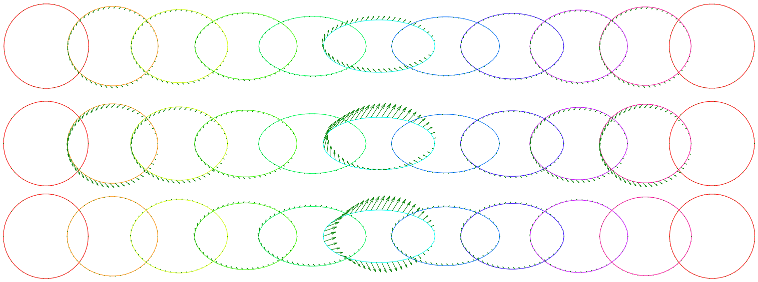


Figure: Gradient of the energy functional along the path depicted in Fig. ?? for $a = 1$ (upper line), $a = 5$ (middle line) and $a = 50$ (lower line) and $b = 1$.

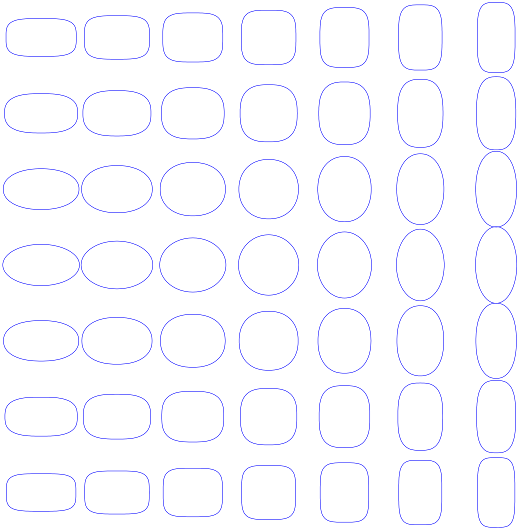


Figure: 2-parameter family of variations of the middle shape of a path connecting a circle to the same circle

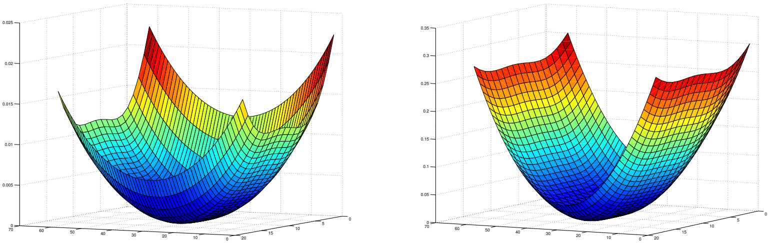


Figure: Energy functional for the 2-parameter family of paths whose middle shape is one of the shapes depicted in Fig. **??**. The left upper picture corresponds to $a = 0.01$, $b = 1$ and the right upper picture to $a = 100$, $b = 1$.

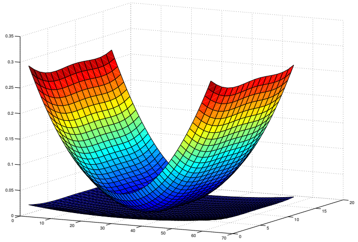
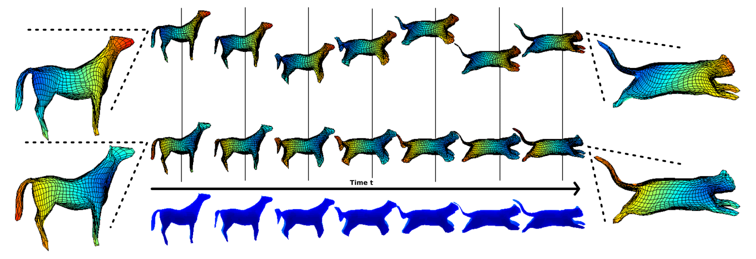
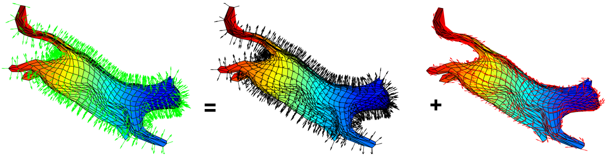
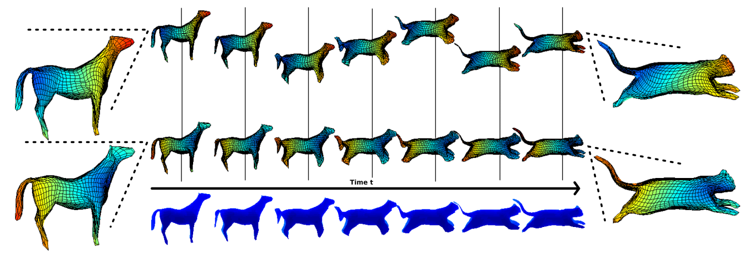


Figure: Energy functional for the 2-parameter family of paths whose middle shape is one of the shapes depicted in Fig. **??**.

2. Gauge invariant metric







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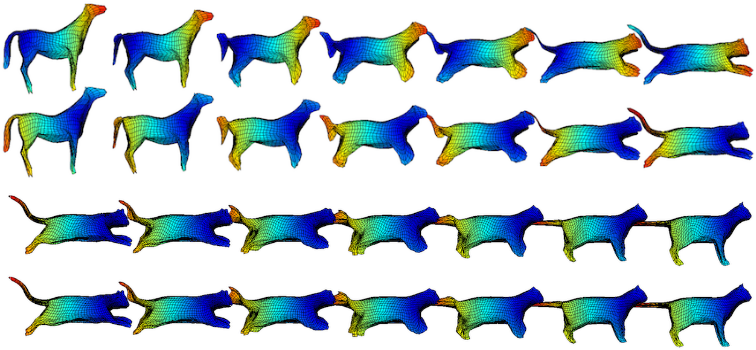
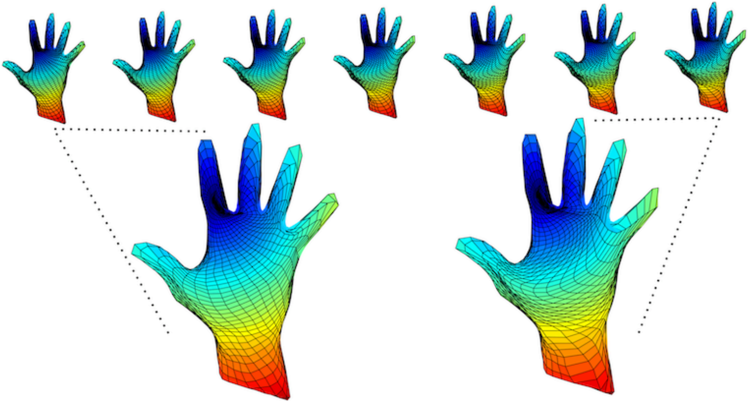


Figure: Pairs of paths projecting to the same path in Shape space, but with different parametrizations. The energies of these paths, as computed by our program, are respectively (from the upper row to the lower row): $E_\Delta = 225.3565$, $E_\Delta = 225.3216$, $E_\Delta = 180.8444$, $E_\Delta = 176.8673$.



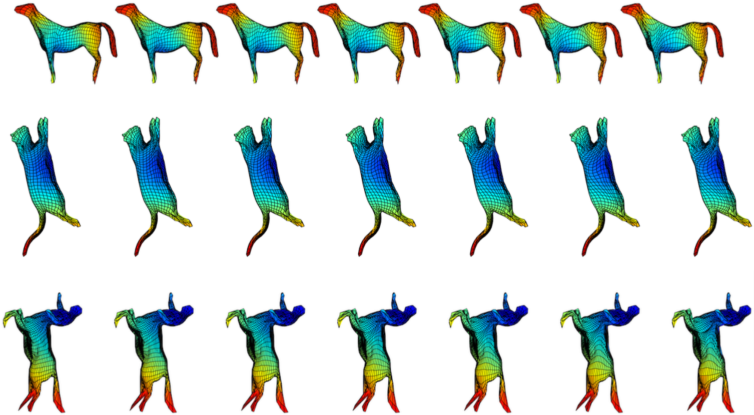


Figure: Four Paths connecting the same shape but with a parametrization depending smoothly on time. The energy computed by our program is respectively $E_\Delta = 0$ for the path of hands, $E_\Delta = 0.1113$ for the path of horses, $E_\Delta = 0$ for the path of cats, and $E_\Delta = 0.0014$ for the path of Centaurs.

3. Induced metric on a section

We will consider the 2-parameter family of elastic metrics on $\mathcal{C}_1(I)$ introduced by Mio et al. :

$$G^{a,b}(w,w) = \int_0^1 \left(a\,(D_s w \cdot \mathrm{v})^2 + b\,(D_s w \cdot \mathrm{n})^2\right) |\gamma'(t)|\, dt, \qquad (5)$$

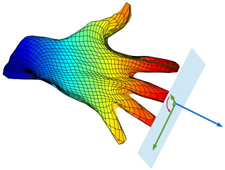
where a and b are positive constants, γ is any parameterized curve in $\mathcal{C}_1(I)$, w is any element of the tangent space $T_\gamma\mathcal{C}_1(I)$, with $D_s w = \frac{w'}{|\gamma'|}$ denoting the arc-length derivative of w, $\mathrm{v} = \gamma'/|\gamma'|$ and $\mathrm{n} = \mathrm{v}^\perp$.

Since the set of arc-length parameterized $\mathscr{A}_1([0,1])$ curves is a submanifold of $\mathcal{C}_1([0,1])$, it inherites a riemannian metric induced by metric (**??**), which we will denote by $G^{a,b}_{|\mathscr{A}_1}$:

$$G^{a,b}_{|\mathscr{A}_1}(w,w) = G^{a,b}(w,w)$$

where w is tangent to $\mathscr{A}_1([0,1])$.

Canonical parameterizations of surfaces?



Genus-0 surfaces of $\mathbb{R}^3$ are *Riemann surfaces*. Since they are compact and simply connected, the Uniformization Theorem says that they are conformally equivalent to the unit sphere. This means that, given a spherical surface, there exists a homeomorphism, called the *uniformization map*, which preserves the angles and transforms the unit sphere into the surface.

$\Rightarrow$ This gives a canonical parameterization of the surface modulo the choice of 3 points. (or unique modulo the action of $PSL(2, \mathbb{C})$).

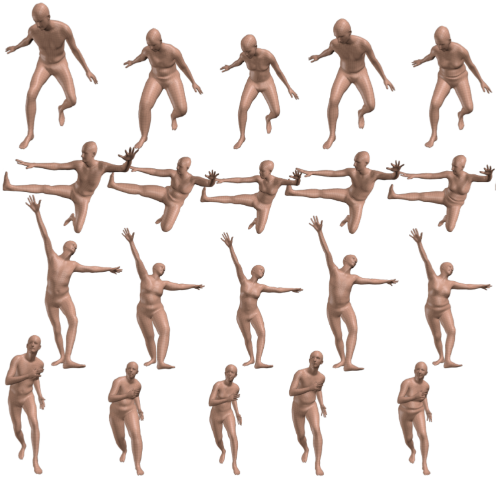


Figure 13: Animating SMPL. *Decomposition of SMPL parameters into pose and shape: Shape parameters, $\vec{\beta}$, vary across different subjects from left to right, while pose parameters, $\vec{\theta}$, vary from top to bottom for each subject.*

Figure: Representation of the manifold of Human bodies and poses, see SMPL: a skinned multi-person linear model

Final remarks: 3 different ways to put a metric on shape space

1. Quotient metric
   - in order to compute the geodesic distance between two shapes one needs to minimized over the group of diffeomorphisms
   - the horizontal space may be difficult to compute
   - a lot of quotient metric studied in the litterature
2. Gauge invariant metric
   - one may need to reparametrized during optimization
3. induced metric on a section
   - a section may be difficult to define
   - the tangent space to the section may be difficult to compute
   - the algorithm of minimization of energy has to preserve the section

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