PREREQUISITES OF HOMOLOGICAL ALGEBRA

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Contents

1. Introduction

Functors and short exact sequences. Let $\mathcal{A}$ be an abelian category, for example the category of modules over a ring R. To analyze the structure of an object $A \in \mathcal{A}$, we often cut it into smaller pieces: we find a subobject $A' \subset A$, and we study the properties of A' and of the quotient A/A'. Then we try to recover some information on A from the information we have on A' and A/A'.

Short exact sequences formalize the decomposition of an object into smaller pieces. They are diagrams in $\mathcal{A}$ of the following form, where (i) f is a

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monomorphism, (ii) the kernel of g is equal to the image of f and (iii) g is an epimorphism:

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0 \ .$$

The information we want to know about objects of $\mathcal{A}$ is often encoded by a functor from $\mathcal{A}$ to an other abelian category $\mathcal{B}$. For example, $\mathcal{A} = \mathrm{Ab}$ is the category of abelian groups and we want to determine the n-torsion part of an abelian group. In this case, F is the functor

$$\begin{array}{rcl} \mathrm{Ab} & \to & \mathbb{Z}/n\mathbb{Z}\text{-Mod} \\ A & \mapsto & {}_nA := \{a \in A \, ; na = 0\} \ . \end{array}$$

A functor $F : \mathcal{A} \to \mathcal{B}$ is *exact* if it sends short exact sequences in $\mathcal{A}$ to short exact sequences in $\mathcal{B}$. If F is exact and if we know two of the objects $F(A')$, $F(A)$, $F(A'')$, then we have good chances to recover the third one.

Unfortunately, many interesting functors are not exact, but only semi-exact. For example, the n-torsion functor is only *left exact*, that is it sends a short exact sequence is to a sequence

$$0 \to {}_nA' \xrightarrow{{}_nf} {}_nA \xrightarrow{{}_ng} {}_nA''$$

where (i) ${}_nf$ is injective, (ii) the image of ${}_nf$ equals the kernel of ${}_ng$. But ${}_ng$ is not surjective in general as one easily sees by applying the n-torsion functor to the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$. In that case, to reconstruct one of the objects $F(A')$, $F(A)$ and $F(A'')$ from the two other ones, we need the theory of derived functors.

Derivation of semi-exact functors. By a semi exact functor, we mean a functor which is left exact or right exact. When a functor F is left exact we can (if the category $\mathcal{A}$ has enough injectives, cf. section 2) define its right derived functors R^iF, $i \geq 1$, which are designed so that the left exact sequence

$$0 \to F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'')$$

fits into a *long exact sequence* (i.e. the kernel of each map equals the image of the preceding map)

$$0 \to F(A') \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(A'') \xrightarrow{\delta} R^1F(A') \xrightarrow{R^1F(f)} R^1F(A) \xrightarrow{R^1F(g)} \dots$$
$$\dots \xrightarrow{\delta} R^iF(A') \xrightarrow{R^iF(f)} R^iF(A) \xrightarrow{R^iF(g)} R^iF(A'') \xrightarrow{\delta} \dots$$

In particular, the cokernel of $F(g)$ equals the kernel of $R^1F(f)$, so the derived functors of F definitely help to recover one of the objects $F(A')$, $F(A)$ and $F(A'')$ from the two other ones.

Similarly, when a functor $G : \mathcal{A} \to \mathcal{B}$ is right exact (i.e. it sends short exact sequences $0 \to A' \to A \to A'' \to 0$ to right exact sequences $G(A') \to G(A) \to G(A'') \to 0$) we can define its left derived functors L^iG, $i \geq 1$ fitting into long exact sequences

$$\dots L_1G(A) \to L_1G(A'') \to G(A') \to G(A) \to G(A'') \to 0 \ .$$

(Co)homology theories as derived functors. As we will see in section 2, the theory of derived functors of semi-exact functors provides a unified conceptual framework to study many (co)homology theories. For example the (co)homology of discrete groups or categories which appear in the lectures of A. Djament in this volume, or the cohomology of algebraic groups which appears in the lectures of W. van der Kallen in this volume.

Derived functors of non-additive functors. A functor $F : \mathcal{A} \to \mathcal{B}$ is *additive* if it commutes with direct sums, i.e. the canonical inclusions $A \hookrightarrow A \oplus A'$ and $A' \hookrightarrow A \oplus A'$ induce an isomorphism

$$F(A \oplus A') \simeq F(A) \oplus F(A') .$$

It is easy to show (see exercise 2.3) that semi-exact functors are additive, as well as their derived functors. So the theory of derivation of semi-exact functors lives in the world of additive functors.

Unfortunately, there is a profusion of non additive functors in algebra and topology. For example, the group ring $\mathbb{Z}A$ an abelian group A may be thought of as a non-additive functor $\mathrm{Ab} \to \mathrm{Ab}$. Similarly the homology $H_*(A, \mathbb{Z})$ of an abelian group A may be thought of as a non additive functor with variable A.

For such general functors, Dold and Puppe provided a theory of derivation which we explain in section 3. This theory provides a conceptual framework to study non additive functors. For example, the homology of an abelian group A with coefficients in $\mathbb{Z}$ may be interpreted as derived functors of the group ring functor. Such derived functors appear in the lecture of R. Mikhailov in this volume.

Spectral sequences. Spectral sequences are a technical but essential tool in homological algebra. They play a crucial role to study and compute effectively derived functors. In section 4 we provide an introduction to spectral sequences, with a focus on standard examples appearing in the remainder of the book.

2. Derived functors of semi exact functors

2.1. Basic notions of homological algebra.

2.1.1. *Definitions related to complexes.* Let $\mathcal{A}$ be an abelian category, e.g. the category R-Mod (resp. Mod-R) of left (resp. right) R-modules over a ring R.

A complex in $\mathcal{A}$ is a collection of objects $(C_i)_{i\in\mathbb{Z}}$ together with morphisms $d_C : C_i \to C_{i-1}$, satisfying $d_C \circ d_C = 0$. The i-th homology object of C is the subquotient of C_i defined by:

$$H_i(C) = \mathrm{Ker}(d_C : C_i \to C_{i-1})/\mathrm{Im}(d_C : C_{i+1} \to C_i) .$$

If C and D are complexes, a map $f : C \to D$ is a collection of morphisms $f_i : C_i \to D_i$, satisfying $f_i \circ d = d \circ f_{i+1}$. It induces morphisms $H_i(f) : H_i(C) \to H_i(D)$ on the level of the homology. Two maps $f, g : C \to D$ are homotopic if there exists collection of morphisms $h_i : C_i \to D_{i+1}$, such that $f_i - g_i = d_D \circ h_i + h_{i-1} \circ d_C$ for all $i \geq 0$. Homotopic chain maps induce the same morphism in homology. A homotopy equivalence is a chain map

$f : C \to D$ such that there exists $g : D \to C$ with $g \circ f$ and $f \circ g$ homotopic to the identity.

We will use the word *chain complex* to indicate a complex concentrated in nonnegative degrees, i.e. $C_i = 0$ for $i < 0$. We denote by $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ the category of chain complexes and chain maps. We will use the word *cochain complex* to indicate a complex concentrated in nonpositive degrees, i.e. $C_i = 0$ for $i > 0$. By letting $C^i = C_{-i}$, we can see a cochain complex in a more familiar way as a collection of objects C^i, $i \geq 0$ with differentials $d_C : C^i \to C^{i+1}$ raising the degree by one. We denote by $\mathrm{Ch}^{\geq 0}(\mathcal{A})$ the category of cochain complexes.

For a nonnegative integer n, the n-fold suspension of a chain complex C is the chain complex $C[n]$ defined by $(C[n])_i = C_{i+n}$ and $d_{C[n]} = (-1)^n d_C$. This yields a functor $[n] : \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$.

2.1.2. *Additive functors.* Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. The functor F is called additive if it satisfies one of the following equivalent conditions.

(1) For all objects A, A' of $\mathcal{A}$ the canonical maps $A \to A \oplus A'$ and $A' \to A \oplus A'$ induce an isomorphism $F(A) \oplus F(A') \simeq F(A \oplus A')$.
(2) For all $f, g : A \to A'$, we have $F(f + g) = F(f) + F(g)$.

If F is an additive functor, then the functors $F : \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{B})$ and $F : \mathrm{Ch}^{\geq 0}(\mathcal{A}) \to \mathrm{Ch}^{\geq 0}(\mathcal{B})$ obtained by applying degreewise F preserve chain homotopies.

2.1.3. *δ-functors.*

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A *homological* δ-functor is a family of functors $G_i : \mathcal{A} \to \mathcal{B}$, $i \geq 0$, together with connecting morphisms $\delta_n : G_n(A'') \to G_{n-1}(A')$ defined for each exact sequence $0 \to A' \to A \to A'' \to 0$ such that:

(1) For each exact sequence $0 \to A' \to A \to A'' \to 0$ the diagram in $\mathcal{A}$ below is a long exact sequence (by convention, $G_i = 0$ if $i < 0$):

$$\dots \xrightarrow{\delta_{i+1}} G_i(A') \to G_i(A) \to G_i(A'') \xrightarrow{\delta_i} G_{i-1}(A') \to \dots .$$

(2) For each morphism of exact sequences, that is for each triple (f', f, f'') such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array} ,$$

we have a morphism of between the corresponding long exact sequences, that is the following diagram commutes:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & G_i(A') & \longrightarrow & G_i(A) & \longrightarrow & G_i(A'') & \longrightarrow & G_{i-1}(A') & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle G_i(f')} & & \downarrow{\scriptstyle G_i(f)} & & \downarrow{\scriptstyle G_i(f'')} & & \downarrow{\scriptstyle G_{i-1}(f')} & & \\ \cdots & \longrightarrow & G_i(B') & \longrightarrow & G_i(B) & \longrightarrow & G_i(B'') & \longrightarrow & G_{i-1}(B') & \longrightarrow & \cdots \end{array} .$$

A morphism of δ-functors from $(G_i,\delta_i)_{i\geq 0}$ to $(H_i,\delta_i)_{i\geq 0}$ is a family of natural transformations $\theta_i : G_i \to H_i$, $i \geq 0$ such that for each short exact sequence in $\mathcal{A}$, the θ_i induce a morphism between the associated long exact sequences.

The name 'homological' δ-functor comes from the fact that the connecting morphisms decrease the degrees by one. Cohomological δ-functors are defined similarly with connecting morphisms raising the degrees by one.

Definition 2.2. A *cohomological δ-functor* is a family of functors $F^i : \mathcal{A} \to \mathcal{B}$, $i \geq 0$, equipped with connecting homomorphisms $\delta^i : F^i(A'') \to F^{i+1}(A')$ defined for each short exact sequence $0 \to A' \to A \to A'' \to 0$, such that we have long exact sequences:

$$0 \to F^0(A') \to F^0(A) \to F^0(A'') \xrightarrow{\delta^0} F^1(A') \to \dots$$

and such that each morphism of short exact sequences induces a morphism between the corresponding long exact sequences.

The following exercise shows that the functors appearing as a component of a (co)homological δ-functor are additive.

Exercise 2.3. Let $G : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories, such that for all short exact sequences $0 \to A' \to A \to A'' \to 0$ the image of $G(A') \to G(A)$ is equal to the kernel of $G(A) \to G(A'')$. Show that G is additive. (Hint: use the split exact sequence $0 \to A \to A \oplus B \to B \to 0$).

2.1.4. *Projective resolutions.*

Definition 2.4. Let $\mathcal{A}$ be an abelian category. An object P of $\mathcal{A}$ is *projective* if the following functor is exact:

$$\begin{array}{rcccc} \mathrm{Hom}_{\mathcal{A}}(P,-) & : & \mathcal{A} & \to & \mathrm{Ab} \\ & & A & \mapsto & \mathrm{Hom}_{\mathcal{A}}(P,A) \end{array}$$

Exercise 2.5. Show that the projective objects of the category R-Mod of modules over a ring R are the direct summands of free R-modules. Show that there might be projective R-modules which are not free (example: consider the ring $\mathbb{Z} \times \mathbb{Z}$).

Definition 2.6. An abelian category $\mathcal{A}$ has *enough projectives* if all objects A admit a projective resolution. This means that for all A, there exists a chain complex P^A of projective objects such that $H_i(P^A) = A$ if $i = 0$ and zero otherwise.

Exercise 2.7. Let R be a ring. Show that R-Mod and Mod-R have enough projectives.

It is not true that all abelian categories have enough projectives. The category of rational GL_n-modules in section 2.6 provides a counter-example. To compare the various projective resolutions which might occur in an abelian category $\mathcal{A}$, we can use the following fundamental lemma, whose proof is left as an exercise.

Lemma 2.8. *Let A, B be objects of an abelian category $\mathcal{A}$. Let P^A and P^B be projective resolutions of A and B in $\mathcal{A}$. Then for all $f : A \to B$, there* *exists a chain map $\overline{f} : P^A \to P^B$ such that $H_0(\overline{f}) = f$. Such a chain map is unique up to homotopy.*

In particular a projective resolution of A is unique up to a homotopy equivalence.

2.1.5. *Injective coresolutions.*

Definition 2.9. Let $\mathcal{A}$ be an abelian category. An object J of $\mathcal{A}$ is *injective* if the following contravariant functor is exact:

$$\begin{array}{rcccc} \mathrm{Hom}_{\mathcal{A}}(-,J) & : & \mathcal{A} & \to & \mathrm{Ab} \\ & & A & \mapsto & \mathrm{Hom}_{\mathcal{A}}(A,J) \end{array}$$

Exercise 2.10. Show that an object of $\mathcal{A}$ is injective if and only if it is projective in the opposite category[1].

Definition 2.11. An abelian category $\mathcal{A}$ has *enough injectives* if all objects A admit an injective coresolution. This means that for all A, there exists a cochain complex J_A of injective objects, such that $H^i(J_A) = A$ if $i = 0$ and zero otherwise.

Not all abelian categories have enough injectives. With a little work, one can prove:

Proposition 2.12. *The categories R-Mod (resp. Mod-R) of left (resp. right) modules over a ring R have enough injectives.*

Sketch of proof. We do the proof for R-Mod. The proof decomposes in several steps.

(1) We first observe that to prove that an abelian category $\mathcal{A}$ has enough injective objects, it suffices to prove that for all objects A, we can find an injective object and an injective map $A \hookrightarrow J$.

(2) $\mathbb{Q}/\mathbb{Z}$ **is injective in** Ab**.** An abelian group G is *divisible* if for all $x \in G$ and all $n \in \mathbb{Z} \setminus \{0\}$, there exists $x' \in G$ such that nx'.

   To prove that $\mathbb{Q}/\mathbb{Z}$ is injective, we first prove that being injective is equivalent to being divisible. Then we check that $\mathbb{Q}/\mathbb{Z}$ is divisible.

(3) $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ **is injective in R-mod.** We consider the abelian group $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ as an R-module, with action of R on a function $f : R \to \mathbb{Q}/\mathbb{Z}$ given by $rf(x) := f(xr)$. There is an isomorphism of abelian groups, natural with respect to the R-module M

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})) \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \ . \quad (*)$$

   In particular, since $\mathbb{Q}/\mathbb{Z}$ is injective, $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is also injective.

(4) We observe that an arbitrary product of injective R-modules is still injective.

(5) Finally, let M be an R-module. Let J the injective R-module obtained as the product of copies of $J_R = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$, indexed by the set $\mathcal{M} = \mathrm{Hom}_R(M, J_R)$. There is a canonical R-linear map $\phi : M \to J$.

[1] the opposite category of $\mathcal{A}$ is the category $\mathcal{A}^{\mathrm{op}}$ with the same objects as $\mathcal{A}$, with $\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(A, B) := \mathrm{Hom}_{\mathcal{A}}(B, A)$, and the composition in $\mathcal{A}^{\mathrm{op}}$ is defined by $f \circ g := g \circ_{\mathcal{A}} f$, where $\circ_{\mathcal{A}}$ denotes the composition in $\mathcal{A}$.

To be more specific, the coordinate of ϕ indexed by $f \in \mathcal{M}$ is the map $\phi_f : M \to J_R$ which sends $m \in M$ to $f(m)$. To prove that ϕ is an injection, it suffices to find for all $m \in M$ an element $f \in \mathcal{M}$ such that $f(m) \neq 0$ or equivalently (cf isomorphism $(*)$) a morphism of abelian groups $\overline{f} : M \to \mathbb{Q}/\mathbb{Z}$ such that $\overline{f}(m) \neq 0$. Such an $\overline{f}$ can be produced using the injectivity of $\mathbb{Q}/\mathbb{Z}$.

□

The following fundamental lemma can be formally deduced from lemma 2.8 and exercise 2.10.

Lemma 2.13. *Let A, B be objects of an abelian category $\mathcal{A}$. Let J_A and J_B be injective resolutions of A and B in $\mathcal{A}$. Then for all $f : A \to B$, there exists a chain map $\overline{f} : J_A \to J_B$ such that $H^0(\overline{f}) = f$. Such a chain map is unique up to homotopy.*

2.2. **Derivation of semi-exact functors.**

2.2.1. *Derivation of right exact functors.* If $(G_i, \delta_i)_{i\geq 0}$ is a homological δ-functor, the 'end' of the long exact sequences

$$\cdots \to G_0(A') \to G_0(A) \to G_0(A'') \to 0$$

imply that G_0 is a right exact functor. In particular, the assignment $(G_i, \delta_i)_{i\geq 0} \mapsto G_0$ defines a functor from the category of homological δ-functors, to the category of right exact functors (with natural transformations as morphisms):

$$\left\{ \begin{array}{c} \text{homological } \delta\text{-functors} \\ \text{from } \mathcal{A} \text{ to } \mathcal{B} \end{array} \right\} \to \left\{ \begin{array}{c} \text{right exact functors} \\ \text{from } \mathcal{A} \text{ to } \mathcal{B} \end{array} \right\}.$$

Derivation of functors yields an operation going in the other way, which enables (in good cases, cf. condition (ii) in theorem 2.14) to reconstruct the homological functor $(G_i, \delta_i)_{i\geq 0}$ from G_0.

Theorem 2.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, and let $G : \mathcal{A} \to \mathcal{B}$ be a right exact functor. Assume that $\mathcal{A}$ has enough projectives. There exists a homological δ-functor $(L_iG, \delta_i) : \mathcal{A} \to \mathcal{B}$ such that:*

(i) $L_0G = G$,
(ii) *for all $i > 0$, $L_iG(P) = 0$ if P is projective.*

Such a δ-functor is unique up to isomorphism. Moreover, if G and G' are right exact functors, a natural transformation $\theta : G \to G'$ extends uniquely into a morphism of homological δ-functors $\theta_i : L_iG \to L_iG'$ such that $\theta_0 = \theta$.

Definition 2.15. The functors L_iG, $i \geq 0$ from theorem 2.14 are called the left derived functors of G.

We now outline the main ideas of the proof of theorem 2.14.

(1) If $(G_i, \delta_i)_{i\geq 0}$ and $(G'_i, \delta'_i)_{i\geq 0}$ are homological δ-functors satisfying condition (ii), then a natural transformation $\theta : G_0 \to G'_0$ may be uniquely extended into a morphism of δ-functors θ_i, such that $\theta_0 = \theta$.

This results from the following 'dimension shifting argument'. Let $\theta_i : G_i \to G'_i$ be a morphism of δ-functors. For all $M \in \mathcal{A}$ we can find a projective P and an epimorphism $P \twoheadrightarrow M$. Let K be the kernel of this epimorphism. The long exact sequences of G_i and G'_i provide commutative diagrams:

$$\begin{array}{ccccc} G_1(M) & \xhookrightarrow{\delta_1} & G_0(K_M) & \to & \cdots \\ \downarrow{\scriptstyle \theta_{1\,M}} & & \downarrow{\scriptstyle \theta_{0\,K}} & & \\ G_1(M) & \xhookrightarrow{\delta'_1} & G_0(K_M) & \to & \cdots \end{array} \quad , \quad \text{and} \quad \begin{array}{ccc} G_{i+1}(M) & \xrightarrow[\simeq]{\delta_{i+1}} & G_i(M) \\ \downarrow{\scriptstyle \theta_{i+1\,M}} & & \downarrow{\scriptstyle \theta_{i\,K}} \\ G'_{i+1}(M) & \xrightarrow[\simeq]{\delta'_{i+1}} & G_i(M) \end{array} \qquad i \geq 1.$$

So the natural transformations θ_i, $i \geq 0$ are completely determined by θ_0. Whence the uniqueness of the extension of θ.

For the existence, one checks that taking $\theta_{1\,M}$ as the restriction of θ_K to $G_1(M)$ and $\theta_{i+1\,M} = (\delta'_{i+1})^{-1} \circ \theta_{i\,K} \circ \delta_{i+1}$ provides a well defined morphism of δ-functors extending θ.

(2) The uniqueness of $(L_iG, \delta_i)_{i\geq 0}$ satisfying (i) and (ii) follows the existence and uniqueness of the extension of the natural transformation $\mathrm{Id} : G \to G$.

(3) The construction of a homological δ-functor $(L_iG, \delta_i)_{i\geq 0}$ satisfying (i) and (ii) is achieved by the following recipe. We first fix for each object A a projective resolution P^A. Then L_iG is defined on objects by:

$$L_iG(A) = H_i(G(P^A))$$

To define L_iG on morphisms, we use the fundamental lemma 2.8. Each map $f : A \to B$ induces a chain map $\overline{f} : P^A \to P^B$, and we define $L_iG(f)$ as the map

$$H_i(G(\overline{f})) : H_i(G(P^A)) \to H_i(G(P^B)) \ .$$

Since $\overline{f}$ is unique up to homotopy and G is additive (see exercise 2.3), $G(\overline{f})$ is unique up to homotopy, hence different choices of $\overline{f}$ induce the same morphism on homology. So $H_i(G(\overline{f}))$ is well defined. To finish the proof theorem 2.14 it remains to check that the functors L_iG actually form a δ-functor. This verification is rather long so we refer the reader to the classical references [ML, Wei].

Exercise 2.16. Assume that $\mathcal{A}$ has enough projectives. Prove that a right exact functor G is exact if and only if its left derived functors L_iG are zero for $i > 1$.

Exercise 2.17. Find two non-isomorphic δ-functors $(G_i, \delta_i)_{i\geq 0}$ and $(G'_i, \delta'_i)_{i\geq 0}$ such that $G_0 \simeq G'_0$.

2.2.2. *Derivation of left exact functors.* Derivation of right exact functors has the following analogue in the case of left exact functors.

Theorem 2.18. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, and let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume that $\mathcal{A}$ has enough injectives. There exists a cohomological δ-functor $(R^iG, \delta^i) : \mathcal{A} \to \mathcal{B}$ such that:*

(i) *$R^0F = F$,*

(ii) *for all $i > 0$, $R^iF(J) = 0$ if J is injective.*

Such a δ-functor is unique up to isomorphism. Moreover, if F and F' are left exact functors, a natural transformation $\theta : F \to F'$ extends uniquely into a morphism of δ-functors $\theta^i : R^iF \to R^iF'$ such that $\theta^0 = \theta$.

The proof of theorem 2.18 is completely similar to the case of right exact functors, so we omit it. Let us just mention that the derived functor R^iF sends an object A to the homology group $H^i(F(J_A))$, and for $f : A \to B$, the morphism $R^i(f)$ is equal to $H^i(F(\overline{f}))$, where $\overline{f} : J_A \to J_B$ is a lifting of f to the injective coresolutions.

Exercise 2.19. Prove that a left exact functor $F : \mathcal{A} \to \mathcal{B}$ is equivalent to a right exact functor $\mathcal{A}^{\mathrm{op}} \to B^{\mathrm{op}}$, and that a cohomological δ-functor $(F^i, \delta^i)_{i\geq 0} : \mathcal{A} \to \mathcal{B}$ is equivalent to an homological δ-functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$. Deduce theorem 2.18 from the statement of theorem 2.14.

2.3. **Ext and Tor.** The most common examples of derived functors are the functors Tor and Ext. Basic references for this section are [ML, III] and [Wei, Chap 3].

2.3.1. *Ext.* Let $\mathcal{A}$ be an abelian category. Assume that $\mathcal{A}$ is enriched over a commutative ring $\Bbbk$ (one also says that A is $\Bbbk$-linear). This means that Hom groups in $\mathcal{A}$ are $\Bbbk$-modules, and composition in $\mathcal{A}$ is $\Bbbk$-bilinear. For example, the category of left (or right) modules over a $\Bbbk$-algebra R are of that kind. And abelian categories are by definition enriched over $\mathbb{Z}$. Then for objects M, N in $\mathcal{A}$ we have left exact functors:

$$\mathrm{Hom}_{\mathcal{A}}(M,-) : \mathcal{A} \to \Bbbk\text{-Mod}\ , \qquad \mathrm{Hom}_{\mathcal{A}}(-,N) : \mathcal{A}^{\mathrm{op}} \to \Bbbk\text{-Mod}\ .$$

If $\mathcal{A}$ has enough injectives, we define Ext-functors by the formula:

$$\mathrm{Ext}^i_{\mathcal{A}}(M,-) = R^i(\mathrm{Hom}_{\mathcal{A}}(M,-))\ .$$

If $\mathcal{A}^{\mathrm{op}}$ has enough injectives (this is equivalent to the fact that $\mathcal{A}$ has enough projectives) then we define Ext-functors by the formula

$$\mathrm{Ext}^i_{\mathcal{A}}(M,-) = R^i(\mathrm{Hom}_{\mathcal{A}}(-,N))\ .$$

Remark 2.20. Assume that $\mathcal{A}$ has enough injectives and enough projectives. Then the notation $\mathrm{Ext}^i_{\mathcal{A}}(M,N)$ might have two different meanings: either the value of $R^i(\mathrm{Hom}_{\mathcal{A}}(M,-))$ on N or the value of $R^i(\mathrm{Hom}_{\mathcal{A}}(-,N))$ on M. It can be proved that these two definitions coincide. That is, $\mathrm{Ext}^i_{\mathcal{A}}(M,N)$ can be indifferently computed as the i-th homology of the cochain complex $\mathrm{Hom}_{\mathcal{A}}(P^M,N)$ where P^M is a projective resolution of M or as the i-th homology of the cochain complex $\mathrm{Hom}_{\mathcal{A}}(M,J_N)$ where J_N is an injective resolution of N.

The notation Ext is the abbreviation of 'Extension groups'. If A, B are objects of $\mathcal{A}$, an extension of A by B in $\mathcal{A}$ is a short exact sequence $0 \to B \to C \to A \to 0$. Two extensions are isomorphic if we have a commutative diagram (in this case f is automatically an isomorphism):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & C' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

An extension is trivial when it is isomorphic to $0 \to A \to A \oplus B \to B \to A$. We let $\mathcal{E}(A,B)$ be the set of isomorphism classes of extensions of A by B, pointed by the class of trivial extensions. The following proposition is the reason for the name 'Ext' (such an approach of Exts is used in R. Mikhailov's lectures).

Proposition 2.21. *There is a bijection of pointed sets (on the left hand side* $\mathrm{Ext}^1_{\mathcal{A}}(A,B)$ *is pointed by* 0*):*

$$\mathrm{Ext}^1_{\mathcal{A}}(A,B) \simeq \mathcal{E}(A,B)\ .$$

Actually much more can be said: we can describe explicitly the abelian group structure in terms of operations on $\mathcal{E}(A,B)$, and higher Ext also have analogous interpretations. We refer the reader to [ML, III] for more details on these topics (and for the proof of proposition 2.21).

Exercise 2.22. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories with enough projectives and enough injectives. Let $G:\mathcal{B}\to\mathcal{A}$ be a right adjoint to $F:\mathcal{A}\to\mathcal{B}$. Recall that this means that there is a bijection, natural in x,y:

$$\mathrm{Hom}_{\mathcal{B}}(F(x),y)\simeq \mathrm{Hom}_{\mathcal{A}}(x,G(y))\ .$$

(1) Show that F is right exact and that G is left exact.
(2) Show that G is exact if and only if F preserves the projectives (and similarly F is exact if and only if G preserves the injectives).
(3) Show that F and G are both exact if and only if there are isomorphisms $\mathrm{Ext}^*_{\mathcal{B}}(F(x),y)\simeq \mathrm{Ext}^*_{\mathcal{A}}(x,G(y))$ for all x,y.

2.3.2. *Tor.* In this section, we fix a commutative ring $\Bbbk$ and a $\Bbbk$-algebra R (if $\Bbbk=\mathbb{Z}$, a $\Bbbk$-algebra is nothing but an ordinary ring). If M is a right R-module and N is a left R-module, the tensor product $M\otimes_R N$ is the $\Bbbk$-module generated by the symbols $m\otimes n$, $m\in M$, $n\in N$, with the following relations:

$$\begin{aligned}
&m\otimes n+m'\otimes n=(m+m')\otimes n\ ,\\
&m\otimes n+m\otimes n'=(m+m')\otimes n\ ,\\
&\lambda(m\otimes n)=(m\lambda)\otimes n=m\otimes(\lambda n)\ \text{for }\lambda\in\Bbbk\ ,\\
&(mr)\otimes n=m\otimes(rn)\ \text{for }r\in R\ .
\end{aligned}$$

Tensor product over the $\Bbbk$-algebra R yields functors:

$$M\otimes_R : R\text{-Mod}\to \Bbbk\text{-Mod}\ ,\qquad \otimes_R N:\text{Mod-}R\to\Bbbk\text{-Mod}\ .$$

Exercise 2.23. Prove that we have isomorphisms, natural with respect to $M\in$ Mod-R, $N\in R$-Mod, and $K\in\Bbbk$-Mod:

$$\begin{aligned}
\mathrm{Hom}_{\Bbbk}(M\otimes_R N,Q)&\simeq \mathrm{Hom}_{\text{Mod-}R}(M,\mathrm{Hom}_{\Bbbk}(N,Q))\ ,\\
\mathrm{Hom}_{\Bbbk}(M\otimes_R N,Q)&\simeq \mathrm{Hom}_{R\text{-Mod}}(N,\mathrm{Hom}_{\Bbbk}(M,Q))\ .
\end{aligned}$$

Deduce[2] from these isomorphisms the following properties of the functors $M\otimes_R$ and $\otimes_R N$: (i) they are right exact, (ii) they commute with arbitrary sums, and (iii) if M (resp. N) is projective, then $M\otimes_R$ (resp. $\otimes_R N$) is exact.

By exercise 2.23, $M\otimes_R$ and $\otimes_R N$ are right exact. Tor-functors are defined by the following formulas (recall the the category of left or right R-modules has enough projectives so that left derived functors are well defined).

$$\mathrm{Tor}^R_i(-,N)=L_i(\otimes_R N)\ ,\quad \mathrm{Tor}^R_i(M,-)=L_i(M\otimes_R)\ .$$

[2]This question requires the Yoneda lemma, see section 2.5.2.

Remark 2.24. The notation $\mathrm{Tor}_i^R(M,N)$ has two different meanings: on the one hand it is the value on M of $L_i(\otimes_R N)$ and on the other hand it is the value of $L_i(M \otimes_R)$ on N. However, we shall see in example 4.22 that they coincide. Thus, the $\mathbb{k}$-module $\mathrm{Tor}_i^R(M,N)$ can be indifferently be computed as the i-th homology of the complex $P^M \otimes_R N$ or of the complex $N \otimes_R P^N$, where P^X denotes a projective resolution of X.

The name 'Tor' is the abbreviation of 'Torsion', and it is justified by the case of abelian groups, as the following exercise shows it.

Exercise 2.25. Let A and B be abelian groups. Show that $\mathrm{Tor}_i^{\mathbb{Z}}(A,B) = 0$ for $i > 1$. If $B = \mathbb{Z}/n\mathbb{Z}$, show that $\mathrm{Tor}_1^{\mathbb{Z}}(A, \mathbb{Z}/n\mathbb{Z})$ is the functor sending an abelian group A to its n-torsion part ${}_nA = \{a \in A\ : na = 0\}$.

The following exercise explains why so many examples of left derived functors can be interpreted as functors $\mathrm{Tor}_*^R(M,-)$.

Exercise 2.26. Let R be a $\mathbb{k}$-algebra, and let $G : R\text{-Mod} \to \mathbb{k}\text{-Mod}$ be a right exact functor, commuting with arbitrary sums.
1. Show that functoriality endows the $\mathbb{k}$-module $G(R)$ with the structure of a right R-module.
2. Show that for all free R-modules F, there is an isomorphism of $\mathbb{k}$-modules, natural with respect to F: $G(R) \otimes_R F \simeq G(F)$.
3. Deduce that for all $i \geq 0$, $L_iG(-) \simeq \mathrm{Tor}_i^R(G(R),-)$.

2.3.3. *Bar complexes.* Let R be a $\mathbb{k}$-algebra, let $M \in \text{Mod-}R$ and $N \in R\text{-Mod}$. The double sided bar complex $B(M,R,N)$ is the complex of $\mathbb{k}$-modules defined as follows. As a graded $\mathbb{k}$-module, we have (tensor products are taken over $\mathbb{k}$):

$$B(M,R,N)_k := M \otimes R^{\otimes k} \otimes N\ .$$

An element $m \otimes x_1 \otimes \cdots \otimes x_k \otimes n \in B(M,R,N)_k$ is denoted by $m[x_1|\dots|x_k]n$, and an element of $B(M,R,N)_0$ is denoted by $m[\ ]n$. This handy notation is the origin of the name 'bar resolution'. The differential $d : B(M,R,N)_k \to B(M,R,N)_{k-1}$ is the $\mathbb{k}$-linear map defined by

$$\begin{array}{rl} d(m[x_1,\dots,x_k]n) = & mx_1[x_2|\dots|x_k]n \\ & +\sum_{i=1}^{k-1}(-1)^i m[x_1|\dots|x_ix_{i+1}|\dots|x_k]n \\ & +(-1)^n m[x_1|\dots|x_{k-1}]x_kn \end{array} .$$

If $M = R$, then $B(R,R,N)$ becomes a complex in the category of R-modules. The action of R on $B(R,R,N)_k$ is given by the formula:

$$x \cdot (m[x_1|\dots|x_k]n) := (xm)[x_1|\dots|x_k]n\ .$$

Proposition 2.27. *$B(R,R,N)$ is a resolution of N in the category of left R-modules. Furthermore, if R and N are projective as $\mathbb{k}$-modules, then it is a projective resolution of N in R-Mod.*

Proof. We have already seen that $B(R,R,N)$ is a complex of R-modules. Furthermore, if N and R are projective as $\mathbb{k}$-modules, then the R-modules $B(R,R,N)_k$ are projective R-modules. So, to prove proposition 2.27 we want to prove that $H_i(B(R,R,N)) = N$ if $i = 0$ and zero otherwise. Let us consider N as a complex of R-modules concentrated in degree zero. The multiplication

$$\begin{array}{rrcl} \lambda: & R\otimes N = B(R,R,N)_0 & \to & N \\ & x\otimes n & \mapsto & xn \end{array}$$

yields a morphism of complexes of R-modules $B(R,R,N)\xrightarrow{\lambda} N$. So, to prove proposition 2.27, it suffices to prove that $H_*(\lambda)$ is an isomorphism.

To prove this, we forget the action of R and consider λ as a morphism of complexes of $\Bbbk$-modules. We are actually going to prove that λ is a homotopy equivalence of complexes of $\Bbbk$-modules. The inverse of λ is the map $\eta : N \to B(R,R,N)$ which sends an element $n\in N$ to $1\otimes N\in R\otimes N = B(R,R,N)_0$ (Notice that η is not R-linear, it is only a morphism of complexes of $\Bbbk$-modules). Then $\epsilon\circ\eta = \mathrm{Id}$, and $\eta\circ\epsilon$ is homotopic to the identity via the homotopy h defined by $h(x[x_1|\dots|x_k]n) = 1[x|x_1|\dots|x_k]n$. □

Similarly, we can prove that if R and M are projective as $\Bbbk$-modules, then $B(M,R,R)$ is a projective resolution of M in Mod-R. As a consequence of proposition 2.27, we obtain a nice explicit complex to compute Tors.

Corollary 2.28. *Let R be a $\Bbbk$-algebra, let $M\in$ Mod-R and $N\in R$-Mod. Assume that R, and at least one of the two R-modules M and N are projective as $\Bbbk$-modules. Then the homology of $B(M,R,N)$ equals $\mathrm{Tor}_*^R(M,N)$.*

Proof. Assume that R and N are projective $\Bbbk$-modules. Then $B(R,R,N)$ is a projective resolution of N. Hence $M\otimes_R B(R,R,N)$ computes $\mathrm{Tor}_*^R(M,N)$. But $M\otimes_R B(R,R,N)$ equals $B(M,R,N)$. □

We can also use bar complexes to compute extension groups.

Corollary 2.29. *Let R be a $\Bbbk$-algebra, let M,N be left R-modules. Assume that R and M are projective as $\Bbbk$-modules. Then $\mathrm{Ext}^*_R(M,N)$ equals the homology of the complex $\mathrm{Hom}_R(B(R,R,M),N)$.*

2.4. **Homology and cohomology of discrete groups.** As basic references for the (co)homology of groups, the reader can consult [Wei, Chap 6], [ML, IV], [Br] or [Ben, Chap 2].

2.4.1. *Definitions.* Let G be a group, and let $\Bbbk$ be a commutative ring. A $\Bbbk$-linear representation of G is a $\Bbbk$-module M, equipped with an action of G by $\Bbbk$-linear morphisms. A morphism of representations $f: M\to N$ is a $\Bbbk$-linear map which commutes with the action of G, i.e. $f(gm) = gf(m)$ for all $m\in M$ and all $g\in G$.

Let $\Bbbk G$ be the group algebra[3] of G over $\Bbbk$. Then the category of $\Bbbk$-linear representations of G is isomorphic to the category of (left) $\Bbbk G$-modules. In particular, it has enough injectives and projectives.

Let $\Bbbk^{\mathrm{triv}}$ be the trivial $\Bbbk G$-module, that is the $\Bbbk$-module $\Bbbk$, acted on by G by $gx = x$. If M is a representation of G, its fixed points (or invariants) under the action of G is the $\Bbbk$-module M^G defined by:

$$M^G = \{m\in M\ , gm = m\ \forall g\in G\} = \mathrm{Hom}_{\Bbbk G}(\Bbbk^{\mathrm{triv}}, M)\ .$$

[3]The group algebra $\Bbbk G$ is the free $\Bbbk$-module with basis $(b_g)_{g\in G}$, with product defined by $b_g b_{g'} := b_{gg'}$. The unit of the $\Bbbk$-algebra $\Bbbk G$ is the element $1 = b_e$ corresponding to the identity element $e\in G$.

Its coinvariants is the $\Bbbk$-module M_G defined by:

$$M_G = M/\langle gm - m\ ,\ m \in M, g \in G\rangle = \Bbbk^{\mathrm{triv}} \otimes_{\Bbbk G} M\ .$$

The (co)homology of G is defined by deriving the functor of (co)invariants.

$$H^i(G, -) = R^i(-^G) = \mathrm{Ext}^i_{\Bbbk G}(\Bbbk^{\mathrm{triv}}, -)\ .$$
$$H_i(G, -) = L_i(-_G) = \mathrm{Tor}_i^{\Bbbk G}(\Bbbk^{\mathrm{triv}}, -)\ .$$

We don't mention the ring $\Bbbk$ in the notations $H^i(G, M)$ and $H_i(G, M)$. Since a $\Bbbk G$-module is also a $\mathbb{Z}G$-module, these notations have two interpretations, e.g. $H^i(G, M)$ means $\mathrm{Ext}^i_{\Bbbk G}(\Bbbk^{\mathrm{triv}}, M)$ as well as $\mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}^{\mathrm{triv}}, M)$. The following exercise shows that these two interpretations coincide.

Exercise 2.30. If M is a $\mathbb{Z}G$-module and N is a $\Bbbk G$-module, there is a canonical structure of $\Bbbk G$-module on $\Bbbk \otimes_{\mathbb{Z}} M$, and a canonical structure of $\Bbbk$-module on $\mathrm{Hom}_{\mathbb{Z}G}(M, N)$. Prove that there is an isomorphism of $\Bbbk$-modules

$$\mathrm{Hom}_{\mathbb{Z}G}(M, N) \simeq \mathrm{Hom}_{\Bbbk G}(\Bbbk \otimes_{\mathbb{Z}} M, N)\ .$$

Recall the bar complex from section 2.3.3. Show that $\Bbbk \otimes_{\mathbb{Z}} B(\mathbb{Z}G, \mathbb{Z}G, \mathbb{Z}^{\mathrm{triv}})$ is isomorphic to $B(\Bbbk G, \Bbbk G, \Bbbk^{\mathrm{triv}})$ as a complex of $\Bbbk G$-modules. Deduce that there is an isomorphism of $\Bbbk$-modules $\mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}^{\mathrm{triv}}, M) \simeq \mathrm{Ext}^i_{\Bbbk G}(\Bbbk^{\mathrm{triv}}, M)$. Similarly, prove the isomorphism $\mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}^{\mathrm{triv}}, M) \simeq \mathrm{Tor}_i^{\Bbbk G}(\Bbbk^{\mathrm{triv}}, M)$.

The following exercise gives a relation between the homology of a group and the homology of its subgroups.

Exercise 2.31 (Shapiro's lemma)**.** Let H be a subgroup of G. Let us denote by $\mathrm{res}^G_H : \Bbbk G\text{-Mod} \to \Bbbk H\text{-Mod}$ the restriction functor. If M is a $\Bbbk H$-module, we denote by $\mathrm{ind}^G_H M$ the $\Bbbk$-module $\Bbbk G \otimes_{\Bbbk H} M$, acted on by G by the formula: $g(x \otimes m) := gx \otimes m$. We denote by $\mathrm{coind}^G_H M$ the $\Bbbk$-module $\mathrm{Hom}_{\Bbbk H}(\Bbbk G, M)$, acted on by G by the formula $(gf)(x) := f(xg)$.

1. Prove that ind^G_H (resp. coind^G_H) is left (resp. right) adjoint to res^G_H.
2. Prove that res^G_H is exact and preserves injectives and projectives, that ind^G_H is exact and preserves projectives, and that coind^G_H is exact and preserves injectives (use exercise 2.22).
3. Prove Schapiro's lemma, namely we have isomorphisms, natural with respect to the $\Bbbk H$-module N:

$$H_*(H, N) \simeq H_*(G, \mathrm{ind}^G_H N)\ , \quad H^*(H, N) \simeq H^*(G, \mathrm{coind}^G_H N)\ .$$

2.4.2. *Products in cohomology.* If A is a $\Bbbk G$-algebra, the cohomology $H^*(G, A)$ is equipped with a so-called cup product, which makes it into a graded $\Bbbk$-algebra. Moreover, if A is commutative, then $H^*(G, A)$ is graded commutative, i.e. the cup product of homogeneous elements x and y satisfies

$$x \cup y = (-1)^{\deg(x)\deg(y)} y \cup x\ .$$

This cup product may be defined in many different ways (there are four different definitions in [ML]). We present here a description of cup products using the complex $\mathrm{Hom}_{\Bbbk G}(B(\Bbbk G, \Bbbk G, \Bbbk^{\mathrm{triv}}), M)$ from corollary 2.29. We first rewrite this complex under a more convenient form.

Proposition 2.32. *The complex* $\mathrm{Hom}_{\Bbbk G}(B(\Bbbk G, \Bbbk G, \Bbbk^{\mathrm{triv}}), M)$ *is isomorphic to the complex of* $\Bbbk$*-modules* $C^*(G, M)$ *defined by*

$$C^n(G, M) = \mathrm{Map}(G^{\times n}, M)\ ,$$

where $\mathrm{Map}(G^{\times n}, M)$ *means the maps from the set* $G^{\times n}$ *to the* $\Bbbk$*-module* M*, and* $G^{\times 0}$ *should be understood as a set with one element, and with differential* $\partial : C^n(G, M) \to C^{n+1}(G, M)$ *defined by:*

$$\begin{array}{rl}(\partial f)(g_1, \dots, g_{n+1}) := & g_1 f(g_2, \dots, g_{n+1}) \\ & + \sum_{i=1}^n (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ & +(-1)^{n+1} f(g_2, \dots, g_n)\end{array} .$$

Proof. There is an isomorphism of $\Bbbk$-modules:

$$\mathrm{Map}(G^{\times n}, M) \simeq \mathrm{Hom}_{\Bbbk G}(\Bbbk G \otimes (\Bbbk G)^{\otimes n}, M)$$

which sends a map f to the $\Bbbk G$-linear morphism $\widetilde{f}$ defined by:

$$\widetilde{f}(b_{g_0}[b_{g_1}|\dots|b_{g_n}] := g_0 f(g_1, \dots, g_n)\ .$$

We check that this isomorphism is compatible with the differentials. □

We define the cup product on the complex $C^*(G, A)$:

$$\begin{array}{ccc} C^n(G, A) \times C^m(G, A) & \to & C^{m+n}(G, A) \\ f_1, f_2 & \mapsto & f_1 \cup f_2 \end{array}$$

by the formula (where $\cdot_A$ denotes the product of the $\Bbbk G$ algebra A)

$$(f_1 \cup f_2)(g_1, \dots, g_{n+m}) := f_1(g_1, \dots, g_n) \cdot_A [(g_1 \dots g_n) f_2(g_{n+1}, \dots, g_{n+m})]\ .$$

This products makes $(C^*(G, A), \cup, \partial)$ into a differential graded algebra, that is, the differential ∂ of $C^*(G, A)$ acts on products by the formula:

$$\partial(f_1 \cup f_2) = (\partial f_1) \cup f_2 + (-1)^{\deg(f_1)} f_1 \cup (\partial f_2)\ .$$

Since $(C^*(G, A), \cup, \partial)$ is a differential graded algebra, there is an induced cup product on the level of cohomology, defined by the formula (where the brackets stand for the cohomology class represented by a cycle):

$$[f_1] \cup [f_2] := [f_1 \cup f_2]\ .$$

Remark 2.33. If A is commutative, the differential graded algebra $C^*(G, A)$ is *not* graded commutative, so it is not clear with our definition why $H^*(G, A)$ should be graded commutative. One has to show that the difference $f_1 \cup f_2 - (-1)^{\deg(f_1)\deg(f_2)} f_2 \cup f_1$ is a boundary in $C^*(G, A)$, which is left as an exercise to the courageous reader.

2.5. **Homology and cohomology of categories.** As a reference for the cohomology of categories, the reader can consult [FP], or the article [Mit].

2.5.1. *From groups to categories.* A group G can be thought of as a category with one object, say $*$, with $\mathrm{Hom}(*, *) = G$ and the composite of morphisms g and h is the product gh. Let us denote by $*_G$ the category corresponding to the group G. With this description of a group, a $\Bbbk$-linear representation of G is the same as a functor

$$F : *_G \to \Bbbk\text{-Mod}\ .$$

The $\Bbbk$-module on which G acts is $F(*)$ and an element $g \in G$ acts on $F(*)$ by the $\Bbbk$-linear endomorphism $F(g)$. Keeping this in mind, the following definition is quite natural.

Definition 2.34. Let $\mathcal{C}$ be a small[4] category and let $\Bbbk$ be a commutative ring. A left $\Bbbk$-linear representation of $\mathcal{C}$ is a functor $F : \mathcal{C} \to \Bbbk\text{-Mod}$, and a right $\Bbbk$-linear representation is a functor $F : \mathcal{C}^{\mathrm{op}} \to \Bbbk\text{-Mod}$. A morphism of representations $\theta : F \to G$ is a natural transformation of functors.

Since the (co)homology of groups was defined in terms of Tor and Ext in the category of representations of G, the (co)homology of categories should be similarly defined in terms of Tor and Ext in the category of representations of $\mathcal{C}$. This motivates the study of functor categories.

2.5.2. *The Yoneda lemma.* The Yoneda lemma is an elementary result, but it plays a so crucial role in the study of categories and functors that it deserves its own section. Let $\mathcal{C}$ be a category and let x be an object of $\mathcal{C}$. We denote by $h^x : \mathcal{C} \to \mathrm{Set}$ the functor $c \mapsto \mathrm{Hom}_{\mathcal{C}}(x, c)$.

Lemma 2.35 (The Yoneda lemma)**.** *Let $F : \mathcal{C} \to \mathrm{Set}$ be a functor, and let x be an object of $\mathcal{C}$. Then the natural transformations from h^x to F form a set, and the following map is a bijection:*

$$\begin{array}{ccc} \mathrm{Nat}(h^x, F) & \to & F(x) \\ \theta & \mapsto & \theta_x(\mathrm{Id}_x) \end{array} .$$

Proof. First, a natural transformation θ is uniquely determined by $\theta_x(\mathrm{Id}_x)$, since for all objects y in $\mathcal{C}$ and for all $f \in h^x(y)$

$$\theta_y(f) = (\theta_y \circ h^x(f))\,(\mathrm{Id}_x) = (F(f) \circ \theta_x)\,(\mathrm{Id}_x) = F(f)(\,\theta_x(\mathrm{Id}_x)\,)\ .$$

Hence the natural transformation forms an set and the map of lemma 2.35 is injective. To prove it is surjective, we observe that if $\alpha \in F(x)$, the maps

$$\begin{array}{cccc} \theta^\alpha_y : & h^x(y) & \to & F(y) \\ & f & \mapsto & F(f)(\alpha) \end{array}$$

define a natural transformation θ^α in the preimage of α. □

Exercise 2.36. Let $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$. Show that the maps

$$\begin{array}{cccc} (h^f)_z : & h^y(z) & \to & h^x(z) \\ & \alpha & \mapsto & \alpha \circ f \end{array}$$

define a natural transformation $h^f : h^y \to h^x$. Let $\mathcal{H}$ be the category with the functors h^x as objects, and with natural transformations as morphisms. Show that the functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{H}$, $x \mapsto h^x$ is an equivalence of categories.

[4]A *small* category is a category whose objects form a set. For example, the category of all sets is not a small category (cf. the classical paradox of the set of all sets).

2.5.3. *The structure of functor categories.* Let $\mathcal{C}$ be a small category and let $\Bbbk$ be a commutative ring. We denote by $\mathcal{C}$-Mod the category[5] whose objects are functors $F : \mathcal{C} \to \Bbbk$-Mod, and whose morphisms are natural transformations of functors.

Direct sums, products, kernels, cokernel, quotients of functors are defined in the target category. Specifically, this means that the direct sum of two functors F, G is the functor $F \oplus G$ defined by $(F \oplus G)(c) = F(c) \oplus G(c)$ for all $c \in \mathcal{C}$. Similarly, the kernel of a natural transformation $\theta : F \to G$ is defined by $(\ker \theta)(c) := \ker[\theta_c : F(c) \to G(c)]$ and so on. The following lemma is an easy check.

Lemma 2.37. *The category $\mathcal{C}$-Mod is an abelian category.*

Short exact sequences of functors are the diagrams of functors $0 \to F' \to F \to F'' \to 0$ such that for all $c \in \mathcal{C}$, evaluation on c yields a short exact sequence of $\Bbbk$-modules:

$$0 \to F'(c) \to F(c) \to F''(c) \to 0 \ .$$

Since we consider functors with values in $\Bbbk$-mod, a linear combination of natural transformations is a natural transformation. So, the Hom-set in $\mathcal{C}$-Mod are actually $\Bbbk$-modules. Moreover, composition of natural transformations is bilinear. In other words, the functor category $\mathcal{C}$-Mod is enriched over $\Bbbk$ (we also say a $\Bbbk$-linear category).

2.5.4. *Homological algebra in functor categories.* We want to do homological algebra in $\mathcal{C}$-Mod. Let us first define the standard projectives P_x for $x \in \mathcal{C}$. These functors will play the same role in $\mathcal{C}$-Mod as the representation $\Bbbk G$ does in the category $\Bbbk G$-Mod.

Fix an object $x \in \mathcal{C}$. We define P_x as the $\Bbbk$-linearization of the functors h^x from section 2.5.2. To be more specific, if X is a set, we let $\Bbbk X$ be the free $\Bbbk$-module with basis $(b_x)_{x \in X}$ indexed by the elements of X. Then $P_x \in \mathcal{C}$-Mod is defined by

$$P_x(c) := \Bbbk \operatorname{Hom}_{\mathcal{C}}(x, c) \ .$$

Lemma 2.38 ($\Bbbk$-linear Yoneda lemma)**.** *Let $x \in C$ and let $F \in \mathcal{C}$-Mod. Let us denote by $1_x \in \Bbbk\mathrm{End}_{\mathcal{C}}(x)$ the basis element indexed by $\mathrm{Id}_x \in \mathrm{End}_C(x)$. The following map is an isomorphism of $\Bbbk$-modules, natural with respect to F and x:*

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}\text{-}Mod}(P_x, F) & \to & F(x) \\ \theta & \mapsto & \theta_x(1_x) \end{array} .$$

Proof. The canonical inclusion of sets $X \hookrightarrow \Bbbk X$ induces an isomorphism natural with respect to the set X and the $\Bbbk$-module M:

$$\operatorname{Hom}_{\Bbbk\text{-Mod}}(\Bbbk X, M) \simeq \operatorname{Hom}_{\mathrm{Set}}(X, M) \ .$$

Hence the canonical natural transformation of functors $h^x \to P^x$ induces an isomorphism $\operatorname{Hom}_{\mathcal{C}\text{-Mod}}(P_x, F) \simeq \mathrm{Nat}(h^x, F)$. Now the $\Bbbk$-linear map of

[5] Actually we are cheating a bit here. It is a priori not clear that $\mathcal{C}$-Mod is a genuine category. Indeed, it is not clear that the collection of natural transformations between two functors form a set. However, this is the case when the source category $\mathcal{C}$ is small, and this is exactly why we impose this condition on $\mathcal{C}$.

lemma 2.38 is the composite of this isomorphism and the standard Yoneda isomorphism of lemma 2.35, hence it is bijective. □

Exercise 2.39. Let $\mathcal{C}$ be a small category, and let x, y be objects of $\mathcal{C}$. Write an explicit basis of $\mathrm{Hom}_{\mathcal{C}\text{-Mod}}(P_x, P_y)$.

Let us spell out the homological consequences of the Yoneda lemma.

Lemma 2.40. *Let $x \in \mathcal{C}$. The functor P_x is projective.*

Proof. Projectivity of P_x means exactness of the functor $\mathrm{Hom}_{\mathcal{C}\text{-Mod}}(P_x, -)$. By the Yoneda lemma, the latter is isomorphic to the functor $F \mapsto F(x)$, which is exact by the definition of short exact sequences of functors. □

Lemma 2.41. *The category $\mathcal{C}$-Mod has enough projectives, and the family of functors $(P_x)_{x\in\mathcal{C}}$ is a projective generator. That is, all $F \in \mathcal{C}$-Mod can be written as a quotient of a direct sum of these functors.*

Proof. By the Yoneda lemma, for all $x \in \mathcal{C}$ and all $\alpha \in F(x)$, there is a unique natural transformation $\widetilde{\alpha} : P_x \to F$ which sends $1_x \in P_x(x)$ to $\alpha \in F(x)$. Taking the sum of all the $\widetilde{\alpha}$ we obtain a surjective natural transformation: $\bigoplus_{x\in\mathcal{C},\alpha\in F(x)} P_x \twoheadrightarrow F$. □

Exercise 2.42. Prove that any projective object in $\mathcal{C}$-Mod can be seen as a direct summand of a direct sum of P_x.

The category $\mathcal{C}$-Mod also has enough injectives, but it is slightly more complicated to prove.

Exercise 2.43. Let J be an injective generator of $\Bbbk$-Mod (i.e. every $\Bbbk$-module embeds into a product of copies of J). For all $x \in \mathcal{C}$, let us denote by $I_{x,J} \in \mathcal{C}$-Mod the functor defined by:

$$I_{x,J}(c) := \mathrm{Hom}_{\Bbbk}\left(\Bbbk\,\mathrm{Hom}_{\mathcal{C}}(c, x), J\right) .$$

Show that $I_{x,J}$ is injective, and that the family $(I_{x,J})_{x\in\mathcal{C}}$ is an injective generator of $\mathcal{C}$-Mod. (Find inspiration from the proof of proposition 2.12.)

2.5.5. *Homology and cohomology of categories.* Let $\mathcal{C}$ be a small category, let $\Bbbk$ be a commutative ring and let $F : \mathcal{C} \to \Bbbk$-Mod be a $\Bbbk$-linear representation of $\mathcal{C}$. Let us denote by $\Bbbk$ the constant functor with value $\Bbbk$. The cohomology of $\mathcal{C}$ with coefficients in F is the graded $\Bbbk$-module $H^*(\mathcal{C}, F)$ defined by

$$H^*(\mathcal{C}, F) := \mathrm{Ext}^*_{\mathcal{C}\text{-Mod}}(\Bbbk, F) .$$

To define the homology $H_*(\mathcal{C}, F)$ of $\mathcal{C}$ with coefficients in the representation F, we need a generalization of tensor products to the framework of functors. Let us denote the category $\mathcal{C}^{\mathrm{op}}$-Mod of contravariant functors with source $\mathcal{C}$ and target $\Bbbk$-Mod by the more suggestive notation Mod-$\mathcal{C}$. Such contravariant functors will play the role of right modules in our tensor product definition.

Let $G \in$ Mod-$\mathcal{C}$ and let $F \in \mathcal{C}$-Mod. The tensor product $G \otimes_{\mathcal{C}} F$ is the $\Bbbk$-module generated by the symbols $m \otimes n$, for $m \in G(x)$ and $n \in F(x)$, for all $x \in \mathcal{C}$, subject to the relations:

$$m \otimes n + m' \otimes n = (m + m') \otimes n \; ,$$
$$m \otimes n + m \otimes n' = (m + m') \otimes n \; ,$$
$$\lambda(m \otimes n) = (m\lambda) \otimes n = m \otimes (\lambda n) \text{ for } \lambda \in \Bbbk \; ,$$
$$(G(f)(m)) \otimes n = m \otimes (F(f)(n)) \text{ for all morphisms } f \text{ in } \mathcal{C} \; .$$

Exercise 2.44. Prove that $P_x \otimes_{\mathcal{C}} F \simeq F(x)$ naturally with respect to F, x and similarly that $G \otimes_{\mathcal{C}} P_x \simeq G(x)$, naturally with respect to G, x.

Exercise 2.45. Prove that the isomorphisms, natural with respect to $G \in$ Mod-$\mathcal{C}$, $F \in \mathcal{C}$-Mod and $M \in \Bbbk$-Mod:

$$\mathrm{Hom}_{\Bbbk\text{-Mod}}(G \otimes_{\mathcal{C}} F, M) \simeq \mathrm{Hom}_{\text{Mod-}\mathcal{C}}(G, \mathrm{Hom}_{\Bbbk}(F, M)) \; ,$$
$$\mathrm{Hom}_{\Bbbk\text{-Mod}}(G \otimes_{\mathcal{C}} F, M) \simeq \mathrm{Hom}_{\mathcal{C}\text{-Mod}}(F, \mathrm{Hom}_{\Bbbk}(G, M)) \; .$$

Deduce from these isomorphisms the following properties of the functors $F\otimes_{\mathcal{C}}$ and $\otimes_{\mathcal{C}}F$: (i) they are right exact, (ii) they commute with arbitrary sums, and (iii) if F (resp. G) is projective, then $G\otimes_{\mathcal{C}}$ (resp. $\otimes_{\mathcal{C}}F$) is exact.

As in the case of modules over an algebra, we define the Tor functors by deriving the right exact functors $G\otimes_{\mathcal{C}} : \mathcal{C}\text{-Mod} \to \Bbbk\text{-Mod}$ and $\otimes_{\mathcal{C}}F : \text{Mod-}\mathcal{C} \to \Bbbk\text{-Mod}$. As in the case of modules over a ring, the notation has two interpretations, but we can prove that they coincide. Thus we have:

$$\mathrm{Tor}_i^{\mathcal{C}}(G, F) = L_i(G\otimes_{\mathcal{C}})(F) = L_i(\otimes_{\mathcal{C}}F)(G) \; .$$

The homology $H_*(\mathcal{C}, F)$ of a small category $\mathcal{C}$ with coefficients in the functor $F \in \mathcal{C}$-Mod is then defined by:

$$H_*(\mathcal{C}, F) := \mathrm{Tor}_*^{\mathcal{C}}(\Bbbk, F) \; .$$

2.6. **Cohomology of linear algebraic groups.** We present three equivalent viewpoints on linear algebraic groups and their representations. We may view a linear algebraic groups as a Zariski closed subgroup of matrices (this viewpoint requires that the ground field $\Bbbk$ is infinite[6], and the reader may take [Bo],[Hum] and [S] as references), as a commutative finitely generated Hopf $\Bbbk$-algebra, or as a representable functor from finitely generated $\Bbbk$-algebras to groups (we refer to [Wa] for the latter viewpoints, which are valid over an arbitrary ground field). Then we define their cohomology in terms of Exts. We recall the definition of the Hochschild complex, and we use it to define the cohomological product. References for the cohomology of algebraic groups are [F] and [J].

2.6.1. *Linear algebraic groups and their representations.* Let $\Bbbk$ be an infinite field. A *linear algebraic group over* $\Bbbk$ is a Zariski closed subgroup of some $SL_n(\Bbbk)$, i.e. which is defined as the zero set of a family of polynomials with coefficients in $\Bbbk$ and with variables in the entries of the matrices $[x_{i,j}] \in SL_n(\Bbbk)$.

[6]It is usually required that the ground field $\Bbbk$ is algebraically closed, but for the basic definitions an infinite field is sufficient, and this is what we need here.

Example 2.46. Besides finite groups, we have the following common examples of algebraic groups:

$$\begin{aligned}
SL_n(\Bbbk) &= \{M \in M_n(\Bbbk)\ ;\ \det M = 1\}\ ,\\
GL_n(\Bbbk) &= \left\{ \begin{bmatrix} * & 0 \\ 0 & M \end{bmatrix} \in SL_{n+1}(\Bbbk) \right\}\ ,\\
\mathbb{G}_a &= \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \in M_2(\Bbbk) \right\}\ ,\\
O_n(\Bbbk) &= \{M \in M_n(\Bbbk)\ ;\ M^t M = I\}\ ,\\
Sp_n(\Bbbk) &= \{M \in M_{2n}(\Bbbk)\ ;\ M^t \Omega M = I\}\ ,
\end{aligned}$$

where Ω is the matrix in the canonical basis (e_i) of $\Bbbk^{2n}$ of the bilinear form ω_n defined by $\omega_n(e_i, e_j) = 1$ if $j = i + n$, $\omega_n(e_i, e_j) = -1$ if $i + n = j$, and $\omega_n(e_i, e_j)$ is zero otherwise.

A *rational (or algebraic) representation* of a linear algebraic group G is a $\Bbbk$-vector space V, equipped with an action $\rho : G \to \mathrm{End}_{\Bbbk}(V)$, satisfying the following condition.

(C) If V is finite dimensional, choose a basis of V and let $\rho_{k,\ell}(g)$ be the coordinates of $\rho(g)$ in the corresponding basis of $\mathrm{End}_{\Bbbk}(V)$. Then we require that the maps $g \mapsto \rho_{k,\ell}(g)$ are polynomials in the entries of the matrices $[g_{ij}] \in G$. (One says that ρ is *regular*).

(C') If V is infinite dimensional, we require that V is the union of finite dimensional sub-representations satisfying (C).

Example 2.47. (1) If V is a $\Bbbk$-vector space, the representation V^{triv} (i.e. G acts trivially) is a rational representation.

(2) If $G \subset SL_n(\Bbbk)$, then $\Bbbk^n$ acted on by G by matrix multiplication is a rational representation.

(3) If $G \subset SL_n(\Bbbk)$ is an algebraic group, let $\Bbbk[G]$ be the regular functions on G. That is, $\Bbbk[G]$ is the quotients of the algebra $\Bbbk[x_{i,j}]_{1 \le i,j \le n}$ by the ideal of the polynomials vanishing on G. The action of G on $\Bbbk[G]$ by left translations $(gf)(x) := f(g^{-1}x)$ is a rational action.

(4) If V and W are rational representations of G, the diagonal action on $V \otimes_{\Bbbk} W$ (that is $g(v \otimes w) := (gv) \otimes (gw)$ makes it into a rational representation of G.

We denote by rat-G-Mod the category whose objects are rational representations of G and morphisms are G-equivariant maps. Thus, rat-G-Mod is a full subcategory of $\Bbbk G$-mod. It is not hard to check that direct sums, subrepresentations and quotients of rational representations are rational representations, so rat-G-Mod is actually a full abelian subcategory of $\Bbbk G$-Mod. The following exercise provides an explanation for the name 'rational G-module'.

Exercise 2.48. Show that a morphism of monoids $\rho : GL_n(\Bbbk) \to M_m(\Bbbk)$ defines a rational representation of $GL_n(\Bbbk)$ if and only if its coordinates $\rho_{k,\ell}$ satisfy the following property. There exists a polynomial $P_{k,\ell}(x_{i,j})$ with n^2 variables $(1 \le i, j \le n)$ and an integer d such that for all $[g_{i,j}]_{1 \le i,j \le n} \in$

$GL_n(\Bbbk)$ we have:

$$\rho_{k,\ell}([g_{i,j}]) = \frac{P(g_{i,j})}{(\det[g_{i,j}])^d} \ .$$

2.6.2. *From algebraic groups to Hopf algebras.* The definitions of section 2.6.1 are not relevant over general fields $\Bbbk$. For example, if $\Bbbk$ is a finite field, then all subsets of $M_n(\Bbbk)$ are Zariski closed, and all maps $M_n(\Bbbk) \to M_m(\Bbbk)$ may be expressed as values of polynomials. Thus, representations of algebraic groups would have the same meaning as representations of finite groups. This is certainly not what we want.

To bypass this problem, we use the coordinate algebra $\Bbbk[G]$ of a linear algebraic group $G \subset M_n(\Bbbk)$. It is the finitely generated commutative reduced (i.e. without nilpotents) $\Bbbk$-algebra obtained as the quotient of $\Bbbk[x_{i,j}, 1 \le i,j \le n]$ by the ideal of polynomials vanishing on G. It may be interpreted a the algebra of regular maps $G \to \Bbbk$, that is set-theoretic maps which can be obtained as restrictions of polynomials with n^2 variables. The $\Bbbk$-algebra $\Bbbk[G]$ characterizes G as a Zariski closed set but says nothing about the group structure of G.

Proposition 2.49. *Let G be a linear algebraic group over an infinite field $\Bbbk$. The group structure of G yields a Hopf algebra structure on $\Bbbk[G]$.*

Recall that a *Hopf $\Bbbk$-algebra H* is a $\Bbbk$-algebra H equipped with morphisms of algebras (respectively called the comultiplication, the counit and the antipode):

$$H \xrightarrow{\Delta} H^{\otimes 2}\ , \quad H \xrightarrow{\epsilon} \Bbbk\ , \quad H \xrightarrow{\chi} H\ ,$$

which satisfy the following axioms (where all tensor products are taken over $\Bbbk$ and $\eta : \Bbbk \to H$ is the morphism $\lambda \mapsto \lambda 1$):

(i) Δ is coassociative, i.e: $(\Delta \otimes \mathrm{Id}_H) \circ \Delta = (\mathrm{Id}_H \otimes \Delta) \circ \Delta$,
(ii) Δ is counital, i.e: $(\mathrm{Id}_H \otimes \epsilon) \circ \Delta = \eta \circ \epsilon = (\epsilon \otimes \mathrm{Id}_H) \circ \Delta$,
(iii) Δ has a coinverse, i.e: $m \circ (\chi \otimes \mathrm{Id}_H) \circ \Delta = \mathrm{Id}_H = m \circ (\mathrm{Id}_H \otimes \chi) \circ \Delta$.

A *morphism of Hopf algebras $f : H \to H'$* is a $\Bbbk$-algebra morphism which commutes with comultiplications, counits and antipodes:

$$(f \otimes f) \circ \Delta = \Delta' \circ f\ , \quad \epsilon' \circ f = \epsilon\ , \quad \chi' \circ f = f \circ \chi\ .$$

Proof of proposition 2.49. We respectively define Δ, χ and ϵ as precomposition of regular functions on G by the multiplication, the inverse and the unit of G. The axioms (i), (ii), (iii) follow directly from the associativity of m, the fact that m has a unit, and the equation $g \cdot g^{-1} = 1 = g^{-1} \cdot g$. □

Example 2.50. Let $\Bbbk$ be an infinite field. The $\Bbbk$-algebra of regular functions of the algebraic group $GL_n(\Bbbk)$ is

$$\Bbbk[x_{i,j}, t]/ < t \det[x_{i,j}] - 1 >\ ,$$

where there are n^2 variables $x_{i,j}$ $(1 \le i,j \le n)$, and the brackets on the right refer to the ideal generated by the polynomial $t\det[x_{i,j}] - 1$. The comultiplication, the counit and the antipode are given by:

$$\Delta(x_{i,j}) = \sum_{k=1}^{n} x_{i,k} \otimes x_{k,j} \;, \qquad \Delta(t) = t \otimes t \;,$$
$$\epsilon(x_{i,j}) = \delta_{i,j} \;, \qquad \epsilon(t) = 1 \;,$$
$$\chi(x_{i,j}) = t(-1)^{ij}\mathrm{M}_{i,j} \qquad \chi(t) = \det[x_{i,j}] \;,$$

where $\mathrm{M}_{i,j}$ refers to the polynomial of degree $(n-1)^2$ obtained as the (i,j)-minor of the $n \times n$ matrix $[x_{k,\ell}]_{1\le k,\ell\le n}$.

Rational representations of linear algebraic groups can be translated in the language of Hopf algebras.

Proposition 2.51. *Let $\Bbbk$ be an infinite field, and let G be an algebraic group scheme over $\Bbbk$. The category of rational G-modules is equivalent to the category Comod-$\Bbbk[G]$ of right $\Bbbk[G]$-comodules.*

Recall that if $(H, \Delta, \epsilon, \chi)$ is a Hopf algebra over $\Bbbk$, a *right H-comodule* is a $\Bbbk$-vector space V equipped with a coaction morphism $\Delta_V : V \to V \otimes_{\Bbbk} H$ satisfying the following axioms:

(i) compatibility with Δ, i.e: $(\mathrm{Id}_V \otimes \Delta) \circ \Delta_V = (\Delta_V \otimes \mathrm{Id}_H) \circ \Delta_V$,
(ii) compatibility with ϵ, i.e: $(\mathrm{Id}_V \otimes \epsilon) \circ \Delta_V = \mathrm{Id}_V$.

A *morphism of comodules* is a $\Bbbk$-linear morphism $f : V \to W$ which commutes with the coaction, i.e. $(f \otimes \mathrm{Id}_H) \circ \Delta_V = \Delta_W \circ f$. If H is a Hopf algebra over a field $\Bbbk$, the category Comod-H of right H-comodules is abelian.

Proof of proposition 2.51. We first prove that the category of finite dimensional rational representations of G is equivalent to the category of finite dimensional $\Bbbk[G]$-comodules. We let $(b_i)_{0\le i\le n}$ be the canonical basis of $\Bbbk^n$.

1. Let $\rho : G \to M_n(\Bbbk)$ be a rational representation, and let $\rho_{k,\ell}$, $1 \le k, \ell \le n$ denote its coordinates. We check that the following formula defines a $\Bbbk[G]$-comodule map on $\Bbbk^n$:

$$\Delta_\rho(b_\ell) = \textstyle\sum_{i=1}^n b_k \otimes \rho_{k,\ell} \;.$$

   Moreover, we check that $\Bbbk$-linear map $f : \Bbbk^n \to \Bbbk^m$ is G equivariant if and only if it is a comodule map for the corresponding coactions. So, by sending a G-module ρ to the comodule given by Δ_ρ (and by acting identically on Hom-sets) we get a fully faithful functor:

$$\Phi : \{\text{fin. dim. rat-}G\text{-Mod}\} \to \{\text{fin. dim. } \Bbbk[G]\text{-comodules}\}$$

2. Conversely, if $\Delta : \Bbbk^n \to \Bbbk^n \otimes \Bbbk[G]$ is a comodule structure on $\Bbbk^n$, then $\Delta_{\Bbbk^n}(b_j)$ can be uniquely written as a sum $\sum_{i=1}^n b_i \otimes \Delta_{i,j}$ with $\Delta_{i,j} \in \Bbbk[G]$. We check that the map $\rho : G \to M_n(\Bbbk)$ defined by $\rho(g) = [\Delta_{i,j}(g)]$ is a morphism of groups. Thus Φ is essentially surjective, hence it is an equivalence of categories.

Now we prove that rat-G-mod is equivalent to Comod-$\Bbbk[G]$.

3. Assume that V is the union of finite dimensional vector subspaces $(V_\alpha)_{\alpha\in A}$, and that there are comodule maps $\Delta_\alpha : V_\alpha \to V_\alpha \otimes \Bbbk[G]$ such that the inclusions $V_\alpha \subset V_\beta$ are morphisms of comodules. Then

we define a comodule structure $V \to V \otimes \Bbbk[G]$ by letting $\Delta(v) = \Delta_\alpha(v)$ for any α such that $v \in V_\alpha$ ($\Delta(v)$ does not depend on α).

In particular, the functor Φ extends to a fully faithful functor:

$$\overline{\Phi} : \text{rat-}G\text{-Mod} \to \text{Comod-}\Bbbk[G]\ .$$

4. To prove that $\overline{\Phi}$ is an equivalence of categories, it remains to prove that it is essentially surjective, which is equivalent to prove that all $\Bbbk[G]$-comodules can be obtained as the union of their finite dimensional subcomodules. This finiteness property actually holds for comodules over arbitrary finitely generated Hopf algebras, see [Wa, Chap 3.3].

This concludes the proof. □

To sum up, a linear algebraic group G over an infinite field $\Bbbk$ yields a Hopf algebra $\Bbbk[G]$, and the study of the rational representations of G is equivalent to the study of the right $\Bbbk[G]$-comodules. Now when $\Bbbk$ is an arbitrary field, the definitions of section 2.6.1 are not relevant, but their Hopf algebra version still is. So we just think of algebraic groups as commutative finitely generated Hopf $\Bbbk$-algebra, and of rational modules as right comodules.

Example 2.52. If $\Bbbk$ is an arbitrary field, the formulas of example 2.50 define a commutative finitely generated Hopf $\Bbbk$-algebra, which we denote $\Bbbk[GL_n]$. The category of rational representations of the general linear group over $\Bbbk$ is defined as the category of right $\Bbbk[GL_n]$-comodules.

Remark 2.53. If the ground field $\Bbbk$ is infinite, the coordinate algebra $\Bbbk[G]$ of a linear algebraic group G is reduced, i.e. without nilpotent elements. So by considering *all* commutative finitely generated Hopf $\Bbbk$-algebras H, we introduce new objects. For example, if $\Bbbk$ has characteristic $p > 0$, the finite dimensional Hopf algebra $\Bbbk[(GL_n)_r]$ obtained as a quotient of $\Bbbk[GL_n]$ by the relations $x_{i,j}^{p^r} = \delta_{i,j}$ has nilpotent elements. These Hopf algebras (called Frobenius kernels of GL_n in the group scheme terminology) play a central role in the study of the rational representations of the general linear group, even when the ground field $\Bbbk$ is algebraically closed, see [J].

2.6.3. *From Hopf algebras to affine group schemes.* Let $\Bbbk$ be a field. An affine group scheme over $\Bbbk$ is a representable functor

$$G : \{\text{fin. gen. com. } \Bbbk\text{-Alg}\} \to \{\text{Groups}\}\ .$$

Representable means that there exists a finitely generated commutative $\Bbbk$-algebra B such that G is the functor $\mathrm{Hom}_{\Bbbk\text{-Alg}}(B, -)$. The algebra B representing G is often denoted by $\Bbbk[G]$. A morphism of affine group schemes is a natural transformation.

If H is a Hopf algebra we endow the set $\mathrm{Hom}_{\Bbbk\text{-Alg}}(H, A)$ with a group structure by letting the product of f and g be $m_A \circ (f \otimes g) \circ \Delta$, where m_A denotes the multiplication of A. (The unit element is the map $1_A\epsilon$, and the inverse of f is $f \circ \chi$). This group structure is natural with respect to A, hence the functor $\mathrm{Sp}(H) = \mathrm{Hom}_{\Bbbk\text{-Alg}}(H, -)$ has values in groups. This yields a functor

$$\mathrm{Sp} : \{\text{fin. gen. com. Hopf } \Bbbk\text{-alg}\}^{\mathrm{op}} \to \{\text{Affine Group Schemes over } \Bbbk\}\ .$$

As a consequence of the Yoneda lemma 2.35, this functor is an equivalence of categories (compare with exercise 2.36). Thus affine group schemes are a geometric way to view Hopf algebras.

Exercise 2.54. Show that $\mathrm{Sp}(\Bbbk[GL_n])$ is the functor which sends a $\Bbbk$-algebra A to the group $GL_n(A)$.

Now we translate comodules over Hopf algebras in the language of affine group schemes. If V is a $\Bbbk$-vector space, we denote by End_V the functor from finitely generated $\Bbbk$-algebras to monoids, which sends an algebra A to $\mathrm{End}_A(V \otimes_\Bbbk A)$, where $V \otimes_\Bbbk A$ is considered as a A-module by the formula $a \cdot (v \otimes b) := v \otimes (ab)$. A representation of a group scheme G is simply a natural transformation of functors $\rho : G \to \mathrm{End}_V$. A $\Bbbk$-linear morphism $f : V \to W$ is a morphism of representations if for all A, the A-linear map $f \otimes \mathrm{Id}_A$ is $G(A)$-equivariant. We denote by G-Mod the abelian category of representations of G. Let H be a finitely generated commutative Hopf $\Bbbk$-algebra. Using the Yoneda lemma once again, we prove an equivalence of categories:

$$\text{Comod-}H \simeq \mathrm{Sp}(H)\text{-Mod} .$$

2.6.4. *Cohomology of algebraic groups or group schemes.* We have several ways to think of representations of linear algebraic groups or group schemes. For example we have three equivalent categories for the general linear group: the representations of the group scheme GL_n, the comodules over the Hopf algebra $\Bbbk[GL_n]$, and the rational representations of the linear algebraic group $GL_n(\Bbbk)$ (the latter is well defined over an infinite field only)

$$GL_n\text{-Mod} \;\simeq\; \text{Comod-}\Bbbk[GL_n] \;\simeq\; \text{rat-}GL_n(\Bbbk)\text{-mod} .$$

The reader may work with his favourite category. From now on, we fix an affine group scheme G and we focus on the category G-Mod in the statements, although we might use another version of this category in the proofs.

Proposition 2.55. *The abelian category G-Mod has enough injectives.*

Proof. To prove the proposition, it suffices to prove that all $\Bbbk[G]$-comodule V can be embedded into an injective $\Bbbk[G]$-comodule.

1. Let us denote by $V^{\mathrm{triv}} \otimes \Bbbk[G]$ the $\Bbbk$-vector space $V \otimes \Bbbk[G]$ equipped with the coaction $\mathrm{Id}_V \otimes \Delta$. The assignment $f \mapsto (f \otimes \mathrm{Id}_{\Bbbk[G]}) \circ \Delta_W$ defines an isomorphism, natural with respect to $V \in \Bbbk$-mod and $W \in \text{Comod-}\Bbbk[G]$:
$$\mathrm{Hom}_\Bbbk(W, V) \simeq \mathrm{Hom}_{\text{Comod-}\Bbbk[G]}(W, V^{\mathrm{triv}} \otimes \Bbbk[G]) .$$
Since $\Bbbk$ is a field, the left hand side is an exact functor with variable W. Whence the injectivity of $V^{\mathrm{triv}} \otimes \Bbbk[G]$.
2. For all $\Bbbk[G]$-comodules V, the $\Bbbk$-linear map $\Delta_V : V \to V^{\mathrm{triv}} \otimes \Bbbk[G]$ is injective (by the compatibility with ϵ) and it is a morphism of comodules (by the compatibility with Δ).

Thus V embeds in the injective comodule $V^{\mathrm{triv}} \otimes \Bbbk[G]$. □

As in the case of discrete groups, there is a fixed point functor

$$\begin{array}{cccc} -^G : & G\text{-Mod} & \to & \Bbbk\text{-vect} \\ & V & \mapsto & V^G = \mathrm{Hom}_{G\text{-Mod}}(\Bbbk^{\mathrm{triv}}, V) \end{array}$$

Since G-Mod has enough injectives, the right derived functors of the fixed point functor are well defined and we call them the rational cohomology of G (or simply the cohomology of G).

$$H^i(G,-) = R^i(-^G) = \mathrm{Ext}^i_{G\text{-Mod}}(\Bbbk^{\mathrm{triv}},-)\ .$$

Remark 2.56. The category G-Mod has usually not enough projectives [J, I, section 3.18]. So in general we cannot hope to derive a functor of coinvariants to define a notion of rational homology of an algebraic group.

If G is a linear algebraic group over an infinite field $\Bbbk$ (as in section 2.6.1), the notation $H^*(G,V)$ is ambiguous. Indeed, this notation might stand for its rational cohomology $\mathrm{Ext}^*_{\text{rat-}G\text{-Mod}}(\Bbbk^{\mathrm{triv}},V)$, or for the cohomology of the discrete group G in the sense of section 2.4, i.e. $\mathrm{Ext}^*_{\Bbbk G\text{-Mod}}(\Bbbk^{\mathrm{triv}},V)$. These Exts need not be the same, since they are not computed in the same categories: rat-G-Mod is only a subcategory of $\Bbbk\,G$-Mod. It is usually clear from the context which notion is used. The following exercise should warn the reader of the dangers of misinterpreting the notation $H^*(G,M)$.

Exercise 2.57. Let $\Bbbk$ be an infinite field.

(1) Let $\rho : GL_1(\Bbbk)\to GL_n(\Bbbk)$ be a representation of the multiplicative group. Prove that there exists a family of $n\times n$-matrices $(M_k)_{k\in\mathbb{Z}}$ which are all zero but a finite number of them such that $\rho(x) = \sum x^k M_k$. Show that the M_k satisfy the relations

$$\sum M_k = I\ ,\quad M_kM_\ell = 0 \text{ if } k\neq \ell\ ,\quad M_k^2 = M_k\ .$$

Deduce that the representation splits as a direct sum of simple representations of dimension 1.

(2) Prove that for all rational representations V of $GL_1(\Bbbk)$,

$$\mathrm{Ext}^*_{\text{rat-}GL_1(\Bbbk)\text{-Mod}}(\Bbbk^{\mathrm{triv}},V) = 0 \text{ if } i>0.$$

(3) If $\Bbbk=\mathbb{R}$ or $\mathbb{C}$, prove that the following representation of the discrete group $GL_1(\Bbbk)$ provides a non-split extension of $\Bbbk^{\mathrm{triv}}$ by $\Bbbk^{\mathrm{triv}}$:

$$\begin{array}{cccc}\rho: & GL_1(\Bbbk) & \to & \mathrm{End}(\Bbbk^2)\\ & z & \mapsto & \begin{bmatrix}1 & \ln(|z|)\\ 0 & 1\end{bmatrix}\end{array}.$$

Deduce that $\mathrm{Ext}^1_{\Bbbk\, GL_1(\Bbbk)\text{-Mod}}(\Bbbk^{\mathrm{triv}},\Bbbk^{\mathrm{triv}})\neq 0$.

2.6.5. *Hochschild complex and cohomology algebras.* The Hochschild complex computes the rational cohomology $H^*(G,V)$ of an affine group scheme G with coefficients in V. This complex does not require specific knowledge about G or V to be written down (but of course, some knowledge is needed if you want to compute its homology!). It is often used to describe the general (i.e. independent of G and V) properties of rational cohomology. As an example, we will use it to define cohomology cup products.

The Hochschild complex is similar to the complex of proposition 2.32 (computing the cohomology of discrete groups), but set-theoretic maps $G^{\times n}\to V$ have to be replaced by their algebro-geometric counterpart. To be more specific, if V is a finite dimensional vector space, we can consider it as a scheme, that is a representable functor

$$\{\text{fin. gen. com. } \Bbbk\text{-Alg}\} \to \{\text{Sets}\} .$$

Its value on A is the set $V\otimes_{\Bbbk} A$, and it is represented by $\Bbbk[V]$. We shall write $\mathrm{Hom}(G^{\times n}, V)$ for the set of natural transformations from $G^{\times n}$ to V. If V is infinite dimensional, we let $\mathrm{Hom}(G^{\times n}, V)$ be the set of natural transformations taking values in a finite dimensional subspace of V. By the Yoneda lemma, we have a $\Bbbk$-linear isomorphism:

$$V\otimes \Bbbk[G]^{\otimes n} \simeq \mathrm{Hom}(G^{\times n}, V) .$$

We can now define the Hochschild complex (compare with proposition 2.32). It is the complex of $\Bbbk$-modules $C^*(G, V)$ with:

$$C^n(G, V) = \mathrm{Hom}(G^{\times n}, V) \simeq V \otimes (\Bbbk[G]^{\otimes n}) ,$$

and the differential $\partial : C^n(G, V) \to C^{n+1}(G, V)$ sends a morphism f to the morphism ∂f defined by

$$\begin{array}{rl}(\partial f)(g_1, \dots, g_{n+1}) := & g_1 f(g_2, \dots, g_{n+1}) \\ & + \sum_{i=1}^{n}(-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ & +(-1)^{n+1} f(g_2, \dots, g_n)\end{array} .$$

The proof of the following proposition is dual to the proof of proposition 2.27 and corollary 2.28, see [J, I section 4.16]:

Proposition 2.58. *The homology of $C^*(G, V)$ is equal to $H^*(G, V)$.*

Let G be a group scheme and let A be a $\Bbbk G$-algebra. By this we mean that A is a $\Bbbk$-algebra, equipped with an action of G by algebra automorphisms. Then the rational cohomology of G with coefficients in A is equipped with a cup product. This cup product can be defined on the level of the Hochschild complex exactly as in the case of discrete groups. Namely, if $\cdot_A$ denotes the multiplication in the algebra A, we define the cup product of $f_1 \in C^n(G, A)$ and $f_2 \in C^m(G, A)$ by:

$$(f_1 \cup f_2)(g_1, \dots, g_{n+m}) := f_1(g_1, \dots, g_n) \cdot_A [(g_1 \cdots g_n) f_2(g_{n+1}, \dots, g_{n+m})] .$$

This product makes the Hochschild complex into a differential graded algebra, hence it induces a product on homology. As in the case of discrete groups, one can prove that if A is commutative, then the graded algebra $H^*(G, A)$ is graded commutative.

3. Derived functors of non additive functors

Many functors between abelian categories are not additive. In this section, we explain how to derive arbitrary functors between abelian categories. This theory was invented by Dold and Puppe, and later generalized by Quillen's homotopical algebra. Dold-Puppe theory of derived functors relies heavily on simplicial techniques, for which the reader can take [GJ, May, ML, Wei] as references. The best reference for derived functors is the seminal article [DP].

3.1. Simplicial objects.

3.1.1. *The categorical definition.* Let Δ denote the category whose objects are the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n \geq 0$, and whose morphisms are the non-decreasing monotone maps.

Definition 3.1. Let $\mathcal{C}$ is a category, a *simplicial object in* $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. A *morphism of simplicial objects* is a natural transformation. We denote by $s\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$.

As a functor category, $s\mathcal{C}$ inherits the properties of $\mathcal{C}$. For example, if $\mathcal{C}$ has products then so does $s\mathcal{C}$, the product $X \times Y$ is defined by $(X\times Y)([n]) := X([n]) \times Y([n])$. If $\mathcal{C}$ is an abelian category, so is $s\mathcal{C}$. If $F : \mathcal{C} \to \mathcal{D}$ is a functor, postcomposition by F induces a functor $s\mathcal{C} \to s\mathcal{D}$.

3.1.2. *The geometric definition.* The categorical definition of simplicial objects is practical to derive some of their abstract properties, but there is an equivalent geometrical definition which gives a more concrete picture of what simplicial objects are.

For $0 \leq i \leq n$ we denote by $d^i : [n-1] \to [n]$ the unique injective morphism of Δ which misses $i \in [n]$, and by $s^i : [n+1] \to [n]$ the unique surjective map in Δ with two elements sent to $i \in [n]$. It can be proved that the d^i and the s^i generate the category Δ, that is all morphism of Δ can be written as a composite of these maps. Thus, a simplicial object X in $\mathcal{C}$ is uniquely determined by the sequence of objects $X_n = X([n])$, $n \geq 0$, and by the morphisms

$$d_i := X(d^i) : X([n]) \to X([n-1])\,, \quad s_i := X(s^i) : X([n]) \to X([n+1])\,.$$

However, the category Δ is not freely generated by the d^i and the s^i: relations hold between various compositions of these maps. So not all data of objects $(X_n)_{n\geq 0}$ and morphisms $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ come from simplicial objects. One can prove that the ones corresponding to simplicial objects are the ones satisfying the following simplicial identities.

$$\begin{aligned} &d_i d_j = d_{j-1} d_i \text{ if } i < j,\\ &s_i s_j = s_{j+1} s_i \text{ if } i \leq j,\\ &d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j,\\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1,\\ s_j d_{j-1} & \text{if } i > j+1. \end{cases} \end{aligned}$$

This leads us to the geometric definition of simplicial objects.

Definition 3.2. Let $\mathcal{C}$ be a category. A *simplicial object in* $\mathcal{C}$ is a sequence of objects $(X_n)_{n\geq 0}$ of $\mathcal{C}$, together with face operators $d_i : X_n \to X_{n-1}$ and degeneracy operators $s_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$, satisfying the simplicial identities listed above. A *morphism of simplicial sets* is a sequence of maps $f_n : X_n \to Y_n$, $n \geq 0$ commuting with the face and the degeneracy operators.

Of most interest for us are the category sSet of simplicial sets and the sR-Mod of simplicial R-modules.

3.1.3. *Examples of simplicial sets.* The *standard n-simplex* $\Delta[n]$ is the simplicial set defined by:

$$\Delta[n] := \mathrm{Hom}_{\Delta}(-, [n])\,.$$

The map $d^i : [n-1] \to [n]$ in Δ induces a morphism of simplicial sets $d^i : \Delta[n-1] \to \Delta[n]$. Its image is a simplicial subset of $\Delta[n]$ called the i-th face of $\Delta[n]$ and denoted by $\partial_i \Delta[n]$. The boundary $\partial\Delta[n] \subset \Delta[n]$ is the simplicial subset $\bigcup_{0\le i\le n} \partial_i\Delta[n]$.

The Yoneda lemma 2.35 yields bijections $\mathrm{Hom}_{s\mathrm{Set}}(\Delta[n], X) \simeq X_n$ for all simplicial sets X. The terminology 'standard n-simplex' for $\Delta[n]$ and 'face' for $\partial_i\Delta[n]$ is justified by the following geometrical interpretation.

Geometrical interpretation 3.3. Let $(v_0, \dots, v_n)$ be the canonical basis of $\mathbb{R}^{n+1}$. The geometric n-simplex Δ^n is the subspace of $\mathbb{R}^{n+1}$ which is the convex hull of the $n+1$ points v_i. The v_i are the vertices of Δ^n. For each morphism $f : [n] \to [m]$ in Δ we can define affine maps $f : \Delta^n \to \Delta^m$ by letting $f(v_k) = v_{f(k)}$. This yields an isomorphism between the full subcategory of sSet with objects the standard n-simplices onto the category whose objects are the geometric n-simplices, and whose morphisms are the affine maps sending vertices to vertices and preserving the order of the vertices.

Topological spaces provide examples of simplicial sets. Let T be a topological space. The *singular simplicial set of* T is the simplicial set $S(T)$ with $S(T)_n = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, T)$. The face operator $d_i : S(T)_n \to S(T)_{n-1}$ is induced by precomposing an n-simplex $\sigma : \Delta^n \to T$ by the affine map $d^i : \Delta^{n-1} \to \Delta^n$ (cf example 3.3). The degeneracy operator $s_i : S(T)_n \to S(T)_{n+1}$ is obtained by precomposing a simplex by the affine map $s^i : \Delta^{n+1} \to \Delta^n$. The following exercise shows that not all simplicial sets can be obtained as singular simplicial sets of topological spaces.

Exercise 3.4. Let $\Lambda^i[n]$ be the simplicial subset of $\Delta[n]$ obtained as as the boundary of $\Delta[n]$ with the i-th face removed:

$$\Lambda^i[n] = \textstyle\bigcup_{k\neq i} \partial_k\Delta[n] \;.$$

(1) Show that the singular simplicial set of a topological space T satisfies the following 'Kan condition'. For all $n \ge 0$, for all $0 \le i \le n$ and for all morphisms $\sigma : \Lambda^i[n] \to X$, there exists morphisms $\overline{\sigma} : \Delta[n] \to X$ whose restriction to $\Lambda^i[n]$ equals σ.
(2) Show that $\Delta[n]$ and $\Lambda^i[n]$ do not satisfy the Kan condition.

Categories provide another source of simplicial sets. Let $\mathcal{C}$ be a small category. An n-chain of composable morphisms is a chain of n morphisms

$$c_0 \xleftarrow{f_1} c_1 \xleftarrow{f_2} \dots \xleftarrow{f_n} c_n \;.$$

We shall denote such a chain as a n-tuple $(f_1, \dots, f_n)$. The *nerve* of $\mathcal{C}$ is the simplicial set $B\mathcal{C}$ defined as follows. $B\mathcal{C}_0$ is the set of objects of $\mathcal{C}$, $B\mathcal{C}_1$ is the set of morphisms of $\mathcal{C}$, and more generally for all $n \ge 1$, $B\mathcal{C}_n$ is the set of n-chains of composable morphisms. For $n = 1$, the face operators $d_0, d_1 : B\mathcal{C}_1 \to B\mathcal{C}_0$ are defined by

$$d_0 f = \mathrm{source}(f) \;, \qquad d_1(f) = \mathrm{target}(f) \;.$$

For $n \ge 2$, the face operators are given by dropping or composing morphisms:

$$d_i(f_1, \dots, f_n) = \begin{cases} (f_2, \dots, f_n) & \text{if } i = 0, \\ (f_1, \dots, f_i f_{i+1}, \dots, f_n) & \text{if } 0 < i < n, \\ (f_1, \dots, f_{n-1}) & \text{if } i = n. \end{cases}$$

The degeneracy operators s_i are given by inserting the identity morphism in i-th position. The following exercise characterizes the simplicial sets obtained as nerves of categories, and shows that the theory of (small) categories may be seen as a special case of the theory of simplicial sets.

Exercise 3.5. Let us denote by Cat the category whose objects are the small categories and whose morphisms are the functors.

(1) Show that the nerve defines a functor $B : \mathrm{Cat} \to s\mathrm{Set}$.
(2) The edge $[i, i+1]$ of $\Delta[n]$ is the image of $\Delta[1]$ by the morphism $\Delta[1] \hookrightarrow \Delta[n]$ induced by the map $[1] \to [n]$, $x \mapsto i+x$. The backbone of $\Delta[n]$ is the simplicial subset $[0,1] \cup \cdots \cup [n-1,n]$. Show that a simplicial set X is isomorphic to the nerve of a category if and only if it satisfies the following 'backbone condition'. For all morphisms of simplicial sets $\sigma : [0,1] \cup \cdots \cup [n-1,n] \to X$ there exists a unique simplex $\overline{\sigma} : \Delta[n] \to X$ whose restriction to the backbone of $\Delta[n]$ equals σ.
(3) Show that B induces an equivalence of categories between Cat and the full subcategory of simplicial sets satisfying the 'backbone condition'.

3.1.4. *Examples of simplicial R-modules and chain complexes.* Let R be a ring. If X is a set, we denote by RX the free R-module with basis X. This yields a 'free R-module' functor

$$\begin{array}{ccc} \mathrm{Set} & \to & R\text{-Mod} \\ X & \mapsto & RX \end{array} .$$

Interesting examples of simplicial R-modules are provided by applying the free R-module functor to simplicial sets. Before giving illustrations of this, we need to introduce one more definition, relating simplicial R-modules to complexes of R-modules.

Definition 3.6. If M is a simplicial R-module, we define the associated chain complex $\mathcal{C}M$ by letting $(\mathcal{C}M)_n = M_n$ and the differential $d : (\mathcal{C}M)_n \to (\mathcal{C}M)_{n-1}$ is the sum $\sum_{i=0}^{n}(-1)^i d_i$. This defines a functor:

$$\mathcal{C} : s(R\text{-Mod}) \to \mathrm{Ch}_{\geq 0}(R\text{-Mod}) .$$

Exercise 3.7. Check that $\mathcal{C}M$ is indeed a chain complex.

Many of the most usual chain complexes are obtained from simplicial R-modules in this way. For example, let X be a topological space and let $S(X)$ be its singular simplicial set. We make it into a chain complex of R-modules $RS(X)$. The associated chain complex $\mathcal{C}RS(X)$ is the usual singular chain complex of X [Hat, Chap 2], which computes the singular homology $H_*(X, R)$ of X with coefficients in R. Another example is provided by the following exercise.

Exercise 3.8. Let G be a discrete group, considered as a category with one object (cf. section 2.5.1), and let BG be its nerve as defined in exercise 3.5. Show that the chain complex of $\Bbbk$-modules $\mathcal{C}\Bbbk BG$ is equal to the bar complex $B(\Bbbk^{\mathrm{triv}}, \Bbbk G, \Bbbk^{\mathrm{triv}})$ defined in section 2.3.3.

The chain complex $\mathcal{C}M$ admits the following variant, which is a smaller chain complex.

Definition 3.9. Let M be a simplicial R-module. The *normalized chain complex* $\mathcal{N}M$ is the subcomplex of $\mathcal{C}M$ defined by $(\mathcal{N}M)_n = \bigcap_{0\le i<n} \ker d_i$, with differential $d = (-1)^n d_n : (\mathcal{N}M)_n \to (\mathcal{N}M)_{n-1}$. This defines a functor:

$$\mathcal{N} : s(R\text{-Mod}) \to \mathrm{Ch}_{\ge 0}(R\text{-Mod}) .$$

Proposition 3.10. *The normalized chain complex $\mathcal{N}M$ is a direct summand of the complex $\mathcal{C}M$. Moreover the inclusion $\mathcal{N}M \hookrightarrow \mathcal{C}M$ is a homotopy equivalence.*

Proof. Let us denote by DM the graded subobject of $\mathcal{C}M$ generated by the degeneracy operators, that is $(DM)_0 = 0$ and $(DM)_n = \sum_{0\le i\le n-1} s_i(M_{n-1})$. The simplicial identities imply that $(DM)_n \oplus (\mathcal{N}M)_n = (\mathcal{C}M)_n$, and that the differential of $\mathcal{C}M$ sends $(DM)_n$ into $(DM)_{n-1}$. Hence $\mathcal{C}M$ decomposes as the direct sum of complexes $\mathcal{C}M = \mathcal{N}M \oplus DM$.

To prove the homotopy equivalence $\mathcal{N}M \hookrightarrow \mathcal{C}M$, it suffices to prove that DM is homotopy equivalent to the zero complex, i.e. that Id_{DM} is homotopic to zero. For this, we write DM as an increasing union of subcomplexes

$$0 = D_{-1}M \subset D_0M \subset D_1M \subset \cdots \subset D_kM \subset \cdots \subset DM .$$

where $(D_kM)_n = \sum_{0\le i\le \min(k,n-1)} s_i(M_{n-1})$ for $n \ge 1$ (we use the simplicial identities to check that the D_kM are indeed subscomplexes of DM). We define $t_k : (DM)_n \to (DM)_{n+1}$ by $t_k(m) = (-1)^k s_k(m)$ if $k \le \deg(m)$, 0 if $k > \deg(m)$, and we define

$$f_k := \mathrm{Id}_{DM} - dt_k - t_kd ,$$

where d is the differential of DM. By its construction, $f_k : DM \to DM$ is a chain map, which is homotopic to Id_{DM} (via the homotopy t_k), and which coincides with Id_{DM} in low degrees: $f_k(m) = m$ if $\deg(m) < k$. The simplicial identities imply that for all $k \ge 0$:

$$f_k(D_kM) \subset D_{k-1}M , \quad f_k(D_jM) \subset D_jM \text{ if } j < k. \qquad (*)$$

Now we form the composite $f = f_0f_1f_2 \dots f_k \dots$. This composition is finite in each degree (since only a finite number of f_ks are not equal to the identity in a given degree), and is equal to the zero map by $(*)$. Now each f_k is homotopic to the identity, hence so is the composite f. □

3.1.5. *Simplicial sets and algebraic topology.* We now make a very quick digression on the relations between simplicial sets and algebraic topology. Although this section is not needed to understand the concept of derived functors of non-additive functors, it explains some notations and definitions of simplicial R-modules which originate from algebraic topology.

First of all, **simplicial sets are a combinatorial model for topological spaces**. To be more specific, from a simplicial set X we can construct a topological space $|X|$, the *realization of* X, by the following recipe. Consider each X_n as a discrete space, and recall the geometric n-simplex Δ^n described in the geometrical interpretation 3.3. Then $|X|$ is the topological space defined as the quotient

$$\Big(\bigsqcup_{n\ge 0} X_n \times \Delta^n\Big)/\sim ,$$

where $\sim$ is the equivalence relation generated by $(x, d_i(\sigma)) \sim (d^i(x), \sigma)$ for $x \in \Delta^n$ and $\sigma \in X_{n+1}$ and $(x, s_i(\sigma)) \sim (s^i(x), \sigma)$, for $x \in \Delta^n$ and $\sigma \in X_{n-1}$ (and the affine maps $d^i : \Delta^n \to \Delta^{n+1}$ and $s^i : \Delta^n \to \Delta^{n-1}$ are the ones determined by $d^i : [n] \to [n+1]$ and $s^i : [n] \to [n-1]$ as explained in the geometrical interpretation 3.3). If Top denotes the category of topological paces and continuous maps, we have a realization functor:

$$| - | : s\mathrm{Set} \to \mathrm{Top}\ .$$

Among the important properties of the realization functor is the fact that the realization $|S(T)|$ of the singular simplicial set of a topological space T is always weakly homotopy equivalent to T itself (for more complete statements, we refer the reader e.g. to [GJ, Chap I, Thm 11.4] or [Hov, Thm 3.6.7]). This means that all the homotopy-theoretic information (like its homotopy groups or its singular homology groups) of the topological space T is encoded in the simplicial set $S(T)$.

In particular, **the classical invariants of topological spaces have their combinatorial counterpart in the world of simplicial sets**.

(1) We can define the homology of a simplicial set X with coefficient in a ring R as the homology of the complex of R-modules $\mathcal{C}RX$. The homology of X is then isomorphic to the singular homology of its realization $|X|$ (when $X = S(T)$, this was already observed in section 3.1.4).
(2) The homotopy groups $\pi_n(X, x)$ of a simplicial set X with basepoint $x \in X_0$ can also be defined, at least when X is *fibrant*, that is when X satisfies the 'Kan condition' of exercise 3.4 (the exercise shows that $S(T)$ is fibrant). We refer the reader to [Wei, Def 8.3.1], [GJ, Chap I.7] of [Hov, Def 3.4.4] for the definition. As for homology, one has an isomorphism $\pi_n(X, x) \simeq \pi_n(|X|, x)$.

Now a simplicial R-module M can be considered as a simplicial set by forgeting that the M_n are R-module, and just considering the underlying sets. This yields a a forgetful functor

$$s(R\text{-Mod}) \to s\mathrm{Set}\ .$$

In particular, we can consider the homotopy groups of a simplicial R-module M, i.e. the homotopy groups of the underlying simplicial set. In this situation the combinatorial definition of homotopy groups specializes as $\pi_n(M, 0) = H_n(\mathcal{N}M)$. This is why the homology groups of the (normalized) chain complex associated to a simplicial R-module M is commonly denoted by $\pi_n(M)$.

We can also define simplicial homotopies between morphisms of simplicial R-modules. This yields definition 3.12 below. In particular, simplicially homotopic maps $f, g : M \to N$ yield homotopic continuous maps $|f|, |g| : |M| \to |N|$ after realization.

A topological space T is called an Eilenberg-Mac-Lane space of type $K(A, n)$, with $n \geq 1$ and A an abelian group, if its homotopy groups are all zero except $\pi_n(T, *) \simeq A$. As another example of application of the viewpoint above, we call 'Eilenberg-Mac Lane space of type $K(A, n)$' a simplicial abelian group M whose homotopy groups are all zero, except $\pi_n(M) = A$.

The Dold-Kan correspondence provides a way to build such Eilenberg-Mac Lane spaces, see remark 3.16 below.

3.2. **Derived functors of non-additive functors.** Let $F : R\text{-Mod} \to S\text{-Mod}$ be a functor. If F is right exact we can derive it by using the classical theory of derived functors recalled in section 2. The left derived functors of F are then defined by:

$$L_iF(M) = H_i(F(P^M)) \ , \quad L_iF(f) = H_i(F(\overline{f})) \ ,$$

where P^M is a projective resolution of the S-module M, and $\overline{f} : P^M \to P^N$ is a lifting of $f : M \to N$ at the level of the projective resolutions. The chain map $\overline{f}$ is only defined up to homotopy, but the definition is well-founded because F sends homotopic chain maps to homotopic chain maps. This fact relies on the crucial fact that, being right exact, F is actually an additive functor, cf. exercise 2.3.

Now if F is not additive, it does not send homotopic chain maps to homotopic chain maps (hence homotopy equivalences to homotopy equivalences), as the following example shows it. So we cannot use the formulas above to define its derived functors.

Example 3.11. Let $\Bbbk$ be a field, and let $\Lambda^2 : \Bbbk\text{-Vect} \to \Bbbk\text{-Vect}$ be the functor which sends a $\Bbbk$-vector space to its second exterior power. Let P and Q be the projective resolutions of the dimension one vector space respectively defined by: $P_i = 0$ if $i \neq 0$ and $P_0 = \Bbbk$; $Q_i = 0$ if $i > 1$, $Q_1 = \Bbbk$, $Q_0 = \Bbbk^2$ and $d : Q_1 \to Q_0$ sends λ to $(\lambda, 0)$. By lemma 2.8, P and Q are homotopy equivalent. However $H_0(\Lambda^2(P)) = 0$ and $H_0(\Lambda^2(Q)) = \Bbbk$.

The suitable generalization of derived functors for arbitrary functors $F : R\text{-Mod} \to S\text{-Mod}$ was found by Dold and Puppe [DP], and relies on the Dold-Kan correspondence.

3.2.1. *The Dold-Kan correspondence.* We have seen in definition 3.9 the normalized chain functor

$$\mathcal{N} : s(R\text{-Mod}) \to \mathrm{Ch}_{\geq 0}(R\text{-Mod}) \ .$$

We first establish a fundamental property of this functor, namely the preservation of homotopies.

Definition 3.12. Two morphisms of simplicial R-modules $f, g : M \to N$ are simplicially homotopic if there are morphisms $h_i : M_n \to N_{n+1}$ for $0 \leq i \leq n$ satisfying the following identities:

$$\begin{aligned} d_0h_0 &= f, \\ d_{n+1}h_n &= g, \\ d_ih_j &= \begin{cases} h_{j-1}d_i & \text{if } i < j \\ d_ih_{i-1} & \text{if } i = j \neq 0 \\ h_jd_{i-1} & \text{if } i > j+1 \end{cases} , \\ s_ih_j &= \begin{cases} h_{j+1}s_i & \text{if } i \leq j \\ h_js_{i-1} & \text{if } i > j \end{cases} . \end{aligned}$$

Proposition 3.13. *Let $f, g : M \to N$ be two simplicially homotopic morphisms. Then the induced chain maps at the level of normalized chain complexes $\mathcal{N}(f), \mathcal{N}(g) : \mathcal{N}M \to \mathcal{N}N$ are chain homotopic.*

Proof. Let $h_i : M_n \to N_{n+1}$, $0 \le i \le n$ be the maps defining the homotopy from f to g (so $d_0 h_0 = f$, $d_{n+1} h_n = g$). We let $t_n : M_n \to N_{n+1}$ be the map defined by

$$t_n(m) = \sum_{j=0}^{n} (-1)^j (h_j(m) - f s_j(m)) \ .$$

We denote by $t_n^{\le i}(m)$ and $t_n^{\ge i}$ the partial sums obtained when the summation index j varies from 0 to i, resp. from i to n (so $t_n = t_n^{\le i} + t_n^{\ge i+1}$). The definition of simplicial homotopies imply that:

$$\begin{aligned} d_0 t_n &= -t_{n-1} d_0 \\ d_i t_n &= t_{n-1}^{\le i-2} d_{i-1} - t_{n-1}^{\ge i} d_i \\ d_{n+1} t_n &= t_{n-1} d_n + (-1)^n (g - f) \end{aligned}$$

As a consequence, t_n maps $(\mathcal{N}M)_n$ to $(\mathcal{N}N)_{n+1}$, and the differential d of $\mathcal{N}M$ satisfies $t_n d + d t_{n+1} = f - g$. □

Remark 3.14. Let us denote by $\pi_n(M)$ the n-th homotopy group of a simplicial R-module M, that is the n-th homology group of its normalized chain complex $\mathcal{N}M$. Proposition 3.13 shows that simplicially homotopic maps induce the same map at the level of homotopy groups. This should not surprise the reader after the discussion of section 3.1.5

Theorem 3.15 (The Dold-Kan correspondence)**.** *The normalized chain functor $\mathcal{N}$ is an equivalence of categories. Moreover, it has an inverse $\mathcal{K} : \mathrm{Ch}_{\ge 0}(R\text{-}Mod) \to s(R\text{-}Mod)$ which preserves homotopies.*

The proof of this theorem is quite long and we will skip it. The reader can consult [DP, Section 3], [GJ, Chap III.2] or [Wei, Section 8.4] for a proof. We just insist on the fact that the functor $\mathcal{K}$ has a very explicit combinatorial expression. For example, let $M[1]$ denote the chain complex which is zero except in degree 1, where it equals the R-module M. Then the value of $\mathcal{K}$ on the complex $M[1]$ is just the simplicial R-module with $\mathcal{K}(M[1])_n = M^{\oplus n}$, the face operators $d_i : M^{\oplus n} \to M^{\oplus n-1}$ are defined by

$$\begin{aligned} d_0(x_1, \dots, x_k) &= (x_2, \dots, x_n) \ , \\ d_i(x_1, \dots, x_k) &= (x_1, \dots, x_i + x_{i+1}, \dots, x_n) \text{ if } 0 < i < k \ , \\ d_n(x_1, \dots, x_k) &= (x_1, \dots, x_{n-1}) \ , \end{aligned}$$

and the degeneracy operators s_i insert a zero in i-th position.

Remark 3.16. Let $M[n]$ denote the chain complex which is zero except in degree n, where it equals the R-module M. Then $\mathcal{K}(M[n])$ is a simplicial Eilenberg-Mac Lane space, i.e. a simplicial R-module whose homotopy groups are zero, except $\pi_n(\mathcal{K}(M[n])) = M$. In particular, the functor $\mathcal{K}$ may be used as a constructor of Eilenberg-Mac Lane spaces.

3.2.2. *Derived functors.* Let $F : R\text{-Mod} \to S\text{-Mod}$ be a functor. As already observed, the induced functor $F : \mathrm{Ch}_{\geq 0}(R\text{-Mod}) \to \mathrm{Ch}_{\geq 0}(S\text{-Mod})$ by applying F degreewise does not preserve chain homotopies if F is not additive. The key observation is that even if F is not additive, the induced functor $F : s(R\text{-Mod}) \to s(S\text{-Mod})$ does preserve simplicial homotopies. Indeed, simplicial homotopies do not involve additions in their definition. Since the Dold-Kan correspondence preserves homotopies, the following composite functor preserves chain homotopies:

$$\mathrm{Ch}_{\geq 0}(R\text{-Mod}) \xrightarrow[\sim]{\mathcal{K}} s(R\text{-Mod}) \xrightarrow{F} s(S\text{-Mod}) \xrightarrow[\sim]{\mathcal{N}} \mathrm{Ch}_{\geq 0}(S\text{-Mod}) \; .$$

So, we can make the following well-founded definition.

Definition 3.17 ([DP])**.** Let $F : R\text{-Mod} \to S\text{-Mod}$ be a functor, and let n be a nonnegative integer. The i-th derived functor of F with height n is defined by

$$L_iF(M;n) = H_i(\mathcal{N}F\mathcal{K}(P^M[n])) \; , \quad L_iF(f;n) = H_i(\mathcal{N}F\mathcal{K}(\overline{f})[n]) \; ,$$

where P^M is a projective resolution of the S-module M, $\overline{f} : P^M \to P^N$ is a lifting of $f : M \to N$ at the level of the projective resolutions, and $[n]$ refers to the n-fold suspension functor $[n] : \mathrm{Ch}_{\geq 0}(R\text{-Mod}) \to \mathrm{Ch}_{\geq 0}(R\text{-Mod})$.

The following proposition shows that definition 3.17 is a generalization of the classical definition of derived functors of semi-exact functors.

Proposition 3.18. *Let $F : R$-Mod $\to S$-Mod be a right-exact functor and let $n \geq 0$. For all $i \geq 0$, there is a natural isomorphism (with the convention that derived functors with negative indexes are zero):*

$$L_iF(M;n) \simeq L_{i-n}F(M)$$

Proof. Since F is right exact, it is additive. In particular, if X is a simplicial R-module, the complexes $F(\mathcal{C}X)$ and $\mathcal{C}F(X)$ are equal (by definition, $(F(\mathcal{C}X))_n = F(X_n) = (\mathcal{C}F(X))_n$ and by additivity, $F(\sum(-1)^i d_i) = \sum(-1)^i F(d_i)$, hence $F(\mathcal{C}X)$ and $\mathcal{C}F(X)$ have the same differentials). Additivity of F also ensures that F preserves homotopy equivalences and commutes with suspension. Thus we have a chain of homotopy equivalences of chain complexes:

$$\begin{aligned}\mathcal{N}F\mathcal{K}(P^M[n]) \simeq \mathcal{C}F\mathcal{K}(P^M[n]) = F\mathcal{C}\mathcal{K}(P^M[n]) \; , \\ \simeq F\mathcal{N}\mathcal{K}(P^M[n]) \simeq F(P^M[n]) = F(P^M)[n] \; .\end{aligned}$$

Taking the homology of the corresponding complexes, we get the result. □

The definition of derived functors can be generalized so that the derived functors are evaluated on complexes of R-modules, instead of complexes of the form $M[n]$. To be more specific, if C is a complex of R-modules, we denote by $LF(C)$ the simplicial S-module

$$LF(C) := F(\mathcal{K}(C')) \; ,$$

where C' is a complex of projective R-modules which is quasi-isomorphic to C. Such a C' always exists, and is unique up to a chain homotopy equivalence [Wei, section 5.7]. Recall that the homotopy groups of a simplicial R-module

M are defined by $\pi_i M := H_i(\mathcal{N}M)$. Then the values of the left derived functor of F on C are just the homotopy groups

$$L_iF(C) := \pi_i LF(C) \,.$$

In particular, the left derived functors with height n of definition 3.17 are given by $L_iF(M,n) := \pi_i LF(M[n])$.

Remark 3.19. Quillen has put the definition of Dold and Puppe into perspective, in the framework of his homotopical algebra [Q]. There is a model structure on $s(R\text{-mod})$ where the weak equivalences are the morphisms $f : M \to N$ inducing isomorphisms on the level of homotopy groups, and cofibrant objects are simplicial R-modules M with M_n R-projective for all $n \geq 0$. We denote by $\mathbf{D}_{\geq 0}(R\text{-mod})$ the localization of $s(R\text{-Mod})$ at weak equivalences. By the Dold-Kan correspondence, it is isomorphic to the localization of the category $\mathrm{Ch}_{\geq 0}(R\text{-mod})$ at quasi-isomorphisms. So we can indifferently consider the objects of $\mathbf{D}_{\geq 0}(R\text{-mod})$ as chain complexes or as simplicial R-modules. If $F : R\text{-Mod} \to S\text{-Mod}$ is a functor, the induced functor $F : s(R\text{-Mod}) \to s(S\text{-Mod})$ admits a Quillen derived functor

$$LF : \mathbf{D}_{\geq 0}(R\text{-mod}) \to \mathbf{D}_{\geq 0}(S\text{-mod})$$

whose value on a complex C is precisely the simplicial object $LF(C)$ defined above.

3.2.3. *Examples of derived functors.* For the sake of concreteness, we finish the section by giving two topological situations where derived functors of non-additive functors appear.

First, derived functors of non-additive functors are related to the singular homology of Eilenberg-Mac Lane spaces. To be more specific, let $\mathbb{Z}A$ denote the group ring of an abelian group A. We can consider the group ring as a functor

$$\mathbb{Z}- : \mathrm{Ab} \to \mathrm{Ab} \,.$$

Then it follows from the discussion of section 3.1.5 and remark 3.16 that the derived functors of the group ring functor with height n compute the singular homology of the Eilenberg-Mac Lane space $K(A, n)$; there is an isomorphism of functors of the abelian group A:

$$L_i\mathbb{Z}(A; n) \simeq H_i(K(A, n); \mathbb{Z}) \,.$$

The case $n = 1$ might be particularly interesting, since the singular homology of $K(A, 1)$ is isomorphic to the homology of an abelian group A [Br, Chap II, Section 4]. In particular, there is an isomorphism of functors of the abelian group A:

$$L_i\mathbb{Z}(A; 1) \simeq H_i(A; \mathbb{Z}) = \mathrm{Tor}_i^{\mathbb{Z}A}(\mathbb{Z}^{\mathrm{triv}}, \mathbb{Z}^{\mathrm{triv}}) \,.$$

Let us denote by $S^n(A)$ the n-th symmetric power of an abelian group A. We can consider the n-th symmetric power as a functor

$$S^n : \mathrm{Ab} \to \mathrm{Ab} \,.$$

The symmetric power has an analogue for topological spaces; the n-th symmetric product $SP^n(X)$ of a topological space X is the space of orbits of $X^{\times n}$ under the action of the symmetric group $\mathfrak{S}_n$ acting on $X^{\times n}$ by permuting the factors. Dold proved [Do] that if X is a CW-complex, the singular homology of $SP^n(X)$ can be computed in terms of the homology of X and the derived functors of S^n (evaluated on the graded object $H_*(X;\mathbb{Z})$, considered as a complex with trivial differential) :

$$\pi_i LS^n(H_*(X;\mathbb{Z})) \simeq H_i(SP^n(X);\mathbb{Z}) \ .$$

4. Spectral sequences

Spectral sequences are a powerful tool to compute derived functors. They can be thought of as an optimal way to organize long exact sequences in computations.

4.1. **A quick overview.** Let $\mathcal{A}$ be an abelian category, e.g. the category of R-modules. Spectral sequences appear in the following typical situation. We want to compute a graded object in $\mathcal{A}$, which we denote by K_* (e.g. the singular homology of a topological space, some derived functors, etc.). We don't know K_* but we can break it into smaller pieces which we understand. Roughly speaking, a spectral sequence is a kind of algorithm, which takes the pieces of K_* as an input, and which computes K_*. Let us state some formal definitions.

Definition 4.1. Let r_0 be a positive integer.

A homological spectral sequence in $\mathcal{A}$ is a sequence of chain complexes in $\mathcal{A}$, $(E^r_*, d^r)_{r\ge r_0}$, such that for all $r \ge r_0$, $E^{r+1}_* = H(E^r_*)$.

A cohomological spectral sequence in $\mathcal{A}$ is a sequence of cochain complexes in $\mathcal{A}$, $(E_r^*, d_r)_{r\ge r_0}$ such that for all $r \ge r_0$, $E_{r+1}^* = H(E_r^*)$.

Cohomology spectral sequences can be converted into homology spectral sequences (and vice-versa) by the usual trick on complexes $E^i_r = E^r_{-i}$. So we concentrate on homological spectral sequences and leave to the reader the translation for cohomological spectral sequences. The term E^r_* of a homological spectral sequence is called the r-th page of the spectral sequence, and d^r is called the differential of the r-th page. The term $E^{r_0}_*$ is called the initial term of the spectral sequence.

In the sequel, we shall often omit to mention the category $\mathcal{A}$ in which the spectral sequence lives. Thus spectral sequences in $\mathcal{A}$ will simply called spectral sequences, etc. This will cause no confusion since the category $\mathcal{A}$ is transparent in all the definitions, we only need the fact that it is abelian.

Definition 4.2. A spectral sequence $(E^r_*, d^r)_{r\ge 0}$ is stationary if for all $k \in \mathbb{Z}$, there is an integer $r(k)$ such that for $r \ge r(k)$ we have $E^r_k = E^{r(k)}_k$. In that case we define its E^∞-page by the formula $E^\infty_k := E^{r(k)}_k$ for all $k \in \mathbb{Z}$.

The following definition of convergence of a spectral sequence is not the most general one[7], but it is sufficient for our applications.

Definition 4.3 (Convergence)**.** Let K_* be a graded object. We say that a spectral sequence $(E^r_*, d^r)_{r\ge 0}$ converges to K_* if (i) the spectral sequence is stationary, and (ii) there exists a filtration of each K_k: $\cdots \subset F_{p-1}K_k \subset F_pK_k \subset \cdots \subset K_k$, which is exhaustive: $\bigcup_p F_pK_k = K_k$, Hausdorff: $\bigcap_p F_pK_k = 0$, and there is a graded isomorphism $\mathrm{Gr}(K_*) \simeq E^\infty_*$.

[7]It is possible to define convergence (in particular the E^∞-term) for non-stationnary spectral sequences. We refer the reader to [Boa] for the most general statements.

Typical theorems for spectral sequences assert the existence of a spectral sequence with an explicit initial term, say E^2_*, converging to what we want to compute: K_*. When such a spectral sequence exists, if we know the initial page, we can run the algorithm provided by the spectral sequence (i.e. compute the successive homologies) to obtain the E^∞-page, hence $\mathrm{Gr}(K_*)$, hence K_*.

There are many other possible uses of spectral sequences. For example, we can use spectral sequences backwards: we know the abutment K_* and we want to compute the initial object E^2_*, so we use the spectral sequence to obtain information on E^2_*. Spectral sequences can also be used to propagate properties. For example, let us take the case of a spectral sequence of abelian groups (E^r_*, d^r). Assume that the graded abelian group E^2_* is finitely generated in each degree. Then all its subquotients are finitely generated in each degree, so E^∞_* is finitely generated in each degree. Thus we deduce that the abutment K_* is also finitely generated in each degree.

4.2. **Bigraded spectral sequences.** In practice, most spectral sequences are bigraded. In this section, we rewrite the definitions of the previous section in this more complicated setting.

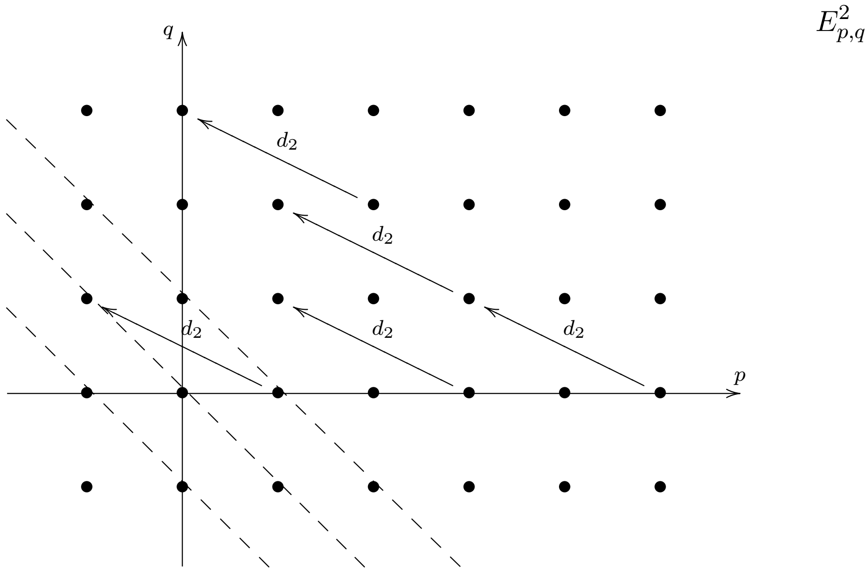
Definition 4.4. A homological spectral sequence is a sequence $(E^r_{*,*}, d^r)_{r\geq r_0}$ of bigraded objects $E^r_{*,*}$, equipped with differentials d^r of bidegree $(-r, r-1)$[8], i.e. d_r restricts to maps $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$. The terms of the spectral sequence are required to satisfy $E^{r+1}_{*,*} = H(E^r_{*,*})$.

Let $E^r_k = \bigoplus_{p+q=k} E^r_{p,q}$ be the summand of total degree k of the r-th page. Then the differential of the spectral sequence lowers the total degree by 1. So bigraded spectral sequences are a refinement of definition 4.1.

Bigraded spectral sequences are often depicted by a diagram of the following type (here we represent the second page, the summands $E^2_{p,q}$ of the second page are represented by dots, and the dots on the same dashed lines

[8]Other conventions on the bidegrees of the differentials are possible, but this convention is the most common one.

have the same total degree):



Definition 4.5. A spectral sequence is stationary if for all $p, q \in \mathbb{Z}^2$ there is an integer $r(p, q)$ such that $E^r_{p,q} = E^{r(p,q)}_{p,q}$ for $r \geq r(p, q)$. For a stationary spectral sequence, we define $E^\infty_{p,q} := E^r_{p,q}$ for $r \gg 0$.

Bigraded spectral sequences converge to *graded* objects. Roughly speaking, a convergent bigraded spectral sequence converges to K_* if for all k, K_k is filtered, and there is an isomorphism $E^\infty_k = \bigoplus_{p+q=k} E^\infty_{p,q} \simeq \mathrm{Gr}(K_k)$. Thus, only the total degree in the spectral sequence is really meaningful to recover the abutment. The bidegree should be considered as a technical refinement of the total degree, useful for intermediate computations[9]. Having this explained, we can now state the full definition of the convergence of a spectral sequence.

Definition 4.6. Let K_* be a graded object. A spectral sequence $(E^r_{*,*}, d^r)$ converges to K_* if (i) it is stationary, and (ii) for all k there is an exhaustive Hausdorff filtration:

$$0 = \left(\bigcap_p F_pK_k\right) \subset \cdots \subset F_{p-1}K_k \subset F_pK_k \subset \cdots \subset \left(\bigcup_p F_pK_k\right) = K_k\ ,$$

together with isomorphisms: $E^\infty_{p,q} \simeq F_pK_{p+q}/F_{p-1}K_{p+q}$.

The convergence of a spectral sequence with initial page r_0 is often written in the symbolic way:

$$E^{r_0}_{p,q} \Longrightarrow K_{p+q}\ .$$

4.3. **A practical example.** The archetypal theorem for spectral sequences is the following theorem[10].

[9]In definition 4.6, the bigrading on $E^\infty_{*,*}$ gives additional information on the isomorphism $E^\infty_* \simeq \mathrm{Gr}(K_*)$. But in many cases, this additional information is useless, so the reader can keep definition 4.3 in mind.

[10]In this theorem, the letter H refers to the singular homology of a topological space. The reader unfamiliar with algebraic topology can think of a fibration $X \to B$ with fiber F as a 'topological space X which is an extension of B by F'. The hypothesis that B is

Theorem 4.7 (Leray-Serre). *Let $f : X \to B$ be a fibration between topological spaces. Assume that the base space B is simply connected, and denote by $F = f^{-1}(\{b\})$ the fiber over an element $b \in B$. There is a homological spectral sequence of abelian groups*

$$E^2_{p,q} = H_p(B, H_q(F)) \Longrightarrow H_{p+q}(X) .$$

In the statement, the spectral sequence is bigraded, and starts at the second page. The bidegree of the differential is not indicated (so the reader should think that the r-th differential d^r has bidegree $(-r, r-1)$ as it is most often the case). The summands $E^2_{p,q}$ are non zero only if $p \geq 0$ and $q \geq 0$: such a spectral sequence is called a *first quadrant* spectral sequence.

Thus, assume for example that we know $H_*(B)$ and $H_*(F)$, hence the bigraded abelian group $H_*(B, H_*(F))$. Then we can use the Leray-Serre spectral sequence as an algorithm to compute $H_*(X)$. Unfortunately, we immediately encounter three practical problems.

4.3.1. *Problem 1: does the spectral sequence stop ?* To compute the E^∞-page of the Leray-Serre spectral sequence, we must start from the initial page, compute its homology $E^3_{p,q}$, then compute the homology $E^4_{p,q}$ of the third page, and so on. Since the spectral sequence is convergent, we know that for all indexes (p, q) the process stops at some point: $E^r_{p,q} = E^\infty_{p,q}$ for r big enough. But it may happen that the process does not stop for all the indices at the same time. This is a problem, because in that case we need infinitely many successive computations to compute completely $E^\infty_{*,*}$ (doing these computations would exceed a mathematician's lifetime).

Definition 4.8. A spectral sequence $(E^r_{*,*}, d^r)$ stops at the s-th page if for all $r \geq s$, the differential d^r is zero. (Thus $E^\infty_{*,*} = E^s_{*,*}$.)

In some cases, stopping follows from the shape of the initial page of the spectral sequence. For example, assume that F and B are topological spaces with bounded singular homology. Then $E^2_{p,q}$ might be non-zero only in some rectangle $0 \leq p \leq p_0$, $0 \leq q \leq q_0$. Thus, for all r big enough ($r \geq \min(p_0 + 1, q_0 + 2)$), either the target or the source of d^r is zero: the differentials d^r are 'too long'. So the spectral sequence stops at the $\min(p_0 + 1, q_0 + 2)$-th page. Here is another situation where stopping follows from the shape of the spectral sequence.

Exercise 4.9. Let $(E^r_{p,q}, d^r)_{r\geq 2}$ be a homological spectral sequence. Assume that $E^r_{p,q} = 0$ if p is odd or if q is odd. Show that $E^2_{p,q} = E^\infty_{p,q}$.

Stopping can also occur under more subtle hypotheses.

Exercise 4.10. Let $(E^r_{p,q}, d^r)_{r\geq 2}$ be a spectral sequence. Assume that there is a graded ring R such that each $(E^r_{p,q}, d_r)$ is a complex of graded R-modules, and $E^2_{p,q}$ is a noetherian R-module. Prove that the spectral sequence stops. (Hint: consider the sequences of R-submodules of the second page:

$$B^2 \subset B^3 \subset \cdots \subset B^i \subset \dots \quad \subset \quad \cdots \subset Z^i \subset \cdots \subset Z^3 \subset Z^2 ,$$

---

simply connected is not necessary, but in the general case, the statement is slightly more complicated.

where $Z^2 = \ker d^2$, $B^2 = \operatorname{Im} d^2$ and the Z^i, B^i for $i > 2$ are defined inductively by $Z^{i+1} = \pi^{-1}(\ker d^{i+1})$, $B^{i+1} = \pi^{-1}(\operatorname{Im} d^{i+1})$ where π is the map $Z^i \twoheadrightarrow Z^i/B^i = E^{i+1}$.)

4.3.2. *Problem 2: differentials are not explicit.* Theorem 4.7 asserts that the spectral sequence exist, but does not give any formula for the differential! So, even if we know what the second page is, it is not possible to compute the third page in general. **It is the main and recurrent problem for spectral sequences**.

To bypass this problem, one can use additional algebraic structure on the spectral sequence (see section 4.4). Alternatively, some differentials, or some algebraic maps related to the spectral sequence might have a geometric interpretation, which can be used to compute them. For example, the full version of the Leray-Serre spectral sequence comprises the following statement in addition to theorem 4.7.

Theorem 4.11 (Leray-Serre - continued)**.** *If $e_B : E^\infty_{p,0} \hookrightarrow E^2_{p,0}$ denotes the canonical inclusion, the composite*

$$H_p(X) \twoheadrightarrow F_pH_p(X)/F_{p-1}H_p(X) \simeq E^\infty_{p,0} \xrightarrow{e_B} E^2_{p,0} \simeq H_p(B, H_0(F)) \simeq H_p(B)$$

is equal to the map $H_p(f) : H_p(X) \to H_p(B)$. Similarly, if $e_F : E^2_{0,q} \twoheadrightarrow E^\infty_{0,q}$ denotes the canonical surjection, the composite

$$H_q(F) \simeq H_0(B, H_q(F)) \simeq E^2_{0,q} \xrightarrow{e_F} E^\infty_{0,q} \simeq F_0H_n(X) \hookrightarrow H_n(X)$$

equals the map $H_n(F) \to H_n(E)$ induced by the inclusion $F \subset X$.

The composites appearing in the second part of theorem 4.11 are called the *edge maps.* Edge maps are a common feature of first quadrant spectral sequences, and the 'geometric' interpretation of edge maps can sometimes be used to compute some differentials. For example, let us assume that f has a section, i.e. there exists a map $s : B \to X$ such that $f \circ s = \mathrm{Id}_B$. Then $H_*(f)$ (hence e_B) is onto. Thus, e_B is an isomorphism. Therefore, all the elements of $E^2_{*,0}$ survive to the E^∞ page. That is, all the differentials starting from $E^2_{*,0}$ must be zero.

4.3.3. *Problem 3: extension problems.* Assume now that we have (finally) succeeded in computing $E^\infty_{*,*}$. We have not finished yet! Indeed the E^∞-page of the spectral sequence is isomorphic to $\mathrm{Gr}(H_*(X))$, not to $H_*(X)$. This can make a big difference. For example, if E^∞_k equals $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then $H_k(X)$ could equal $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.

In the case of spectral sequences of $\Bbbk$-vector spaces (or more generally if the E^∞-page is a bigraded projective $\Bbbk$-module) this problem vanishes thanks to the following lemma.

Lemma 4.12. *Let M be a $\Bbbk$-module, equipped with a filtration*

$$0 = F_mM \subset \cdots \subset F_{p-1}M \subset F_pM \subset \cdots \subset \left(\bigcup_{p\ge m} F_pM\right) = M\ .$$

Assume that $\mathrm{Gr}(M)$ is a projective $\Bbbk$-module. Then there is an isomorphism of $\Bbbk$-modules $M \simeq \mathrm{Gr}(M)$.

Proof. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $\Bbbk$-modules. If M'' is projective, we can find a section of the map $M \to M''$ so M splits as a direct sum: $M \simeq M' \oplus M''$. We build the isomorphism $M \simeq \mathrm{Gr}(M)$ by iterative uses of this fact. □

To solve extension problems in the general case, we often use additional structure on the spectral sequence, or additional information on the abutment $H_*(X)$, obtained independently from the spectral sequence.

4.4. **Additional structure on spectral sequences.** Many spectral sequences bear more structure than what is stated in definition 4.4. This additional structure is usually of great help in effective computations. In this section we present two properties among the most frequent and useful ones, namely algebra structures (the pages of the spectral sequences are bigraded algebras), and naturality (the spectral sequence depends functorially on the object it is built from).

4.4.1. *Spectral sequences of algebras.*

Definition 4.13. Let $(E^r_{*,*}, d^r)$ be a spectral sequence of R-modules, converging to K_*. We say that it is a spectral sequences of algebras if the following conditions are satisfied.

(1) For all r there is a bigraded product $E^r_{p_1,q_1} \otimes_R E^r_{p_2,q_2} \to E^r_{p_1+p_2,q_2+q_2}$, satisfying a Leibniz relation
$$d^r(x_1 \cdot x_2) = d^r(x_1) \cdot x_2 + (-1)^{p_1+q_1} x_1 \cdot d^r(x_2) \quad \text{for } x_i \in E^r_{p_i,q_i}\,.$$
In particular $H(E^r_{*,*})$ is a bigraded algebra.
(2) The isomorphism $H(E^r_{*,*}) \simeq E^{r+1}_{*,*}$ is compatible with products.
(3) The abutment K_* is a filtered graded algebra (i.e. the filtration satisfies $F_pK_i \cdots F_qK_j \subset F_{p+q}K_{i+j}$) and the isomorphism $E^\infty_* \simeq \mathrm{Gr}(K_*)$ is an isomorphism of graded algebras.

Spectral sequences of algebras are interesting for many purposes. First, one might very well be interested in computing the graded R-algebra K_* rather than the graded R-module K_*. In that case, we need a spectral sequence of algebras to do the job. But even if we are not interested in the algebra structure of the abutment, the algebra structure on a spectral sequence $(E^r_{*,*}, d^r)$ is an extremely useful tool to compute the differentials. Indeed, by the Leibniz rule, it suffices to determine $d^r(x)$ for generators x of $E^r_{*,*}$ to completely determine d^r on $E^r_{*,*}$! Examples of spectral sequences of algebras are given in section 4.5.

4.4.2. *Naturality of spectral sequences.* Let $\mathcal{A}$ be an abelian category. To define the category of homological spectral sequences in $\mathcal{A}$, we need to define morphisms of spectral sequences.

Definition 4.14. Let $(E^r_{*,*}, d^r)$ and $({E'}^r_{*,*}, {d'}^r)$ be two spectral sequences converging to K_* and K'_* respectively. A morphism of spectral sequences is a sequence of bigraded maps $f^r : E^r_{*,*} \to {E'}^r_{*,*}$, which commute with the differentials: ${d'}^r(f(x)) = f(d^r(x))$, and such that $H(f^r) = f^{r+1}$. (In particular, the morphism induces a map $f^\infty : E^\infty_{*,*} \to {E'}^\infty_{*,*}$.)

From a practical point of view, morphisms of spectral sequences $f^r : E^r_{*,*} \to E'^r_{*,*}$ are very useful for explicit computations. Indeed, if the differentials in the spectral sequence $E^r_{*,*}$ are known, then one can use f^r to prove that $d'^r(x) = 0$, for x in the image of f^r, or to prove that some some x in the image of f^r are boundaries.

In practice, spectral sequences usually come from a functor

$$E : \mathcal{C} \to \text{Spectral sequences} ,$$

where $\mathcal{C}$ is a given category. One also says that the spectral sequence is natural with respect to the objects $C \in \mathcal{C}$. In this situation, morphisms in $\mathcal{C}$ induce maps of spectral sequences. The Leray-Serre spectral sequence is a typical example ($\mathcal{C}$ is the category of fibrations).

Theorem 4.15 (Leray-Serre - continued)**.** *The Leray-Serre spectral sequence is natural with respect to the fibration f, and the convergence is natural with respect to f.*

The precise meaning of theorem 4.15 is the following. If (g_B, g_X, g_F) is a morphism of fibrations, i.e. there is a commutative diagram:

$$\begin{array}{ccc} F & \xrightarrow{g_F} & F' \\ \downarrow & & \downarrow \\ X & \xrightarrow{g_X} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{g_B} & B' \end{array} ,$$

then (g_B, g_X, g_F) induces a morphism $g^r : E^r_{*,*} \to E'^r_{*,*}$ of spectral sequences between the two Leray-Serre spectral sequences. On the level of the second page, g^2 coincides with the morphism $H_*(g_B, H_*(g_F))$. The map $H_*(g_X)$ preserves the filtrations on the abutments and $\mathrm{Gr}(H_*(g_X))$ equals the map $g^\infty : E^\infty_{*,*} \to E'^\infty_{*,*}$.

4.5. **Examples of spectral sequences.** In this section we give some of the most common examples of spectral sequences. We shall not explain in details how these spectral sequences are constructed, since their construction is quite involved and usually useless for practical computations. We refer the reader to [Wei, Chap. 5], [ML, XI], [Ben, Chap. 3] or [MC] for more information on their construction.

4.5.1. *Filtered complexes.* Let C be a chain complex in an abelian category $\mathcal{A}$. A filtration of C is a family of subcomplexes $\cdots \subset F_pC \subset F_{p+1}C \subset \dots$ of C. The filtration is *bounded below* if for all k, there is an integer p_k such that $F_{p_k}C_k = 0$. It is *exhaustive* if $\bigcup_{p\in\mathbb{Z}} F_pC = C$.

If C is a filtered complex, we denote by Gr_pC the quotient complex: $\mathrm{Gr}_pC = F_pC/F_{p-1}C$. The filtration of C induces a filtration on its homology, defined by

$$F_pH_*(C) := \mathrm{Im}\big(H_*(F_pC) \to H_*(C)\big) .$$

The following theorem provides a spectral sequence which recovers the homology of C from the homology of the complexes Gr_pC.

Theorem 4.16. *Let C be a filtered chain complex, whose filtration is bounded below and exhaustive. There is a homological spectral sequence*

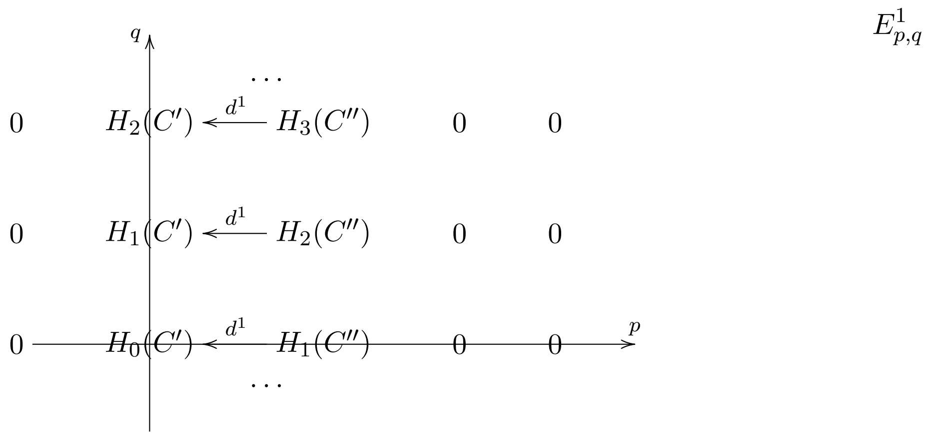
$$E^1_{p,q}(C) = H_{p+q}(\mathrm{Gr}_p C) \Longrightarrow H_{p+q}(C)\ .$$

More explicitly, there are isomorphisms

$$E^\infty_{p,q}(C) \simeq F_p H_{p+q}(C)/F_{p-1}H_{p+q}(C)\ .$$

If $f : C \to C'$ is a chain map preserving the filtrations (i.e. it sends F_pC into F_pC'), then f induces a morphism of spectral sequences with $E^1_{p,q}(f) = H_{p+q}(\mathrm{Gr}_p f)$, and $E^\infty_{p,q}(f)$ coincides on the abutment with $\mathrm{Gr}_p H_{p+q}(f)$.

Example 4.17. Let us test theorem 4.16 on the simplest example. Let $0 \to C' \to C \to C'' \to 0$ be a short exact sequence of chain complexes. This short exact sequence is equivalent to saying that C is filtered with $F_0C = C'$ and $F_1C = C$. Thus we have a spectral sequence whose first page is of the form:



The shape of the E^1-page implies that the differentials d_r are zero for $r > 1$ (these differentials are 'too long': their source or their target must be zero), so $E^2_{p,q} = E^\infty_{p,q}$. As a result we have exact sequences:

$$0 \to E^\infty_{1,q} \to H_{q+1}(C'') \xrightarrow{d_1} H_q(C') \to E^\infty_{0,q} \to 0$$

Now the convergence of the spectral sequence is equivalent to the data of short exact sequences:

$$0 \to E^\infty_{0,q+1} \to H_{q+1}(C) \to E^\infty_{1,q} \to 0\ .$$

Splicing all these exact sequences together, we recover the classical homology long exact sequence associated with the short exact $0 \to C' \to C \to C'' \to 0$.

Theorem 4.16 has a cohomological analogue, which we now state explicitly for the convenience of the reader. A cochain complex C is filtered when it is equipped with a family of subcomplexes $\dots \subset F^pC \subset F^{p-1}C \subset \dots$ (beware of the indices, which are decreasing for filtrations of cohomological complexes). The filtration is bounded below if for each k there is an integer p_k such that $F^{p_k}C^k = 0$, and it is *exhaustive* if $\bigcup_{p\in\mathbb{Z}} F^pC = C$. The cohomology of a filtered cochain complex C is filtered by

$$F^pH^*(C) := \mathrm{Im}\big(H^*(F^pC) \to H^*(C)\big)\ .$$

Theorem 4.18. *Let C be a filtered cochain complex, whose filtration is bounded below and exhaustive. There is a cohomological spectral sequence*

$$E_1^{p,q}(C) = H^{p+q}(F^pC/F^{p+1}C) \Longrightarrow H^{p+q}(C)\ .$$

To be more explicit, the differentials are maps $d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}$ and there are isomorphisms:

$$E_\infty^{p,q}(C) \simeq F^pH^{p+q}(C)/F^{p+1}H^{p+q}(C)\ .$$

If $f : C \to C'$ preserves the filtrations, then f induces a morphism of spectral sequences with $E_1^{p,q}(f) = H^{p+q}(\mathrm{Gr}^p f)$, and $E_\infty^{p,q}(f)$ coincides on the abutment with $\mathrm{Gr}^pH^{p+q}(f)$.

The following instructive exercise is suggested by W. van der Kallen. It provides a conceptual interpretation of the terms of the r-th page of the spectral sequence of a filtered complex.

Exercise 4.19. Let $f : C \to D$ be a map of cochain complexes. If $f^i : C^i \to D^i$ is surjective and $f^{i+1} : C^{i+1} \to D^{i+1}$ is injective, show that $H^i(f)$ is surjective and $H^{i+1}(f)$ is injective.

If C is a filtered complex, for the sake of brevity, we let $C^\bullet_{\geq i} = F^iC^\bullet$, and $C^\bullet_{i/j} = F^iC^\bullet/F^jC^\bullet$ if $j > i$. Let a, b be integers with $a < b$. We have a map ϕ of spectral sequences from the spectral sequence of the filtered complex $C^\bullet$ to the spectral sequence of the filtered complex $C^\bullet/C^\bullet_{\geq b}$, filtered by the images of the $C^\bullet_{\geq i}$. Show that ϕ_r^{pq} is surjective for $p+r \leq b$ and injective for $p < b$.

Similarly we have a map ψ of spectral sequences from the spectral sequence of the suitably filtered complex $C^\bullet_{\geq a}/C^\bullet_{\geq b}$ to the spectral sequence of the filtered complex $C^\bullet/C^\bullet_{\geq b}$. Show that ψ_r^{pq} is surjective for $p \geq a$ and injective for $p - r + 1 \geq a$.

Show that the spectral sequence for $C^\bullet_{\geq a}/C^\bullet_{\geq b}$ has vanishing E_r^{pq} for $p < a$ and for $p \geq b$.

Now prove the formula $E_r^{pq} = (H^{p+q}(C^\bullet_{p-r+1/p+r}))_{p/p+1}$ by taking $a = p - r + 1$, $b = p + r$.

4.5.2. *Bicomplexes.* A *first quadrant chain bicomplex* is a bigraded object $C = \bigoplus_{p\geq 0, q\geq 0} C_{p,q}$, together with differentials $d : C_{p,q} \to C_{p-1,q}$ and $\partial : C_{p,q} \to C_{p,q-1}$ which commute: $d \circ \partial = \partial \circ d$. The *total complex* associated such a bicomplex is the complex $\mathrm{Tot}\, C$ with $(\mathrm{Tot}\, C)_n = \bigoplus_{p+q=n} C_{p,q}$ and whose differential maps an element of $x \in C_{p,q}$ to $dx + (-1)^p\partial x$.

Example 4.20. Let C and D be nonnegative chain complexes of R-modules. Their tensor product $C \otimes_R D$ is equal to $\mathrm{Tot}\,(C \boxtimes_R D)$, where $C \boxtimes_R D$ is the first quadrant chain bicomplex defined by $(C\boxtimes_R D)_{p,q} = C_p \otimes D_q$, $d = d_C \otimes \mathrm{Id}$ and $\partial = \mathrm{Id} \otimes d_D$.

If C is a first quadrant bicomplex, we can obtain a spectral sequence converging to the homology of $\mathrm{Tot}\, C$ as a particular case of theorem 4.16. Let us sketch the construction. We first define a filtration of $\mathrm{Tot}\, C$ by

$$F_n \mathrm{Tot}\, C := \bigoplus_{0\leq p,\, 0\leq q\leq n} C_{p,q}\ .$$

Thus F_0C equals the 0-th column of the bicomplex C, with differential ∂, and more generally F_n is the 'totalization' of the sub-bicomplex of C formed by the objects of columns 0 to n. So there is an isomorphism of complexes:

$$F_n\mathrm{Tot}\,(C)/F_{n-1}\mathrm{Tot}\,(C) \simeq (C_{n,*-n},\partial)\;.$$

By theorem 4.16, the filtered complex $\mathrm{Tot}\,C$ gives birth to a spectral sequence starting with page $E^1_{p,q}(C) = H_q(C_{p,*},\partial)$, converging to $\mathrm{Tot}\,(C)_{p+q}$. Looking at the construction of the spectral sequence of theorem 4.16, one finds that the first differential $d^1 : E^1_{p,q}(C) \to E^1_{p-1,q}(C)$ is equal to the map $H_q(C_{p,*},\partial) \to H_q(C_{p-1,*},\partial)$ induced by the differential $d : C_{p,q} \to C_{p-1,q}$. Thus the second page is given by:

$$E^2_{p,q}(C) = H_p(H_q(C,\partial),d)\;.$$

This is summarized in the following statement.

Theorem 4.21. *Let (C,d,∂) be a first quadrant chain bicomplex. There is a homological spectral sequence*

$$E^2_{p,q}(C) = H_p(H_q(C,\partial),d) \Longrightarrow H_{p+q}(\mathrm{Tot}\,C)\;.$$

To be more specific, we have isomorphisms:

$$E^\infty_{p,q}(C) \simeq F_pH_{p+q}(\mathrm{Tot}\,C)/F_{p-1}H_{p+q}(\mathrm{Tot}\,C)\;.$$

Moreover, the spectral sequence is natural with respect to morphisms of bicomplexes.

First quadrant chain bicomplexes are 'symmetric', that is, the transposed bicomplex C^t defined by $C^t_{p,q} = C_{q,p}$ gives rise to another spectral sequence with second page $E^2_{p,q}(C^t) = H_q(H_p(C,d),\partial)$ and which converges to $H_{p+q}(C^t) \simeq H_{p+q}(C)$. So there are actually two spectral sequences associated to a bicomplex.

Example 4.22. As an application of theorem 4.21 we prove that $\mathrm{Tor}^R_*(M,N)$ is 'well-balanced', i.e. it can be computed using indifferently a projective resolution of M or a projective resolution of N.

Let P (resp. Q) be a projective resolution of M (resp. N). We denote by ${}^\ell\mathrm{Tor}^R_*(M,N)$ the homology of the complex $P\otimes_R N$, and by ${}^r\mathrm{Tor}^R_*(M,N)$ the homology of $M\otimes_R Q$. We form the bicomplex

$$(C,d,\partial) = (P\boxtimes_R Q, d_P\otimes \mathrm{Id}, \mathrm{Id}\otimes d_Q)\;.$$

Theorem 4.21 yields a spectral sequence $(E^r_{p,q}(C),d_r)_{r\ge 2}$, converging to the homology of $\mathrm{Tot}\,(C)$. Since the objects of P are projectives $P_p\otimes_R$ is an exact functor so $H_q(P_p\boxtimes_R Q,\partial) = P_p\otimes_R H_q(Q) = P_p\otimes N$ if $q=0$ and zero otherwise. Thus:

$$E^2_{p,q}(C) = {}^\ell\mathrm{Tor}_p(M,N) \text{ if } q=0 \text{ and zero if } q\neq 0.$$

The shape of the second page implies that the differentials d^r, for $r\ge 2$ must be zero (their source or their target is zero), so $E^2_{p,q}(C) = E^\infty_{p,q}(C)$ and since the spectral sequence abuts to $H_*(\mathrm{Tot}\,C)$, we obtain an isomorphism

$${}^\ell\mathrm{Tor}^R_*(M,N) \simeq H_*(\mathrm{Tot}\,C)\;.$$

The same argument applied to the transposed bicomplex C^t yields an isomorphism:

$$ {}^r\mathrm{Tor}_*^R(M,N) \simeq H_*(\mathrm{Tot}\, C^t) \simeq H_*(\mathrm{Tot}\, C) \;. $$

This proves that $\mathrm{Tor}_*^R(M,N)$ is well-balanced. (If $\mathcal{C}$ is a small category, the same reasoning also shows that $\mathrm{Tor}_*^{\mathcal{C}}(G,F)$ is well balanced).

Exercise 4.23. (1) Let R be an algebra over a commutative ring $\Bbbk$. Let M be a right R-module and let P be a complex of projective left R-modules. Show that there is a first quadrant homological spectral sequence of $\Bbbk$-modules

$$ E^2_{p,q} = \mathrm{Tor}_p(M, H_q(P)) \Longrightarrow H_{p+q}(M \otimes_R P) \;. $$

(hint: consider the bicomplex $Q^M \otimes_R P$, where Q^M is a projective resolution of M)

(2) Assume that M has a projective resolution of length 2 (i.e. of the form $0 \to Q_1 \to Q_0$). Show that if P is a complex of projective R-modules, there are short exact sequences of $\Bbbk$-modules:

$$ 0 \to M \otimes_R H_q(P) \to H_q(M \otimes_R P) \to \mathrm{Tor}_1^R(M, H_{q-1}(P)) \to 0 \;. $$

Theorem 4.21 has the following cohomological analogue for first quadrant cochain bicomplexes (i.e. bigraded objects $C = \bigoplus_{p\ge 0, q\ge 0} C^{p,q}$ equipped with differentials $d : C^{p,q} \to C^{p+1,q}$ and $\partial : C^{p,q} \to C^{p,q+1}$ which commute.).

Theorem 4.24. *Let (C,d,∂) be a first quadrant cochain bicomplex. There is a cohomological spectral sequence*

$$ E_2^{p,q}(C) = H^p(H^q(C,\partial),d) \Longrightarrow H^{p+q}(\mathrm{Tot}\, C) \;. $$

To be more specific, the r-th differential is a map $E_r^{p,q} \to E_r^{p+r,q+1-r}$ and we have isomorphisms:

$$ E_\infty^{p,q}(C) \simeq F^p H^{p+q}(\mathrm{Tot}\, C)/F^{p+1} H^{p+q}(\mathrm{Tot}\, C) \;. $$

Moreover, the spectral sequence is natural with respect to morphisms of bicomplexes.

4.5.3. *Grothendieck spectral sequences.* Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be abelian categories, and let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be right exact functors. On can wonder wether it is possible to recover the derived functors $L_i(G \circ F)$ of the composition from the derived functors L_iF and L_jG. The Grothendieck spectral sequence answers this question under an acyclicity assumption. One says that $N \in \mathcal{B}$ is *G-acyclic* if $L_iG(N) = 0$ for $i > 0$.

Theorem 4.25. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. Assume that we have right exact functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ such that F sends projective objects to G-acyclic objects. Then for all $M \in \mathcal{A}$, there is a homological spectral sequence:*

$$ E^2_{p,q} = L_pG \circ L_qF(M) \Longrightarrow L_{p+q}(G \circ F)(M) \;. $$

The Grothendieck spectral sequence has a cohomological analogue for right derived functors.

Theorem 4.26. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Assume that we have left exact functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ such that F sends injective objects to G-acyclic objects. Then for all $M \in \mathcal{A}$, there is a cohomological spectral sequence, with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}$:*

$$E_2^{p,q} = R^pG \circ R^qF(M) \Longrightarrow R^{p+q}(G \circ F)(M) \ .$$

Example 4.27. Let us illustrate the Grothendieck Spectral sequence by a situation from the algebraic group setting. Let G be an algebraic group over a field $\Bbbk$ and let H be a closed subgroup. The restriction functor:

$$\mathrm{res}_H^G : \{\text{rat. } G\text{-mod}\} \to \{\text{rat. } H\text{-mod}\}$$

admits a right adjoint, namely the induction functor[11]

$$\mathrm{ind}_H^G : \{\text{rat. } H\text{-mod}\} \to \{\text{rat. } G\text{-mod}\} \ .$$

By exercise 2.22, ind_H^G is left exact and preserves the injectives. Moreover, $H^*(H, M)$ is the derived functor of $M \mapsto \mathrm{Hom}_{\text{rat. } H\text{-mod}}(\Bbbk, M) \simeq \mathrm{Hom}_{\text{rat. } G\text{-mod}}(\Bbbk, \mathrm{ind}_H^G M)$ so theorem 4.26 yields a spectral sequence:

$$E_2^{p,q} = H^p(G, R^q\mathrm{ind}_H^G(M)) \Longrightarrow H^{p+q}(H, M) \ .$$

In particular, if by chance the H-module M satisfies the vanishing condition $R^q\mathrm{ind}_H^G(M) = 0$ for $q > 0$, the spectral sequence stops at the second page (since it is concentrated on the 0-th row) and yields an isomorphism:

$$H^*(G, \mathrm{ind}_H^G M) \simeq H^*(H, M) \ .$$

The Kempf vanishing theorem, a cornerstone for the representation theory of algebraic groups, asserts that the vanishing condition is satisfied when G is reductive, H is a Borel subgroup of G and $M = \Bbbk_\lambda$ is the one dimensional representation of H given by a character χ_λ of H, associated to a dominant weight λ[12]. In this case, the representation $\mathrm{ind}_H^G(\Bbbk_\lambda)$ is called 'the costandard module with highest weight λ' and denoted by $\nabla_G(\lambda)$, or $H^0(\lambda)$ [J, II.2].

4.5.4. *Filtered differential graded algebras.* A differential graded algebra is a graded algebra A equipped with a differential d satisfying the Leibniz rule: $d(xy) = d(x)y + (-1)^{\deg x} x d(y)$. A filtration of a differential graded algebra A is a filtration $\cdots \subset F_pA \subset F_{p+1}A \subset \dots A$ of the complex (A, d) compatible with products, i.e. the product sends $F_pA \otimes F_qA$ into $F_{p+q}A$.

If A is a filtered differential algebra, the product of A induces bigraded algebra structures on $\bigoplus_{p,q} H_q(\mathrm{Gr}_pA)$ and $\bigoplus_{p,q} \mathrm{Gr}_pH_q(A)$. Thus, the first page and the ∞-page of the spectral sequence of theorem 4.16 are bigraded algebras. More is true: in this situation, the spectral sequence is a spectral sequence of algebras.

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[11]Let us consider the regular functions $\Bbbk[G]$ as a $G \times H$-module, where G acts by left translations and H by right translations on G. The induction functor is defined by the formula: $\mathrm{ind}_H^G(M) = (\Bbbk[G] \otimes M)^H$.

[12]For example, $G = GL_n(\Bbbk)$, $H = B_n(\Bbbk)$ is the subgroup of upper triangular matrices and $M = \Bbbk_\lambda$ is the one dimensional vector space $\Bbbk$ acted on by $B_n(\Bbbk)$ by multiplication with a scalar $\chi_\lambda([g_{i,j}]) = \prod g_{i,i}^{\lambda_i}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Theorem 4.28. *Let A be a filtered differential graded algebra, whose filtration is bounded below and exhaustive. There is a homological spectral sequence of algebras*

$$E^1_{p,q}(A) = H_{p+q}(\mathrm{Gr}_p A) \Longrightarrow H_{p+q}(A)\ .$$

More explicitly, there are bigraded algebra isomorphisms

$$E^\infty_{p,q}(A) \simeq F_pH_{p+q}(A)/F_{p-1}H_{p+q}(A)\ .$$

If $f : A \to A'$ is a morphism of differential graded algebras preserving the filtrations (i.e. it sends F_pA into F_pA'), then f induces a morphism of spectral sequences with $E^1_{p,q}(f) = H_{p+q}(\mathrm{Gr}_p f)$, and $E^\infty_{p,q}(f)$ coincides on the abutment with $\mathrm{Gr}_pH_{p+q}(f)$.

Theorem 4.28 has an obvious cohomological analogue, whose formulation is left to the reader.

4.5.5. *The Lyndon-Hochschild-Serre spectral sequence.* Let G be a discrete group, let H be a normal subgroup of G, and let M be a $\Bbbk G$-module ($\Bbbk$ is a commutative ring). The Lyndon-Hochschild-Serre spectral sequence allows to reconstruct the cohomology $H^*(G, M)$ from some cohomology groups of H and G/H. Let us be more specific. We can restrict the action on M to G to obtain a $\Bbbk H$-module, still denoted by M.

Lemma 4.29. *The quotient group G/H acts on $H^i(H, M)$ for all $i \geq 0$.*

Proof. Let $[g]$ denote the class of $g \in G$ in G/H. Then G/H acts on M^H by the formula: $[g]m := gm$. Consider the cohomological δ-functor $(F^i, \delta^i)_{i\geq 0}$, where the F^i are the functors

$$\begin{array}{ccc} \Bbbk G\text{-Mod} & \to & \Bbbk\text{-Mod} \\ M & \mapsto & H^i(H,M) \end{array} .$$

Restriction from $\Bbbk G$-modules to $\Bbbk H$-modules preserves injectives (cf. exercise 2.31), so $F^i = R^i(F^0)$. Each $[g] \in G/H$ defines a natural transformation $[g] : F^0 \to F^0$ so it extends uniquely into a morphism of δ-functors from $(F^i, \delta^i)_{i\geq 0}$ to $(F^i, \delta^i)_{i\geq 0}$. By uniqueness, since the axioms of a group action are satisfied on $F^0(M)$, they are satisfied on all $F^i(M)$, $i \geq 0$. □

We are now ready to describe the Lyndon-Hochschild-Serre spectral sequence.

Theorem 4.30. *Let H be a normal subgroup of G, and let M be a $\Bbbk G$-module. There is a first quadrant cohomological spectral sequence of $\Bbbk$-modules, with differentials $d^r : E^{p,q}_r \to E^{p+r,q+r-1}_r$:*

$$E^{p,q}_2 = H^p(G/H, H^q(H, M)) \Longrightarrow H^{p+q}(G, M)\ .$$

This spectral sequence is natural with respect to M. Moreover, if M is a $\Bbbk G$-algebra then the spectral sequence is a spectral sequence of algebras. (On the second page of the spectral sequence, the product is the cup product of the cohomology algebra $H^*(G/H, A)$ where A is the cohomology algebra $H^*(H, M)$.)

A similar Lyndon-Hochschild-Serre spectral sequence exists for the homology of discrete groups, and the rational cohomology of algebraic groups.

Exercise 4.31. Let H be a normal subgroup of G and let $M \in \Bbbk G$-mod ($\Bbbk$ is a commutative ring). Derive from the Lyndon-Hochschild-Serre spectral sequence the five terms exact sequence:

$$0 \to H^1(G/H, M^H) \to H^1(G, M) \to H^1(H, M)^{G/H} \to H^2(G/H, M^H) \to H^2(G, M)\ .$$

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