Frobenius twists and the cohomology of algebraic groups

Dave Benson's Birthday Conference

Antoine Touzé

Frobenius twists and the cohomology of algebraic groups

Main character of the story :

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F, G_{E_r})$$

Main contributors :
Chałupnik (2015), Touzé (2012 & 2013)

1. A motivation
2. Strict polynomial functors
3. The formula and its uses
4. Relation to finite generation theorems

$GL_n(\mathbb{F})$ $\quad \mathbb{F}$ finite field of char p.

$GL_n(\mathbb{F})$ $\mathbb{F}$ finite field of char p.
has two meanings :

Group scheme $GL_{n,\mathbb{F}}$	Finite group $GL_n(\mathbb{F})$

$GL_n(\mathbb{F})$ $\mathbb{F}$ finite field of char p.
has two meanings :

Group scheme $GL_{n,\mathbb{F}}$	Finite group $GL_n(\mathbb{F})$
	$GL_n(\mathbb{F})$-Mod Obj $= (V, \rho)$ V : $\mathbb{F}$-vect. space $\rho : GL_n(\mathbb{F}) \to GL(V)$ morph. gps.

$GL_n(\mathbb{F})$ $\mathbb{F}$ finite field of char p.
has two meanings :

Group scheme $GL_{n,\mathbb{F}}$	Finite group $GL_n(\mathbb{F})$
$GL_{n,\mathbb{F}}$-Mod	$GL_n(\mathbb{F})$-Mod
Obj $= (V, \rho)$ V : $\mathbb{F}$-vect. space $\rho : GL_{n,\mathbb{F}} \to GL_V$ morph. gp sch.	Obj $= (V, \rho)$ V : $\mathbb{F}$-vect. space $\rho : GL_n(\mathbb{F}) \to GL(V)$ morph. gps.

$GL_n(\mathbb{F})$ $\mathbb{F}$ finite field of char p.
has two meanings :

Group scheme $GL_{n,\mathbb{F}}$	Finite group $GL_n(\mathbb{F})$
$GL_{n,\mathbb{F}}$-Mod	$GL_n(\mathbb{F})$-Mod
Obj $= (V,\rho)$ V : $\mathbb{F}$-vect. space $\rho : GL_{n,\mathbb{F}} \to GL_V$ morph. gp sch.	Obj $= (V,\rho)$ V : $\mathbb{F}$-vect. space $\rho : GL_n(\mathbb{F}) \to GL(V)$ morph. gps.

Repres. theories related by taking $\mathbb{F}$-points :

$$\begin{array}{ccc} GL_{n,\mathbb{F}_p}\text{-Mod} & \xrightarrow{\mathbb{F}\text{-points}} & GL_n(\mathbb{F})\text{-Mod} \\ V & \mapsto & V\otimes\mathbb{F} \end{array}$$

Taking $\mathbb{F}$-points in cohomology

$$\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V\quad,W\quad)\otimes\mathbb{F}\to\mathrm{Ext}^i_{GL_n(\mathbb{F})}(V\otimes\mathbb{F},W\otimes\mathbb{F})$$

Taking $\mathbb{F}$-points in cohomology

$$\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V \quad, W \quad) \otimes \mathbb{F} \longrightarrow \mathrm{Ext}^i_{GL_n(\mathbb{F})}(V \otimes \mathbb{F}, W \otimes \mathbb{F})$$

Frobenius twist of (V, ρ)

$$V + \qquad\qquad GL_{n,\mathbb{F}_p} \xrightarrow{\rho} GL_V$$

Taking $\mathbb{F}$-points in cohomology

$$\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V \quad, W \quad) \otimes \mathbb{F} \longrightarrow \mathrm{Ext}^i_{GL_n(\mathbb{F})}(V \otimes \mathbb{F}, W \otimes \mathbb{F})$$

Frobenius twist of (V, ρ)

$$V^{(r)} = V + \begin{array}{ccccc} GL_{n,\mathbb{F}_p} & \xrightarrow{\mathrm{Frob}^r} & GL_{n,\mathbb{F}_p} & \xrightarrow{\rho} & GL_V \\ [a_{ij}] & \mapsto & [a_{ij}^{p^r}] & & \end{array}$$

Taking $\mathbb{F}$-points in cohomology

$$\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V \quad , W \quad) \otimes \mathbb{F} \longrightarrow \mathrm{Ext}^i_{GL_n(\mathbb{F})}(V \otimes \mathbb{F}, W \otimes \mathbb{F})$$

Frobenius twist of (V, ρ)

$$V^{(r)} = V + \begin{array}{ccccc} GL_{n,\mathbb{F}_p} & \xrightarrow{\mathrm{Frob}^r} & GL_{n,\mathbb{F}_p} & \xrightarrow{\rho} & GL_V \\ [a_{ij}] & \mapsto & [a_{ij}^{p^r}] & & \end{array}$$

NB : take $\mathbb{F}$-points, if $\mathrm{card}\,\mathbb{F}$ divides p^r,
then : $\mathrm{Frob}^r = \mathrm{Id}_{GL_n(\mathbb{F})}$.

$\rightarrow$ the decoration $^{(r)}$ vanishes after taking $\mathbb{F}$-points.

Taking $\mathbb{F}$-points in cohomology

$$\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)}) \otimes \mathbb{F} \longrightarrow \mathrm{Ext}^i_{GL_n(\mathbb{F})}(V \otimes \mathbb{F}, W \otimes \mathbb{F})$$

Frobenius twist of (V, ρ)

$$V^{(r)} = V + \begin{array}{ccccc} GL_{n,\mathbb{F}_p} & \xrightarrow{\mathrm{Frob}^r} & GL_{n,\mathbb{F}_p} & \xrightarrow{\rho} & GL_V \\ [a_{ij}] & \mapsto & [a_{ij}^{p^r}] & & \end{array}$$

NB : take $\mathbb{F}$-points, if $\mathrm{card}\,\mathbb{F}$ divides p^r,
then : $\mathrm{Frob}^r = \mathrm{Id}_{GL_n(\mathbb{F})}$.

$\rightarrow$ the decoration $^{(r)}$ vanishes after taking $\mathbb{F}$-points.

Taking $\mathbb{F}$-points in cohomology

Thm : (Cline Parshall Scott Van der Kallen 77 for GL_n)
If r and $\mathrm{card}\,\mathbb{F}$ big enough (explicit bounds) :

$$\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)}) \otimes \mathbb{F} \xrightarrow{\simeq} \mathrm{Ext}^i_{GL_n(\mathbb{F})}(V \otimes \mathbb{F}, W \otimes \mathbb{F})$$

Frobenius twist of (V, ρ)

$$V^{(r)} = V + \quad \begin{array}{ccccc} GL_{n,\mathbb{F}_p} & \xrightarrow{\mathrm{Frob}^r} & GL_{n,\mathbb{F}_p} & \xrightarrow{\rho} & GL_V \\ [a_{ij}] & \mapsto & [a_{ij}^{p^r}] & & \end{array}$$

NB : take $\mathbb{F}$-points, if $\mathrm{card}\,\mathbb{F}$ divides p^r,
then : $\mathrm{Frob}^r = \mathrm{Id}_{GL_n(\mathbb{F})}$.

$\rightarrow$ the decoration $^{(r)}$ vanishes after taking $\mathbb{F}$-points.

Taking $\mathbb{F}$-points in cohomology

Thm : (Cline Parshall Scott Van der Kallen 77 for GL_n)
If r and $\mathrm{card}\,\mathbb{F}$ big enough (explicit bounds) :

$$\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)}) \otimes \mathbb{F} \xrightarrow{\simeq} \mathrm{Ext}^i_{GL_n(\mathbb{F})}(V \otimes \mathbb{F}, W \otimes \mathbb{F})$$

Taking $\mathbb{F}$-points in cohomology

Thm : (Cline Parshall Scott Van der Kallen 77 for GL_n)
If r and $\operatorname{card}\mathbb{F}$ big enough (explicit bounds) :

$$\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)}) \otimes \mathbb{F} \xrightarrow{\simeq} \mathrm{Ext}^i_{GL_n(\mathbb{F})}(V \otimes \mathbb{F}, W \otimes \mathbb{F})$$

Example : $V = W = \mathbb{k}$.

Taking $\mathbb{F}$-points in cohomology

Thm : (Cline Parshall Scott Van der Kallen 77 for GL_n)
If r and $\mathrm{card}\,\mathbb{F}$ big enough (explicit bounds) :

$$\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)}) \otimes \mathbb{F} \xrightarrow{\simeq} \mathrm{Ext}^i_{GL_n(\mathbb{F})}(V \otimes \mathbb{F}, W \otimes \mathbb{F})$$

Example : $V = W = \mathbb{k}$.

Then $V^{(r)} = W^{(r)} = \mathbb{k}$.

Taking $\mathbb{F}$-points in cohomology

Thm : (Cline Parshall Scott Van der Kallen 77 for GL_n)
If r and $\mathrm{card}\,\mathbb{F}$ big enough (explicit bounds) :

$$\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)}) \otimes \mathbb{F} \xrightarrow{\simeq} \mathrm{Ext}^i_{GL_n(\mathbb{F})}(V \otimes \mathbb{F}, W \otimes \mathbb{F})$$

Example :$V = W = \mathbb{k}$.

Then $V^{(r)} = W^{(r)} = \mathbb{k}$.
Kempf vanishinig Thm $\Rightarrow \mathrm{Ext}^{>0}_{GL_{n,\mathbb{k}}}(\mathbb{k}, \mathbb{k}) = 0$

Taking $\mathbb{F}$-points in cohomology

Thm : (Cline Parshall Scott Van der Kallen 77 for GL_n)
If r and $\mathrm{card}\,\mathbb{F}$ big enough (explicit bounds) :

$$\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)}) \otimes \mathbb{F} \xrightarrow{\simeq} \mathrm{Ext}^i_{GL_n(\mathbb{F})}(V \otimes \mathbb{F}, W \otimes \mathbb{F})$$

Example : $V = W = \Bbbk$.

Then $V^{(r)} = W^{(r)} = \Bbbk$.
Kempf vanishinig Thm $\Rightarrow \mathrm{Ext}^{>0}_{GL_{n,\Bbbk}}(\Bbbk, \Bbbk) = 0$

Corollary of CPSVdK for GL_n :

$H^i(GL_n(\mathbb{F}), \mathbb{F}) = 0$ if $\mathbb{F}$ is big enough. (Quillen)

Taking $\mathbb{F}$-points in cohomology

Thm : (Cline Parshall Scott Van der Kallen 77 for GL_n)
If r and $\mathrm{card}\,\mathbb{F}$ big enough (explicit bounds) :

$$\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)}) \otimes \mathbb{F} \xrightarrow{\simeq} \mathrm{Ext}^i_{GL_n(\mathbb{F})}(V \otimes \mathbb{F}, W \otimes \mathbb{F})$$

Example : $V = W = \mathbb{k}$.

Then $V^{(r)} = W^{(r)} = \mathbb{k}$.
Kempf vanishinig Thm $\Rightarrow \mathrm{Ext}^{>0}_{GL_{n,\mathbb{k}}}(\mathbb{k}, \mathbb{k}) = 0$

Corollary of CPSVdK for GL_n :

$H^i(GL_n(\mathbb{F}), \mathbb{F}) = 0$ if $\mathbb{F}$ is big enough. (Quillen)

Question :
Are the extensions $\mathrm{Ext}^i_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)})$ accessible to computations in general ?

Understanding extensions of the form :
$\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)})$

$\Downarrow$ (CPSVdK 77)

Understanding the link :
$GL_{n,\mathbb{F}_p}$-cohomology $\leftrightarrow$ $GL(\mathbb{F})$-cohomology

Understanding extensions of the form :
$\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)})$

$\Downarrow$ (CPSVdK 77)

Understanding the link :
$GL_{n,\mathbb{F}_p}$-cohomology $\leftrightarrow$ $GL(\mathbb{F})$-cohomology

A qualitative result :

Thm : [CPS 83] $\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V, W) \subset \mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)})$.

Understanding extensions of the form :
$\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)})$

$\Downarrow$ (CPSVdK 77)

Understanding the link :
$GL_{n,\mathbb{F}_p}$-cohomology $\leftrightarrow$ $GL(\mathbb{F})$-cohomology

A qualitative result :

Thm : [CPS 83] $\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V, W) \subset \mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)})$.

A quantitative result ?

Understanding extensions of the form :
$\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)})$

$\Downarrow$ (CPSVdK 77)

Understanding the link :
$GL_{n,\mathbb{F}_p}$-cohomology $\leftrightarrow$ $GL(\mathbb{F})$-cohomology

A qualitative result :

Thm : [CPS 83] $\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V, W) \subset \mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)})$.

A quantitative result ?
$\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)})$ as a functor of $\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V, W)$?

$\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)})$ as a functor of $\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V, W)$?

$$\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)}) \text{ as a functor of } \mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V, W)\,?$$

- This can be realized for V, W stable polynomial representations.

$$\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)}) \text{ as a functor of } \mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V, W)\,?$$

- This can be realized for V, W stable polynomial representations.
- Formulation in terms of strict polynomial functors (Friedlander and Suslin)

$$\mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V^{(r)}, W^{(r)}) \text{ as a functor of } \mathrm{Ext}^*_{GL_{n,\mathbb{F}_p}}(V, W)\,?$$

- This can be realized for V, W stable polynomial representations.
- Formulation in terms of strict polynomial functors (Friedlander and Suslin)
- The formula is also part of another story : cohom finite generation for red. gp schemes.

Friedlander and Suslin 97 : $\mathbb{k}$ a field

Friedlander and Suslin 97 : $\mathbb{k}$ a field

- Strict polynomial functor F (of degree d) $=$

  $F : \mathbb{k}\text{-vect} \to \mathbb{k}\text{-vect}$

  $+$ additional structure : polynomials (of degree d)

Friedlander and Suslin 97 : $\mathbb{k}$ a field

- Strict polynomial functor F (of degree d) $=$

  $F : \mathbb{k}\text{-vect} \to \mathbb{k}\text{-vect}$
  $+$ additional structure : polynomials (of degree d)

  $F_{V,W} : \hom_{\mathbb{k}}(V, W) \to \hom_{\mathbb{k}}(F(V), F(W))$
  which give the action of F on morphisms

Friedlander and Suslin 97 : $\mathbb{k}$ a field

- Strict polynomial functor F (of degree d) $=$
  $F : \mathbb{k}\text{-vect} \to \mathbb{k}\text{-vect}$
  $+$ additional structure : polynomials (of degree d)
- Examples : S^d, Λ^d, Γ^d.

Friedlander and Suslin 97 : $\mathbb{k}$ a field

- Strict polynomial functor F (of degree d) $=$
  $F : \mathbb{k}\text{-vect} \to \mathbb{k}\text{-vect}$
  $+$ additional structure : polynomials (of degree d)
- Examples : S^d, Λ^d, Γ^d.
- Let $\mathbb{k}^n$ be the std. rep. of $GL_{n,\mathbb{k}}$.
  $\to$ $F(\mathbb{k}^n)$ is a polyn rep. of $GL_{n,\mathbb{k}}$ (Schur, Green).

Friedlander and Suslin 97 : $\mathbb{k}$ a field

- Strict polynomial functor F (of degree d) $=$
  $F : \mathbb{k}\text{-vect} \to \mathbb{k}\text{-vect}$
  $+$ additional structure : polynomials (of degree d)
- Examples : S^d, Λ^d, Γ^d.
- Let $\mathbb{k}^n$ be the std. rep. of $GL_{n,\mathbb{k}}$.
  $\to$ $F(\mathbb{k}^n)$ is a polyn rep. of $GL_{n,\mathbb{k}}$ (Schur, Green).

Friedlander and Suslin 97 : $\mathbb{k}$ a field

- Strict polynomial functor F (of degree d) $=$
  $F : \mathbb{k}\text{-vect} \to \mathbb{k}\text{-vect}$
  $+$ additional structure : polynomials (of degree d)
- Examples : S^d, Λ^d, Γ^d.
- Let $\mathbb{k}^n$ be the std. rep. of $GL_{n,\mathbb{k}}$.
  $\to$ $F(\mathbb{k}^n)$ is a polyn rep. of $GL_{n,\mathbb{k}}$ (Schur, Green).
- $\mathcal{P}_{\mathbb{k}} =$ cat. of strict polyn functors

Friedlander and Suslin 97 : $\mathbb{k}$ a field

- Strict polynomial functor F (of degree d) =
  $F : \mathbb{k}\text{-vect} \to \mathbb{k}\text{-vect}$
  $+$ additional structure : polynomials (of degree d)
- Examples : S^d, Λ^d, Γ^d.
- Let $\mathbb{k}^n$ be the std. rep. of $GL_{n,\mathbb{k}}$.
  $\to$ $F(\mathbb{k}^n)$ is a polyn rep. of $GL_{n,\mathbb{k}}$ (Schur, Green).
- $\mathcal{P}_\mathbb{k}$= cat. of strict polyn functors

$$\mathrm{Ext}^*_{\mathcal{P}_\mathbb{k}}(F,G) \qquad \mathrm{Ext}^*_{GL_{n,\mathbb{k}}}(F(\mathbb{k}^n),G(\mathbb{k}^n))$$

Friedlander and Suslin 97 : $\mathbb{k}$ a field

- Strict polynomial functor F (of degree d) $=$
  $F : \mathbb{k}\text{-vect} \to \mathbb{k}\text{-vect}$
  $+$ additional structure : polynomials (of degree d)
- Examples : S^d, Λ^d, Γ^d.
- Let $\mathbb{k}^n$ be the std. rep. of $GL_{n,\mathbb{k}}$.
  $\to$ $F(\mathbb{k}^n)$ is a polyn rep. of $GL_{n,\mathbb{k}}$ (Schur, Green).
- $\mathcal{P}_\mathbb{k}$= cat. of strict polyn functors

$$\mathrm{Ext}^*_{\mathcal{P}_\mathbb{k}}(F,G) \longrightarrow \mathrm{Ext}^*_{GL_{n,\mathbb{k}}}(F(\mathbb{k}^n), G(\mathbb{k}^n))$$

Friedlander and Suslin 97 : $\mathbb{k}$ a field

- Strict polynomial functor F (of degree d) $=$
  $F : \mathbb{k}\text{-vect} \to \mathbb{k}\text{-vect}$
  $+$ additional structure : polynomials (of degree d)
- Examples : S^d, Λ^d, Γ^d.
- Let $\mathbb{k}^n$ be the std. rep. of $GL_{n,\mathbb{k}}$.
  $\to$ $F(\mathbb{k}^n)$ is a polyn rep. of $GL_{n,\mathbb{k}}$ (Schur, Green).
- $\mathcal{P}_\mathbb{k}$= cat. of strict polyn functors

  Thm : if $n \geq \deg F, \deg G$

$$\mathrm{Ext}^*_{\mathcal{P}_\mathbb{k}}(F, G) \xrightarrow{\simeq} \mathrm{Ext}^*_{GL_{n,\mathbb{k}}}(F(\mathbb{k}^n), G(\mathbb{k}^n))$$

Friedlander and Suslin 97 (continued)

- $\mathcal{P}_{\mathbb{k}}$ = category of strict polyn. functors
- Γ^d, S^d, Λ^d.
- $F(\mathbb{k}^n)$ = 'polynomial rep' of $GL_{n,\mathbb{k}}$
- $\mathrm{Ext}^*_{\mathcal{P}_{\mathbb{k}}}(F, G) \simeq \mathrm{Ext}^*_{GL_{n,\mathbb{k}}}(F(\mathbb{k}^n), G(\mathbb{k}^n))$ if n big enough (stable)

Friedlander and Suslin 97 (continued)

- $\mathcal{P}_{\Bbbk}$ = category of strict polyn. functors
- Γ^d, S^d, Λ^d.
- $F(\Bbbk^n)$ = 'polynomial rep' of $GL_{n,\Bbbk}$
- $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F, G) \simeq \mathrm{Ext}^*_{GL_{n,\Bbbk}}(F(\Bbbk^n), G(\Bbbk^n))$ if n big enough (stable)

- $I^{(r)}$ = r-th Frobenius twist functor

Friedlander and Suslin 97 (continued)

- $\mathcal{P}_{\Bbbk}$ = category of strict polyn. functors
- Γ^d, S^d, Λ^d.
- $F(\Bbbk^n)$ = 'polynomial rep' of $GL_{n,\Bbbk}$
- $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F,G) \simeq \mathrm{Ext}^*_{GL_{n,\Bbbk}}(F(\Bbbk^n), G(\Bbbk^n))$ if n big enough (stable)

- $I^{(r)}$ = r-th Frobenius twist functor
  - ▷ $I^{(r)}(V) = \left\langle v^{p^r} \mid v \in V \right\rangle \subset S^{p^r}(V)$

Friedlander and Suslin 97 (continued)

- $\mathcal{P}_{\Bbbk}$ = category of strict polyn. functors
- Γ^d, S^d, Λ^d.
- $F(\Bbbk^n)$ = 'polynomial rep' of $GL_{n,\Bbbk}$
- $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F, G) \simeq \mathrm{Ext}^*_{GL_{n,\Bbbk}}(F(\Bbbk^n), G(\Bbbk^n))$ if n big enough (stable)

- $I^{(r)}$ = r-th Frobenius twist functor
  - ▷ $I^{(r)}(V) = \left\langle v^{p^r} \mid v \in V \right\rangle \subset S^{p^r}(V)$
  - ▷ Precomposing by $I^{(r)} \leftrightarrow$ twisting polynomial rep.

Friedlander and Suslin 97 (continued)

- $\mathcal{P}_{\Bbbk}$ = category of strict polyn. functors
- Γ^d, S^d, Λ^d.
- $F(\Bbbk^n)$ = 'polynomial rep' of $GL_{n,\Bbbk}$
- $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F, G) \simeq \mathrm{Ext}^*_{GL_{n,\Bbbk}}(F(\Bbbk^n), G(\Bbbk^n))$ if n big enough (stable)

- $I^{(r)}$ = r-th Frobenius twist functor
  - ▷ $I^{(r)}(V) = \left\langle v^{p^r} \mid v \in V \right\rangle \subset S^{p^r}(V)$
  - ▷ Precomposing by $I^{(r)} \leftrightarrow$ twisting polynomial rep.

    $F^{(r)} := F \circ I^{(r)}$

    If $V = F(\Bbbk^n)$, then $V^{(r)} = F^{(r)}(\Bbbk^n)$.

Friedlander and Suslin 97 (continued)

- $\mathcal{P}_{\Bbbk}$ = category of strict polyn. functors
- Γ^d, S^d, Λ^d.
- $F(\Bbbk^n)$ = 'polynomial rep' of $GL_{n,\Bbbk}$
- $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F,G) \simeq \mathrm{Ext}^*_{GL_{n,\Bbbk}}(F(\Bbbk^n), G(\Bbbk^n))$ if n big enough (stable)

- $I^{(r)}$ = r-th Frobenius twist functor
  - ▷ $I^{(r)}(V) = \left\langle v^{p^r} \mid v \in V \right\rangle \subset S^{p^r}(V)$
  - ▷ Precomposing by $I^{(r)} \leftrightarrow$ twisting polynomial rep.

    $F^{(r)} := F \circ I^{(r)}$
    If $V = F(\Bbbk^n)$, then $V^{(r)} = F^{(r)}(\Bbbk^n)$.

⇒Question becomes : understand $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)})$

Aim : understand $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)})$

Aim : understand $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)})$

Parametrization of a functor G :
V finite dim $\Bbbk$-vector space

$$G_V := W \mapsto G(V \otimes W)$$

Aim : understand $\mathrm{Ext}^*_{\mathcal{P}_{\mathbb{k}}}(F^{(r)}, G^{(r)})$

Parametrization of a functor G :
V finite dim $\mathbb{k}$-vector space

$$G_V := W \mapsto G(V \otimes W)$$

If V is graded, then G_V inherits a grading

Aim : understand $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)})$

Parametrization of a functor G :
V finite dim $\Bbbk$-vector space

$$G_V := W \mapsto G(V \otimes W)$$

If V is graded, then G_V inherits a grading

Example : $G = S^d$, $V = \Bbbk[0] \oplus \Bbbk[2]$.

$$G_V(W) = S^d(V \otimes W) = S^d(W[0] \oplus W[2])$$

Aim : understand $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F^{(r)}, G^{(r)})$

Parametrization of a functor G :
V finite dim $\Bbbk$-vector space

$$G_V := W \mapsto G(V \otimes W)$$

If V is graded, then G_V inherits a grading

Example : $G = S^d$, $V = \Bbbk[0] \oplus \Bbbk[2]$.

$$G_V(W) = S^d(V \otimes W) = S^d(W[0] \oplus W[2])$$
$$= \bigoplus_{i=0}^{d} S^{d-i}(W) \otimes S^i(W)[2i]$$

Aim : understand $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)})$

Aim : understand $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)})$

The vector space E_r :

$E_r := \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(I^{(r)}, I^{(r)})$

Aim : understand $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)})$

The vector space E_r :

$E_r := \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(I^{(r)}, I^{(r)})$

Explicitly computed by FS 97 :

$$(E_r)^i = \begin{cases} \Bbbk & i \text{ even and } < 2p^r \\ 0 & \text{otherwise} \end{cases}$$

Aim : understand $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)})$

The vector space E_r :

$E_r := \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(I^{(r)}, I^{(r)})$

Explicitly computed by FS 97 :

$$(E_r)^i = \begin{cases} \Bbbk & i \text{ even and } < 2p^r \\ 0 & \text{otherwise} \end{cases}$$

Thm : (take total degree on right hand side)

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r}) \; .$$

Thm : $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

Thm : $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I =$ Identity functor :

$$I_{E_r}(V) = V \otimes E_r = I(V) \otimes E_r$$

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(I^{(r)}, I^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(I, I_{E_r})$$

Thm : $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I =$ Identity functor :

$I_{E_r}(V) = V \otimes E_r = I(V) \otimes E_r$

$$\begin{aligned}\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(I^{(r)}, I^{(r)}) &\simeq \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(I, I_{E_r})\\ &= \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(I, I) \otimes E_r\end{aligned}$$

Thm : $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I =$ Identity functor :

$I_{E_r}(V) = V \otimes E_r = I(V) \otimes E_r$

$$\begin{aligned}\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(I^{(r)}, I^{(r)}) &\simeq \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(I, I_{E_r})\\ &= \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(I, I) \otimes E_r\\ &= \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(I, I) \otimes E_r = E_r\end{aligned}$$

Thm : $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I =$ Identity functor :

$I_{E_r}(V) = V \otimes E_r = I(V) \otimes E_r$

$$\begin{aligned}\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(I^{(r)}, I^{(r)}) &\simeq \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(I, I_{E_r}) \\ &= \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(I, I) \otimes E_r \\ &= \mathrm{Hom}_{\mathcal{P}_\Bbbk}(I, I) \otimes E_r = E_r\end{aligned}$$

$\rightarrow$ Recover FS 97 computation

Thm : $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I \to$ FS 97

Thm : $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I \to$ FS 97
- $F = \Gamma^d,\ G = S^d$:

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(\Gamma^{d\,(r)}, S^{d\,(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(\Gamma^d, S^d_{E_r})$$

Thm : $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I \to$ FS 97
- $F = \Gamma^d,\ G = S^d$:

$$\begin{aligned}\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(\Gamma^{d\,(r)}, S^{d\,(r)}) &\simeq \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(\Gamma^d, S^d_{E_r}) \\ &= \mathrm{Hom}^*_{\mathcal{P}_\Bbbk}(\Gamma^d, S^d_{E_r})\end{aligned}$$

Thm : $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I \to$ FS 97
- $F = \Gamma^d,\ G = S^d$:

$$\begin{aligned}\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(\Gamma^{d\,(r)}, S^{d\,(r)}) &\simeq \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(\Gamma^d, S^d_{E_r})\\ &= \mathrm{Hom}^*_{\mathcal{P}_{\Bbbk}}(\Gamma^d, S^d_{E_r})\\ &\simeq S^d(E_r)\end{aligned}$$

Thm : $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I \to$ FS 97
- $F = \Gamma^d,\ G = S^d$:

$$\begin{aligned}\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(\Gamma^{d\,(r)}, S^{d\,(r)}) &\simeq \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(\Gamma^d, S^d_{E_r}) \\ &= \mathrm{Hom}^*_{\mathcal{P}_\Bbbk}(\Gamma^d, S^d_{E_r}) \\ &\simeq S^d(E_r)\end{aligned}$$

$\to$ Fundamental computation of Franjou-Friedlander-Suslin-Scorichenko 99

Thm : $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I \to$ FS 97
- $F = \Gamma^d,\ G = S^d \to$ FFSS 99

Thm : $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I \to$ FS 97
- $F = \Gamma^d,\ G = S^d \to$ FFSS 99
- $F = W_\lambda,\ G = S_\mu \to$ Chałupnik 05

Thm : $\mathrm{Ext}^*_{\mathcal{P}_{\mathbb{k}}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I \to$ FS 97
- $F = \Gamma^d$, $G = S^d \to$ FFSS 99
- $F = W_\lambda$, $G = S_\mu \to$ Chałupnik 05
- Let $R_d = \mathbb{k} \oplus \mathbb{k}[2d] \oplus \cdots \oplus \mathbb{k}[d(2p^r - 2)]$

Thm : $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I \to$ FS 97
- $F = \Gamma^d,\ G = S^d \to$ FFSS 99
- $F = W_\lambda,\ G = S_\mu \to$ Chałupnik 05
- Let $R_d = \Bbbk \oplus \Bbbk[2d] \oplus \cdots \oplus \Bbbk[d(2p^r - 2)]$

  Then $G_{E_r} = R_d \otimes G \ \oplus$ smthg

Thm : $\mathrm{Ext}^*_{\mathcal{P}_{\mathbb{k}}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I \to$ FS 97
- $F = \Gamma^d,\ G = S^d \to$ FFSS 99
- $F = W_\lambda,\ G = S_\mu \to$ Chałupnik 05
- Let $R_d = \mathbb{k} \oplus \mathbb{k}[2d] \oplus \cdots \oplus \mathbb{k}[d(2p^r - 2)]$

  Then $G_{E_r} = R_d \otimes G \ \oplus$ smthg , so that :

  $$\mathrm{Ext}^*_{\mathcal{P}_{\mathbb{k}}}(F^{(r)}, G^{(r)}) = R_d \otimes \mathrm{Ext}^*_{\mathcal{P}_{\mathbb{k}}}(F, G) \ \oplus \ \text{smthg}$$

Thm : $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*(F, G_{E_r})$.

Examples :

- $F = G = I \to$ FS 97
- $F = \Gamma^d,\ G = S^d \to$ FFSS 99
- $F = W_\lambda,\ G = S_\mu \to$ Chałupnik 05
- Let $R_d = \Bbbk \oplus \Bbbk[2d] \oplus \cdots \oplus \Bbbk[d(2p^r - 2)]$

  Then $G_{E_r} = R_d \otimes G \ \oplus$ smthg , so that :

  $$\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F^{(r)}, G^{(r)}) = R_d \otimes \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F, G) \oplus \text{smthg}$$

  $\to$ Explains many repetitions/periodicities previously observed in computations

Thm (TvdK 10) : G be reductive alg. gp over field $\mathbb{k}$.
$G \curvearrowright A$, A comm. f.g. $\mathbb{k}$-algebra, $H^*(G, A)$ f.g.

Thm (TvdK 10) : G be reductive alg. gp over field $\Bbbk$.
$G \curvearrowright A$, A comm. f.g. $\Bbbk$-algebra, $H^*(G, A)$ f.g.

Ex of G : finite groups, finite gp schemes, $GL_{n,\Bbbk}$, $Sp_{n,\Bbbk}\ldots$

4. Relation to CFG theorem (1)

Thm (TvdK 10) : G be reductive alg. gp over field $\mathbb{k}$.
$G \curvearrowright A$, A comm. f.g. $\mathbb{k}$-algebra, $H^*(G, A)$ f.g.

Ex of G : finite groups, finite gp schemes, $GL_{n,\mathbb{k}}$, $Sp_{n,\mathbb{k}}\dots$

One of the key ingredients in the proof :

Thm : There are classes $c[d] \in H^{2d}(GL_{n,\mathbb{k}}, \Gamma^d(\mathfrak{gl}_n^{(1)}))$ such that

- $c[1] \neq 0$
- $\Delta_*(c[d]) = c[1]^{\cup d}$

Thm (TvdK 10) : G be reductive alg. gp over field $\mathbb{k}$.
$G \curvearrowright A$, A comm. f.g. $\mathbb{k}$-algebra, $H^*(G, A)$ f.g.

Ex of G : finite groups, finite gp schemes, $GL_{n,\mathbb{k}}$, $Sp_{n,\mathbb{k}}\ldots$

One of the key ingredients in the proof :

Thm : There are classes $c[d] \in H^{2d}(GL_{n,\mathbb{k}}, \Gamma^d(\mathfrak{gl}_n^{(1)}))$ such that

- $c[1] \neq 0$
- $\Delta_*(c[d]) = c[1]^{\cup d}$

In the remaining time :

$\exists$ classes $\Leftrightarrow$ The formula computing Ext between twisted functors

Observation : $\Gamma^d(\mathfrak{gl}_n^{(1)})$ is NOT a polynomial representation.

4. Relation to CFG theorem (2)

Observation : $\Gamma^d(\mathfrak{gl}_n^{(1)})$ is NOT a polynomial representation.

Strict polynomial bifunctors

- $\mathcal{P}_{\mathbb{k}}(1,1) =$ category of strict polyn bifunctors (1 contravariant variable, 1 covariant variable)
- Evaluating bifunctors on $(\mathbb{k}^n, \mathbb{k}^n)$ yield $GL_{n,\mathbb{k}}$-modules

Observation : $\Gamma^d(\mathfrak{gl}_n^{(1)})$ is NOT a polynomial representation.

Strict polynomial bifunctors

- $\mathcal{P}_{\Bbbk}(1,1) =$ category of strict polyn bifunctors (1 contravariant variable, 1 covariant variable)
- Evaluating bifunctors on $(\Bbbk^n, \Bbbk^n)$ yield $GL_{n,\Bbbk}$-modules

Functor	value on (V, W)	$GL_{n,\Bbbk}$-module
gl	$Hom_{\Bbbk}(V, W)$	$\mathfrak{gl}_n$
$F(gl \quad)$	$F(Hom_{\Bbbk}(V \quad , W \quad))$	$F(\mathfrak{gl}_n \quad)$

Observation : $\Gamma^d(\mathfrak{gl}_n^{(1)})$ is NOT a polynomial representation.

Strict polynomial bifunctors

- $\mathcal{P}_{\mathbb{k}}(1,1) =$ category of strict polyn bifunctors (1 contravariant variable, 1 covariant variable)
- Evaluating bifunctors on $(\mathbb{k}^n, \mathbb{k}^n)$ yield $GL_{n,\mathbb{k}}$-modules

Functor	value on (V, W)	$GL_{n,\mathbb{k}}$-module
gl	$Hom_{\mathbb{k}}(V, W)$	$\mathfrak{gl}_\mathfrak{n}$
$F(gl^{(1)})$	$F(Hom_{\mathbb{k}}(V^{(1)}, W^{(1)}))$	$F(\mathfrak{gl}_n^{(1)})$

4. Relation to CFG theorem (2)

Observation : $\Gamma^d(\mathfrak{gl}_n^{(1)})$ is NOT a polynomial representation.

<u>Strict polynomial bifunctors</u>

- $\mathcal{P}_{\Bbbk}(1,1)$ = category of strict polyn bifunctors (1 contravariant variable, 1 covariant variable)
- Evaluating bifunctors on $(\Bbbk^n, \Bbbk^n)$ yield $GL_{n,\Bbbk}$-modules

Functor	value on (V, W)	$GL_{n,\Bbbk}$-module
gl	$Hom_{\Bbbk}(V, W)$	$\mathfrak{gl}_\mathfrak{n}$
$F(gl^{(1)})$	$F(Hom_{\Bbbk}(V^{(1)}, W^{(1)}))$	$F(\mathfrak{gl}_n^{(1)})$

- **Thm :** (Franjou Friedlander) $\deg B = (d, d)$ and $n \geq d$

$$H^*(B) := \mathrm{Ext}^*_{\mathcal{P}(1,1)}(\Gamma^d(gl), B) \simeq H^*(GL_{n,\Bbbk}, B(\Bbbk^n, \Bbbk^n)) \;.$$

Example relevant for the universal classes :

$$H^*(\Gamma^d(gl^{(1)})) \simeq H^*(GL_{n,\Bbbk}, \Gamma^d(\mathfrak{gl}_\mathfrak{n}{}^{(1)}))$$

4. Relation to CFG theorem (3)

(C) Universal classes exist

(F) $\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r})$ holds

$(Fweak)$ $\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r})$ holds, up to filtration

$(Bweak)$ $H^*(B^{(r)}) \simeq H^*(B_{E_r})$ holds, up to a filtration

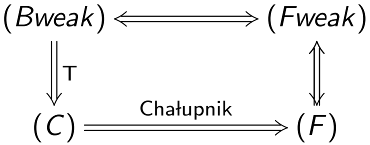
4. Relation to CFG theorem (3)

(C) Universal classes exist

(F) $\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r})$ holds

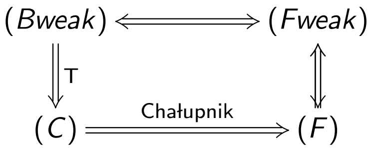
$(Fweak)$ $\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r})$ holds, up to filtration

$(Bweak)$ $H^*(B^{(r)}) \simeq H^*(B_{E_r})$ holds, up to a filtration



4. Relation to CFG theorem (3)

(C)	Universal classes exist
(F)	$\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r})$ holds
$(Fweak)$	$\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r})$ holds, up to filtration
$(Bweak)$	$H^*(B^{(r)}) \simeq H^*(B_{E_r})$ holds, up to a filtration



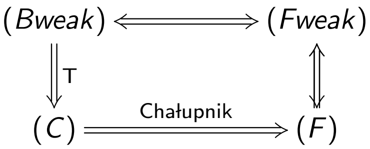
4. Relation to CFG theorem (3)

(C) Universal classes exist

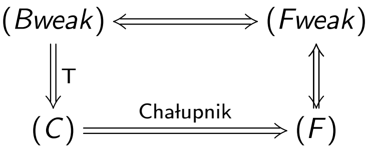
(F) $\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r})$ holds

$(Fweak)$ $\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r})$ holds, up to filtration

$(Bweak)$ $H^*(B^{(r)}) \simeq H^*(B_{E_r})$ holds, up to a filtration



$$
\begin{array}{rl}
(C) & \text{Universal classes exist} \\
(F) & \mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r}) \text{ holds} \\
(Fweak) & \mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\mathcal{P}}(F, G_{E_r}) \text{ holds, up to filtration} \\
(Bweak) & H^*(B^{(r)}) \simeq H^*(B_{E_r}) \text{ holds, up to a filtration}
\end{array}
$$



4. Relation to CFG theorem (4)

Touzé (2013) : Formula (Bweak) holds $\Rightarrow$ Classes exist

- First produce $c[1] \neq 0$ (easy).
- Using (Bweak), it suffices to find $c[d]$ in $H^*(\Gamma^d(gl_{E_1}))$ such that

$$c[1]^{\cup d} = \Delta_*(c[d])$$

  This is an equality in $H^*(gl_{E_r}^{\otimes d})$.

- Observe : $H^*(gl_{E_r}^{\otimes d}) = H^0(gl_{E_r}^{\otimes d})$ (!)

  and $c[1]^{\cup d} \in H^0(gl_{E_r}^{\otimes d})^{\Sigma_d}$

  So $c[d]$ exists by left exactness of H^0 (!)

Chalupnik (2015) : Classes exist $\Rightarrow$ Formula (F) holds

- Adjunction idea
  Precomposition by Frobenius twist has a right adjoint r

$$\mathbf{D}(\mathcal{P}_{\Bbbk}) \xrightarrow{-\circ I^{(1)}} \mathbf{D}(\mathcal{P}_{\Bbbk}) \;, \quad r(F)(V) = \mathbf{R}Hom((\Gamma^{d,V})^{(1)}, F)$$

  $\rightarrow$ Formula (F) follows if we can prove formality :

$$\ell(F^{(1)}) \simeq F_{E_1} \qquad (*)$$

- We know both sides of $(*)$ have same homology.

  Main problem : produce a map $F_{E_1} \to r(F^{(1)})$

  This map is produced in two steps :

$$F_{E_1} \xrightarrow{\eta} r(F^{(1)}_{E_r}) \xrightarrow{\text{classes}} r(F^{(1)})$$

Thank you for your attention !