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Computations and applications of some homological constants for polynomial representations of GL_n

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Abstract. In this paper, we review the applications of the homological constants for polynomial representations of GL_n defined in [*Connectedness of cup products for polynomial representations of GL_n*, Annals of K-theory, to appear]. We also give new applications of these constants, in particular to the cohomology of classical groups. We make further progress on the problem of computing these constants for polynomial modules of interest.

1. Introduction

This article deals with polynomial representations of the general linear group over a field $\Bbbk$ of positive characteristic p, and with their restrictions to the classical matrix subgroups (i.e. symplectic and orthogonal groups). We let $\mathrm{Pol}_{n,d}$ be the category of homogeneous polynomial representations of degree d of GL_n, see Green's book [**8**] or [**7**, Section 3] for basic definitions. Such polynomial representations also belong to the realm of representations of finite dimensional $\Bbbk$-algebras. Indeed, one of the first basic results is the equivalence between $\mathrm{Pol}_{n,d}$ and the category of modules[1] over the Schur algebra $S(n,d)$.

Simple objects in $\mathrm{Pol}_{n,d}$ are indexed by partitions λ in at most n parts, of size $|\lambda| = \sum_{0\le i\le n} \lambda_i = d$. We fix a simple object L_λ for each partition λ and we let P_λ, resp. J_λ, be its projective cover, resp. its injective hull. Recall that a partition in at most n parts is called p^r-restricted if $\lambda_n < p^r$ and for all $i < n$, $\lambda_i - \lambda_{i+1} < p^r$. Such partitions appear naturally when studying the structure of polynomial modules, e.g. in the statement of the Steinberg tensor product theorem [**9**, II.3.17]. We introduced the following definition in [**20**].

Definition. Given a polynomial representation M and a nonnegative integer r, we let $p(M,r) \in \mathbb{N}\cup\{+\infty\}$ be the maximal integer k such that M admits a projective resolution

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$$

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[1] Unlike [**8**] or [**7**], we do not assume here that our modules have finite dimension, but for our purposes it is a rather cosmetic change. Indeed, since the Schur algebra is finite dimensional, infinite dimensional modules have nice finiteness properties (they are direct limits of their finite dimensional submodules, they have finite socle length,...). Thus, most of our arguments working for finite dimensional modules equally work with infinite dimensional ones.

whose first k terms $P_0, \dots, P_{k-1}$ are direct sums of indecomposable projectives P_λ with p^r-restricted λ. Similarly, we let $i(M,r)$ be the maximal integer k such that M has an injective resolution whose first k terms $J^0, \dots, J^{k-1}$ are products of J_λ with p^r-restricted λ.

We are not aware that the homological constants $i(M,r)$ and $p(M,r)$ have been considered before. In [**20**], we found several applications of these cohomological constants. They are related to the behavior of cup products in the cohomology of GL_n, to generalizations and variants of Steinberg tensor product theorem, as well as to the homological behaviour of the Schur functor which compares the cohomology of GL_n with that of $\mathfrak{S}_n$. We also established basic computations for these cohomological constants. The purpose of this paper is to continue the work started in [**20**]. Namely, we give new applications of the constants $i(M,r)$ and $p(M,r)$, and we make further progress on the problem of computing these constants for concrete polynomial modules.

The paper is organized as follows. Section 2 concentrates on the applications of our cohomological constants. We review the applications already proved in [**20**] and we prove additional applications. The most noteworthy new result here is theorem 2.1, which shows that the surprising behavior of cup products proved in [**20**] is not specific to GL_n, but also holds for other classical matrix groups (othogonal or symplectic type). Section 3 concentrates on the problem of computing these constants for polynomial modules of interest. We prove a new characterization of $i(M,r)$ and $p(M,r)$ in proposition 3.4, related to the (derived) adjoint of the tensor product by $\Bbbk^{n\,(r)}$ (i.e. the r-th Frobenius twist of the defining representation of GL_n). With this new tool at our disposal, we undertake to study $i(M,r)$ and $p(M,r)$ for some new families of examples, including tilting modules and costantard modules associated to hooks or to some thin shaped partitions.

Thus, this article reviews some known results and establishes new ones. In order to make a clear distinction between what material is new and what is not, only new theorems and propositions are numbered.

Review of strict polynomial functors. Our applications of the homological constants $p(M,r)$ and $i(M,r)$ all assume that the polynomial modules in play are *stable* polynomial modules, i.e. they work for objects of $\mathrm{Pol}_{n,d}$ for $n \geq d$. The natural home for stating and proving theorems involving stable polynomial modules is the category of strict polynomial functors. Indeed, the categories $\mathrm{Pol}_{n,d}$ for $n \geq d$ are all equivalent to one another, and strict polynomial functors provide a model for these categories where the parasite integer n does not appear.

The remainder of the article is written in the language of strict polynomial functors, so we end this introduction by reviewing briefly some points of this theory. The reader may have full details and additional references by reading the seminal article of Friedlander and Suslin [**7**, Section 2], or [**12**] or [**20**, Section 2].

We denote by $\mathcal{P}_d$ the category of homogeneous strict polynomial functors of degree d over $\Bbbk$ (with values in arbitrary $\Bbbk$-vector spaces), and we let $\mathcal{P} = \bigoplus_{d\geq 0} \mathcal{P}_d$ be the category of strict polynomial functors of bounded degree. If $\Bbbk$ is an infinite field, a strict polynomial functor of degree d is simply a functor F from finite dimensional vector spaces to all vector spaces, such that the function $\mathrm{Hom}_\Bbbk(U,V) \to \mathrm{Hom}_\Bbbk(F(U),F(V))$ determined by F is a polynomial of degree d. In characteristic zero these functors appear in MacDonald's classical book [**13**], where they are called 'polynomial functors'. As already mentioned above, over any field $\Bbbk$, $\mathcal{P}_d$ is equivalent to the category $S(n,d)-\mathrm{Mod}$ of modules over the Schur algebra, or to the category $\mathrm{Pol}_{n,d}$, for any $n \ge d$. The equivalence is obtained by evaluating a functor F on $\Bbbk^n$ [**7**, Thm 3.2].

Many usual functors of $\Bbbk$-vector spaces are strict polynomial functors. To fix notations, we now provide a list of examples of strict polynomial functors which will be considered in this article.

- We let Γ^d be the d-th divided power functor, that is $\Gamma^d(V) = (V^{\otimes d})^{\mathfrak{S}_d}$ is the subspace of invariants of $V^{\otimes d}$ under the action of $\mathfrak{S}_d$ which permutes the factors of the tensor product (in particular Γ^0 is the constant functor with value $\Bbbk$). We let S^d be the d-th symmetric power functor, that is $S^d(V) = (V^{\otimes d})_{\mathfrak{S}_d}$ is the quotient space of coinvariants. More generally, if $\lambda = (\lambda_1, \dots, \lambda_n)$ is any n-tuple of nonnegative integers of weight $|\lambda| = \sum \lambda_i = d$, we let
$$\Gamma^\lambda = \Gamma^{\lambda_1} \otimes \cdots \otimes \Gamma^{\lambda_n}\ , \qquad \text{and} \qquad S^\lambda = S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_n}\ .$$
The functors Γ^λ, resp. S^λ, for all tuples λ of weight d, form a projective generator, resp. injective cogenerator, of $\mathcal{P}_d$. We also use a similar notation Λ^λ indexed by tuples λ for tensor products of exterior power functors. If $(1^{\{d\}}) = (1, \dots, 1)$ (d terms equal to one), we will denote by $\otimes^d$ the tensor product
$$\otimes^d := \Gamma^{(1^{\{d\}})} = \Lambda^{(1^{\{d\}})} = S^{(1^{\{d\}})}\ .$$
- $\mathcal{P}_d$ is a highest weight category [**4**] with simple objects indexed by the poset of partitions λ of weight (or size) $|\lambda| = d$, equipped with the dominance order $\trianglerighteq$. In particular we have the following functors attached to a partition λ. In the first four examples, λ is called the *highest weight* of the functor F_λ, because the evaluation $F_\lambda(\Bbbk^n)$ with $n \ge d$ is a polynomial module with highest weight λ.
  (1) We let L_λ be the simple functor indexed by λ.
  (2) We let S_λ be the costandard object indexed by λ. This functor is nothing but the Schur functor defined by Akin Buchsbaum and Weyman [**2**]. The notation used in [**2**] is different from ours, see [**17**, Section 6.1.1] for the conversion between notations. In particular $S_{(1^{\{d\}})} = \Lambda^d$ and $S_{(d)} = S^d$.
  (3) We let W_λ be the standard object indexed by λ. The functors W_λ are called Weyl functors and can be obtained from the Schur functors by duality: $W_\lambda = S_\lambda^\sharp$, where the dual $F^\sharp$ of a functor F satisfies $F^\sharp(V) = F(V^\vee)^\vee$ with '$\vee$' denoting the duality of $\Bbbk$ vector spaces. In particular $W_{(1^{\{d\}})} = \Lambda^d$ and $W_{(d)} = \Gamma^d$.
  (4) We let T_λ be the indecomposable tilting object indexed by λ. Each T_λ is characterized by indecomposability, self-duality, and by the existence of a filtration whose subquotients are Schur functors S_μ satisfying $\mu \trianglelefteq \lambda$, such that S_λ appears exactly once as a subquotient. The functor T_λ is a direct summand of $\Lambda^{\widetilde{\lambda}}$, where $\widetilde{\lambda}$ is the conjugate partition of λ.
  (5) We let P_λ, resp. J_λ, be the projective cover, resp. injective envelope, of L_λ. Then P_λ is a direct summand of Γ^λ and J_λ is a direct summand of S^λ.

- Finally, given a nonnegative integer r, we let $I^{(r)}$ be the r-th Frobenius twist. Thus $I^{(r)}(V)$ is the subspace of $S^{p^r}(V)$ generated by the elements of the form v^{p^r}, $v \in V$. The functor $I^{(0)} = S^1 = \Lambda^1 = \Gamma^1$ is often simply denoted by I (it is the identity functor of $\Bbbk$-vector spaces). For an arbitrary F, we denote by $F^{(r)}$ the composition $F \circ I^{(r)}$.

REMARK 1.1. In our notations, upper partitions or tuples mean tensor products, while lower partitions mean an index related to the highest weight category structure. For example if $\lambda = (3,2,2)$, then $S^\lambda = S^3 \otimes S^2 \otimes S^2$ is a tensor product of symmetric powers, while S_λ is the costandard (Schur) functor with highest weight $(3,2,2)$.

The homological constants $i(M,r)$ and $p(M,r)$ have the following alternative definition. We say that a tuple of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_n)$ is p^r-bounded if for all i, $\lambda_i < p^r$.

PROPOSITION ([**20**, Prop 4.1]). *Let $F \in \mathcal{P}_d$. Then $p(F,r)$ is the maximal (possibly infinite) integer k such that F admits a projective resolution*

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to F \to 0$$

in which the first k terms $P_0, \dots, P_{k-1}$ are direct sums of functors Γ^λ with p^r-bounded λ. Similarly, $i(F,r)$ is the maximal (possibly infinite) integer k such that F admits a injective resolution in which the first k terms $J^0, \dots, J^{k-1}$ are products of functors S^λ with p^r-bounded λ.

2. Applications of the constants $i(F,r)$ and $p(F,r)$

2.1. Stable cup products for GL_n and applications. Let us say that a quadruple of homogeneous functors (F, F', G, G') satisfies the Künneth condition if F has values in finite dimensional vector spaces and F' or G also has values in finite dimensional vector spaces. This technical condition ensures that the tensor product induces an isomorphism

$$\mathrm{Hom}_{\mathcal{P}}(F,G) \otimes \mathrm{Hom}_{\mathcal{P}}(F',G') \simeq \mathrm{Hom}_{\mathcal{P}(2)}(F \boxtimes F', G \boxtimes G')$$

where $\mathcal{P}(2)$ refers to the category of strict polynomial bifunctors. It is automatically satisfied if all the functors have values in finite dimensional vector spaces, as for the examples of strict polynomial functors given in the introduction. Our first use of the homological constants $i(F,r)$ and $p(F,r)$ is the following result on cup products.

THEOREM ([**20**, Thm 3.6]). *Let (F, G, F', G') be a quadruple of homogeneous strict polynomial functors satisfying the Künneth condition, and let $r \geq 0$. The cup product induces a graded injective map:*

$$\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F,G) \otimes \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F'^{(r)}, G'^{(r)}) \hookrightarrow \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F \otimes F'^{(r)}, G \otimes G'^{(r)}) .$$

Moreover, this graded injective map is an isomorphism in degree k in the following situations.

(1) *When* $\deg F < \deg G$, *and* $k < i(G,r)$.
(2) *When* $\deg F > \deg G$, *and* $k < p(F,r)$.
(3) *When* $\deg F = \deg G$, *and* $k < p(F,r) + i(G,r)$.

This theorem may seem surprising. Indeed, it does not hold in the finite group (scheme) cohomology setting. For example, the cohomological support variety of a representation M of a finite group scheme G is the variety defined from the kernel of the cup product with $\mathrm{Id}_M \in \mathrm{Ext}^0_G(M,M)$:

$$-\cup \mathrm{Id}_M : \mathrm{Ext}^*_G(\Bbbk,\Bbbk) \to \mathrm{Ext}^*_G(M,M)\ .$$

What makes cohomological support non trivial is precisely the fact that cup products are not injective in this case.

Since precomposition by $I^{(r)}$ induces an isomorphism in Ext-degrees zero and one, we can even remove the Frobenius twist when working in these degrees. We then obtain the following corollary. This corollary allows to reduce some Ext^1 computations between simple functors to Ext^1-computations between simple functors with p-restricted highest weight. This should be compared to [**9**, II 10.16 and 10.17].

Corollary ([**20**, Cor 3.7]). *Let F and G be two homogeneous functors of the same degree with values in finite dimensional vector spaces. Assume that both the head of F and the socle of G are direct sums of functors L_λ with λ p^r-restricted. There are isomorphisms:*

$$\mathrm{Hom}_{\mathcal{P}}(F,G)\otimes \mathrm{Hom}_{\mathcal{P}}(F',G') \simeq \mathrm{Hom}_{\mathcal{P}}(F\otimes F'^{(r)}, G\otimes G'^{(r)})\ , \tag{2.1}$$

$$\begin{matrix} \mathrm{Hom}_{\mathcal{P}_\Bbbk}(F,G)\otimes \mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(F',G') \\ \oplus\, \mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(F,G)\otimes \mathrm{Hom}_{\mathcal{P}_\Bbbk}(F',G') \end{matrix} \simeq \mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(F\otimes F'^{(r)}, G\otimes G'^{(r)})\ . \tag{2.2}$$

In [**20**], we use this corollary to investigate the structure of functors or the form $F\otimes G^{(r)}$ where all the composition factors of F are p^r-restricted. For example the subfunctor lattice of such tensor products is essentially determined by the subfunctor lattice of F and the subfunctor lattice of G [**20**, Cor 5.12 and 5.13]. Another example is the next corollary, which can be thought of as a categorical version of the Steinberg tensor product theorem.

Corollary ([**20**, Cor 5.14]). *Let λ be a p^r restricted partition, and denote by $L_\lambda\otimes\mathcal{P}^{(r)}$ the full subcategory of $\mathcal{P}$ whose objects are tensor products of the form $L_\lambda\otimes F^{(r)}$ for any F. Then $L_\lambda\otimes\mathcal{P}^{(r)}$ is a localizing and colocalizing subcategory of $\mathcal{P}$. Moreover, the functor $\mathcal{P}\to L_\lambda\otimes\mathcal{P}^{(r)}$, $F\mapsto L_\lambda\otimes F^{(r)}$ is an equivalence of categories.*

2.2. Stable cup products for other classical types. In this section, we show that the property of cup products of polynomial representations described in section 2.1 has an analogue for the other classical matrix groups. To be more specific, if $G=Sp_{2n}\subset GL_{2n}$ or $G=SO_n\subset GL_n$, any polynomial representation of GL_n restricts to a representation of G. If M and N are polynomial representations, and $r\ge 0$, there is a cup product

$$H^*(G,M)\otimes H^*(G,N^{(r)})\to H^*(G,M\otimes N^{(r)})\ .$$

We will show that when the rank of G is big enough with respect to the degree of the polynomial representations in play, this cup product is injective and it is an isomorphism in low degrees. The result is stated in theorem 2.1.

As with cup products for GL_n, the natural home for stating and proving theorem 2.1 is the category of strict polynomial functors. So we first recall the connection between classical groups and strict polynomial functors proved in [**16**] (and improved in [**19**, section 7.4]). Let us fix a strict polynomial functor X. Then for all F, we let $H^*_X(F)$ be the extension groups:

$$H^i_X(F) := \bigoplus_{k\geq 0} \mathrm{Ext}^i_{\mathcal{P}}(\Gamma^k \circ X, F)\ .$$

For all k, ℓ, there is a natural inclusion $\Gamma^{k+\ell}(V) \to \Gamma^k(V) \otimes \Gamma^\ell(V)$. Replacing V by $X(-)$, we obtain a morphism of strict polynomial functors:

$$\Delta_{k,\ell} : \Gamma^{k+\ell} \circ X \to (\Gamma^k \circ X) \otimes (\Gamma^\ell \circ X)\ .$$

We can now define an associative cup product as the composite

$$H^i_X(F) \otimes H^j_X(G) \xrightarrow{\otimes} \bigoplus_{k,\ell\geq 0} \mathrm{Ext}^{i+j}_{\mathcal{P}}((\Gamma^k \circ X) \otimes (\Gamma^\ell \circ X), F \otimes G) \xrightarrow{\Delta^*_{k,\ell}} H^{i+j}_X(F \otimes G)\ .$$

The next theorem gives the link with the cohomology of symplectic groups.

THEOREM ([**16**, Thm 3.17], [**19**, Thm 7.24]). *There is a graded map, which is natural with respect to $F \in \mathcal{P}_{\Bbbk}$ and compatible with cup products:*

$$\phi_{F,2n} : H^*_{\Lambda^2}(F) \to H^*(Sp_{2n}, F(\Bbbk^{2n}))\ .$$

Moreover it is a graded isomorphism if $2n \geq \deg F$.

There are similar results for orthogonal groups and special orthogonal groups, with Λ^2 replaced by S^2. See [**16**, Thm 3.24] and [**19**, Thm 7.27 and Cor 7.31]. We can now describe the behavior of cup products for symplectic and orthogonal types.

THEOREM 2.1. *Let $X = \Lambda^2$ or $X = S^2$, let F and G be two homogeneous strict polynomial functors and let $r \geq 0$. The cup product*

$$\cup\ :\ H^*_X(F) \otimes H^*_X(G^{(r)}) \to H^*_X(F \otimes G^{(r)})$$

is injective. Moreover, it is an isomorphism in degree k if $k < i(F, r)$.

In the remainder of the section, we outline the proof of theorem 2.1. This proof is a modification of the proof for GL_n given in [**20**]. We will treat only the case when F and G both have even degrees. If one of the two functors has odd degree, then the proof is similar, and actually easier. For example, if F or G has odd degree, then the source of the cup product map in theorem 2.1 is zero for degree reasons, so injectivity is trivial.

So in the remainder of section 2.2 we assume that F has degree $2d$ and G has degree $2e$. The proof decomposes in several steps.

Step 1: reduction to a connectedness statement. Injectivity of the cup product was already proved in [**16**], and we recall the proof here. For all vector spaces V, W and all nonnegative integers n we have a canonical decomposition

$$\Gamma^n(X(V \oplus W)) \simeq \bigoplus_{k+\ell+m=n} \Gamma^k(X(V)) \otimes \Gamma^\ell(V \otimes W) \otimes \Gamma^m(X(W))\ .$$

Thus $\Gamma^d(X(V)) \otimes \Gamma^e(X(W))$ is a direct summand on $\Gamma^{d+e}(X(V \oplus W))$, and there is a commutative diagram, in which $\mathcal{P}(V, W)$ denotes the category of strict polynomial *bifunctors* with variables V and W (see e.g. [**20**, Section 2], [**16**, Section 2]), the top vertical arrow is induced by the canonical projection of $\Gamma^{d+ep^r}(X(V \oplus W))$

onto $\Gamma^d(X(V)) \otimes \Gamma^{ep^r}(X(W))$, and the bottom vertical arrow is induced by the sum diagonal adjunction map

$$\begin{array}{ccc}
\begin{array}{c}\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^d \circ X, F)\\ \otimes \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^{ep^r} \circ X, G^{(r)})\end{array} & \xrightarrow[\simeq]{\otimes} & \mathrm{Ext}^*_{\mathcal{P}(V,W)}(\Gamma^d(X(V)) \otimes \Gamma^{ep^r}(X(W)), F(V) \otimes G^{(r)}(W)) \\
 & & \big\downarrow \\
 & & \mathrm{Ext}^*_{\mathcal{P}(V,W)}(\Gamma^{d+ep^r}(X(V \oplus W)), F(V) \otimes G^{(r)}(W)) \\
 & \searrow^{\cup} & \big\downarrow \simeq \\
 & & \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^{d+ep^r} \circ X, F \otimes G^{(r)})
\end{array}$$

Thus, the cup product is injective. Moreover, the cokernel of the cup product map can be identified from the diagram above. Namely, it is isomorphic to:

$$\mathrm{Ext}^*_{\mathcal{P}(V,W)} \left(\bigoplus \Gamma^k(X(V)) \otimes \Gamma^\ell(V \otimes W) \otimes \Gamma^m(X(W)), F(V) \otimes G^{(r)}(W) \right) ,$$

where the sum is taken over all triples (k, ℓ, m) satisfying $k + \ell + m = d + ep^r$ and $k > 0$, $m > 0$. Since there is no nontrivial extensions between homogeneous bifunctors of different bidegrees, it is also isomorphic to the direct sum of the graded vector spaces for $i > 0$:

$$\mathrm{Ext}^*_{\mathcal{P}(V,W)} \left(\Gamma^{d-i}(X(V)) \otimes \Gamma^{2i}(V \otimes W) \otimes \Gamma^{ep^r-i}(X(W)), F(V) \otimes G^{(r)}(W) \right) .$$

We denote for short by $E^*(i, F, G^{(r)})$ these graded vector spaces. So to prove theorem 2.1, it remains to prove the following connectedness statement:

$$E^k(i, F, G^{(r)}) = 0 \text{ for } i > 0 \text{ and } k < i(F, r). \qquad (*)$$

Step 2: proof of assertion $(*)$ in a special case. We now assume that $F = S^\mu$ for a p^r-bounded tuple μ, and $G = S^\nu$ for an arbitrary tuple ν. We observe that $S^\mu(V \oplus V')$ can be written as a direct sum of terms of the form $S^{\mu^1}(V) \otimes S^{\mu^2}(V')$ where both μ^1 and μ^2 are p^r bounded. Thus, using sum-diagonal adjunction (see e.g. [**20**, Section 2] or [**16**, Section 5]), we obtain that $E^*(i, S^\mu, S^{\nu\,(r)})$ is isomorphic to a finite direct sum of terms of the form

$$\mathrm{Ext}^*_{\mathcal{P}(V,W,V',W')} \Big(\Gamma^{d-i}(X(V)) \otimes \Gamma^{2i}(V' \otimes W') \otimes \Gamma^{ep^r-i}(X(W')) ,$$
$$S^{\mu^1}(V) \otimes S^{\mu^2}(V') \otimes S^{\nu^1\,(r)}(W) \otimes S^{\nu^2\,(r)}(W') \Big) ,$$

where μ^1 and μ^2 are p^r-bounded, and $\mathcal{P}(V, W, V', W')$ is the category of strict polynomial quadrifunctors with variables V, W, V', W'. By the Künneth theorem, such an Ext is isomorphic to $E^*_1 \otimes E^*_2 \otimes E^*_3$ with

$$E^*_1 = \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^{d-i} \circ X, S^{\mu^1}) ,$$
$$E^*_2 = \mathrm{Ext}^*_{\mathcal{P}(V',W')}(\Gamma^{2i}(V' \otimes W'), S^{\mu^2}(V') \otimes S^{\nu^1\,(r)}(W')) ,$$
$$E^*_3 = \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^{ep^r-i} \circ X, S^{\nu^2\,(r)}) .$$

We claim that E^*_2 is zero in all degrees. Indeed, replacing V' by $U^\vee$, where U is a finite dimensional vector space and '$\vee$' denote $\Bbbk$-linear duality, yields an equivalence of categories between $\mathcal{P}(V', W')$ and $\mathrm{Bif}(U, W')$ where the latter denotes the category of strict polynomial bifunctors, contravariant in the first variable U and covariant with the second variable W'. With this substitution, E_2^* is isomorphic to the graded vector space:

$$\mathrm{Ext}^*_{\mathrm{Bif}(U,W')}(\Gamma^{2i}(\mathrm{Hom}_{\Bbbk}(U,W')),\mathrm{Hom}_{\Bbbk}(\Gamma^{\mu^2}(U),S^{\nu^1\,(r)}(W'))\;.$$

By [**6**, Thm 1.5], the latter is isomorphic to $\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^{\mu^2},S^{\nu^1\,(r)})$, which is zero in positive degrees because Γ^{μ^2} is projective, and which is zero in degree zero by [**20**, Lm 3.10]. Thus $E_1^*\otimes E_2^*\otimes E_3^*$ is zero, which implies assertion $(*)$ in our special case.

Step 3: proof of assertion $(*)$ **in general.** We now return to the general case. The functor F has an injective coresolution J_F whose first $i(F,r)$ terms are products of functors S^μ for p^r-bounded tuples μ. Let J_G be an injective coresolution of G. Then $J_F(V)\otimes J_G^{(r)}(W)$ is a coresolution of $F(V)\otimes G(W)$, whose first $i(F,r)$ terms are finite direct sums of functors of the form $(\prod S^{\mu^i}(V))\otimes(\prod S^{\nu^j\,(r)}(W))$ where the μ^i are p^r-bounded tuples of integers.

Lemma 2.2. *Let μ^i be p^r-bounded tuples of integers and let ν^j be arbitrary tuples of integers. Then $(\prod S^{\mu^i}(V))\otimes(\prod S^{\nu^j}(W))$ is a direct summand of a product of bifunctors of the form $S^\mu(V)\otimes S^\nu(W)$ for p^r-bounded tuples μ and arbitrary tuples ν.*

Proof. First, by [**20**, Lm 3.3(iii)] $(\prod S^{\mu^i})$ is a direct summand of a direct sum $\bigoplus_k S^{\alpha^k}$ for p^r-bounded tuples α^k. Similarly, $\prod S^{\nu^j}$ is a direct summand of a direct sum of a direct sum $\bigoplus_\ell S^{\beta^\ell}$. So $(\prod S^{\mu^i}(V))\otimes(\prod S^{\nu^j}(W))$ is a direct summand of $\bigoplus_{k,\ell} S^{\alpha^k}(V)\otimes S^{\beta^\ell}(W)$. The latter is an injective bifunctor, hence by injectivity it is a direct summand of $\prod_{k,\ell} S^{\alpha^k}(V)\otimes S^{\beta^\ell}(W)$. □

Using lemma 2.2 and applying the r-th Frobenius twist on the variable W, we obtain that in the complex $J_F(V)\otimes J_G^{(r)}(W)$, the terms $(J_F(V)\otimes J_G^{(r)}(W))^s$ with $0\le s<i(F,r)$ are direct summands of products of bifunctors of the form $S^\mu(V)\otimes S^{\nu\,(r)}(W)$ for p^r-bounded tuples μ and arbitrary tuples ν. Thus using the special case of assertion $(*)$ proved in step 2, together with the spectral sequence

$$E_1^{s,t}=\mathrm{Ext}^t_{\mathcal{P}(V,W)}(B(V,W),(J_F(V)\otimes J_G^{(r)}(W))^s)\\ \Longrightarrow \mathrm{Ext}^{s+t}_{\mathcal{P}(V,W)}(B(V,W),F(V)\otimes G^{(r)}(W))$$

which we apply to the bifunctor

$$B(V,W)=\Gamma^{d-i}(X(V))\otimes\Gamma^{2i}(V\otimes W)\otimes\Gamma^{ep^r-i}(X(W))$$

we obtain assertion $(*)$ in the general case. This finishes the proof of theorem 2.1.

2.3. On the closed monoidal structure of $\mathcal{P}_d$. As explained in [**12**], $\mathcal{P}_d$ has a closed symmetric monoidal structure. We denote by $\underline{\otimes}$ the internal tensor product, and by $\underline{\mathrm{Hom}}$ the internal tensor product. Since $\mathcal{P}=\bigoplus_{d\ge0}\mathcal{P}_d$, we formally extend this symmetric monoidal structure to the whole category $\mathcal{P}$ by letting for all F, G with homogeneous components F^d, G^d of degree d :

$$\underline{\mathrm{Hom}}(F,G)=\bigoplus_{d\ge0}\underline{\mathrm{Hom}}(F^d,G^d)\;,\qquad F\underline{\otimes}G=\bigoplus_{d\ge0}F^d\underline{\otimes}G^d\;.$$

The properties of cup products described in section 2.1 have consequences for the symmetric monoidal structure of $\mathcal{P}_d$. This is what is behind the proof of [**20**, Prop 6.1], and we explain everything in more details here.

The internal Hom can be concretely described in terms of parametrizations. To be more specific, we denote by F^V and F_V the strict polynomial functors $U \mapsto F(\mathrm{Hom}_{\Bbbk}(V,U))$, $U \mapsto F(U \otimes V)$. Then one has isomorphisms of functors of the variable V, natural with respect to F, G:

$$\mathrm{Hom}_{\mathcal{P}}(F^V, G) \simeq \underline{\mathrm{Hom}}(F,G)(V) \simeq \mathrm{Hom}_{\mathcal{P}}(F, G_V)\ .$$

We let $\underline{\mathrm{Ext}}^*(F,G)$ be the value on G of the derived functor of $\underline{\mathrm{Hom}}(F,-) : \mathcal{P} \to \mathcal{P}$. Since parametrization is exact and preserves projectives and injectives, one has

$$\mathrm{Ext}^*_{\mathcal{P}}(F^V, G) \simeq \underline{\mathrm{Ext}}^*(F,G)(V) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_V).$$

Such parametrized extension groups appear naturally in computations of strict polynomial functors, see e.g. [**17**,**18**]. It is easy to see [**20**, Prop 7.3 (a)] that for all V one has $p(F,r) = p(F^V, r)$ and $i(G,r) = i(G_V, r)$. Thus we have a parametrized version of [**20**, Thm 3.6].

Proposition 2.3. *Let (F,G,F',G') be a quadruple of homogeneous strict polynomial functors satisfying the Künneth condition, and let $r \geq 0$. Cup product induces a graded injective map:*

$$\underline{\mathrm{Ext}}^*(F,G) \otimes \underline{\mathrm{Ext}}^*(F'^{(r)}, G'^{(r)}) \hookrightarrow \underline{\mathrm{Ext}}^*(F \otimes F'^{(r)}, G \otimes G'^{(r)})\ .$$

Moreover, this graded injective map is an isomorphism in degree k in the following situations.

1. *When $\deg F < \deg G$, and $k < i(G,r)$.*
2. *When $\deg F > \deg G$, and $k < p(F,r)$.*
3. *When $\deg F = \deg G$, and $k < p(F,r) + i(G,r)$.*

We let $\underline{\mathrm{Tor}}_*(F,G)$ be the value on G of the derived functors of $F \underline{\otimes} - : \mathcal{P} \to \mathcal{P}$. Internal Tor are related to internal Ext by isomorphisms of functors, natural with respect to F, G:

$$\underline{\mathrm{Tor}}_i(F,G)^\sharp \simeq \underline{\mathrm{Ext}}^i(F, G^\sharp)\ ,$$

where $^\sharp$ refers to Kuhn (or contragedient) duality: $F^\sharp(V) = F(V^\vee)^\vee$ with $^\vee$ denoting the duality of $\Bbbk$-vector spaces. Internal Tor are not equipped with a cup product but rather with a coproduct. The construction of the coproduct in degree zero is explained in [**20**, Section 6.1], and after deriving, this coproduct induces a graded map:

$$\underline{\mathrm{Tor}}_*(F \otimes F', G \otimes G') \to \underline{\mathrm{Tor}}_*(F,G) \otimes \underline{\mathrm{Tor}}_*(F', G')\ .$$

One has the following connectedness property for internal Tor.

Proposition 2.4. *Let F, G, F', G' be homogeneous strict polynomial functors and let $r \geq 0$. The coproduct induces a graded surjective map:*

$$\underline{\mathrm{Tor}}_*(F \otimes F'^{(r)}, G \otimes G'^{(r)}) \twoheadrightarrow \underline{\mathrm{Tor}}_*(F,G) \otimes \underline{\mathrm{Tor}}_*(F'^{(r)}, G'^{(r)})\ .$$

Moreover, this graded surjective map is an isomorphism in degree k in the following situations.

1. *When $\deg F < \deg G$, G is finite and $k < p(G,r)$.*
2. *When $\deg F > \deg G$, F is finite and $k < p(F,r)$.*
3. *When $\deg F = \deg G$, F and G are finite and $k < p(F,r) + p(G,r)$.*

Proof. We first prove the statement when F, F', G, G' are finite (i.e their values are finite dimensional vector spaces). As noted in the proof of [**20**, Prop 6.1], in this case the coproduct is just the dual (for the duality $^\sharp$) of the cup product for internal Ext for the quadruple $(F, F'^{(r)}, G^\sharp, G'^{(r)\sharp})$. Since $G'^{(r)\sharp} \simeq (G'^\sharp)^{(r)}$, and $i(G^\sharp, r) = p(G, r)$, the statement follows from proposition 2.3. For the general case, we use that each functor is the filtered colimit of its finite subfunctors, and that tensor products and internal Tor commute with filtered colimits. □

Remark 2.5. The finiteness hypotheses in proposition 2.4 can actually be removed, but then our short proof using dualization does not work anymore. One rather needs to redo all the calculations of the proof of [**20**, Thm 3.6], replacing the Hom spaces by tensor products over the source category or by its parametrized variant $\underline{\otimes}$.

Proposition 2.4 is already interesting in the cases where one has isomorphism in degree zero. For example, the following application is analogous to the Steinberg tensor product theorem in that it allows to reduce the computation of the internal tensor product between simples to that of internal tensor products between simples with p-restricted highest weights.

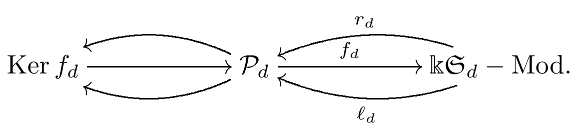
Theorem ([**20**, Thm 6.2, Cor 6.5]). *Let $\lambda^0, \dots, \lambda^r$ and $\mu^0, \dots, \mu^s$ be p-restricted partitions, and let $\lambda = \sum p^i \lambda^i$ and $\mu = \sum p^i \mu^i$.*

(1) *If $r = s$ and μ^i and λ^i have the same weight for all i, then $L_\lambda \underline{\otimes} L_\mu$ is nonzero and there is an isomorphism:*

$$L_\lambda \underline{\otimes} L_\mu \simeq (L_{\lambda^0} \underline{\otimes} L_{\mu^0}) \otimes (L_{\lambda^1} \underline{\otimes} L_{\mu^1})^{(1)} \otimes \cdots \otimes (L_{\lambda^r} \underline{\otimes} L_{\mu^r})^{(r)} .$$

(2) *Otherwise, $L_\lambda \underline{\otimes} L_\mu$ is zero.*

2.4. Connectivity of the Schur functor. Given a degree d homogenous strict polynomial functor F, we let $f_d(F) = \mathrm{Hom}_{\mathcal{P}}(\otimes^d, F)$. The action of the symmetric group $\mathfrak{S}_d$ on $\otimes^d$ makes $f_d(F)$ into a $\Bbbk\mathfrak{S}_d$-module. The functor f_d is known as the Schur functor, and it is part of a recollement of abelian categories:



The interaction between cohomology of the symmetric group and cohomology of GL_n has been studied by the means of the Grothendieck spectral sequences associated to the composition of f_d and its adjoints [**5**, **10**]. We observe that $i(F, 1)$ can be understood as the maximal integer k such that F has an injective resolution J such that the first k terms of the complex $f_d(J)$ are injective $\Bbbk\mathfrak{S}_d$-modules (and $p(F, r)$ has a similar description). So we deduce the following comparison result.

Theorem ([**20**, Thm 8.2]). *Let F and G be homogeneous functors of degree d. The map induced by the Schur functor:*

$$\mathrm{Ext}^k_{\mathcal{P}}(F, G) \to \mathrm{Ext}^k_{\Bbbk\mathfrak{S}_d}(f_d(F), f_d(G))$$

is an isomorphism in degrees $k < p(F, 1) + i(G, 1) - 1$, and it is injective in degree $k = p(F, 1) + i(G, 1) - 1$.

For some classes of functors, it is not to hard to compute lower bounds for $p(F,1)$ or $i(G,1)$, and we recover in this way various known results on the cohomology of symmetric groups.

- The right adjoint r_d of f_d is left exact and sends injective $\Bbbk\mathfrak{S}_d$-modules to p-bounded injectives, hence one has $i(r_d M, 1) \geq 2$ for all $\Bbbk\mathfrak{S}_d$-modules M. This remark together with the previous theorem encompasses most of the Ext^1-computations and comparison results between representations of Schur algebras and symmetric groups obtained in [**5**].
- Using Ringel duality for strict polynomial functors we proved [**20**, Prop 7.6] that $i(W^d_\lambda, 1) \geq p-1+i(S_{\widetilde{\lambda}}, 1)$ where W_λ, resp. $S_{\widetilde{\lambda}}$ denotes the Weyl functor associated to λ, resp. the Schur functor associated to the conjugate partition $\widetilde{\lambda}$. This computation and the previous theorem encompass all the applications in [**10**, Section 6], except [**10**, Theorem 6.1].
- We compute in corollary 3.12 below that $i(S_\lambda, 1) \geq 2$ when λ is a partition with $\lambda_1 \leq p-2$. Thus for all p-restricted partitions μ such that $\mu \not\trianglerighteq \lambda$ one has (the first equality comes from highest weight category theory):
$$0 = \mathrm{Ext}^1_{\mathcal{P}}(L_\mu, S_\lambda) \simeq \mathrm{Ext}^1_{\Bbbk\mathfrak{S}_d}(f_d L_\mu, f_d S_\lambda)\ .$$
This result is similar (but not equivalent) to the main result of [**11**].

3. Computations of $i(F,r)$ and $p(F,r)$

The results of section 2 motivates us to investigate how to compute the homological constants $i(F,r)$ and $p(F,r)$. In this section, we give basic tools to compute $i(F,r)$ and $p(F,r)$, and partial results regarding the values of these constants for simples L_λ, Schur functors S_λ, Weyl functors W_λ and tilting functors T_λ.

3.1. General computation rules and reductions. We first recall some general facts established in [**20**, Section 7].

PROPOSITION. *Let F and G be two strict polynomial functors, let λ be a partition and $\widetilde{\lambda}$ its conjugate partition. The following holds.*

$$p(F,r) = i(F^\sharp, r)\ , \tag{3.1}$$
$$i(F,r) = i(F^{(s)}, r+s)\ , \tag{3.2}$$
$$i(F\otimes G, r) = \min\{\, i(F,r)\,,\, i(G,r)\,\}\ , \tag{3.3}$$
$$i(W_\lambda, r) \geq p^r - 1 + i(S_{\widetilde{\lambda}}, r)\ , \tag{3.4}$$
$$i(\Lambda^d, r) = p^r - 1 \text{ if } d \geq p^r, \tag{3.5}$$
$$i(\Gamma^d, r) = 2p^r - 2 \text{ if } d \geq p^r, \tag{3.6}$$
$$i(F,r) = +\infty \text{ if } \deg F < p^r. \tag{3.7}$$

In view of these results we will restrict ourselves to understand $i(F,r)$ for the following functors in the sequel.

- Functors $F = L_\lambda$ with p-restricted highest weight λ. Indeed, the value for arbitrary λ can then be obtain by equality (3.2) and (3.3) and the Steinberg tensor product theorem.
- Functors $F = S_\lambda$, with λ arbitrary. Indeed, we then obtain lower bounds for W_λ by inequality (3.4). Note that (3.5) and (3.6) show that inequality

(3.4) is an equality for $\lambda = (1, \dots, 1)$ and for $\lambda = (d)$. We don't know if equality holds in general.

- Functors $F = T_\lambda$ with λ arbitrary.

3.2. The integer $i(F,r)$ as a connectedness bound. We now recall a homological criterion to compute $i(F,r)$. Given a tuple of nonnegative integers $(d_0, \dots, d_k)$ we let

$$T^{(d_0,\dots,d_k)} := (I^{(0)})^{\otimes d_0} \otimes (I^{(1)})^{\otimes d_1} \otimes \cdots \otimes (I^{(k)})^{\otimes d_k} \ .$$

Thus $T^{(d_0,\dots,d_k)}$ is a homogeneous strict polynomial functor of degree $\sum_{i=0}^k p^i d_i$. We let $T(d,r)$ be the (finite) direct sum of all the $T^{(d_0,\dots,d_k)}$ of degree d which have at least one factor $I^{(s)}$ with $s \geq r$ (in particular $T(d,r) = 0$ as soon as $d < p^r$). The next proposition shows that we can interpret $i(F,r)$ as a connectedness bound. For brevity we will use the following notation.

Definition 3.1 (Notation). If E^* is a graded vector space, or a graded functor, we let

$$\operatorname{conn} E^* = \inf \left\{ k \in \mathbb{N} \mid E^k \neq 0 \right\} \ .$$

In particular $\operatorname{conn} E^* = +\infty$ if and only if $E^* = 0$.

Proposition ([**20**, Prop 7.1]). *For all F homogeneous of degree d:*

$$i(F,r) = \operatorname{conn} \operatorname{Ext}^*_{\mathcal{P}}(T(d,r), F) \ .$$

We will now prove another characterization of $i(F,r)$ which will be easier to check in practice. This characterization, given in proposition 3.4 below, uses the right adjoint of the functor $- \otimes I^{(r)} : \mathcal{P} \to \mathcal{P}$. This adjoint exists for formal reasons [**14**, Chap V], but we need an explicit formula. The reasonning works for the adjoint of $- \otimes F$ for any degree d homogeneous strict polynomial functor F, so we present the results in this generality. Given a strict polynomial functor G, we form the bifunctor $(V,W) \mapsto G(V \oplus W)$. As for every strict polynomial bifunctor, there is a decomposition

$$G(V \oplus W) = \bigoplus_{k \geq 0} G^{(k,*)}(V,W)$$

where each $G^{(k,*)}(V,W)$ is homogeneous of degree k with respect to the variable V. We let $[F : -] : \mathcal{P} \to \mathcal{P}$ be the functor defined by:

$$[F : G](W) = \operatorname{Hom}_{\mathcal{P}(V)} \left(F(V), G^{(d,*)}(V,W) \right) \ ,$$

where $\operatorname{Hom}_{\mathcal{P}(V)}(-,-)$ means that we are computing Hom between strict polynomial functors of the variable V.

In the decomposition of $G(V \oplus W)$, the direct summands $G^{(k,*)}(V,W)$ have degree less or equal to $\deg G - k$ as strict polynomial functors of the variable W and they are zero if $\deg G - k$ is negative. Thus $\deg [F : G] \leq \deg G - d$ and $[F : G]$ is zero if $\deg G - d$ is negative. Moreover $G^{(\deg G, *)}(V,W) = G(V)$, hence if $\deg G = d$ then $[F : G]$ is a constant functor with value $\operatorname{Hom}_{\mathcal{P}}(F,G)$.

Lemma 3.2. *The functor $[F : -] : \mathcal{P} \to \mathcal{P}$ is right adjoint to $- \otimes F : \mathcal{P} \to \mathcal{P}$.*

PROOF. We have a chain of natural isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}}(F\boxtimes H,G) &\simeq \mathrm{Hom}_{\mathcal{P}(V,W)}(F(V)\otimes H(W),G(V\oplus W))\\ &\simeq \mathrm{Hom}_{\mathcal{P}(V,W)}(F(V)\otimes H(W),G^{(d,*)}(V,W))\\ &\simeq \mathrm{Hom}_{\mathcal{P}(W)}(H(W)\,,\,\mathrm{Hom}_{\mathcal{P}(V)}(F(V),G^{(d,*)}(V,W))\,)\\ &= \mathrm{Hom}_{\mathcal{P}(W)}(H(W),[F:G](W))\;.\end{aligned}$$

The first isomorphism is by sum-diagonal adjunction, the second one follows from the fact that there is no Hom between homogeneous strict polynomial bifunctors of different bidegrees, the third one is [**14**, Prop 1 p. 37]. □

Next we consider the right derived functors $\mathbf{R}^i[F:-]:\mathcal{P}\to\mathcal{P}$. If $\deg G<d$ then $\mathbf{R}^*[F:G]$ is zero, and if $\deg G=d$ then $\mathbf{R}^*[F:G]$ is a constant functor with value $\mathrm{Ext}^*_{\mathcal{P}}(F,G)$. The next lemma gives basic rules to compute $\mathbf{R}^*[F:G]$ when the degree of G is greater than d.

LEMMA 3.3. *Assume that F is homogeneous of degree d with finite dimensional values. For all i, there is a functor $\Phi^i:\mathcal{P}^{\mathrm{op}}\times\mathcal{P}\times\mathcal{P}\to\mathcal{P}$ and a canonical decomposition, natural with respect to F,G,H*

$$\mathbf{R}^i[F:G\otimes H]\simeq G\otimes\mathbf{R}^i[F:H]\;\oplus\;\Phi^i(F,G,H)\;\oplus\;\mathbf{R}^i[F:G]\otimes H\;.$$

Moreover, for all $r\geq 0$, $\Phi^i(I^{(r)},G,H)=0$ for all G, H. Finally, let $E^*=S^*$, Λ^*, or any exponential functor with each E^k homogeneous of degree k. Let $\Delta_{k,\ell}:E^{k+\ell}\to E^k\otimes E^\ell$ and $m_{k,\ell}:E^k\otimes E^\ell\to E^{k+\ell}$ be the components of the multiplication and comultiplications of E^*. Then for all $k\geq 0$ the following compositions (with $E^j=0$ for $j<0$) are isomorphisms

$$E^{k-d}\otimes\mathbf{R}^i[F:E^d]\hookrightarrow\mathbf{R}^i[F:E^{k-d}\otimes E^d]\xrightarrow{\mathbf{R}^i[F:m_{k-d,\ell}]}\mathbf{R}^i[F:E^k]\;,$$

$$\mathbf{R}^i[F:E^k]\xrightarrow{\mathbf{R}^i[F:\Delta_{k-d,\ell}]}\mathbf{R}^i[F:E^{k-d}\otimes E^d]\twoheadrightarrow E^{k-d}\otimes\mathbf{R}^i[F:E^d]\;.$$

PROOF. We have a direct sum decomposition:

$$(G\otimes H)^{(d,*)}(V,W)=\bigoplus_{i=0}^{d}G^{(i,*)}(V,W)\otimes H^{(d-i,*)}(V,W)\;.\qquad(*)$$

The summands indexed by $i=0$ and $i=d$ are respectively equal to

$$G(W)\otimes H^{(d,*)}(V,W)\text{ and }G^{(d,*)}(V,W)\otimes H(W)\;.$$

The direct sum decomposition is obtained by applying $\mathrm{Ext}^i_{\mathcal{P}(V)}(F(V),-)$ to this decomposition, and using the canonical isomorphisms (which hold since F has finite dimensional values hence it has a resolution by finitely generated projectives):

$$\mathrm{Ext}^i_{\mathcal{P}(V)}(F(V),G(W)\otimes H^{(d,*)}(V,W))\simeq G(W)\otimes\mathrm{Ext}^i_{\mathcal{P}(V)}(F(V),H^{(d,*)}(V,W))\;,$$
$$\mathrm{Ext}^i_{\mathcal{P}(V)}(F(V),G^{(d,*)}(V,W)\otimes H(W))\simeq\mathrm{Ext}^i_{\mathcal{P}(V)}(F(V),G^{(d,*)}(V,W))\otimes H(W)\;.$$

If $F=I^{(r)}$, Pirashvili's vanishing lemma [**7**, Thm 2.13] shows that the summands of $(*)$ indexed by $i=0$ and $i=d$ are the only ones which may have a nonzero contribution after applying $\mathrm{Ext}^i_{\mathcal{P}(V)}(I^{(r)}(V),-)$. Finally, the last statement follows from the fact that there is an isomorphism:

$$(E^k)^{(d,*)}(V,W)\xrightarrow{\simeq}E^d(V)\otimes E^{k-d}(W)$$

induced by the comultiplication of E, with inverse induced by the multiplication of E. $\square$

PROPOSITION 3.4. *For all F homogeneous of degree d one has*

$$i(F,r) = \min_{r\le s\le \log_p(d)} \operatorname{conn} \mathbf{R}^*[I^{(s)} : F]\ .$$

PROOF. For all G, there is a Grothendieck spectral sequence

$$E_2^{k,\ell} = \operatorname{Ext}^k_{\mathcal{P}}(G, \mathbf{R}^\ell[I^{(s)} : F]) \Longrightarrow \operatorname{Ext}^{k+\ell}_{\mathcal{P}}(G\otimes I^{(s)}, F)\ .$$

Let E_2^* denote the second page with total grading. Then

$$\operatorname{conn} \operatorname{Ext}^*_{\mathcal{P}}(G\otimes I^{(s)}, F) \ge \operatorname{conn} E_2^* \ge \operatorname{conn} \mathbf{R}^*[I^{(s)} : F]\ . \quad (*)$$

Moreover, if $c = \operatorname{conn}\mathbf{R}^*[I^{(s)} : F]$ and $\operatorname{Hom}_{\mathcal{P}}(G, \mathbf{R}^c[I^{(s)} : F]) \ne 0$, then $E_2^{0,c} = E_\infty^{0,c}$ for degree reasons, so the inequalities in $(*)$ are equalities.

This being said, let m denote the right hand side of the equality of proposition 3.4. Since any direct summand $T^{(d_0,\dots,d_k)}$ of $T(d,r)$ is of the form $G\otimes I^{(s)}$ for some $s\in[r,\log_p(d)]$, inequality $(*)$ implies that $i(F,r)\ge m$. Let t be such that $\operatorname{conn}\mathbf{R}^*[I^{(t)} : F] = m$. Since all simple functors are quotients of some $T^{(d_0,\dots,d_\ell)}$ (this is a consequence of the Steinberg tensor product theorem and Clausen and James' theorem, see e.g [**20**, Cor 4.3]), one may find a tuple $(d_0,\dots,d_\ell)$ such that $\operatorname{Hom}_{\mathcal{P}}(T^{(d_0,\dots,d_\ell)}, \mathbf{R}^m[I^{(t)} : F])$ is nonzero. Thus we have

$$i(F,r) \le \operatorname{conn}\operatorname{Ext}^*_{\mathcal{P}}(T^{(d_0,\dots,d_\ell)}\otimes I^{(t)}, F) = \operatorname{conn}\mathbf{R}^*[I^{(t)} : F] = m\ .$$

This finishes the proof of proposition 3.4. $\square$

3.3. Computations for tilting functors.

LEMMA 3.5. *Let λ be a partition of weight d. The following assertions are equivalent.*

(i) *λ is p^r-restricted,*
(ii) *$\operatorname{Hom}_{\mathcal{P}}(T(d,r), J_\lambda)$ is zero,*
(iii) *$\operatorname{Hom}_{\mathcal{P}}(\otimes^{d-p^r}\otimes I^{(r)}, J_\lambda)$ is zero.*

PROOF. (i)⇒(ii) since no composition factor of $T(d,r)$ is p^r-restricted, and (ii)⇒(iii) since $\otimes^{d-p^r}\otimes I^{(r)}$ is a direct summand of $T(d,r)$. It remains to prove (iii)⇒(i). But (iii) implies that L_λ is not a composition factor of $\otimes^{d-p^r}\otimes I^{(r)}$. Thus, to prove (i), it suffices to prove that all L_μ with μ of weight d and not p^r-restricted appear as composition factors of $\otimes^{d-p^r}\otimes I^{(r)}$. If μ is not p^r-restricted then by Clausen and James' theorem and the Steinberg tensor product theorem L_μ is a quotient of a tensor product $T^{(d_0,\dots,d_k)}$ with $d_k>0$ for some $k\ge r$. Observe that for all s, $I^{(s+1)}$ is a subfunctor of $S^{p^{s+1}}$, hence a subquotient of $(I^{(s)})^{\otimes p}$. Using this observation repeatedly, we find that $T^{(d_0,\dots,d_k)}$ is a subquotient of $\otimes^{d-p^r}\otimes I^{(r)}$, hence that L_μ is a composition factor of $\otimes^{d-p^r}\otimes I^{(r)}$. $\square$

PROPOSITION 3.6. *Let λ be an arbitrary partition. Then $i(T_\lambda,r)\in\{p^r-1,\infty\}$. Moreover $i(T_\lambda,r) = +\infty$ if and only if $\widetilde{\lambda}$ is p^r-restricted.*

PROOF. We use Ringel duality, in the same way as in the proof of [**20**, Lemma 7.5]. To be more specific, Ringel duality is an equivalence of categories $\Theta : D(\mathcal{P}) \to D(\mathcal{P})$, which sends $J_{\widetilde{\lambda}}$ to T_λ and $T^{(d_0,\dots,d_k)}$ to $T^{(d_0,\dots,d_k)}[-s]$, where $s = \sum_{i=0}^{k} d_i(p^i - 1)$. In particular

$$\mathrm{Ext}^*_{\mathcal{P}}(T^{(d_0,\dots,d_k)}, J_{\widetilde{\lambda}}) \simeq \mathrm{Ext}^{*+s}_{\mathcal{P}}(T^{(d_0,\dots,d_k)}, T_\lambda)\;. \qquad (*)$$

Since $J_{\widetilde{\lambda}}$ is injective, the left hand side of $(*)$ is zero in positive degrees. Now if $\widetilde{\lambda}$ is p^r-restricted, then $\mathrm{Hom}_{\mathcal{P}}(T(d,r), J_{\widetilde{\lambda}})$ is zero by lemma 3.5, hence $\mathrm{Ext}^*_{\mathcal{P}}(T(d,r), T_\lambda)$ is zero by $(*)$. If $\widetilde{\lambda}$ is not p^r-restricted, then $\mathrm{Hom}_{\mathcal{P}}(\otimes^{d-p^r} \otimes I^{(r)}, J_\lambda)$ is nonzero by lemma 3.5. Since the degree shift s is minimal for the summand $\otimes^{d-p^r} \otimes I^{(r)}$ of $T(d,r)$, we obtain

$$i(T_\lambda, r) = \mathrm{conn}\,\mathrm{Ext}^*_{\mathcal{P}}(\otimes^{d-p^r} \otimes I^{(r)}, T_\lambda) = p^r - 1\;.$$

□

3.4. Computations for Schur functors indexed by hooks. Schur functors indexed by hooks are often quite easy to deal with. Indeed they appear as the kernels of some Koszul complexes which have a very simple description. To be more specific, for $d \geq 1$ we let κ^d be the complex:

$$\Lambda^d \to \Lambda^{d-1} \otimes S^1 \to \cdots \to \Lambda^1 \otimes S^{d-1} \to S^d$$

where each $\Lambda^{d-i} \otimes S^i$ is placed in homological degree $d-i$, and the differential $d : \Lambda^{d-i} \otimes S^i \to \Lambda^{d-i-1} \otimes S^{i+1}$ is the composition (recall that $S^1 = \Lambda^1 = I$):

$$\Lambda^{d-i} \otimes S^i \xrightarrow{\mathrm{comult} \otimes S^i} \Lambda^{d-i-1} \otimes I \otimes S^i \xrightarrow{\Lambda^{d-i-1} \otimes \mathrm{mult}} \Lambda^{d-i-1} \otimes S^{i+1}\;.$$

Then κ^d is an exact complex, and for all i one has (see e.g. [**3**, p. 80]):

$$S_{(i,1^{\{d-i\}})} \;=\; \ker\left[\Lambda^{d-i} \otimes S^i \to \Lambda^{d-i-1} \otimes S^{i+1}\right]\;.$$

All of what is recalled above is valid for $d > 0$. If $d = 0$, we let κ^0 be the complex equal to $\Bbbk$ concentrated in degree zero.

For all nonnegative i, we may apply the functor $\mathbf{R}^i[I^{(r)} : -]$ termwise to the Koszul complex κ^d to obtain another complex of strict polynomial functors, which we denote by $\mathbf{R}^i[I^{(r)} : \kappa^d]$. In the next lemma we compute these complexes. As usual, the shift $C[s]$ of a complex C is a complex satisfying $C[s]_n = C_{s+n}$.

LEMMA 3.7. *Let $d \geq p^r$. We have*

$$\mathbf{R}^i[I^{(r)} : \kappa^d] \simeq \begin{cases} \kappa^{d-p^r} & \text{if } i = 0,\\ \kappa^{d-p^r}[p^r] & \text{if } i = p^r - 1,\\ 0 & \text{else.}\end{cases}$$

PROOF. As computed in [**7**, (4.5.1) p. 241], $\mathbf{R}^i[I^{(r)} : \Lambda^{p^r}] = \mathrm{Ext}^i_{\mathcal{P}}(I^{(r)}, \Lambda^{p^r})$ equals $\Bbbk$ if $i = p^r - 1$ and zero if $i \neq p^r - 1$. Also, $\mathbf{R}^i[I^{(r)} : S^{p^r}] = \mathrm{Ext}^i_{\mathcal{P}}(I^{(r)}, S^{p^r})$ is zero in positive degrees and equals $\Bbbk$ in degree zero. Thus, by lemma 3.3, $\mathbf{R}^i[I^{(r)} : \kappa^d]$ is zero for $i \notin \{0, p^r - 1\}$. Also, we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathbf{R}^0[I^{(r)}, \Lambda^{d-i} \otimes S^i] & \longrightarrow & \mathbf{R}^0[I^{(r)}, \Lambda^{d-i-1} \otimes I \otimes S^i] & \longrightarrow & \mathbf{R}^0[I^{(r)}, \Lambda^{d-i-1} \otimes S^{i+1}] \\
\simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow \\
\Lambda^{d-i} \otimes S^{i-p^r} \otimes \Bbbk & \longrightarrow & \Lambda^{d-i-1} \otimes I \otimes S^{i-p^r} \otimes \Bbbk & \longrightarrow & \Lambda^{d-i-1} \otimes S^{i+1-p^r} \otimes \Bbbk
\end{array}$$

where the horizontal maps are induced by the comultiplication of the exterior algebra and the multiplication of the symmetric algebra, and the vertical isomorphisms are compositions of the form (for suitable values of G, j) as in lemma 3.3:

$$G\otimes S^j\otimes \Bbbk = G\otimes S^j\otimes \mathbf{R}^0[I^{(r)},S^{p^r}]\simeq G\otimes \mathbf{R}^0[I^{(r)},S^{j+p^r}]\to \mathbf{R}^0[I^{(r)},G\otimes S^{j+p^r}]\,.$$

Thus $\mathbf{R}^0[I^{(r)}:\kappa^d]\simeq \kappa^{d-p^r}$. One identifies $\mathbf{R}^{p^r-1}[I^{(r)}:\kappa^d]$ similarly. □

LEMMA 3.8. *Let $\lambda=(i,1^{\{d-i\}})$ be a hook-shaped partition of weight d and let $1\le r\le \log_p(d)$. If $d<p^r$ then $\mathbf{R}^*[I^{(r)},S_\lambda]$ is zero. If $d=p^r$ then*

$$\mathbf{R}^k[I^{(r)},S_\lambda]=\begin{cases}\Bbbk & \text{if } k=(d-i),\\ 0 & \text{else.}\end{cases}$$

If $d>p^r$ then

$$\mathbf{R}^k[I^{(r)},S_\lambda]=\begin{cases}S_{(i-p^r,1^{\{d-i\}})} & \text{if } k=0 \text{ and } i>p^r,\\ S_{(i,1^{\{d-i-p^r\}})} & \text{if } k=p^r-1 \text{ and } d-i\ge p^r,\\ 0 & \text{else.}\end{cases}$$

PROOF. If $d<p^r$ vanishing occurs for degree reasons ($[I^{(r)}:F]$ is zero on functors of degree less than p^r). So we may assume that $d\ge p^r$. Let $\kappa^d_{\le d-i}$ be the stupid truncation of κ^d:

$$\kappa^d_{\le d-i}=\Lambda^{d-i}\otimes S^i\to\cdots\to\Lambda^1\otimes S^{d-1}\to S^d\,.$$

The homology of $\kappa^d_{\le d-i}$ is $S_{(i,1^{\{d-i\}})}$ in degree $d-i$ and zero in other degrees. Thus, we have a hypercohomology spectral sequence:

$$E_1^{-s,t}=\mathbf{R}^t[I^{(r)}:\Lambda^s\otimes S^{d-s}]\Longrightarrow \mathbf{R}^{t-s+(d-i)}[I^{(r)}:S_{(i,1^{\{d-i\}})}]\,.$$

By lemma 3.7 there are only two nonzero rows in the first page, the row $t=0$ and the row $t=p^r-1$, and the complex $(E_1^{*,0},d_1)$ is isomorphic to $\kappa^{d-p^r}_{\le d-i}$ while the complex (E_1^{*,p^r-1},d_1) is isomorphic to $\kappa^{d-p^r}_{\le d-i-p^r}[p^r]$. Thus we can compute explicitly page E_2 of the spectral sequence. Assume first that $d=p^r$. Then $E_2^{0,0}=\Bbbk$ is the only nonzero term of the second page. Thus $E_2^{*,*}=E_\infty^{*,*}$ and we obtain $\mathbf{R}^*[I^{(r)}:S_\lambda]$ as described in the lemma. Assume now that $d>p^r$. Then the only possibly nonzero terms of page E_2 are $E_2^{-(d-i),0}$ and $E_2^{-(d-i),p^r-1}$. Thus

$$\mathbf{R}^*[I^{(r)}:S_{(i,1^{\{d-i\}})}]=E_\infty^{-(d-i),*}=E_2^{-(d-i),*},$$

and we obtain $\mathbf{R}^*[I^{(r)}:S_\lambda]$ as described in the lemma. □

We are now ready to prove the main result of this section.

PROPOSITION 3.9. *Let $\lambda=(i,1^{\{d-i\}})$ be a hook-shaped partition, and let $r\ge 1$. The following holds.*

(i) *If $d<p^r$, then $i(S_\lambda,r)=+\infty$.*
(ii) *If $d=p^r$, then $i(S_\lambda,r)=(d-i)$.*
(iii) *If $p^r<d<2p^r$ then*

$$i(S_\lambda,r)=\begin{cases}p^r-1 & \text{if } i\le d-p^r,\\ +\infty & \text{if } d-p^r<i\le p^r,\\ 0 & \text{if } p^r<i.\end{cases}$$

(iv) *If* $2p^r \leq d$, *then* $i(S_\lambda, r) = \begin{cases} p^r - 1 & \text{if } i \leq p^r, \\ 0 & \text{if } i > p^r. \end{cases}$

PROOF. Statement (i) follows from general rules of computations recalled in section 3.1. For (ii), (iii) and (iv) we use that $i(S_\lambda, r)$ is the minimal value of the connectedness of $\mathbf{R}^*[I^{(s)}, S_\lambda]$ with $r \leq s \leq \log_p(d)$ by proposition 3.4. If $p^r \leq d < 2p^r$ one only has to consider the connectedness of $\mathbf{R}^*[I^{(s)}, S_\lambda]$ for $s = r$, which is directly given by lemma 3.8, and one obtains (ii) and (iii). For (iv) there might be several integers s to consider. As $p^r \leq d - p^r$ we have by lemma 3.8:

$$\operatorname{conn} \mathbf{R}^*[I^{(r)}, S_\lambda] = \begin{cases} p^r - 1 & \text{if } i \leq p^r, \\ 0 & \text{if } i > p^r. \end{cases}$$

To prove (iv), it remains to prove that if $s > r$, then $\operatorname{conn} \mathbf{R}^*[I^{(s)}, S_\lambda]$ is greater or equal to $\operatorname{conn} \mathbf{R}^*[I^{(r)}, S_\lambda]$. It suffices to check it when $i \leq p^r$. If $s > r$ and $d = p^s$ then $\operatorname{conn} \mathbf{R}^*[I^{(s)}, S_\lambda] = p^s - i$ by lemma 3.8. But $i \leq p^r$ implies that $p^s - i \geq p^s - p^r > p^r - 1$ and we are done. If $s > r$ and $d \neq p^s$ then $i \leq p^r$ implies that $i \leq p^s$, hence by lemma 3.8 $\operatorname{conn} \mathbf{R}^*[I^{(r)}, S_\lambda] \in \{p^s - 1, +\infty\}$ is greater than $p^r - 1$ and we are done. □

3.5. Thin partitions. In this section we do not compute exact values for $i(S_\lambda, r)$ or $i(L_\lambda, r)$, but we rather provide lower bounds for these integers. The philosophy of the results obtained can be roughly explained as follows. First if $\lambda = (1^{\{d\}})$, then $S_\lambda = L_\lambda = \Lambda^d$ and $i(\Lambda^d, r)$ are high numbers ($p^r - 1$ or $+\infty$). Now if λ is a thin partition, that is if λ resembles $(1^{\{d\}})$ then S_λ and L_λ should resemble Λ^d, hence $i(S_\lambda, r)$ and $i(L_\lambda, r)$ should be high numbers, too.

LEMMA 3.10. *Let* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *be a partition of* p^r *in exactly* n *parts. Then*

$$i(S_\lambda, r) = \operatorname{conn} \operatorname{Ext}^*_{\mathcal{P}}(I^{(r)}, S_\lambda) \geq n - 1 \,.$$

PROOF. The equality follows from proposition 3.4 and the fact that λ has weight p^r. By the Pieri rule [**1**, Thm (3)], one has a short exact sequence

$$0 \to S_\lambda \to S^{\lambda_1} \otimes S_{(\lambda_2, \ldots, \lambda_n)} \to C_\lambda \to 0$$

and C_λ has a finite length filtration whose layers are Schur functors indexed by partitions μ of p^r in n or $n-1$ parts, which all strictly dominate λ. Since $\operatorname{Ext}^*(I^{(r)}, S^{\lambda_1} \otimes S_{(\lambda_2, \ldots, \lambda_n)})$ is zero by Pirashvili vanishing lemma [**7**, Thm 2.13], the associated long exact sequence of Ext-groups tells us that

$$i(S_\lambda, r) \geq \min_\mu \; i(S_\mu, r) + 1 \qquad (*)$$

where the minimum is taken over the layers S_μ of the Pieri filtration of $S^{\lambda_1} \otimes S_{(\lambda_2, \ldots, \lambda_n)}$. Now we prove lemma 3.10 inductively. For $n = 1$ there is nothing to prove. Assume that lemma 3.10 holds for partitions of p^r in at most n parts. Then if λ is a partition of p^r in $(n+1)$ parts which is maximal among those partitions with respect to the dominance order, the filtration of C_λ has only layers of the form S_μ for partitions of μ in n parts. Thus, by using $(*)$ lemma 3.10 holds for S_λ. We then propagate the statement of lemma 3.10 by downwards induction on the finite poset (with respect to the dominance order) of partitions of p^r in $(n+1)$ parts. Thus we obtain that lemma 3.10 holds for all partitions of p^r into $(n+1)$ parts. □

Proposition 3.11. *Let λ be a partition. Assume that λ contains no partition α of p^r in less or equal to n parts with trivial p-core. Then $i(S_\lambda, r) \geq n$.*

Proof. By [**2**, Thm II.4.11], $S_\lambda(V \oplus W)$ has a filtration whose layers have the form $S_\alpha(V) \otimes S_{\lambda/\alpha}(W)$ for $\alpha \subset \lambda$. Let $s \geq r$. Restricting to the direct summand of $S_\lambda(V \oplus W)$ which has degree p^s with respect to V, one obtains

$$\operatorname{conn} \mathbf{R}^*[I^{(s)} : S_\lambda] \geq \min_\alpha \left\{ \operatorname{conn} \left(\operatorname{Ext}^*_{\mathcal{P}}(I^{(s)}, S_\alpha) \right) \right\} , \qquad (*)$$

where the minimum is taken over the partitions α of p^s which are contained in λ. If λ contains no partition α of p^r in less or equal to n parts with trivial p-core, then it contains no partition of α of p^s in less or equal to n parts with trivial p-core. Thus by lemma 3.10 the right hand side of $(*)$ is greater or equal to n, so that $\operatorname{conn} \mathbf{R}^*[I^{(s)} : S_\lambda] \geq n$. By proposition 3.4, this implies that $i(S_\lambda, r) \geq n$. □

The hypothesis in proposition 3.11 is not very simple. We give a simpler equivalent formulation for $r = 1$ in corollary 3.12, and a simpler but weaker condition for $r \geq 1$ in corollary 3.13.

Corollary 3.12. *If λ satisfies $\lambda_1 \leq p - n$, then $i(S_\lambda, 1) \geq n$.*

Proof. The partitions of p with trivial p-core are the hooks $(p - i, 1^{\{i\}})$. □

Corollary 3.13. *If λ satisfies $\lambda_1 + \cdots + \lambda_n < p^r$, then $i(S_\lambda, r) \geq n$.*

Proof. The condition implies that λ is too thin to contain any partition of p^r into less or equal to n parts. □

Proposition 3.14. *Let λ be an arbitrary partition. The following holds.*

1. *If λ satisfies $\lambda_1 \leq p - n$ then $i(L_\lambda, 1) \geq n$.*
2. *If λ satisfies $\lambda_1 + \cdots + \lambda_n < p^r$, then $i(L_\lambda, r) \geq n$.*

Proof. We will use the short exact sequences (coming from the highest weight structure of S_λ)

$$0 \to L_\lambda \to S_\lambda \to D_\lambda \to 0 , \qquad (*)$$

where $D_\lambda := S_\lambda / L_\lambda$ has a finite filtration whose layers are L_μ for μ dominated by λ. We prove both statements of proposition 3.14 simultaneously, by induction on the poset of partitions (with respect to the dominance order). First, if λ is minimal in this poset, then $L_\lambda = S_\lambda$ and the result is given by corollaries 3.12 and 3.13. Now let λ be a partition and assume that proposition 3.14 holds for all partitions dominated by λ. The key observation is that if μ is dominated by λ and λ satisfies one of the conditions of proposition 3.14, then μ also satisfies the same condition. In particular, we obtain that $i(D_\lambda, r) \geq n$ by the induction hypothesis. We also have $i(S_\lambda, r) \geq n$ by corollary 3.12 or 3.13. Thus the $\operatorname{Ext}^*_{\mathcal{P}}(T(d, r), -)$-long exact sequence associated to $(*)$ implies that $i(L_\lambda, r) \geq n$. □

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