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Mini-Workshop: Cohomology of Hopf Algebras and Tensor Categories

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Abstract. The mini-workshop featured some open questions about the cohomology of Hopf algebras and tensor categories. Questions included whether the cohomology ring of a finite dimensional Hopf algebra or a finite tensor category is finitely generated, questions about corresponding geometric methods in representation theory, and questions about noetherian Hopf algebras. The workshop brought together mathematicians currently working on these and other open problems.

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Introduction by the Organisers

The workshop brought together experts currently working on open problems about the cohomology of Hopf algebras and tensor categories and applications. The workshop also included experts from related areas. We describe some open questions that were of particular focus for the workshop: In the 1960s, Golod, Venkov, and Evens proved that the cohomology rings of finite groups are finitely generated. This landmark result led to the use of geometric methods in the representation theory of finite groups, via the support varieties introduced by Quillen, Carlson, Avrunin and Scott. This theory of varieties for modules continues to be a very active and fruitful area of research in the representation theory of finite groups. Since that time, finite generation has been proven for the cohomology rings of restricted enveloping algebras (Friedlander and Parshall), of small quantum groups and function algebras (Ginzburg and Kumar and Gordon), of finite-dimensional cocommutative Hopf algebras generally (Friedlander and Suslin), and of many more types of Hopf algebras in the past 20 years. These results and more point to the following question, asked by a number of mathematicians: Is the cohomology ring of a finite dimensional Hopf algebra always finitely generated? In fact, this question is stated as a conjecture in a 2004 paper of Etingof and Ostrik in a more general setting: They conjectured that the cohomology ring of a finite tensor category is always finitely generated. This conjecture remains open. Its importance is attested by the ubiquity of Hopf algebras and tensor categories in many areas of mathematics, including representation theory, combinatorics, topology, and statistical mechanics.

An obstacle to a general proof of finite generation is a lack of general knowledge of the structure of finite dimensional Hopf algebras, and accordingly in recent work, mathematicians focus on specific types of Hopf algebras for which helpful structure results are known. The mini-workshop greatly benefitted many of these mathematicians due to the time provided to meet and compare notes on their techniques and results, facilitating further collaborations. Those Hopf algebras for which cohomology is finitely generated enjoy the rich geometric methods afforded by support variety theory, and the workshop also featured some open questions about support varieties. We considered the same questions for finite tensor categories, where even less is known.

Homological and ring-theoretic properties of infinite dimensional noetherian Hopf algebras have been investigated extensively since Brown's lecture given in Seattle in the summer of 1997. One of the questions proposed in Brown's lecture was whether or not a noetherian Hopf algebra is always Gorenstein, or equivalently, has finite self-injective dimension. This question has been raised repeatedly by Brown, Goodearl and others in recent years, and it is now called the Brown-Goodearl Question. The same question can be asked for other classes of noetherian algebras that are similar to Hopf algebras, for example, for weak Hopf algebras, braided Hopf algebras, and Nichols algebras. In 2004 Andruskiewitsch independently asked the following related question: If a noetherian Nichols algebra is a domain, does it have finite global dimension? The Brown-Goodearl Question is still wide open though it has been verified for all noetherian Hopf algebras that are well-studied. It has been proven that the Brown-Goodearl Question is related to the existence of rigid dualizing complexes over noetherian Hopf algebras and therefore to the twisted Calabi-Yau property of these algebras. An affirmative answer to this question would have many other consequences, especially in the study of ring-theoretic properties of noetherian Hopf algebras. During this workshop several talks (including Brown's survey talk) discussed recent progress on the Brown-Goodearl Question, as well as several other related open problems.

Introductory talks. On the first day there were four introductory talks. Witherspoon gave a historical survey about the finite generation conjectures for the cohomology of finite-dimensional Hopf algebras and tensor categories. The talk of Andruskiewitsch explained the special role of Nichols algebras in the study of finite-dimensional Hopf algebras, including the classification of such algebras and the finite generation problem for cohomology. Pevtsova gave an introduction to support theory for Hopf algebras, explaining the general concepts as well as several examples. The survey of Brown featured homological properties of Hopf algebras which are either noetherian or finitely generated with finite Gelfand-Kirillov dimension.

Research talks. Most participants contributed talks about their recent research, largely devoted to the finite generation problem. Let us mention a few talks that were special. For instance, Touzé gave an expository talk on the cohomology algebras of reductive algebraic groups, with an emphasis on the open problems in this area. Negron's talk discussed some recent progress on the finite generation conjecture for finite tensor categories, explaining a transfer principle involving the Drinfeld double. Many results in this area are based on extensive calculations of examples. A talk of Solberg explained the use of a computer algebra package which he developed over the last years. This was complemented by a computer demonstration.

Finally, in the **evening problem session** on Thursday, many open problems were discussed.

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Mini-Workshop: Cohomology of Hopf Algebras and Tensor Categories

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Abstracts

Finite generation conjecture for finite dimensional Hopf algebras and finite tensor categories

Sarah Witherspoon

Around 1960, Golod [6], Venkov [8], and Evens [2] proved that the cohomology ring of any finite group is finitely generated. Wilkerson [9] noted in 1981 that one generalization of a group algebra is a cocommutative Hopf algebra and stated in *loc. cit.* that it was "the intent of this work to push this analogy as far as possible"; he proved finite generation for finite dimensional graded connected cocommutative Hopf algebras. Friedlander and Suslin [4] proved this for general finite dimensional cocommutative Hopf algebras in 1997 and stated that "we do not know whether it is reasonable to expect" that the cohomology ring of any finite dimensional Hopf algebra is finitely generated. In 2004, Etingof and Ostrik [1] conjectured that more generally, the cohomology ring of a finite tensor category is finitely generated and that the cohomology space of any object is finitely generated as a module over the cohomology ring. These cohomology rings are known to be graded commutative in general, leading to one application of finite generation, when it holds: a good theory of support varieties, that is geometric techniques in representation theory.

Other 20th century results include a proof by Friedlander and Parshall [3] in 1983 that the cohomology ring of a restricted Lie algebra in positive characteristic is finitely generated, preceding the more general result of Friedlander and Suslin [4] for all finite dimensional cocommutative Hopf algebras, and a proof by Ginzburg and Kumar [5] in 1993 that the cohomology ring of a small quantum group is finitely generated, the first noncocommutative examples. Many more results were published or announced in the 21st century, all proofs of finite generation of cohomology rings for various classes of finite dimensional Hopf algebras. The finite generation question remains open in general, one difficulty being that mathematicians do not know enough about the structure of finite dimensional Hopf algebras. A cautionary tale is provided by a counterexample of Xu [7, 10] to an analogous conjecture about Hochschild cohomology rings.

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The role of Nichols algebras in the structure of finite-dimensional Hopf algebras

Nicolás Andruskiewitsch
(joint work with Iván Angiono, Julia Pevtsova, Sarah Witherspoon)

1. Let H be a Hopf algebra with bijective antipode over a field $\Bbbk$, H_0 its coradical [13] and $H_{[0]} = \Bbbk\langle H_0\rangle$ its Hopf coradical, a Hopf subalgebra of H [5]. The classification of those H with finite Gelfand-Kirillov dimension can be organized in four classes:

(a) $H = H_0$, i. e. H is cosemisimple.
(b) $H = H_{[0]} \neq H_0$.
(c) $H \neq H_{[0]} = H_0$.
(d) $H \neq H_{[0]} \neq H_0$.

Nichols algebras are relevant to the study of classes (c) and (d). See [1] for an introduction to Nichols algebras and [3] for recent advances in class (d).

In [8] it was asked whether the cohomology ring $\mathrm{H}(H, \Bbbk)$ is finitely generated–for brevity, H has fgc–when $\dim H < \infty$. Inspired by work of several authors on this question, Etingof and Ostrik, conjectured that the analogous one is true in the setting of finite tensor categories. We discuss how fgc for Nichols algebras impacts in this question. We observe that if the Drinfeld double $D(H)$ of a finite-dimensional H has fgc, then so do the dual Hopf algebra H^* and any cocycle twist of either H or H^*, see [12, Theorem 3.4].

2. Let H be in class (c) i. e. H_0 is a proper Hopf subalgebra. Let $\operatorname{gr} H$ be the graded Hopf algebra associated to the coradical filtration of H; $\operatorname{gr} H \simeq R\# H_0$, where $R = \oplus_{n\in\mathbb{N}_0} R^n$ is a connected graded Hopf algebra in the braided monoidal category ${}^{H_0}_{H_0}\mathcal{YD}$. The assumption implies that R is coradically graded, hence its subalgebra generated by $V := R^1$ is isomorphic to the Nichols algebra $\mathscr{B}(V)$.

Assume from now on that $\Bbbk$ is algebraically closed, that $\operatorname{char} \Bbbk = 0$ and that $\dim H < \infty$. Then H_0 is semisimple by [13, 16.1.2]. By [15, 2.17], $\mathrm{H}(\operatorname{gr} H, \Bbbk) = \mathrm{H}(R, \Bbbk)^{H_0}$. Consequently, if R has fgc, then so does $\operatorname{gr} H$ by [14, Theorem 6.2 (iii)].

3. Assume next that H is pointed, i. e. $H_0 \simeq \Bbbk G$, and that $G := G(H)$ is abelian. Then $\mathscr{B}(V)$ is of diagonal type. By [6], respectively [7], we know that

(e) $R = \mathscr{B}(V)$, i. e. H is generated by group-like and skew-primitive elements.

(f) H is a cocycle deformation of $\operatorname{gr} H$, i. e. $H \simeq (\mathscr{B}(V)\#\Bbbk G)_{\sigma}$.

We announce the following results to appear in [4].

Theorem 1. *The cohomology ring of a finite-dimensional Nichols algebra $\mathscr{B}(V)$ of diagonal type is finitely generated.*

We elaborate and derive from Theorem 1 that the Drinfeld double $D(\mathscr{B}(V)\#\Bbbk G)$ has fgc. Together with previous remarks, we conclude:

Theorem 2. *Let H be a finite dimensional pointed Hopf algebra with G abelian. Then H and H^* have finitely generated cohomology.*

Theorem 2 generalizes the main results of [9, 11]. However the proof has some differences. Via the Anick resolution, we reduce the proof of Theorem 1 to verification in root systems of Nichols algebras of diagonal type that we perform case-by-case in the list of [10] using the information from [2].

If H_0 is an arbitrary semisimple Hopf algebra, then we expect that (e) and (f) still hold; at least this was verified in many examples with $H_0 = \Bbbk G$, G not necessarily abelian. In this way, the question of fgc for finite-dimensional Nichols algebras appears to be crucial.

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Supports: axiomatics and examples, a survey talk

Julia Pevtsova

The purpose of this extended abstract is to give a snapshot of what was covered in a one and a half lecture survey given at the beginning of the mini-workshop on "Cohomology of Hopf Algebras and Tensor Categories". My lectures focused on support theories for finite dimensional Hopf algebras with the key example being the group algebra of a finite group scheme.

We start with some terminology. A finite group scheme G defined over a field k is a representable functor:

$$G : \{\text{comm } k\text{-algebras}\} \to \{\text{groups}\}$$

such that the representing algebra $k[G]$ is finite dimensional as a vector space over k. For what follows, we assume that k has positive characteristic p. Dualizing the coordinate algebra, we get the *group algebra* kG which is a finite dimensional cocommutative Hopf algebra. This correspondence gives an equivalence of categories

$$\left\{ \begin{array}{c} \text{finite group} \\ \text{schemes} \end{array} \right\} \sim \left\{ \begin{array}{c} \text{finite dimensional co-} \\ \text{commutative Hopf algebras} \end{array} \right\} \tag{1}$$

Examples of these structures include finite groups, restricted Lie algebras and Frobenius kernels of algebraic groups. On the other hand, based on the equivalence of categories (1), representations of a finite group scheme G are equivalent to representations of its group algebra kG, and hence, give an example of a finite tensor category. Since any finite dimensional Hopf algebra is Frobenius, one can construct the stable module categories $\operatorname{Stmod} G$ and $\operatorname{stmod} G$ of all and finite dimensional representations of G which are tensor triangulated categories. Hence, on can study tensor triangular geometry in the sense of Balmer in this context.

Another corollary of kG being a finite dimensional Hopf algebra is that the cohomology algebra $H^*(kG, k)$ is graded commutative. The question of finite generation of the cohomology algebra $H^*(A, k)$ for any finite dimensional Hopf algebra A was the central question of the mini-workshop. In the case of $A = kG$ for a G a finite group scheme (or a finite supergroup scheme) the finite generation of cohomology is known thanks to a celebrated theorem of Friedlander and Suslin [4], extended by Drupieski to the super case [3]. These finite generation results opened the door to developing the support for representations of finite (super)group schemes, the theory pioneered by Alperin-Evens and Carlson, which followed the ground breaking work of Quillen on mod p group cohomology.

We proceed with a partial list of finite dimensional Hopf algebras (or algebraic objects whose categories of representations are equivalent to those of some finite dimensional Hopf algebras) for which finite generation of cohomology is known. The support theory has been developed for some of these examples, many open problems are still waiting to be solved for others.

(1) Finite groups
(2) Finite group schemes
(3) Small quantum groups
(4) Pointed Hopf algebras of diagonal type with abelian groups of group like elements (see N. Andruskiewitsch abstract in the same report for more details on this type of Hopf algebras)
(5) Finite supergroup schemes
(6) Finite dimensional Hopf subalgebras of the mod p Steenrod algebra.

In the first part of the survey I reviewed the general construction of the stable module category, the Balmer spectrum and the Benson-Iyengar-Krause local cohomology support theory. The abstract support datum as defined by Balmer is a function from the objects of a tensor triangulated category such as $\mathrm{stmod}\, G$ to an ambient topological space which satisfies a certain natural list of properties ([2]). The concrete Benson-Iyengar-Krause (BIK) theory constructs supports via the action of the graded center of a tensor triangulated category on the graded homs. In the case of kG or any finite-dimensional Hopf algebra A, this produces supports of kG-modules living in $\mathsf{Proj}\, H^*(kG, k)$. The BIK theory, though applicable very generally, lacks an important property required of Balmer's support data, known as the *tensor product property*: for two modules M, N, one would like to have the following formula:

$$\mathrm{supp}(M \otimes N) = \mathrm{supp}\, M \cap \mathrm{supp}\, N.$$

To obtain the tensor product property one needs to develop a different approach to supports - the one via *rank varieties* or *π-points*.

The second half of the survey went through the list of several examples (much more limited than the list of finite dimensional Hopf algebras in the first part) for which such a description is known and the full force support theory exists:

(1) Restricted Lie algebras
(2) Frobenius kernels of algebraic groups
(3) Elementary abelian p-groups
(4) Finite group schemes
(5) Quantum complete intersections (special cases)
(6) Complete intersections (via Avramov-Iyengar hypersurface approach [1])
(7) Finite unipotent supergroup schemes.

There was not much time to focus on the applications of support theories but it was briefly mentioned that they were directly relevant to the determination of the representation type of a Hopf algebra, computations of Balmer's spectrum, and classifications of localizaing and colocalizaing subcategories.

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Homological properties of noetherian Hopf algebras

Ken Brown

This talk was a survey of the key homological properties satisfied by, or conjectured to be satisfied by, noetherian Hopf algebras H. Relations between these properties were discussed, and their connection to various dimensions defined on H-modules, such as the Gel'fand-Kirillov dimension. For reasons of space this last aspect, and its connection to the (commutative) Cohen Macaulay property, is not discussed here.

The discussion has two starting points. On the one hand consider the heirarchy of three - by now, classical - properties which may be satisfied by an affine commutative k-algebra R: namely R might be *regular*, meaning it has finite global dimension, $\mathrm{gl.dim}R < \infty$; or R might be *Gorenstein*, meaning it has finite injective dimension, $\mathrm{inj.dim}R < \infty$; or R could be *Cohen Macaulay*. Here, the first implies the second implies the third; see for example [3] for details. When R is in addition a Hopf algebra, it is always Gorenstein, (with injective dimension equal to its Gel'fand-Kirillov dimension); moreover, R is regular if and only if it is semiprime; when k has characteristic 0 it is always semiprime, hence always regular. See [9] for details.

The second starting point is the 1969 theorem of Larson and Sweedler: a finite dimensional Hopf algebra H is Frobenius [4], (and consequently H has finite global dimension if and only if it is semisimple).

Motivated by these cases and a number of other examples, Brown and Goodearl proposed [1] in 1997 that every noetherian Hopf algebra might have finite injective dimesnion, a possibility which remains open and which has been refined as follows in succeeding years.

Definition 1. *(1) A noetherian Hopf algebra H is* Artin-Schelter Gorenstein (AS-Gorenstein) *if it has finite (left) injective dimension d, and* $\mathrm{Ext}^i_H(k,H) = \delta_{id}k$, *where k here denotes the trivial left H-module; and similarly on the right.*

(2) H is AS-regular *if it is AS-Gorenstein with* $\mathrm{gl.dim}H < \infty$. *(In this case,* $\mathrm{gl.dim}H = d$.*)*

Question 2. *Is every noetherian Hopf algebra AS-Gorenstein?*

At the present time, all known noetherian Hopf algebras are AS-Gorenstein - see [2, §6] for a summary of the state of play in this regard at 2008. The most striking general theorem so far obtained in support of a positive answer is:

Theorem 3. *(Wu, Zhang, 2003* [10]*) Every affine noetherian Hopf algebra satisfying a polynomial identity is AS-Gorenstein.*

A key outgrowth of work on Question 2 has been the generalisation of the *integral* from the finite dimensional case to AS-Gorenstein Hopf algebras by Lu, Wu and Zhang in 2007, as follows.

Definition 4. *Let H be an AS-Gorenstein noetherian Hopf algebra with* inj.dim$H = d$. *The* (left) integral $\int_H^\ell$ *is the one-dimensional H-bimodule* $\mathrm{Ext}_H^d(k, H)$.

Observe that, as a *left* module, $\int_H^\ell$ is simply a copy of the trivial H-module, whereas, on the *right*, the action may be through a non-trivial character χ of H, generalising the fact that a finite dimensional Hopf algebra is not in general unimodular. Brown and Zhang [2] in 2008 used the integral along with a seminal paper of Van den Bergh [8] to connect the AS-Gorenstein condition with the existence of a *balanced dualising complex* for H. They assumed that the antipode S of H was bijective, but this hypothesis was removed in [6], where it was shown that bijectivity of S is a consequence of the AS-Gorenstein condition. Summing up this circle of ideas, we state:

Theorem 5. *Let H be a noetherian Hopf algebra.*

(1) The following are equivalent:
  - *(a) H is AS-Gorenstein, with* inj.dim$H = d$*;*
  - *(b) H has a rigid dualising complex $V[s]$, V an invertible bimodule, $[s]$ denoting the shift, $s \in \mathbb{Z}$;*
  - *(c) H has a rigid dualising complex ${}^{\nu}H^1[d]$, for an algebra automorphism ν of H.*

(2) When the equivalent conditions (a)-(c) hold, S is bijective.

(3) Suppose that (a)-(c) hold, and in addition that gl.dim$H < \infty$. *Then* gl.dim$H = d$ *and H is twisted Calabi-Yau. In particular, H satisfies "twisted Poincaré duality": for all H-bimodules M and for all $i = 0, \ldots, d$,*

$$H^i(H, M) = H_{d-i}(H, {}^{\nu^{-1}}M).$$

The twisting automorphism ν occurring above was christened in [2] the *Nakayama automorphism* of H since it generalises the earlier incarnation for Frobenius algebras. It is determined up to an inner automorphism of H, equalling $S^2 \circ \tau_\chi^\ell$, where τ_χ^ℓ is the left winding automorphism determined by the character χ of the right structure on $\int_H^\ell$. Since the dualising complex depends only on the algebra structure of H, it can be calculated also for H^{cop}. Equating the two answers, one obtains a generalisation of the famous 1976 formula [7] of Radford for S^4 in the finite dimensional case:

Corollary 6. *Let H be an AS-Gorenstein Hopf algebra, and retain the above notation. Then there is an inner automorphism γ of H such that*

$$S^4 = \gamma \circ \tau^{\ell}_{\chi} \circ \tau^{r}_{-\chi}.$$

Question 7. *What is the inner automorphism γ in the corollary?*

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Finite generation of the cohomology of reductive groups

Antoine Touzé

Let G be a reductive group scheme over a field k (for example $G = GL_n$). Let A be a finitely generated commutative k-algebra, acted on by G by algebra automorphisms. Then it is proved in [6] that the cohomology $H^*(G, A)$ is a finitely generated k-algebra. This theorem is a natural generalization of:

(1) the results of Evens [1] and Friedlander and Suslin [2] on the cohomology of finite dimensional cocommutative Hopf algebras, and
(2) the results of classical invariant theory, which assert that the algebra of invariants $H^0(G, A) = A^G$ is finitely generated [4, 5, 3].

In this talk, we explain the connections between this theorem and the ongoing work on the cohomology of finite dimensional Hopf algebras. For example, we underline the role important role of the nontrivial coefficients. Indeed, while $H^*(GL_n, k)$ is a trivial k-algebra (i.e. equal to the field k in degree zero and zero in higher degrees), the cohomology of all finite group schemes can be realized as $H^*(GL_n, A)$ for $A = \mathrm{Coind}_G^{GL_n} k = k[GL_n]^G$.

Then we review the spectral sequence technique used to prove finite generation of cohomology, which goes back to Evens [1]. This technique relies on the computation of permanent cocycles. Such permanent cocycles have to be computed directly, i.e. without the specral sequence. Evens did produce these permanent cocycles for finite groups by using his norm map. In the talk, we explain how Friedlander and Suslin computed the permanent cocycles in the case of finite group schemes [2] from the computation of the cohomology groups $H^*(GL_n, \mathfrak{gl}_n^{(r)})$. The main theorem of [6] relies (among other things) on a more general cohomology computation, of a similar nature.

Finally, we mention some of the many open questions around the cohomology of reductive group schemes. In particular

(1) what is the geometric interpretation of the cohomology rings $H^*(G, A)$? (Geometric invariant theory tells us about the interpretation of $A^G = H^0(G, A)$, the variety corresponding to $H^*(G, A)$ is something bigger and quite mysterious.)
(2) Can we obtain more information on the cohomology rings $H^*(G, A)$? Explicit computations would be welcome, but classical invariant theory already tells us that one should not have too great expectations in this direction. One could also ask for which properties of A transfer to the cohomology.
(3) Finally, one could wonder the role of the cohomology algebras $H^*(G, A)$ for smooth reductive groups (such as $G = GL_n$) with respect to their representation theory.

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Cohomology and support varieties for finite supergroup schemes

Christopher M. Drupieski

(joint work with Jonathan R. Kujawa)

In 1997, Friedlander and Suslin (FS) [5] showed that the cohomology ring $H^*(G, k)$ of a finite group scheme G over a field k—equivalently, a finite-dimensional cocommutative Hopf algebra over k—is a finitely-generated k-algebra. Combined with subsequent work by Suslin, Friedlander, and Bendel (SFB) [6, 7], this opened the door to studying the representation theory of finite group schemes by way of the geometry of the corresponding affine scheme $|G| := Spec(H^*(G,k))$. There are, however, many interesting objects, such as certain finite-dimensional cocommutative $\mathbb{Z}$-graded Hopf algebras, that are 'morally' but not literally equivalent to finite group schemes, and hence to which the FS and SFB theories cannot be applied. More generally, one can consider finite *supergroup* schemes, that is, finite group scheme objects in the braided monoidal category of $\mathbb{Z}/2$-graded ('super') k-vector spaces. In this category, the usual twist map $V \otimes W \cong W \otimes V$ is replaced by a graded analogue in which $v \otimes w \mapsto -w \otimes v$ whenever v and w are both homogeneous of odd superdegree. In this talk we'll discuss recent progress in generalizing the work of FS and SFB to the context of finite supergroup schemes. The first main result is the following theorem of Drupieski [2]:

Theorem 1. *Let G be a finite supergroup scheme (equivalently, a finite-dimensional cocommutative Hopf superalgebra) over the field k, and let M be a finite-dimensional G-supermodule. Then $H^*(G,k)$ is finitely-generated as a k-algebra, and $H^*(G,M)$ is finitely-generated as a module over $H^*(G,k)$.*

In parallel to the work of FS, the theorem is proved by first reducing to $GL_{m|n(r)}$, the r-th infinitesimal Frobenius kernel of the general linear supergroup, and then by constructing certain universal extension classes coming from the category of *strict polynomial superfunctors*. These calculations demonstrate interesting new 'super' phenomena, showing for example that odd degree cohomology classes play a role that is invisible in the classical 'purely even' theory.

The next results, which are joint with Jonathan Kujawa [3], are a description for G *infinitesimal unipotent* of the spectrum $|G|$ and the cohomological support varieties $|G|_M \subseteq |G|$ associated to each finite-dimensional G-supermodule M. For each $r \geq 1$, we define a Hopf superalgebra $\mathbb{P}_r$, and a superalgebra homomorphism $\iota : \mathbb{P}_1 \hookrightarrow \mathbb{P}_r$. Let kG be the group algebra of G, a finite-dimensional cocommutative Hopf superalgebra, and let $V_r(G) := \mathrm{Hom}_{Hopf}(\mathbb{P}_r, kG)$ be the set of Hopf superalgebra homomorphisms $\nu : \mathbb{P}_r \to kG$. We then show:

Theorem 2. *Let G be an infinitesimal unipotent supergroup scheme of height $\leq r$, and let M be a finite-dimensional G-supermodule. Then $V_r(G)$ admits the structure of an affine scheme, and there exists a universal homeomorphism $|G| \simeq V_r(G)$. Under this identification, $|G|_M \simeq \{\nu \in V_r(G) : \mathrm{projdim}_{\mathbb{P}_1}(\iota^*\nu^*M) = \infty\}$.*

The proof of the theorem relies on two major inputs. The first is a generalization to supergroups of the characteristic extension classes constructed by SFB [4]. The second is a detection theorem of Benson, Iyengar, Krause, and Pevtsova [1].

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Cohomology of finite tensor categories: Duality and Drinfeld centers

Cris Negron
(joint work with Eric M. Friedlander, Julia Plavnik)

Let us call a finite tensor category $\mathcal{C}$ of *finite type* (over a given base field k) if the following conditions hold: The self-extensions of the unit $\mathrm{Ext}^*_{\mathcal{C}}(\mathbf{1},\mathbf{1})$ are a finitely generated algebra, and for any object X in $\mathcal{C}$ the extensions $\mathrm{Ext}^*_{\mathcal{C}}(\mathbf{1},X)$ are a finite module over the extensions of $\mathbf{1}$, via Yoneda composition. The question of whether of not all finite tensor categories are in fact of finite type has persisted in the literature at least since the 90's.

Here we study the behavior of the finite type property under certain now-standard operations on finite tensor categories. Namely, we consider duality $\mathcal{C} \rightsquigarrow \mathcal{C}^*_{\mathcal{M}}$ with respect to exact module categories $\mathcal{M}$ and the Drinfeld center construction $\mathcal{C} \rightsquigarrow Z(\mathcal{C})$.

In the case of the representation category $\mathrm{rep}(A)$ of a finite-dimensional Hopf algebra A, one recovers the representations of the linear dual $\mathrm{rep}(A^*)$, as well as arbitrary cocycle twists $\mathrm{rep}(A_\sigma)$, as duals $\mathrm{rep}(A)^*_{\mathcal{M}}$ with respect to specific choices of $\mathcal{M}$. The center construction for $\mathrm{rep}(A)$ recovers the Drinfeld double $Z(\mathrm{rep}(A)) \cong \mathrm{rep}(D(A))$, and can more generally be obtained via a duality $(\mathcal{C} \otimes \mathcal{C}^{cop})^*_{\mathcal{C}} \cong Z(\mathcal{C})$. Whence this categorical $\mathcal{M}$-relative duality is seen to be a generalization of many important constructions for Hopf algebras.

In joint work with J. Plavnik [2] we conjecture that the finite type property for finite tensor categories is preserved under duality. That is, for any finite type tensor category $\mathcal{C}$, the center $Z(\mathcal{C})$ should be of finite type as well, as should arbitrary duals $\mathcal{C}^*_{\mathcal{M}}$. We also conjecture that the Krull dimension of cohomology is preserved under duality. (From a certain perspective, this conjecture only makes sense in characteristic 0. However, we heedlessly proceed without deep consideration of this point.)

We show in [1, 2], with E. Friedlander and J. Plavnik, that for the representation categories $\mathrm{rep}(\mathbb{G}_{(r)})$ of Frobenius kernels in a smooth algebraic group $\mathbb{G}$, in finite characteristic, the Drinfeld center of $\mathrm{rep}(\mathbb{G}_{(r)})$ is in fact of finite type, and that all duals are of finite type as well. We additionally provide a uniform bound on the Krull dimensions of cohomology for arbitrary duals. (Recall that the $\mathrm{rep}(\mathbb{G}_{(r)})$ are themselves of finite type by Friedlander-Suslin.) We similarly show that Conjecture [2] holds for quantum groups, and generalized quantum groups à la

Andruskiewitsch-Schneider. This leads to new examples of finite type tensor categories, including those coming from cocycle twists of function algebras on (some) infinitesimal group schemes, and so-called dynamical quantum groups. Finally, we discuss our proof of Conjecture [2] for braided tensor categories in characteristic 0 with semisimple Müger center, i.e. with semisimple braiding degeneracy.

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Symmetry of the Mackey algebra

Baptiste Rognerud

Mackey functors were introduced as a convenient tool for handling the induction theory of several objects having a similar behavior (group representations; representation rings, group cohomology, etc;). Later, it was proved by Th??venaz and Webb that the category of Mackey functors is equivalent to the category of modules over a finite dimensional algebra called the *Mackey algebra*. The proof is far from being difficult, but this result is of crucial importance : one can study Mackey functors using ring and module theory. It turns out that the Mackey algebra is, in many aspects, similar to the group algebra but there are many interesting differences.

Let k be a field of characteristic p and G be a finite group. We denote by kG the group algebra of G, by $\mu_k(G)$ its Mackey algebra and by $co\mu_k(G)$ the *cohomological* Mackey algebra of G which is an interesting quotient of the Mackey algebra. Here we give some classical properties of these algebras.

Property	kG	$\mu_k(G)$	$co\mu_k(G)$
Dimension	$\|G\|$	independant of k	$\sum_{H,K\leq G} \|H\backslash G/K\|$
Semisimple	$p \nmid \|G\|$	$p \nmid \|G\|$	$p \nmid \|G\|$
Symmetric	Yes	$p^2 \nmid \|G\|$	$p \nmid \|G\|$
Finite Rep type	Sylow Cyclic	$p^2 \nmid \|G\|$	$p^2 \nmid \|G\|$
Gorenstein	Yes	$p^2 \nmid \|G\|$	Sylow cyclic or Dihedral

It is well-known that the group algebras of finite representation type are product of semisimple algberas and *Brauer trees* algebras. This is also true for the Mackey algebras but not for the cohomological Mackey algebras.

Another interesting property of the categories of Mackey functors is that they have a 'canonical' structure of closed symmetric monoidal category. By canonical, we mean that it extends the cartesian product of G-sets. Note that the Mackey algebras are not Hopf algebras in general. The monoidal structure can be constructed by Day convolution or by using an associative tri-module for the Mackey algebra.

Using this monoidal structure, I explained how to build central linear forms for the Mackey algebra. This leads to a characterization of the Mackey algebras that are *symmetric.*

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Cohomology of some local selfinjective algebras

Karin Erdmann

It is open at present whether the cohomology of finite-dimensional Hopf algebras which are not cocommutative, is finitely generated. The paper [3] classifies connected Hopf algebras of dimension p^3 over fields of characteristic p, and in [1] the structure of these algebras, and in particular finite generation of their cohomology is analysed. The algebras come in 24 families. For all but one family, it could be shown in [1] that the cohomology is finitely generated. The missing case is the family called $A5(\beta)$ for characteristic $p > 2$ and $\beta \in k$. We had noted without proof in [1] that its cohomology is finitely generated for $\beta \neq 0$.

This is now settled in [2], by a construction which applies to more general classes of local selfinjective algebras, they need not be Hopf algebras. The input consists of some partial information on the algebra, and with this it is shown that the cohomology is finitely generated. The construction applies in particular to the algebras $A5(\beta)$ for β non-zero.

We consider local selfinjective algebras of the form

$$A = k\langle x, y\rangle/I$$

where k is a field of arbitrary characteristic, and where x, y are independent generators of the radical of A. We impose three conditions on the ideal I. The first two are:

(1) There is a minimal relation $\psi_1 x + \psi_2 y = 0$ with ψ_1, ψ_2 independent modulo the square of the radical of A.

(2) There are minimal relations

$$\sigma_1 x + \sigma_2 y = 0, \quad \theta_1 x + \theta_2 y = 0$$

Moreover, let $\psi = (\psi_1, \psi_2)$, $\sigma = (\sigma_1, \sigma_2)$ and $\theta = (\theta_1, \theta_2)$. Then σ and θ are independent modulo $A\psi$.

The crucial observation is that σ, ψ and θ form a minimal generating set for the A-module $\Omega^2(k)$. Therefore the module $\Omega^2(A)$ is completely determined by specifying elements ρ_i for $1 \leq i \leq 4$ such that

$$x\sigma + \rho_1\psi = 0, \qquad y\sigma + \rho_2\psi = 0$$

$$x\theta + \rho_3\psi = 0, \qquad y\theta + \rho_4\psi = 0.$$

With this, in addition to (1) and (2) we require the following condition:
(3) We have

$$\sigma_1\rho_1 + \sigma_2\rho_2 = 0, \qquad \psi_1\rho_1 + \psi_2\rho_2 = 0,$$
$$\psi_1\rho_3 + \psi_2\rho_4 = 0, \qquad \theta_1\rho_3 + \theta_2\rho_4 = 0.$$

Moreover $\theta_1\rho_2 + \theta_2\rho_2$ is non-zero and lies in the socle of A, and

$$\theta_1\rho_1 + \theta_2\rho_2 + c(\sigma_1\rho_3 + \sigma_2\rho_4) = 0.$$

Assuming only (1) and (2) one can show that the elements listed in (3) must necessarily belong to the socle, and it is possible that (3) is redundant.

Example 1 Let $A := k\langle x, y\rangle/(x^n, y^m, xy - qyx)$ where $n, m \geq 2$ and where q is a non-zero element of k. This includes group algebras of 2-generated finite abelian p-groups, and also 2-generated quantum complete intersections. For these algebras it is easy to establish conditions (1) to (3), taking
$\sigma = (x^{n-1}, 0), \quad \psi = (-qy, x), \quad \theta = (0, y^{m-1})$.

Example 2 The algebra $A = A5(\beta)$ of [3] has presentation

$$A = k\langle z, y\rangle/\langle [y,z]^p, , y^p, [[y,z],y], [[y,z],z], \ z^p + [y,z]^{p-1}y - \beta[y,z]\rangle$$

for $\beta \in k$, we take $\beta \neq 0$. We set $a := yz - zy$, this is a central element and $a^p = 0$. One can see that $z^{p^2-1} \neq 0$ but $z^{p^2} = 0$. We establish conditions (1) to (3) (with z, y instead of x, y) taking $\sigma = (z^{p^2-1}, 0), \quad \psi = (z^{p-1} + \beta y, a^{p-1} - \beta z)$
and $\theta = (0, y^{p-1})$.

Using (1) to (3) we compute an explicit minimal projective resolution, and some cohomology products of elements of degree r with elements of degree 1 and with elements of degree 2, and obtain:

Theorem *The cohomology $H^*(A, k)$ is finitely generated, with generators in degrees ≤ 2.*

A complete presentation of the cohomology $H^*(A, k)$ would depend on the precise details of (1) to (3). Though one can see in the case of $A5$ directly that the cohomology is graded commutative.

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Extension of Brown's table to weak Hopf algebras

James J. Zhang
(joint work with Daniel Rogalski, Robert Won)

Brown-Goodearl Question asks if every noetherian Hopf algebra has finite self-injective dimension. The theme of my talk is about the Brown-Goodearl Question and related homological properties of noetherian weak Hopf algebras, which can be viewed as an extension of Ken Brown's survey talk on "Homological properties of noetherian Hopf algebra" on the first day.

Throughout let W be a weak Hopf algebra that is a finitely generated module over its center and assume that the center of W is an affine algebra over the base field. We show that the Brown-Goodearl question has a positive answer in this case.

Theorem 1: Let W be as above. Then W has finite self-injective dimension. Further, as an algebra, W is a finite direct sum of Artin-Schelter Gorenstein and Cohen-Macaulay algebras.

We also prove a few other results.

Theorem 2: Let W be as above. Then W has a quasi-Frobenius artinian quotient ring.

Let GKdim denote the Gelfand-Kirillov dimension. We say an algebra W is homogeneous if $\text{GKdim}\, L = \text{GKdim}\, W$ for all nonzero left ideals $L \subseteq W$.

Theorem 3: Let W be as above. If W has finite global dimension, then W is a direct sum of prime algebras and each summand is a homogeneous, Artin-Schelter regular, Auslander regular and Cohen-Macaulay algebra.

The following is a version of Nichols-Zoeller Theorem for infinite dimensional weak Hopf algebras.

Theorem 4: Let W_1 and W_2 be two noetherian weak Hopf algebras satisfying the general hypothesis as in the previous theorems, and homogeneous of the same Gelfand-Kirillov dimension. Suppose that W_2 has finite global dimension. If there is an algebra map $f : W_1 \to W_2$ such that W_2 is a finitely generated module over W_1 on both sides, then W_2 is a projective module over W_1 on both sides.

This talk is based on joint work with Daniel Rogalski and Robert Won.

On the existence of permanent cocycles via twisted tensor product and Anick resolutions

Van C. Nguyen
(joint work with Xingting Wang and Sarah Witherspoon)

The cohomology ring of a finite dimensional Hopf algebra is conjectured to be finitely generated. Friedlander and Suslin proved this for cocommutative Hopf algebras, generalizing earlier results of Evens, Golod, and Venkov for finite group algebras and of Friedlander and Parshall for restricted Lie algebras. There are many finite generation results as well for various types of noncocommutative Hopf algebras. Most of these results are in characteristic 0.

In this talk, we prove finite generation of cohomology for classes of noncocommutative Hopf algebras over a field of prime characteristic $p > 2$. These algebras arise in the classification of finite dimensional pointed Hopf algebras in positive characteristic given by Nguyen and Wang [3]. They include bosonizations of Nichols algebras of Jordan type in a general setting. Our proofs are based on an algebra filtration and a lemma of Friedlander and Suslin, using both twisted tensor product resolutions (introduced by Shepler and Witherspoon [4]), and Anick resolutions (introduced by Anick [1, 2]). We describe the constructions of these resolutions and use them to explicitly locate the needed permanent cocycles in the May spectral sequences associated to these algebras. As a result, we obtain:

Main Result. Let k be an algebraically closed field of prime characteristic $p > 2$. Consider the following Hopf algebras over k:

(1) The p^2q-dim bosonization $R\#kC_q$ of a rank two Nichols algebra R of Jordan type over a cyclic group $C_q = \langle g \rangle$, where q is divisible by p. As Hopf algebras, $R\#kC_q$ is isomorphic to $k\langle w, x, y\rangle$, where $w = g-1$, subject to relations

$$w^q,\ x^p,\ y^p,\ yx - xy - \frac{1}{2}x^2,\ xw - wx,\ yw - wy - wx - x.$$

(2) The 27-dim liftings $H := H(\epsilon, \mu, \tau)$ of $R\#kC_q$ when $p = q = 3$, which is isomorphic to $k\langle w, x, y\rangle$ subject to relations

$$w^3 = 0,\ x^3 = \epsilon x,\ y^3 = -\epsilon y^2 - (\mu\epsilon - \tau - \mu^2)y,$$
$$yw - wy = wx + x - (\mu - \epsilon)(w^2 + w),\ xw - wx = \epsilon(w^2 + w),$$
$$yx - xy = -x^2 + (\mu + \epsilon)x + \epsilon y - \tau(w^2 - w),$$

with $\epsilon \in \{0, 1\}$ and $\tau, \mu \in k$.

Then the cohomology rings of $R\#kC_q$ and of H are finitely generated.

Our main result is exclusive for odd characteristic since the Nichols algebra of Jordan type does not appear in characteristic 2. Part (2) of our main result above is only stated for characteristic 3; this is because we use the classification of such Hopf algebras from [3]. There, a complete classification is given only in case $p = 3$ of the Hopf algebra liftings H of $R\#kC_q$, that is the Hopf algebras whose associated graded algebra with respect to the coradical filtration is $R\#kC_q$. We expect our homological techniques will be able to handle liftings in case $p > 3$ once more is known about their structure.

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Connected Hopf algebras and related homological properties

Xingting Wang

Connected (or called irreducible by Sweedler) Hopf algebras are generalizations of universal enveloping algebras of finite dimensional Lie algebras in Lie theory and coordinate rings of unipotent group schemes in affine algebraic groups. Recently, there are many new developments in understanding these non-commutative and non-cocommutative Hopf algebras both in zero and positive characteristic.

First of all, in characteristic zero, connected Hopf algebras of finite Gelfand-Kirillov dimension (GK-dimension) are all deformations of polynomial algebras and hence they possess excellent homological and ring-theoretic properties. This phenomena was first observed by Zhuang in his paper [14] and further he proved the following result: Let H be a connected Hopf algebra of finite GK-dimension over an algebraically closed field of characteristic zero. Then we have

- H is a noetherian domain.
- H is Auslander-regular and Cohen-Macaulay.

One of the famous conjectures about noetherian Hopf algebras is the Brown-Goodearl Conjecture (BGC) [1, Question E] which states that every noetherian Hopf algebra is Artin-Schelter Gorenstein (AS-Gorenstein). As a consequence of Zhuang's result, every connected Hopf algebra of finite GK-dimension in characteristic zero belongs to this nice family of algebras in the sense of Artin-Schelter regularity. Moreover, their structures are interesting to people working in non-commutative invariant theory, non-commutative algebraic geometry, deformation theory, representation theory, and etc. Meanwhile, at least for low GK-dimension, the classification program about possible algebraic structures of connected Hopf algebras over an algebraically closed field of characteristic zero took place simultaneously. The most current status of this classification program can be summarized as follows.

- GK dimension 0: No such Hopf algebras.

- GK dimension 1: Polynomial algebra of one variable.
- GK dimension 2: Universal enveloping algebras of 2-dimensional Lie algebras by Goodearl-Zhang [6].
- GK dimension 3: Universal enveloping algebras of 3-dimensional Lie algebras by Zhuang [14].
- GK dimension 4: Universal enveloping algebras of 4-dimensional Lie algebras by Wang-Zhuang-Zhang [10].
- GK dimension 5: There exists a connected Hopf algebra of GK-dimension 5 that is NOT isomorphic to, as algebras, any universal enveloping algebra by Brown-Gilmartin-Zhang [2].

In the other hand, connected Hopf algebras appearing in positive characteristic $p > 0$ have rich algebra structures even in GK-dimension 0 or finite dimensional case. Wang in [13] showed that these finite dimensional connected Hopf algebras are all deformations of restricted polynomial algebras (truncated polynomials where every variable's pth power vanishes). The classification program about their possible algebra structures in dimension p^n for $n \leq 3$ has been recently accomplished by Nguyen-Wang-Wang in a series of papers [7, 8, 9, 11, 12, 13]. Moreover, Erdmann, Ø. Solberg and X. Wang in [3] showed the representation types of these low-dimensional connected Hopf algebras can only be

- semisimple algebras;
- finite group algebras;
- selfinjective Nakayama algebras;
- restricted enveloping algebras of finite dimensional restricted Lie algebras;
- covering of local algebras;
- other types of local algebras.

In particular, these finite dimensional connected Hopf algebras are Frobenius algebras whose cohomology rings play an important role in their support variety theory. It is conjectured by Etingof and V. Ostrik [4] these cohomology rings are always finitely generated, which is an essential assumption in support variety theory (see for instance [5]). As a result of the previous discussion, the conjecture holds for all low-dimensional connected Hopf algebras.

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Cyclic A_∞-algebras and cyclic homology

Estanislao Herscovich

In [3], the authors gave a new description of Hochschild and cyclic homology in characteristic zero, based on noncommutative (formal) geometry. In particular, they showed that the complex computing the cyclic cohomology of a homologically unitary A_∞-algebra A is quasi-isomorphic to a(n even shift of the) complex $\Omega^2_{\mathrm{cyc,cl}}(A[1])$ of closed cyclic 2-forms. By combining this with a formal version of Darboux's theorem, they showed that a closed cyclic 2-form induces an isomorphism class of symplectic structures on the minimal model $H^\bullet(A)$ of A, if $H^\bullet(A)$ is finite dimensional. On the other hand, C.-H. Cho noted in [1] that a constant closed cyclic 2-form on a finite dimensional A_∞-algebra A is the same as a strict isomorphism of A_∞-bimodules between A and its dual $A^\#$. He also found an equivalent description for the existence of a symplectic structure on a minimal model (see [1], Thm. 4.1, but also [2], Thm. 3.6), and Cho and S. Lee found an explicit description of the quasi-isomorphism of A_∞-bimodules between $H^\bullet(A)$ and its dual $H^\bullet(A)^\#$ stated in [3], that they called *strong homotopy inner product* (see [2]). Their proof is however somehow *ad hoc* as well as computationally highly involved, and the proof of several steps are omitted.

Our goal is to show that the mentioned results in [1] and [2] can be directly deduced from a new description of the complex computing the Hochschild homology of A, which is closer to the complex $\Omega^2_{\mathrm{cyc,cl}}(A[1])$ in [3]. In order to express our results more clearly, we consider the dual noncommutative Cartan calculus of A: in this case, the complex computing the cyclic homology is a(n even shift of the) complex $(\mho^2_{\mathrm{cyc}}(A[1])/\operatorname{Im}(d_{\mathrm{DR}^3})$, dual to $\Omega^2_{\mathrm{cyc}}(A[1])$.

Recall that, given a dg bicomodule N over a dg coalgebra C, with left and right coactions $\rho_l : N \to C \otimes N$ and $\rho_r : N \to N \otimes C$, one defines $N^\natural = \operatorname{Ker}(\rho_l - \tau_{N,C} \circ \rho_r)$. Given an A_∞-algebra A and two A_∞-bimodules M and N, we introduce the *tensor product* $M \otimes^\infty_{A^e} N$ of M and N as the dg vector space

$$\Big(B^u(A,M,A) \,\square^{B(A)^+}\, B^u(A,N,A)\Big)^\natural,$$

where $B^u(A, M, A)$ is the bar construction of M and $B(A)^+$ is the coaugmentation of the bar construction of A. If A is H-unitary, then $A \otimes^{\infty}_{A^e} A$ computes the Hochschild homology $HH_\bullet(A)$ of A. We are interested in the previous complex for the following reason.

Proposition 1. *If A is an H-unitary A_∞-algebra, there is a canonical isomorphism between $H^0((A \otimes^{\infty}_{A^e} A)^{\#})$ and the space of morphisms $\mathrm{Hom}_{\mathcal{D}_\infty(A^e)}(A, A^{\#})$.*

Moreover, we also have the following result.

Proposition 2. *If A is an H-unitary A_∞-algebra, there is a morphism $\overline{\mathrm{sym}}$: $A \otimes^{\infty}_{A^e} A \to (\mho^2_{\mathrm{cyc}}(A[1])/\,\mathrm{Im}(d_{\mathrm{DR}^3})$ of complexes inducing the map $I : HH_\bullet(A) \to HC_\bullet(A)$ from the SBI sequence.*

We obtain as a consequence Thm. 4.1 in [1] (see also [2], Thm. 3.6).

Theorem 3. *Let A be an H-unitary A_∞-algebra having finite dimensional cohomology such that $A^{\#}$ is an H-unitary A_∞-bimodule. Let ${}^uF_{\bullet,\bullet} : A \to A^{\#}[d]$ be a morphism of A_∞-bimodules. Then, ${}^uF_{\bullet,\bullet}$ is a strong homotopy inner product if and only if there is a quasi-isomorphism of A_∞-algebras $G : B \to A$ with B is finite dimensional and ${}^u\tilde{G}^{\#}_{\bullet,\bullet}[d] \circ {}^uF_{\bullet,\bullet} \circ {}^u\tilde{G}_{\bullet,\bullet}$ a d-cyclic structure on B, where ${}^u\tilde{G}_{\bullet,\bullet}$ is the associated morphism of A_∞-bimodules over B.*

As an application of this result we obtain that a strongly smooth pseudo-compact local augmented dg algebra satisfies the (resp., almost exact) d-Calabi-Yau property if and only if its Koszul dual has a (resp., strong) homotopy inner product, extending Thm. 11.1 in [4], but with a completely different proof.

References

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Analyzing Hopf algebras using QPA

Øyvind Solberg

The aim of the talk/the computer demonstration was to show how one can use the deposited GAP-package QPA (`https://github.com/gap-packages/qpa`) to analyze the structure of finite dimensional Hopf algebras in prime characteristic and their cohomology ring. In particular we analyzed the example (A5) from [1] for both $\beta \neq 0$ and $\beta = 0$ with $p = 3$, the algebras (A) and (B) from [2, Theorem

5.1.2] and [2, Theorem 5.2.1], respectively, for $p = 3$ and the latter with $\epsilon = \mu = 1$ and $\tau = 0$, and the super Jordan plane (SJP) given by

$$\mathbb{F}_p \left(x_1 \circlearrowleft 1 \circlearrowright x_2 \right) / \langle x_1^2, [x_1, x_2] - x_{21}x_1, x_{21}^p, x_2^{2p} \rangle$$

with $[x, y] = [xy - yx]$ and $x_{21} = x_1x_2 + x_2x_1$ and $p = 3$. The computer calculations showed that the algebra (A5) for $\beta \neq 0$ and $\beta = 0$ had different algebra structure and cohomology structure, the algebra (A) has $1, 2, 5$ generators in the degrees $0, 1, 2$ and none in the degrees $3, 4$, the algebra (B) is decomposable, and the cohomology algebra of (SJP) has $1, 2, 3$ generators in degrees $0, 1, 2$ and none in the degrees $3, 4, 5$.

References

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Completing the representation ring of a finite dimensional Hopf algebra, I

David Benson

This talk was given both at this mini workshop and at the workshop ID 1913 “Representations of finite groups” three weeks later. It breaks down naturally into two parts, so I have decided to report on the first part here and the second for that meeting.

Let H be a finite dimensional Hopf algebra over a field k; for example, the group algebra of a finite group, or a finite group scheme. We only consider finitely generated H-modules. These are usually of wild representation type, so we shouldn't contemplate classifying the indecomposables. The subject of this talk is the asymptotics of the decomposition of tensor products and tensor powers of modules. We obtain a new invariant of modules reflecting this information, and interpret it as a spectral radius in a suitable Banach algebra. Part of this is joint work with Peter Symonds, and the rest grew out of that work.

The *representation ring* $a(H)$ has generators $[M]$ with M an H-module, and relations $[M] + [N] = [M \oplus N]$ and $[M][N] = [M \otimes_k N]$. Additively, it is the free abelian group on the isomorphism classes $[M]$ of indecomposable H-modules M. This ring encodes information about summands of tensor products. We'll assume that we have isomorphisms $M^{**} \cong M$ and $M \otimes N \cong N \otimes M$, although much can be done without these assumptions.

We regard the projective H-modules as well understood. So we define the *core* of M to be the non-projective part. In other words, writing $M = M' \oplus P$ where P is projective and M' has no non-zero projective summands, $\mathsf{core}(M) = M'$. The question we wish to address is,

how does the dimension of $\mathsf{core}(M^{\otimes n})$ grow with n?

To address this, we form a generating function

$$f_M(t) = \sum_{n=0}^{\infty} t^n \mathsf{c}_n(M), \qquad \mathsf{c}_n(M) = \dim \mathsf{core}(M^{\otimes n}).$$

The radius of convergence of this power series is given by

$$1/r = \limsup_{n\to\infty} \sqrt[n]{\mathsf{c}_n(M)}.$$

We write $\gamma(M) = 1/r$, and regard this as an invariant of the module M.

As an example, let $G = \langle g \rangle \cong \mathbb{Z}/5$, k a field of characteristic five, and M the two dimensional module $g \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. We have

$$\dim \mathsf{core}(M^{\otimes 2n}) \sim \tau^{2n+1}, \qquad \dim \mathsf{core}(M^{\otimes 2n+1}) \sim 2\tau^{2n+1},$$

where $\tau = (1+\sqrt{5})/2 = 2\cos(\pi/5)$ is the golden ratio. In this example, we have $\gamma(M) = \tau$. Asymptotically, most of $M^{\otimes n}$ is projective, but the non-projective part still grows exponentially, like τ^n.

Some properties of the invariant $\gamma(M)$ are as follows.
(1) $\gamma(M) = \lim_{n\to\infty} \sqrt[n]{\mathsf{c}_n(M)} = \inf_{n\to\infty} \sqrt[n]{\mathsf{c}_n(M)}$.
(2) $0 \le \gamma(M) \le \dim M$.
(3) Some $M^{\otimes n}$ has a projective summand $\iff \gamma(M) < \dim M$.
(4) M is projective if and only if $\gamma(M) = 0$. Otherwise $\gamma(M) \ge 1$.
(5) If $1 \le \gamma(M) < \sqrt{2}$ then M is endotrivial, i.e., $M \otimes M^* \cong k \oplus$ (projective). In the case of kG-modules, G a finite group, this implies $\gamma(M) = 1$.
(6) If M is not endotrivial, $\gamma(M) = \sqrt{2} \Rightarrow M \otimes M^* \otimes M \cong M \oplus M \oplus$ (projective).
(7) $\gamma(M) = \gamma(M^*)$.
(8) If $0 \to M_1 \to M_2 \to M_3 \to 0$, each $\gamma(M_i)$ is at most the sum of the other two.
(9) $\max\{\gamma(M), \gamma(N)\} \le \gamma(M \oplus N) \le \gamma(M) + \gamma(N)$.
(10) $\gamma(m.M) = m.\gamma(M)$.
(11) $\gamma(k \oplus M) = 1 + \gamma(M)$.
(12) $\gamma(M \otimes N) \le \gamma(M).\gamma(N)$.
(13) $\gamma(M^{\otimes m}) = \gamma(M)^m$.
(14) In the case of kG-modules, G a finite group, we have $\gamma_G(M) = \max_{E \le G} \gamma_E(M)$, where the maximum is taken over the elementary abelian p-subgroups of G, $p = \mathrm{char}(k)$.

To clarify the input from representation theory, we formulate axioms for an abstract representation ring. We define a *representation ring* $\mathfrak{a}$ to be a commutative ring whose additive group is free abelian on a basis x_i, i in an index set $\mathfrak{I}$. The identity $\mathbb{1} = x_0$ is a basis element, and

$$x_i x_j = \sum_{k \in \mathfrak{I}} [x_i x_j : x_k] x_k$$

with structure constants $[x_i x_j : x_k] \ge 0$. The axioms are as follows.
(1) There is an involution $i \mapsto i^*$ of $\mathfrak{I}$ inducing an involutive automorphism of $\mathfrak{a}$ sending $x = \sum_i a_i x_i$ to $x^* = \sum_i a_i x_{i^*}$.
(2) If $[x_i x_j : \mathbb{1}] > 0$ then $j = i^*$.

(3) If $[x_i x_{i^*} : \mathbb{1}] = 0$ then $[x_i x_{i^*} x_i : x_i] \geq 2$.
(4) There is a dimension function $\dim \colon \mathfrak{a} \to \mathbb{Z}$ with $\dim(x_i) = \dim(x_{i^*}) > 0$.
(5) There is an element $\rho \in \mathfrak{a}_{\succcurlyeq 0}$ such that $\forall x \in \mathfrak{a}$, $x\rho = \dim(x).\rho$.

A *representation ideal* is a proper subset $\mathfrak{X} \subset \mathfrak{I}$ such that
(1) if $i \in \mathfrak{X}$ and $[x_i x_j : x_k] > 0$ then $x_k \in \mathfrak{X}$; and
(2) if $i \in \mathfrak{X}$ then $i^* \in \mathfrak{X}$.

For example, the unique maximal representation ideal $\mathfrak{X}_{\max}$ consists of those $i \in \mathfrak{I}$ such that $[x_i x_{i^*} : \mathbb{1}] = 0$.

If $[\rho : \mathbb{1}] > 0$ then $\mathfrak{a}$ has finite rank and is semisimple; this is called *ordinary representation theory*. In this case, the only representation ideal is $\varnothing$. Otherwise, it is called *modular representation theory*, and the unique minimal non-zero representation ideal is $\mathfrak{X}_{\mathsf{proj}}$, the set of $i \in \mathfrak{I}$ such that $[\rho : x_i] > 0$. There are many other examples of representation ideals, such as those given by support varieties.

Let $\mathfrak{X}$ be a representation ideal in a representation ring $\mathfrak{a}$. If $x = \sum_{i \in \mathfrak{I}} a_i x_i \in \mathfrak{a}_{\succcurlyeq 0}$, we set $\mathsf{core}_{\mathfrak{X}}(x) = \sum_{i \in \mathfrak{I} \setminus \mathfrak{X}} a_i x_i$. Then as before, for $x \in \mathfrak{a}_{\succcurlyeq 0}$ we set

$$\mathsf{c}_n^{\mathfrak{X}}(x) = \dim \mathsf{core}_{\mathfrak{X}}(x^n), \qquad \gamma_{\mathfrak{X}}(x) = \limsup_{n \to \infty} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)}.$$

There is a list of properties of this invariant $\gamma_{\mathfrak{X}}(x)$ just as before.

In the second part of this talk, described separately, we put a Banach space structure on $\mathfrak{a}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{a}$ by setting $\| \sum_{i \in \mathfrak{I}} a_i x_i \| = \sum_{i \in \mathfrak{I}} |a_i| \dim x_i$. The completion of $\mathfrak{a}_{\mathbb{C}}$ with respect to this norm is a commutative Banach $*$-algebra $\hat{\mathfrak{a}}$. If $\mathfrak{X}$ is a representation ideal in $\mathfrak{a}$ then we write $\hat{\mathfrak{a}}_{\mathfrak{X}}$ for the quotient of $\hat{\mathfrak{a}}$ by the closure of the ideal generated by $\mathfrak{X}$, and we interpret $\gamma_{\mathfrak{X}}(x)$ as the spectral radius of the image of x in $\hat{\mathfrak{a}}_{\mathfrak{X}}$.

Reporter: Baptiste Rognerud