Frobenius twists in higher invariant theory

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Université Paris 13

Mini symposium in honor of Wilberd van der Kallen
22/01/2012

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Lots of examples in Nature :

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$G \subset GL_n(\mathbb{k})$ acts on $\mathbb{k}^n$ by matrix multiplication, on $M_n(\mathbb{k})$ by conjugacy, on its Lie algebra $\mathfrak{g}$ ...

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We can iterate : $V^{(r)} = (V^{(r-1)})^{(1)}$

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Notation :

In many places we shall need the following notation :
vdK = Wilberd van der Kallen

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Need for positive results on finite generation !

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- Finite groups, $GL_n(\mathbb{k})$, $SO_n(\mathbb{k})$, $Sp_n(\mathbb{k})$, $SL_n(\mathbb{k})$ are reductive $(\mathbb{k}, +)$, $B_n(\mathbb{k})$ are not reductive

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Higher invariant Theory : can we describe $H^*(G,A)$?

In particular, if A is commutative and f.g. is $H^*(G,A)$ f.g ?

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- **Conjecture** (vdK, 2000) : All red. groups have (CFG) prop !

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- **Conjecture** (vdK, 2000) : All red. groups have (CFG) prop !
  Complete proof of the conjecture in 2010.

Hilbert's XIVth problem and **higher invariant theory** (5)

Invariant theory : G has (FG) prop. if A^G always f.g.
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Cohomology of **finite groups of Lie type** (1)

Situation :

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Examples : $GL_n(\mathbb{F}_q)$, $SO_n(\mathbb{F}_q)$, $Sp_n(\mathbb{F}_q)$, $\mathrm{Spin}(\mathbb{F}_q)$, $SL_n(\mathbb{F}_q)\dots$

They play key role in the theory of finite groups.

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For $i \le n$,

$$H^i(G, M^{(r)}) \to H^i(G(\mathbb{F}_{p^{r+f}}), M^{(r)}) \,.$$

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Such vanishings are really stable phenomena !

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Remark : there exist spectral sequence

$$E_2^{p,q} = Ext^p_{\mathcal{P}_{\Bbbk}}(F, G_{E_r}) \Rightarrow Ext^{p+q}_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)})$$

Conjecture is equivalent to collapsing at E_2-page.

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Cor (T,2011) : new simple proof of the existence of the universal classes.