Contemporary Mathematics
Volume **617**, 2014
http://dx.doi.org/10.1090/conm/617/12299

# Applications of functor (co)homology

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Abstract. This article is a survey of recent applications of functor (co)homology (i.e. Ext and Tor computations in functor categories) to the (co)homology of discrete groups, of group schemes, and to the derived functors in homotopical algebra.

## 1. Introduction

The terms 'Functor (co)homology' in the title refer to Ext and Tor computations in functor categories. In this article, we will focus on two specific functor categories. The first one is the category $\mathcal{F}_R$ of *ordinary functors* over a ring $R$. The objects of this category are simply the functors $F : P(R) \to R$-mod, where $P(R)$ denotes the category of finitely generated projective $R$-modules, and the morphisms are the natural transformations between such functors. The second one is the category $\mathcal{P}_R$ of *strict polynomial functors* over a commutative ring $R$. It is the algebro-geometric analogue of $\mathcal{F}_R$: strict polynomial functors can be seen as ordinary functors $F : P(R) \to R$-mod equipped with an additional scheme theoretic structure (described below).

Some classical homological invariants of rings, groups, or spaces can be interpreted as functor homology. A prototypical example is the case of the Topological Hochschild Homology $\mathrm{THH}(R)$ of a ring $R$, which is weakly equivalent [**13**] to the stable $K$-theory of $R$. By [**31**], $\mathrm{THH}_*(R)$ can be computed as the MacLane Homology $\mathrm{HML}_*(R)$ of $R$. The latter can be interpreted [**27**] as functor homology:

$$\mathrm{THH}_*(R) \simeq \mathrm{HML}_*(R) \simeq \mathrm{Tor}_*^{\mathcal{F}_R}(\mathrm{Id}, \mathrm{Id}) \;.$$

Why do we want such functor homology interpretations? There are several reasons. First, functor homology usually bears a lot of structure. Hence, if we have a functor homology interpretation of a classical invariant, we often get extra structure on this classical invariant (see corollary 3.13 below for an example).

Second, we may use functor homology interpretations to get explicit computations. Indeed, Ext and Tor are reasonably computable in $\mathcal{F}_R$ and in $\mathcal{P}_R$. To be more specific, Ext and Tor computations are complicated enough to contain interesting information. At the same time, these computations are not too complicated: many Ext and Tor groups can be explicitly computed. For example, the functor homology interpretation of $\mathrm{THH}(R)$ allow the explicit computation of Topological Hochschild Homology when $R$ is a finite field [**20**] or the ring of integers [**21**].

2010 *Mathematics Subject Classification.* Primary 18G15; Secondary 18G55, 20G10.

Third, several classical homology invariants can be interpreted as functor homology. So functor homology can be taken as an intermediate concept, giving links between a priori unrelated classical homological invariants.

The aim of this article is to present the reader with functor homology interpretations of some classical (co)homology theories. We do not give proofs, but rather describe the general picture explaining how these interpretations are related to each other.

## 2. The (co)homology of discrete groups and $\mathcal{F}_R$

Let $n$ be a positive integer. Evaluation on the $R$-module $R^n$ yields a functor $\mathrm{ev}_{R^n}$ from the category of ordinary functors over $R$ to the category of modules over $RGL_n(R)$ (the group algebra of $GL_n(R)$ over $R$):

$$\begin{array}{cccc} \mathrm{ev}_{R^n}: & \mathcal{F}_R & \to & RGL_n(R)\text{-Mod} \\ & F & \mapsto & F(R^n)\ . \end{array}$$

The $R$-linear group action $\rho_F : GL_n(R) \to \mathrm{End}_R(F(R))$ is simply given by $\rho_F(g) := F(g)$. Since the evaluation functor is exact, it induces for all pairs of functors $(F, G)$ a graded morphism:

$$\mathrm{ev}_{R^n} : \mathrm{Ext}^*_{\mathcal{F}_R}(F, G) \to \mathrm{Ext}^*_{RGL_n(R)}(F(R^n), G(R^n))\ . \tag{1}$$

A natural question is to ask what the properties of this morphism are. Of course, one cannot expect it to be an isomorphism, since the right hand side depends on $n$, while the left hand side does not. It is also too much to ask for nice properties of this map for all functors $F$, $G$ since ordinary functors can be quite wild objects. The properties of this map will be given for the class of polynomial functors, which we now introduce.

**2.1. Polynomial functors.** Polynomial functors were introduced by Eilenberg and Mac Lane, in the study [**16**] of the homology of the Eilenberg-Mac Lane spaces $K(\pi, n)$. Let $F \in \mathcal{F}_R$ be an ordinary functor. For all $X \in P(R)$, we let $\Delta F(X)$ be the kernel of the morphism $F(X \oplus R) \to F(X)$ induced by the canonical projection $X \oplus R \twoheadrightarrow X$. This actually defines an ordinary functor $\Delta F$. The induced functor

$$\Delta : \mathcal{F}_R \to \mathcal{F}_R$$

is called the difference functor. An ordinary functor $F$ is called *polynomial of degree less or equal to* $n$ if there is an $n$ such that the $(n+1)$-fold iteration $\Delta^{n+1}F$ is zero. It is *polynomial of degree* $n$ if in addition $\Delta^n F$ is nonzero.

Observe the analogy with set-theoretic maps, which justifies the terminology. For all maps $f : \mathbb{Z} \to \mathbb{Z}$, we denote by $\delta f$ the map $x \mapsto f(x+1) - f(1)$. Then $f$ is polynomial i.e. $f(x)$ is of the form $\sum_{i=0}^n a_i x^i$ (with $a_i \in \mathbb{Q}$) if and only if $\delta^{n+1} f$ is constant with value zero.

In the case of functors, $F$ is polynomial of degree zero if and only if it is constant, $F$ is polynomial of degree less or equal to one if and only if it can be written as the sum of an additive functor and of a constant functor. When $R$ is a commutative ring, typical functors of degree $n$ are the tensor product functor $M \mapsto M^{\otimes n}$, the exterior power functor $M \mapsto \Lambda^n(M)$, the symmetric power functor $M \mapsto S^n(M) = (M^{\otimes n})_{\mathfrak{S}_n}$, or the divided power functor $M \mapsto \Gamma^n(M) = (M^{\otimes n})^{\mathfrak{S}_n}$ (all the tensor products are taken over $R$, $M$ is projective an finitely generated and the symmetric group $\mathfrak{S}_n$ acts as usual by permuting the factors of the tensor product). Also, subfunctors and quotients of polynomial functors are polynomial functors (since $\Delta$ is exact).

**2.2. The case of a finite field $R$.**

2.2.1. *The case of general linear groups.* Assume that $R$ is a finite field. Dwyer proved [**14**] that the right hand side of the morphism (1) does not depend on $n$ if $F$ and $G$ are polynomial functors and $n$ is big enough (Dwyer's theorem is actually valid over a PID).

To be more specific, let $\iota_{n+1,n} : GL_n(R) \hookrightarrow GL_{n+1}(R)$ be the embedding of groups sending $M$ to $\begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}$. Then the projection $\pi : R^{n+1} \to R^n$ onto the first $n$ coordinates and the inclusion $\iota : R^n \hookrightarrow R^{n+1}$ into the first $n$ coordinates are both $GL_n(R)$-equivariant. Hence for all functors $F$, $G$ we get a morphism:

$$\phi^i_{n,n+1} : \mathrm{Ext}^i_{RGL_{n+1}(R)}(F(R^{n+1}), G(R^{n+1})) \to \mathrm{Ext}^i_{RGL_n(R)}(F(R^n), G(R^n))$$

by first restricting the actions to $GL_n(R)$, and then restricting attention to $F(R^n)$ and $G(R^n)$, considered as direct summands of $F(R^{n+1})$ and $G(R^{n+1})$ via $F(\iota)$ and $G(\iota)$. If $F$ and $G$ are polynomial functors, then $\phi^i_{n,n+1}$ is an isomorphism if $n$ is big enough with respect to $i$, the degree of $F$ and the degree of $G$. We will use the notation $\mathrm{Ext}^i_{RGL(R)}(F, G)$ for the stable value.

REMARK 2.1. When $F$ and $G$ are polynomial of degree zero (that is $F$ and $G$ are constant functors), the stabilization of the extension groups $\mathrm{Ext}^i_{RGL_n(R)}(F(R^n), G(R^n))$ is equivalent to the stabilization of the cohomology groups $H^i(GL_n(R), R)$, see e.g.[**41**] for a proof of the latter. If $F$ and $G$ are constant functors, the stable value is zero (since for $i > 0$, the stable value of $H^i(GL_n(R), R)$ is zero, as proved by Quillen [**33**]) but for polynomial functors of higher degrees, the stable value can be very far from zero.

The following theorem was proved independently by Betley [**1**] and Suslin [**19**, Appendix].

THEOREM 2.2. *Let $R$ be a finite field, let $F$ and $G$ be polynomial functors. Evaluation on $R^n$ yields a natural isomorphism*

$$\mathrm{Ext}^*_{\mathcal{F}_R}(F, G) \xrightarrow{\simeq} \mathrm{Ext}^*_{RGL(R)}(F, G) .$$

If $M, N$ are $RGL_n(R)$-modules, the vector space $\mathrm{Hom}_R(M, N)$ is endowed with an action of $GL_n(R)$ and there is an isomorphism

$$\mathrm{Ext}^*_{RGL(R)}(M, N) \simeq H^*(GL_n(R), \mathrm{Hom}_R(M, N)) .$$

So theorem 2.2 is indeed a theorem about the stable cohomology of $GL_n(R)$. It can be easily extended to more general coefficients than those of the somewhat restrictive form $\mathrm{Hom}_R(M, N)$ using the category of ordinary bifunctors, see [**1**, **18**].

2.2.2. *The cohomology of orthogonal and symplectic groups.* One can ask for a similar statement for other classical matrix groups. Indeed, there is a similar stabilization of the cohomology $H^*(G_{2n}, F(R^{2n}))$, when $G_{2n} = Sp_n(R)$ is the symplectic group or $G_{2n} = O_{n,n}(R)$ is the orthogonal group. The stabilization with trivial coefficients was proved by Fiedorowicz and Priddy [**17**], following the ideas of Quillen, and the stabilization with $F(R^n)$ as coefficients ($F$ is a polynomial functor, and the action of $G_{2n}$ is obtained by restriction of the action of $GL_n(R)$) was proved by Charney [**4**]. Let us denote by $H^*(G,F)$ the stable value. Djament and Vespa proved [**8**] that this stable value can be interpreted as functor cohomology.

To be more specific, let us denote by $R\Lambda^2$ the ordinary functor which assigns to each $R$-vector space $V$ the $R$-vector space with basis the set $\Lambda^2(V)$. There is an evaluation map

$$\mathrm{Ext}^i_{\mathcal{F}_R}(R\Lambda^2, F) \to \mathrm{Ext}^i_{RSp_n(R)}(R\Lambda^2(R^{2n}), F(R^{2n})) \to H^i(Sp_n(R), F(R^{2n}))$$

obtained by composing the evaluation on $R^{2n}$ and the pullback along the $RSp_n$-equivariant map $R \to R\Lambda^2(R^{2n})$, which sends $\lambda \in R$ to $\lambda w \in R\Lambda^2(R^{2n})$, where $w$ is the invariant element corresponding to the form defining $Sp_n$. When $n$ is big enough, this map lands in the stable cohomology $H^*(Sp(R), F)$, and Djament and Vespa proved [**8**, Cor 3.33]:

THEOREM 2.3. *Let $R$ be a finite field, let $F$ be a polynomial functor. Evaluation induces a natural isomorphism*

$$\mathrm{Ext}^*_{\mathcal{F}_R}(R[\Lambda^2], F) \xrightarrow{\simeq} H^*(Sp(R), F)\ .$$

A remarkable feature in this theorem is that the category $\mathcal{F}_R$, which is already related to the cohomology of $GL_n(R)$, is also related to the cohomology of $Sp_n(R)$. The functor $R[\Lambda^2]$ is reminiscent of the bilinear antisymmetric form defining $Sp_n(R)$. Djament and Vespa proved a similar statement for the orthogonal group, in odd characteristic, when $\Lambda^2$ is replaced by $S^2$ [**8**, Cor 3.33].

As an illustration that group cohomology computations become easier once transposed in functor categories, Djament and Vespa compute the stable cohomology of $O_{n,n}(R)$ and $Sp_n(R)$ with coefficients in many representations, including [**8**, Thm 4.18] $S^d(R^{2n})$, $\Lambda^d(R^{2n})$ or $\Gamma^d(R^{2n})$.

**2.3. The case of an arbitrary ring $R$.** In this section, we use a variant of the category $\mathcal{F}_R$, namely the category $\mathcal{F}_R^{\mathrm{Ab}}$ of functors with source $P(R)$ and target the category Ab of abelian groups. We do this to stick to the framework of the articles of Scorichenko [**36**] and Djament [**9**]. This is rather a cosmetic change for functor homology: the adjunction $R\otimes_{\mathbb{Z}} : \mathrm{Ab} \rightleftarrows R\text{-Mod} : \mathcal{O}$ induces a similar adjunction $\mathcal{F}_R^{\mathrm{Ab}} \rightleftarrows \mathcal{F}_R$ which can be used to translate many computations in $\mathcal{F}_R^{\mathrm{Ab}}$ into computations in $\mathcal{F}_R$ and vice versa.

2.3.1. *Stabilization: homology versus cohomology.* Assume first that $R$ is a finite field. Then the homology of the symplectic, orthogonal and general linear groups enjoys the same stabilization property as their cohomology. These two stabilizations are actually equivalent by the universal coefficient theorem. However, the stable value of the homology has a nice interpretation which is not shared by the cohomology. Indeed, let $H_i(G,F)$ be the stable value of the homology $H_i(G_{2n}, F(R^{2n}))$, where $G_{2n}$ equals $Sp_n(R)$ or $O_{n,n}(R)$. Since homology commutes with filtered colimits, there is an isomorphism:

$$H_*(G,F) \simeq \mathrm{colim}_n\, H_*(G_{2n}, F(R^{2n})) \simeq H_*(G_\infty, F(R^\infty))\ ,$$

where the group $G_\infty$ is the colimit of the groups $G_{2n}$ under suitable block matrices inclusions (analogous to the block matrices inclusion $\iota_{n,n+1}$ of the general linear groups), and $F(R^\infty)$ denotes the colimit of the $F(R^{2n})$. A similar result holds for the general linear group. If $F$ is an ordinary functor, we denote by $F^\vee : P(R)^{\mathrm{op}} \to \mathrm{Ab}$ the functor obtained by precomposing $F$ by $R$-linear duality. Evaluation of $F^\vee$ on $R^n$ yields a *right* $\mathbb{Z}GL_n(R)$-module, and we denote by $\mathrm{Tor}^{GL}_*(F^\vee, G)$ the stable value of the torsion modules $\mathrm{Tor}_*^{\mathbb{Z}GL_n(R)}(F^\vee(R^n), G(R^n))$ (which exists if $F$ and $G$ are polynomial). Then we have an isomorphism:

$$\mathrm{Tor}_*^{GL}(F^\vee, G) \simeq \mathrm{Tor}_*^{\mathbb{Z}GL_\infty(R)}(F^\vee(R^\infty), G(R^\infty)) \ .$$

Assume now that $R$ is an arbitrary ring. Then the homology of the symplectic, orthogonal and general linear groups need not stabilize, even for trivial coefficients. However, the homology groups $H_*(G_\infty, F(R^\infty))$ still makes sense. Thus we have a homological object independent of $n$ and related to our groups, regardless of the stabilization issue. This makes homology nicer to formulate the generalization of theorems 2.2 and 2.3 over an arbitrary ring $R$.

2.3.2. *The homology of classical groups over an arbitrary ring.* Let us start with the case of the general linear group $GL_n(R)$. Recall that for $F \in \mathcal{F}_R^{\mathrm{Ab}}$, the contravariant functor $F^\vee$ is obtained by precomposing $F$ by $R$-linear duality. There is a tensor product functor

$$F^\vee \otimes_{\mathcal{F}_R^{\mathrm{Ab}}} : \mathcal{F}_R^{\mathrm{Ab}} \to \mathrm{Ab} \ .$$

characterized by the following properties. It commutes with arbitrary sums, it is right exact, and if $P^n$ is the ordinary functor which sends a $R$-module $M$ to the free abelian group with basis $\mathrm{Hom}_R(R^n, M)$, there is a natural isomorphism:

$$F^\vee \otimes_{\mathcal{F}_R^{\mathrm{Ab}}} P^n \simeq F^\vee(R^n) \ .$$

The derived functors of $F^\vee \otimes_{\mathcal{F}_R}$ are denoted $\mathrm{Tor}_*^{\mathcal{F}_R^{\mathrm{Ab}}}(F^\vee, -)$.

We want to interpret $\mathrm{Tor}_*^{\mathbb{Z}GL_\infty(R)}(F^\vee(R^\infty), G(R^\infty))$ as functor homology, at least when $F$ and $G$ are nice functors. As a first guess, we might say it is isomorphic to $\mathrm{Tor}_*^{\mathcal{F}_R^{\mathrm{Ab}}}(F^\vee, G)$. However, such an isomorphism cannot hold in general, for if $\mathbb{Z}$ denotes the constant functor with value $\mathbb{Z}$, then $\mathrm{Tor}_i^{\mathcal{F}_R^{\mathrm{Ab}}}(\mathbb{Z}, \mathbb{Z}) = 0$ if $i > 0$, whereas $\mathrm{Tor}_*^{\mathbb{Z}GL_\infty(R)}(\mathbb{Z}, \mathbb{Z}) = H_*(GL_\infty(R), \mathbb{Z})$ is usually non zero.

The situation is actually even worse. Functor homology is a reasonably computable object. By contrast, the homology of $GL_\infty(R)$ with trivial coefficients is a very complicated object, whose computation is out of reach in general. The torsion groups $\mathrm{Tor}_*^{\mathbb{Z}GL_\infty(R)}(F^\vee(R^\infty), G(R^\infty))$ are even more complicated, so it seems hopeless to interpret them as functor homology. The reader should keep this in mind to estimate the value of the following theorem of Scorichenko [**36**].

THEOREM 2.4. *Let $R$ be a ring and let $F, G \in \mathcal{F}_R^{\mathrm{Ab}}$ be polynomial functors. Assume moreover that $F$ takes projective values. There is a natural isomorphism:*

$$\mathrm{Tor}_n^{\mathbb{Z}GL_\infty(R)}(F^\vee(R^\infty), G(R^\infty)) \simeq \bigoplus_{p+q=n} \mathrm{Tor}_p^{\mathcal{F}_R^{\mathrm{Ab}}}(H_q(GL_\infty(R), \mathbb{Z}) \otimes_{\mathbb{Z}} F^\vee, G) \ .$$

This theorem clearly breaks the computation of torsion groups for $GL_\infty(R)$ into two pieces: on the one hand there is the contribution of the homology with trivial coefficients, and on the other hand there is the contribution of the functor homology $\mathrm{Tor}_*^{\mathcal{F}_R^{\mathrm{Ab}}}(F^\vee, G)$. These two pieces of different nature allow to compute the right hand side of the isomorphism (e.g. by the universal coefficient theorem). When $R$ is a finite field, the contribution of the trivial coefficients is zero in positive degrees (by Quillen's computation), so theorem 2.4 yields an homological version of theorem 2.2. Theorem 2.4 is a special case of Scorischenko's theorem, which holds for bifunctor coefficients. Scorischenko's theorem was originally stated in terms of

Waldhausen's stable K-theory (that is, the homology of the homotopy fiber of the canonical map $BGL(R) \to BGL(R)^+$). It is not available as a publication, but an account of its original proof is available in [**22**], see also [**9**, Section 5.2].

Djament reinvestigated the methods of Scorischenko, in particular his powerful vanishing theorem, a key ingredient in the proof of theorem 2.4. He combined them with the arguments of [**8**] to obtain a generalization of theorem 2.4 to unitary groups [**9**, Thm 1]. The general linear groups can be seen as unitary groups, so that theorem 2.4 appears as a particular case of Djament's result. Symplectic groups are also unitary groups, and in this case Djament's theorem specializes to the following statement.

THEOREM 2.5. *Let $R$ be a ring and let $F \in \mathcal{F}_R^{\mathrm{Ab}}$ be a polynomial functor. There is a natural isomorphism:*

$$H_n(Sp_\infty(R), F(R^\infty)) \simeq \mathrm{Tor}_p^{\mathcal{F}_R^{\mathrm{Ab}}}(H_q(Sp_\infty(R),\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}\Lambda^2)^\vee, F)\ .$$

When $R$ is a finite field, the contribution of the trivial coefficients is zero in positive degrees (by the computation of Fiedorowicz and Priddy), so one recovers the isomorphism of Djament and Vespa (theorem 2.3 above) in its homological version [**8**, Cor 3.31].

$$H_n(Sp_\infty(R), F(R^\infty)) \simeq \mathrm{Tor}_n^{\mathcal{F}_R^{\mathrm{Ab}}}((\mathbb{Z}\Lambda^2)^\vee, F)\ .$$

(When $F$ is in $\mathcal{F}_R$, the torsion group on the right hand side is isomorphic to $\mathrm{Tor}_n^{\mathcal{F}_R}((R\Lambda^2)^\vee, F)$). The orthogonal groups are also unitary groups, and there is an analogue of theorem 2.5 for orthogonal groups, with $Sp_\infty(R)$ replaced by $O_{\infty,\infty}(R)$, and $\Lambda^2$ replaced by $S^2$. There are also functor homology interpretations for the homology of some orthogonal groups defined by non-hyperbolic forms, like $O_n(R)$, under some restrictions on the ring $R$, see [**9**, Section 6].

## 3. The cohomology of algebraic groups and $\mathcal{P}_R$

In this section, we describe the algebro-geometric analogue of the setting of section 2. We now think of the symplectic, the orthogonal or the general linear groups as *affine algebraic group schemes*, and we interpret their cohomology (in the algebro-geometric meaning) as functor homology. In this setting, the functor category involved is Friedlander and Suslin's category $\mathcal{P}_R$ of strict polynomial functors. Such functor cohomology interpretations have played a key role to prove the finite generation of the cohomology algebras of finite group schemes [**24**] and more generally reductive group schemes [**40**].

**3.1. Representations of affine group schemes.** In this section, we recall basic facts of the representation theory of affine algebraic group schemes and the relation to the representation theory of discrete groups.

3.1.1. *Definitions.* We fix a commutative ring $R$, and we denote by $R$-alg the category of commutative, unital and finitely generated $R$-algebras. An *affine algebraic group scheme over $R$* is a representable functor

$$G_R : R\text{-alg} \to \text{Groups}\ .$$

For example, we denote by $GL_{n,R}$ the functor which sends a $R$-algebra $A$ to the group $GL_n(A)$ of invertible matrices. It is represented by the $R$-algebra $R[x_{i,j},t]_{1\le i,j\le n}/ < \det[x_{i,j}]t = 1 >$. Orthogonal and symplectic groups can be considered as affine group schemes as well, and we denote them by $O_{n,n,R}$ and

$Sp_{n,R}$. (This notation is intended to distinguish them from the discrete groups $O_{n,n}(R)$ and $Sp_n(R)$ from section 2).

A representation of $G_R$ (or a $G_R$-module) is a $R$-module $M$, endowed with a natural transformation $G_R \to GL_M$, where $GL_M$ denotes the (maybe not representable) functor $R$-alg $\to$ Groups which sends an algebra $A$ to the group $GL_A(A\otimes_R M)$ of $A$-linear invertible endomorphisms of $A \otimes_R M$. A morphism of $G_R$-modules is a linear map $f \in \mathrm{Hom}_R(M,N)$, such that for all $R$-algebras $A$, the map $\mathrm{Id}_A \otimes_R f$ commutes with the action of $G_R(A)$. Under reasonable hypothesis (the algebra representing $G_R$ is flat over $R$), the category $G_R$-Mod is a $R$-linear abelian category with enough injectives, and the cohomology of $G_R$ with coefficients in a representation $M$ is

$$H^*(G_R, M) := \mathrm{Ext}^*_{G_R\text{-Mod}}(R, M) \;,$$

where the left argument in the Ext is the trivial representation.

Remark 3.1. In contrast to the representations of discrete groups, the category $G_R$-Mod almost never has enough projectives (even for $GL_{n,R}$), thus there is no definition for the *homology* of group schemes. We refer the reader to [**23**] for a short introduction to the cohomology of group schemes, and to [**26**] for full details.

3.1.2. *Affine group schemes versus discrete groups.* The cohomology of affine group schemes and the cohomology of discrete groups are related in the following way. Let $G_R$ be an affine algebraic group scheme. For all $R$-algebras $A$, the discrete group $G_R(A)$ is called the group of $A$-points of $G_R$. Evaluation on $A$ yields an exact forgetful functor (where $AG_R(A)$ is the algebra of the discrete group $G_R(A)$ over $A$):

$$G_R\text{-Mod} \to AG_R(A)\text{-Mod} \;.$$

Hence the cohomology of algebraic group schemes is related to the cohomology of discrete groups of points by natural $A$-linear morphisms (induced by evaluation on $A$ and extension of scalars):

$$\begin{aligned} &A \otimes_R \mathrm{Ext}^*_{G_R\text{-Mod}}(M,N) \to \mathrm{Ext}^*_{AG_R(A)}(A\otimes_R M, A\otimes_R N) \;,\\ &A \otimes_R H^*(G_R, M) \to H^*(G_R(A), A\otimes_R M) \;. \end{aligned}$$

The properties of these morphisms are not understood in general. However, the situation is pretty well understood over finite fields, thanks to the work of Cline, Parshall, Scott and van der Kallen [**6**]. Their result applies in particular to the group schemes $GL_{n,\mathbb{F}_p}$, $Sp_{n,\mathbb{F}_p}$ and $O_{n,n,\mathbb{F}_p}$ over a prime field $\mathbb{F}_p$. To describe their result, we have to introduce a few notations. Let $G_{\mathbb{F}_p}$ be a group scheme defined over $\mathbb{F}_p$. We denote by $F^r : G_{\mathbb{F}_p} \to G_{\mathbb{F}_p}$ the morphism of group schemes which sends for all $A$ a matrix $[a_{i,j}] \in G_{\mathbb{F}_p}(A)$ to the matrix $[a_{i,j}^{p^r}] \in G_{\mathbb{F}_p}(A)$.

Remark 3.2. The natural transformation $F^r : G_{\mathbb{F}_p} \to G_{\mathbb{F}_p}$ is not an isomorphism of group schemes. However, for all field $\Bbbk$ of characteristic $p$, it induces an isomorphism of groups $G_{\mathbb{F}_p}(\Bbbk) \to G_{\mathbb{F}_p}(\Bbbk)$.

If $M$ is a $G_{\mathbb{F}_p}$-module, we let $M^{(r)}$ be its $r$-th Frobenius twist. Concretely, $M^{(r)}$ is the $\mathbb{F}_p$-vector space $M$, equipped with the modified action $\rho_{M^{(r)}}$ defined as the composite:

$$G_{\mathbb{F}_p} \xrightarrow{F^r} G_{\mathbb{F}_p} \xrightarrow{\rho_M} GL_M \;.$$

If $\Bbbk$ is a field with $p^r$ elements, we observe that the $\Bbbk G_{\mathbb{F}_p}(\Bbbk)$-modules $\Bbbk \otimes_{\mathbb{F}_p} M$ and $\Bbbk \otimes_{\mathbb{F}_p} M^{(r)}$ are equal. We can now state [**6**, Thm 6.6].

THEOREM 3.3. *Let $G_{\mathbb{F}_p}$ be a reductive algebraic group scheme defined and split over the ground ring $\mathbb{F}_p$, let $M$ be a $G_{\mathbb{F}_p}$-module. Let $i$ be a nonnegative integer. Assume that $r$ is big enough (with respect to $i$) and that $q = p^n$ is big enough (with respect to $i$, and some constant depending on $M$). Then the evaluation map:*

$$\mathbb{F}_q \otimes_{\mathbb{F}_p} H^i(G_{\mathbb{F}_p}, M^{(r)}) \to H^i(G_{\mathbb{F}_p}(\mathbb{F}_q), \mathbb{F}_q \otimes_{\mathbb{F}_p} M^{(r)}) \simeq H^i(G_{\mathbb{F}_p}(\mathbb{F}_q), \mathbb{F}_q \otimes_{\mathbb{F}_p} M)$$

*is an isomorphism.*

The original theorem of Cline, Parshall, Scott and van der Kallen actually gives explicit bounds for $q$ and $r$, which we have omitted for the sake of simplicity. Theorem 3.3 shows a strong connection in positive characteristic between the representation theory of the reductive algebraic groups schemes, and the representation theory of the finite groups of Lie type. Another example of this type of connection is the theorem of Steinberg which relates the simple $G_{\mathbb{F}_p}(\mathbb{F}_q)$-modules to the simple $G_{\mathbb{F}_p}$-modules. (The reader might consult [**25**] for more results of that kind).

REMARK 3.4. If a reductive algebraic group scheme $G_{\mathbb{F}_p}$ is connected, its cohomology with trivial coefficients $H^*(G_{\mathbb{F}_p}, \mathbb{F}_p)$ is zero in positive degrees [**26**, II.4.13] (this follows from Kempf vanishing theorem in sheaf cohomology). Moreover, the Frobenius twist of the trivial representation is the trivial representation. Thus, from theorem 3.3 we retrieve the fact (originally proved by Quillen [**33**] for $GL_n(\mathbb{F}_q)$, and by Fiedorowicz and Priddy [**17**] for other finite groups of Lie type) that the degree $i$ cohomology with trivial coefficients $\mathbb{F}_p$ of the finite groups of Lie type $G_{\mathbb{F}_p}(\mathbb{F}_q)$ vanishes for $0 < i \leq n(q)$, where $n(q)$ is an explicit increasing function of $q$.

**3.2. Strict polynomial functors.** We want to interpret the cohomology of algebraic group schemes in the same fashion as in section 2. So we are first looking for some category of functors, temporarily denoted 'Func$_R$', together with an exact evaluation functor:

$$\mathrm{Func}_R \to GL_{n,R}\text{-Mod}$$

The category of ordinary functors $\mathcal{F}_R$ is not a good choice for Func$_R$. Indeed, if $F$ is an ordinary functor, then $F(R^n)$ has a canonical action of the discrete group $GL_n(R)$, that is of the group of $R$-points of the algebraic group scheme $GL_{n,R}$. But there is no canonical way (and sometimes, no way at all) to extend this action of $GL_n(R)$ to an action of the group scheme $GL_{n,R}$. The solution is to use for Func$_R$ a functor category whose objects are ordinary functors, equipped with some extra data specifying how to extend the action of $GL_n(R)$ into an action of the group scheme $GL_{n,R}$. This leads to the definition of the category of strict polynomial functors $\mathcal{P}_R$ introduced by Friedlander and Suslin in [**24**]. This category fits into a commutative square

(D)
$$\begin{tikzcd} \mathcal{P}_R \arrow[r] \arrow[d, "\mathrm{ev}_n"] & \mathcal{F}_R \arrow[d, "\mathrm{ev}_n"] \\ GL_{n,R}\text{-Mod} \arrow[r] & GL_n(R)\text{-Mod}, \end{tikzcd}$$

where the horizontal arrows are forgetful functors, and the vertical arrows are evaluation maps. The reader may have some intuition of the category $\mathcal{P}_R$ by thinking of diagram $(D)$ as a pullback square defining $\mathcal{P}_R$.

In this section, we state the definition and basic properties of $\mathcal{P}_R$, and some relations to the category of ordinary functors.

3.2.1. *Definitions and basic properties.* In this section, a functor from $R$-alg $\to$ Sets will be called a 'scheme', and natural transformation between such functors will be called a 'morphism of schemes'.[1] Using these names is quite incorrect (actual schemes are required to satisfy additional conditions) but it will simplify the exposition. If $M$ is an $R$-module, we denote by $M^\dagger$ the scheme which assigns an $R$-algebra $A$ to the set $A\otimes_R M$. If $f: M\to N$ is an $R$-linear map, the maps $\mathrm{Id}_A\otimes_R f : A\otimes_R M\to A\otimes_R N$ define a morphism of schemes $f^\dagger: M^\dagger\to N^\dagger$.

The definition of algebraic group schemes and their representations look formally similar to the definition of discrete groups and their representations. The only difference is that sets, and set-theoretic maps in the discrete setting are replaced by schemes and morphisms of schemes in the algebraic group scheme setting.

Definition 3.5 ([**24**, Def 2.1]). A *strict polynomial functor* $F$ is the data of:

(1) for each $M\in P(R)$, an $R$-module $F(M)$,
(2) for each pair $(M,N)\in P(R)^2$, a morphism of schemes:

$$F_{M,N}:\mathrm{Hom}_R(M,N)^\dagger\to \mathrm{Hom}_R(F(M),F(N))^\dagger,$$

satisfying the following conditions:

(i) The set-theoretic map $F_{M,M}(R):\mathrm{End}_R(M)\to\mathrm{End}_R(F(M))$ sends the identity map of $M$ to the identity map of $F(M)$.
(ii) The following diagrams of natural transformations commute (where the horizontal natural transformations are the one induced by composition):

$$\begin{tikzcd}
\mathrm{Hom}_R(N,P)^\dagger\times\mathrm{Hom}_R(M,N)^\dagger \arrow[r, "\circ"] \arrow[d, "F_{N,P}\times F_{M,N}"] & \mathrm{Hom}_R(M,P)^\dagger \arrow[d, "F_{M,P}"] \\
\mathrm{Hom}_R(F(N),F(P))^\dagger\times\mathrm{Hom}_R(F(M),F(N))^\dagger \arrow[r, "\circ"] & \mathrm{Hom}_R(F(M),F(P))^\dagger
\end{tikzcd}$$

A *morphism of strict polynomial functors* is a family of $R$-linear maps $f_M: F(M)\to G(M)$, $M\in P(R)$, such that for all $M,N$ the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_R(M,N)^\dagger \arrow[r, "F_{M,N}"] \arrow[d, "G_{M,N}"] & \mathrm{Hom}_R(F(M),F(N))^\dagger \arrow[d, "(f_N\circ -)^\dagger"] \\
\mathrm{Hom}_R(G(M),G(N))^\dagger \arrow[r, "(-\circ f_M)^\dagger"] & \mathrm{Hom}_R(F(M),G(N))^\dagger
\end{tikzcd}$$

Remark 3.6. Despite their name, strict polynomial functors are not really *functors*, but rather the scheme theoretic equivalent of functors.

For example, symmetric powers $S^d$ can be considered as strict polynomial functors when equipped with the family of natural transformations $S^d_{M,N}$ defined by sending the element

$$a\otimes f\in A\otimes_R \mathrm{Hom}_R(M,N)\simeq \mathrm{Hom}_A(A\otimes_R M, A\otimes_R N)$$

to the element (where $S^d_A$ denote the symmetric power defined over $A$)

$$S^d_A(a\otimes f)\in \mathrm{Hom}_A(S^d_A(A\otimes_R M),S^d_A(A\otimes_R N))\simeq A\otimes_R\mathrm{Hom}_R(S^d(M),S^d(N))\;.$$

Similarly the $d$-th tensor power $\otimes^d$, the $d$-th exterior power $\Lambda^d$ and the $d$-th divided power $\Gamma^d$ can be considered as strict polynomial functors.

[1]Such natural transformations are called 'lois polynomes' in [**35**].

REMARK 3.7. Contrarily to what the case of symmetric powers might suggest, it is not always possible to define a strict polynomial functor from an ordinary functor $F$. Indeed, the assignment $f \mapsto F(f)$ might not be $R$-linear, so the assignment $a \otimes f \mapsto a \otimes F(f)$ is usually not well-defined, and cannot be used to define a family $(F_{M,N})$ satisfying the axioms of definition 3.5. Actually, such a family might not exist for a given ordinary functor $F$.

Let us denote by $\mathrm{Hom}_{M,N} : R\text{-alg} \to \mathrm{Sets}$ the functor which sends an algebra $A$ to the set $\mathrm{Hom}_A(A \otimes_R M, A \otimes_R N)$. Extension of scalars defines a natural transformation $\mathrm{Hom}_R(M,N)^\dagger \to \mathrm{Hom}_{M,N}$, which is an isomorphism if $M \in P(R)$. For all $M \in P(R)$ the composite

$$GL_M \hookrightarrow \mathrm{End}_M \simeq \mathrm{End}_R(M)^\dagger \xrightarrow{F_{M,M}} \mathrm{End}_R(F(M))^\dagger \to \mathrm{End}_{F(M)} \tag{2}$$

defines an action of the group scheme $GL_M$ on $F(M)$.

DEFINITION 3.8. A strict polynomial functor $F$ is homogeneous of degree $d \geq 0$, if for all $M \in P(R)$ the action of the center $GL_{1,R}$ of $GL_M$ on $F(M)$ is given by the formula $\lambda \cdot (a \otimes x) = (\lambda^d a) \otimes x$.

Friedlander and Suslin's category $\mathcal{P}_R$ of strict polynomial functors of finite degree, is the category whose objects are finite sums of homogeneous strict polynomial functors, and whose morphisms are the morphisms of strict polynomial functors. We let $\mathcal{P}_{d,R}$ be the full subcategory of $\mathcal{P}_R$ whose objects are the homogeneous strict polynomial functors of finite degree.

We now list the basic properties of $\mathcal{P}_R$. First, evaluation on a free $R$-module $R^n$ yields an exact functor (the action on $F(R^n)$ is given by the morphism (2))

$$\mathrm{ev}_n : \mathcal{P}_R \to GL_{n,R}\text{-Mod} .$$

Second, $\mathcal{P}_R$ is an abelian category, and $\mathcal{P}_{d,R}$ is an abelian subcategory. Moreover we have a direct sum decomposition:

$$\mathcal{P}_R \simeq \bigoplus_{d \geq 0} \mathcal{P}_{d,R} .$$

This means that each functor decomposes as the direct sum of finitely many homogeneous functors, and that there are no nonzero morphisms between homogeneous functors of different degrees.

Finally, homogeneous strict polynomial functors can be more concretely described as representations of the classical Schur algebras. Recall e.g. from [**30**] that for positive integers $n, d$ the Schur algebra $S(n,d)$ is the algebra $\mathrm{End}_{\mathfrak{S}_d}((R^n)^{\otimes d})$ of $\mathfrak{S}_d$ equivariant endomorphisms of $(R^n)^{\otimes d}$ (where $\mathfrak{S}_d$ acts by permuting the factors of the tensor product as usual). Then all the $GL_{n,R}$-modules in the image of the evaluation functor $\mathrm{ev}_n$ can be equivalently described as $S(n,d)$-modules. For $n \geq d$, the evaluation map actually induces an equivalence of categories

$$\mathcal{P}_{d,R} \simeq S(n,d)\text{-Mod} .$$

The proof is given in [**24**, Thm 3.2] when $R$ is a field but generalizes without change to an arbitrary commutative ring $R$.

3.2.2. *Strict polynomial versus ordinary functors.* By evaluation on the $R$-algebra $R$, a morphism of schemes $\phi : M^\dagger \to N^\dagger$ yields a set theoretic map $\phi(R) : M \to N$. Thus a strict polynomial functor $F = (F(M), F_{M,N})$ yields an ordinary functor $(F(M), F_{M,N}(R))$, which we call the underlying ordinary functor of $F$. For example, the underlying ordinary functor of the $d$-th symmetric power $S^d$ is just the usual $d$-th symmetric power. We obtain in this way a forgetful functor:

$$\mathcal{U} : \mathcal{P}_R \to \mathcal{F}_R\ .$$

It is not hard to see that the forgetful functor sends strict polynomial functors to polynomial functors in the sense of Eilenberg and Mac Lane (i.e. relative to the definition of section 2.1). However, we warn the reader that the two notions do not coincide.

(1) There exist polynomial functors which do not lie in the image of the forgetful functor $\mathcal{U}$.
(2) There exist nonisomorphic strict polynomial functors which are sent to isomorphic functors by $\mathcal{U}$.
(3) The functor $\mathcal{U}$ does not preserve degrees.

This is the reason for the adjective 'strict' in the terminology 'strict polynomial functors'. We shall not go deeply into details about the difference between strict polynomial and polynomial functors, we just give an elementary illustration of both the second and the third phenomenon.

Let $R$ be a field of positive characteristic $p$. We denote by $I^{(r)} \in \mathcal{P}_{p^r,R}$ the $r$-th Frobenius twist functor. To be more specific, $I^{(r)}$ is the intersection of the kernels of the maps $S^{p^r} \to S^i \otimes S^{p^r-i}$, $0 < i < p^r$, induced by the comultiplication of the graded Hopf algebra $S^*$. So for all $R$-vector spaces $V$, $I^{(r)}(V) \subset S^{p^r}(V)$ is the subvector space generated by the $p^r$-th powers of the elements $v \in V$. As a strict polynomial functor, $I^{(r)}$ is homogeneous of degree $p^r$, hence $I^{(r)}$ is not isomorphic to $I^{(\ell)}$ if $r \neq \ell$. However, if $R = \mathbb{F}_{p^r}$, all the underlying ordinary functors $\mathcal{U}I^{(nr)}$, $n \geq 0$ are isomorphic to the identity functor, hence they all have degree one in the sense of Eilenberg and Mac Lane.

Remark 3.9. If $F \in \mathcal{P}_{d,R}$ is a strict polynomial functor, we can precompose it by $I^{(r)}$ to get a strict polynomial functor $F \circ I^{(r)} \in \mathcal{P}_{dp^r,R}$. Let $\mathrm{ev}_n$ be the evaluation functor on $R^n$. Then the $GL_{n,R}$-module $\mathrm{ev}_n(F \circ I^{(r)})$ is isomorphic to the $r$-th Frobenius twist of the $GL_{n,R}$-module $\mathrm{ev}_n(F)$ as defined in section 3.1.2. The fact that the forgetful functor $\mathcal{U}$ sends different functors $I^{(r)}$ to the same ordinary functor (or more generally functors of the form $F \circ I^{(r)}$, for distinct values of $r$, to the same ordinary functor) is the functorial version of the fact already observed in section 3.1.2 that the $RGL_n(R)$-modules $M^{(r)} \otimes R$ and $M \otimes R$ might be equal.

Theorem 3.3 has a functor homology analogue, which was proved by Franjou, Friedlander, Suslin, Scorischenko in [**19**], and by Kuhn in [**29**].

Theorem 3.10. *Let $F, G \in \mathcal{P}_{d,R}$ be homogeneous strict polynomial functors of degree $d$ over a finite field $R$ of cardinal greater or equal to $d$. If $r$ is big enough with respect to $i$, the natural morphism induced by the forgetful map*

$$\mathrm{Ext}^i_{\mathcal{P}_R}(F \circ I^{(r)}, G \circ I^{(r)}) \to \mathrm{Ext}^i_{\mathcal{F}_R}(F \circ I^{(r)}, G \circ I^{(r)}) = \mathrm{Ext}^i_{\mathcal{F}_R}(F, G)$$

*is an isomorphism.*

One of the interests of this theorem is that the left hand side of the isomorphism is easier to compute than the right hand side. The extension groups in $\mathcal{P}_R$ involving functors precomposed by Frobenius twists can be computed from simpler extension groups (without Frobenius twists) [**3**, **39**].

When the field $R$ is replaced by the ring of integers, or by an arbitrary ring commutative, very little is known about the relations between Ext and Tor-groups in $\mathcal{F}_R$ and in $\mathcal{P}_R$.

**3.3. The cohomology of classical groups.** The following theorem was proved by Friedlander and Suslin when $R$ is a field in [**24**, Cor 3.13], and the result generalizes over an arbitrary commutative ring $R$ easily [**37**, Thm 3.10].

THEOREM 3.11. *Let $R$ be a commutative ring, let $F$ and $G$ be strict polynomial functors with values in $P(R)$, and let $n$ be an integer greater than the degree of $F$ and $G$. The evaluation map induces a natural isomorphism*

$$\mathrm{Ext}^*_{\mathcal{P}_R}(F,G) \xrightarrow{\simeq} \mathrm{Ext}^*_{GL_{n,R}}(F(R^n), G(R^n))\;.$$

As in the discrete group case of theorems 2.2 and 2.4, there are bifunctor versions of this theorem [**18**, **37**], allowing more general coefficients for the cohomology of $GL_{n,R}$.

Theorem 3.11 shows major differences between the discrete setting of section 2 and the algebraic setting. First, no limit appears on the right hand side. Actually, as a corollary of theorem 3.11, we obtain that the extensions $\mathrm{Ext}^i_{GL_{n,R}}(F(R^n), G(R^n))$ stabilize when $n$ grows. Observe that in the discrete group case, the analogous theorems 2.2 and 2.4 cannot be used to prove such a stabilization. Moreover, the stabilization is quite brutal: in contrast with the case of discrete groups, stabilization occurs for all commutative ring $R$, and all the $\mathrm{Ext}^i$ stabilize for the same value of $n$. Another difference with the discrete group case is that the cohomology of $GL_{n,R}$ with trivial coefficients does not appear in the statement (in contrast with theorem 2.4). The reason for this is quite simple: unlike the cohomology of discrete groups, the cohomology of connected reductive group schemes (like $GL_{n,R}$) with trivial coefficients is always zero in positive degrees [**26**, II.4.13].

Theorem 3.11 has analogues for other classical groups, proved in [**37**]. Let us explain the symplectic case. First, for all $F \in \mathcal{P}_R$, the $R$-module $F(R^{2n})$ is endowed with an action of the symplectic group by restricting the action of the general linear group to $Sp_{n,R} \subset GL_{2n,R}$. A special role is played by the functors $\Gamma^d \circ \Lambda^2 = ((\Lambda^2)^{\otimes d})^{\mathfrak{S}_d}$. Indeed, for all $d \geq 0$, the antisymmetric bilinear form defining $Sp_{n,R}$ yields an invariant element $w \in \Lambda^2(R^{2n})$, hence an invariant element $w^{\otimes d} \in \Gamma^d(\Lambda^2(R^{2n}))$. So there is a graded morphism

$$\mathrm{Ext}^*_{\mathcal{P}_R}(\Gamma^d \circ \Lambda^2, F) \to \mathrm{Ext}^*_{Sp_{n,R}}(\Gamma^d(\Lambda^2(R^{2n})), F(R^{2n})) \to H^*(Sp_{n,R}, F(R^{2n}))$$

obtained by composing the evaluation on $R^{2n}$ and the pullback along the $Sp_{2n,R}$-equivariant map $R \to \Gamma^d(\Lambda^2(R^{2n}))$, $\lambda \mapsto \lambda w^{\otimes d}$. The following result is obtained in [**37**, Thm 3.17].

THEOREM 3.12. *Let $R$ be a commutative ring, let $F \in \mathcal{P}_{d,R}$ with values in $P(R)$, and let $n$ be an integer greater than half the degree of $F$. Then $H^*(Sp_{n,R}, F(R^{2n}))$ is zero if $d$ is odd, and if $d$ is even, the evaluation map induces an isomorphism:*

$$\mathrm{Ext}^*_{\mathcal{P}_R}(\Gamma^{d/2} \circ \Lambda^2, F) \xrightarrow{\simeq} H^*(Sp_{n,R}, F(R^{2n}))\;.$$

There is a similar result for the orthogonal group schemes $O_{n,n,R}$, when 2 is invertible in $R$, with $\Lambda^2$ replaced by $S^2$ [**37**, Thm 3.24]. Although the statement of theorem 3.12 looks quite similar to the ones of theorems 2.3 and 2.5, the proof is more elementary, and relies on rather different ingredients. It relies on cohomological vanishing results [**26**, II.4.13] (implied by Kempf's vanishing theorem in sheaf cohomology), and on the 'fundamental theorems' of classical invariant theory, proved by Weyl over fields of characteristic zero [**43**] and de Concini and Procesi over an arbitrary ring [**5**].

As in the case of $GL_{n,R}$, theorem 3.12 implies a stabilization result for cohomology. As another example of application, theorem 3.12 can be used to study cup products in the stable range. If $M$ and $N$ are $Sp_{n,R}$-modules, there is an (external) cup product

$$H^i(Sp_{n,R}, M) \otimes H^j(Sp_{n,R}, N) \xrightarrow{\cup} H^{i+j}(Sp_{n,R}, M \otimes_R N) .$$

One might want to use the cup product to construct classes in the cohomology of $Sp_{n,R}$ with coefficients in tensor products. But this does not work so well in general because the cup product $c_1 \cup c_2$ of two nonzero classes might very well be zero. This cancelling might occur even when $c_1$ and $c_2$ are degree zero classes, and even when $R$ is a field of characteristic zero. The functor homology interpretation has the following somewhat surprising consequence [**37**, Cor. 6.2].

COROLLARY 3.13. *Let $R$ be a field. Let $F$ and $G$ be strict polynomial functors of respective degrees $d_1$, $d_2$. If $2n \geq d_1 + d_2$, the cup product induces an injective map*

$$H^*(Sp_{n,R}, F(R^{2n})) \otimes H^*(Sp_{n,R}, G(R^{2n})) \hookrightarrow H^*(Sp_{n,R}, F(R^{2n}) \otimes G(R^{2n})) .$$

## 4. Ringel duality in algebraic topology

The derived category of strict polynomial functors is equipped with a Ringel duality operator $\Theta : \mathbf{D}^b(\mathcal{P}_R) \to \mathbf{D}^b(\mathcal{P}_R)$, which originates from the representation theory of quasi-hereditary algebras and the theory of tilting modules [**15**, **34**]. On the other hand, Dold and Puppe defined [**11**, **12**] a notion of derived functors for ordinary (not necessarily additive) functors, related to algebraic topology computations. If $F \in \mathcal{F}_R$, we denote by $L_iF(M; n)$ the value of its $i$-th derived functor (with height $n$) on a given $R$-module $M$. In this section, we present the connection between the two theories, worked out in [**38**]. Namely, if $F \in \mathcal{P}_{d,R}$ is a strict polynomial functor and $M \in P(R)$, there is an isomorphism:

$$L_{nd-i}F(M; n) \simeq H^i\left(\Theta^n F(M)\right) .$$

As a corollary of this isomorphism, we can provide functor homology interpretations of Dold Puppe derived functors, and therefore provide new links between problems of representation theory and algebraic topology computations.

**4.1. Derived functors of non additive functors.** Let $F : R\text{-Mod} \to R\text{-Mod}$ be an arbitrary functor. For all $R$-modules $M$, there exists a unique (up to weak simplicial homotopy equivalence) simplicial $R$-module $M'$ with each $M'_i$ a projective $R$-module, and such that $\pi_i(M')$ is zero for $i \neq n$, and equals $M$ for $i = n$. The $i$-th derived functor of $F$ with height $n$

$$L_iF(-, n) : R\text{-Mod} \to R\text{-Mod}$$

is the functor sending an $R$-module $M$ to the $i$-th homotopy group of the simplicial $R$-module $F(M')$.

This definition was introduced by Dold and Puppe [**11**, **12**], and later generalized by Quillen [**32**] using model categories. When the functor $F$ is additive, one recovers the usual notion of derived functors in homological algebra [**42**, Chap. 3]. To be more specific, for all nonnegative integers $n$ there is an isomorphism (where the functor on the right hand side is the usual derived functor)

$$L_{i+n}F(M;n) \simeq L_iF(M)\ .$$

Remark 4.1. The existence and uniqueness of $M'$ follows from the Dold-Kan correspondence [**10**], which asserts that the category $sR$-Mod of simplicial $R$-modules is equivalent to the category $\mathrm{Ch}_{\geq 0}(R\text{-Mod})$ of nonnegative chain complexes of $R$-modules, by an equivalence preserving homotopies. The equivalence of categories is the normalized chain functor $N : sR\text{-Mod} \to \mathrm{Ch}_{\geq 0}(R\text{-Mod})$, and it has an explicit inverse $K$. Thus, if $M$ is an $R$-module and $P^M$ a projective resolution of $M$, one may take for $M'$ the simplicial $R$-module $K(P^M[n])$.

The derived functors of non additive functors are related to algebraic topology computations. For example, the derived functors of symmetric powers $S^d$ : $R$-Mod $\to$ $R$-Mod are related to the homology of Eilenberg-Mac Lane spaces [**12**, Satz 4.16] and the homology of symmetric products of spaces [**10**, section 7]. The derived functors of the free Lie functors $\mathcal{L}^d : R\text{-Mod} \to R\text{-Mod}$ are related to the homotopy groups of spheres, via the Curtis spectral sequence [**7**].

We finish the section by two elementary observations, which make the link with strict polynomial functors. First, if $R$ is a commutative ring, and $M$ an $R$-module, then $M' = K(M[n]) = K(R[n]) \otimes_R M$. Thus, if $F$ is a strict polynomial functor (as for example when $F$ is a symmetric power or a free Lie functor), then the derived functors $L_iF(-,n)$ are strict polynomial functors (of the same degree) as a consequence of the following easy lemma.

Lemma 4.2 (First parameterization lemma). *If $F \in \mathcal{P}_{d,R}$, then for all $M \in P(R)$, the functor $F_M : N \to F(M \otimes N)$ is canonically endowed with the structure of a homogeneous strict polynomial functor of degree $d$. Moreover, an $R$-linear map $f : M \to N$ induces a morphism of strict polynomial functors $F_M \to F_N$.*

Second, rather than studying the derived functors $L_iF(M,n)$, we can study the complex formed by the normalized chains of the simplicial object $F(K(R[n])\otimes_R M)$. This yields a bounded (because we have taken *normalized* chains) complex of strict polynomial functors, which we denote by $L(F;n)$. This definition extends when $F$ is replaced by a complex $C$ of strict polynomial functors: we define $L(C;n)(M)$ as the total complex of the bicomplex $C(K(R[n]) \otimes_R M)$. This yields a functor

$$L(-;n) : \mathbf{D}(\mathcal{P}_{d,R}) \to \mathbf{D}(\mathcal{P}_{d,R})\ ,$$

where $\mathbf{D}(\mathcal{P}_{d,R})$ denotes the (unbounded) derived category of homogeneous strict polynomial functors of degree $d$. We can also consider $L(-;n)$ as an endofunctor of the bounded, bounded above or bounded below derived categories.

**4.2. Ringel duality.** Ringel duality was introduced in the context of the representation theory of quasi-hereditary algebras by Ringel [**34**], and worked out for the Schur algebra by Donkin [**15**]. Since $\mathcal{P}_{d,R}$ is equivalent to a category of modules over the Schur algebra, Ringel duality can be transposed in the realm of strict polynomial functors [**2**, **28**, **38**].

If $F$ and $G$ are homogeneous strict polynomial functors of degree $d$, we define the Hom-group with parameter $M \in P(R)$ by

$$\underline{\mathrm{Hom}}_{\mathcal{P}_{d,R}}(F,G)(M) = \mathrm{Hom}_{\mathcal{P}_{d,R}}(F, G_M)\ ,$$

where $G_M$ is the strict polynomial functor $N \mapsto G(M \otimes_R N)$ of lemma 4.2. The following lemma is an easy check.

Lemma 4.3 (Second parameterization lemma). *If $F, G \in \mathcal{P}_{d,R}$, then $\underline{\mathrm{Hom}}_{\mathcal{P}_{d,R}}(F,G)$ is canonically endowed with the structure of a homogeneous strict polynomial functor of degree d. Moreover, morphisms of strict polynomial functors $F \to F'$ and $G \to G'$ induce morphisms of strict polynomial functors between the corresponding parameterized* Hom*s.*

In particular, by placing the $d$-th divided power $\Lambda^d$ as the first argument of parameterized Homs, we get a functor

$$\underline{\mathrm{Hom}}_{\mathcal{P}_{d,R}}(\Lambda^d, -) : \mathcal{P}_{d,R} \to \mathcal{P}_{d,R}\ .$$

If $R$ is a field of characteristic zero, this functor is an equivalence of categories, but this is not the case over a general ring $R$. For example, it is not hard to compute

$$\begin{aligned}
&\underline{\mathrm{Hom}}_{\mathcal{P}_{d,R}}(\Lambda^d, S^d) = \Lambda^d\ ,\\
&\underline{\mathrm{Hom}}_{\mathcal{P}_{d,R}}(\Lambda^d, \Lambda^d) = \Gamma^d\ ,\\
&\underline{\mathrm{Hom}}_{\mathcal{P}_{d,R}}(\Lambda^d, \Gamma^d) = \Gamma^d \text{ if } R \text{ has characteristic 2 and } \Lambda^d \text{ otherwise.}
\end{aligned}$$

The Ringel duality operator $\Theta$ is defined as the right derived functor of the functor $\underline{\mathrm{Hom}}_{\mathcal{P}_{d,R}}(\Lambda^d, -)$:

$$\Theta = \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{P}_{d,R}}(\Lambda^d, -) : \mathbf{D}(\mathcal{P}_{d,R}) \to \mathbf{D}(\mathcal{P}_{d,R})\ .$$

It restricts to an endofunctor of the bounded, bounded above and bounded below derived categories, still denoted by $\Theta$. The following theorem [**2**, **28**] explains the interest of Ringel duality.

Theorem 4.4. *The functor $\Theta$ is an equivalence of categories.*

Remark 4.5. Despite the name 'duality', $\Theta$ is not an involution. The name 'duality' becomes clearer when we look at the seminal paper of Ringel [**34**] written in the context of representations of finite dimensional algebras. To be more specific, some finite dimensional algebras $A$ admit a tilting module $T$ (roughly $T$ is a $A$-module admitting some nice filtrations) and the dual algebra $A'$ is then defined as the endomorphism algebra $\mathrm{End}_A(T)$. The original Ringel duality is an equivalence of categories between $\mathbf{D}^b(A)$ and $\mathbf{D}^b(A')$. Donkin proved [**15**] that the Schur algebra $S(n,d)$, with $n \geq d$ is self-dual. In this case, Ringel duality becomes an auto-equivalence of the category $\mathbf{D}^b(S(n,d))$ (or equivalently of the category $\mathcal{P}_{d,R}$.)

### 4.3. Functor homology interpretation and applications.

4.3.1. *Ringel duality and derived functors.* The following relation between Ringel duality and derivation of functors was proved in [**38**, Thm 5.5].

THEOREM 4.6. *There is an isomorphism of endofunctors of the (unbounded, bounded, bounded above or bounded below) derived category of $\mathcal{P}_{d,R}$*

$$\Theta^n \simeq L(-;n)[-nd]\;,$$

*where $\Theta^n$ denotes the $n$ fold composition of $\Theta$ and $[-nd]$ denotes the suspension. In particular, for all $F \in \mathcal{P}_{d,R}$ and all $M \in P(R)$ there are natural isomorphisms*

$$H^i\left(\Theta^n F\right)(M) \simeq L_{nd-i}F(M;n)\;.$$

In [**38**], we stated theorem 4.6 for the bounded derived category, under the hypothesis that the ground ring $R$ is a PID. This case covered all the applications we had in mind. But this restriction is not really necessary: we used it to keep our presentation of Ringel duality as close as possible to the one in [**2**], and theorem 4.6 above is actually valid over an arbitrary ring $R$.

SKETCH OF PROOF OF THEOREM 4.6. First the simplicial $R$-module $K(R[n])$ is homotopy equivalent to $K(R[1])^{\otimes n}$. Thus $L(C;n)(M)$ is homotopy equivalent to the complex $C(K(R[1])^{\otimes n} \otimes_R M)$. In particular, the operator $L(-;n) : \mathbf{D}(\mathcal{P}_{d,R}) \to \mathbf{D}(\mathcal{P}_{d,R})$ is isomorphic to the $n$-fold composition of $L(-;1)$. Hence, it suffices to prove theorem 4.6 for the case $n = 1$.

For $M \in P(R)$, let us denote by $Q(M)$ the normalized chain complex associated to the cosimplicial object $\Gamma^d(K(R[1])^\vee \otimes_R M)$ (here $K(R[1])^\vee$ is the cosimplicial object which is the $R$-linear dual of the simplicial object $K(R[1])$. So $Q(M)$ looks as follows:

$$\Gamma^d(M) \to \Gamma^{d-1}(M) \otimes M \oplus M \otimes \Gamma^{d-1}(M) \to \cdots \to M^{\otimes d}\;,$$

with $\Gamma^d(M)$ in degree 1 and $M^{\otimes d}$ in degree $d$. Moreover it is well known that the homology of $Q(M)$ is zero everywhere, except in degree $d$ where it equals $\Lambda^d(M)$. Thus $Q[d]$ is a projective resolution of $\Lambda^d$, so that

$$(*) \qquad \Theta(C) \simeq \underline{\mathrm{Hom}}_{\mathcal{P}_{d,R}}(Q[d], C) = \underline{\mathrm{Hom}}_{\mathcal{P}_{d,R}}(Q, C)[d]\;.$$

If $F$ is a functor, and $X$ is a cosimplicial object in $P(R)$, we can parameterize $F$ by the cosimplicial object $X$ as in lemma 4.2 to get a cosimplicial strict polynomial functor $F_X$. For example, $P$ is just the normalized chains of $\Gamma^d_{K(R[1])^\vee}$. If $C$ is a complex of strict polynomial functors, there is an isomorphism of complexes of simplicial strict polynomial functors

$$\underline{\mathrm{Hom}}_{\mathcal{P}_{d,R}}(\Gamma^d_{K(R[1])^\vee}, C) \simeq C_{K(R[1])}\;.$$

So by taking first normalized chains degreewise, and then total complexes, we get an isomorphism:

$$(**) \qquad \underline{\mathrm{Hom}}_{\mathcal{P}_{d,R}}(Q, C) \simeq L(C;1)\;.$$

By composing the isomorphisms $(*)$ and $(**)$ together, we obtain the required isomorphism. □

REMARK 4.7. Tensor products yield monoidal products $\otimes : \mathbf{D}^b(\mathcal{P}_{d,R}) \times \mathbf{D}^b(\mathcal{P}_{e,R}) \to \mathbf{D}^b(\mathcal{P}_{d+e,R})$. It is not hard to prove that $\Theta$ and $L(-;n)$ are monoidal. In

[**38**, Thm 5.5], we also proved the slightly delicate compatibility of the asserted isomorphism with products. This forces to keep track of many signs in the proofs.

4.3.2. *Applications.* Theorem 4.6 leads to functor homology interpretations of derived functors in the sense of Dold and Puppe. Let us denote by $\underline{\mathrm{Ext}}^*_{\mathcal{P}_{d,R}}(F,G)$ the parameterized extension groups $\underline{\mathrm{Ext}}^*_{\mathcal{P}_{d,R}}(F,G)(M) = \mathrm{Ext}^*_{\mathcal{P}_{d,R}}(F,G_M)$, for all $M \in P(R)$. The case $n = 1$ in theorem 4.6 yields an isomorphism:

$$\underline{\mathrm{Ext}}^i_{\mathcal{P}_{d,R}}(\Lambda^d, G)(M) \simeq L_{d-i}G(M;1)\ . \tag{3}$$

Since $\Lambda^d = \Theta^{-1}(\Gamma^d)$, we also have a functor homology interpretation of derived functors with height 2:

$$\underline{\mathrm{Ext}}^i_{\mathcal{P}_{d,R}}(S^d, G)(M) \simeq L_{2d-i}G(M;2)\ . \tag{4}$$

Now the simplicial $R$ module $K(R[1])^{\otimes n}$ is homotopy equivalent to $K(R[n])$, so that $L_*G(M^{\otimes n};n) \simeq L_*(G \circ \otimes^d)(M;1)$ when $M \in P(R)$, so we also have functor homology interpretations of higher derived functors:

$$\underline{\mathrm{Ext}}^i_{\mathcal{P}_{d,R}}(\Lambda^{nd}, G \circ \otimes^n)(M) \simeq L_{nd-i}G(M^{\otimes n};n)\ . \tag{5}$$

For other functor homology interpretations of this kind, see [**38**, Section 6.2]. Such functor homology interpretations enable nontrivial computations of derived functors in the sense of Dold and Puppe. Indeed, some available methods of representation theory for studying the left hand side (such as block theory, or highest weight categories) are not available on the right hand side. Conversely, some algebraic topology computations (like the homology of Eilenberg-Mac Lane spaces) yield new Ext-computations for strict polynomial functors.

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