Algebraic topology : some reminders.

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1) Axioms for homology and cohomology theory
2) The cup product
3) Homotopy groups
4) Cell complexes and Whitehead's theorem
5) Fibration
6) Suspension theorem (Freudenthal)
7) A link between homotopy and homology : Hurewicz' theorem
8) Hopf's invariant

A homology theory with coefficients in a ring R is a functor H_* which associates a graded R-module to any topological pair (X, A) together with a natural transformation ∂_*, called the connecting homomorphism :

$$\begin{array}{ccc} H_{n+1}(X,A) & \xrightarrow[\partial_n]{} & H_n(A,\varnothing) \\ \downarrow & & \downarrow \\ H_{n+1}(Y,B) & \xrightarrow[\partial_n]{} & H_n(B,\varnothing) \end{array}$$

such that the following axioms are satisfied :

0) Homotopy : If $f, g : (X, A) \to (Y, B)$ are homotopic, then $H_*(f) = H_*(g)$.
1) Additivity : the map $\bigoplus_\alpha H_n(X_\alpha) \to H_n(\sqcup_\alpha X_\alpha)$ induced by inclusions maps is an isomorphism.
2) Exactness : $\forall (X, A)$, the following sequence, induced by the inclusions maps, is exact

$$... \longrightarrow H_{n+1}(X, A) \xrightarrow[\partial_n]{} H_n(A) \xrightarrow[i_n]{} H_n(X) \xrightarrow[j_n]{} H_n(X, A)...$$

3) Excision : given an open set U such that $\bar{U} \subset int(A)$, then the map $(X - U, A - U) \to (X, A)$ induces an isomorphism $H(X - U, A - U) \to H(X, A)$
4) Dimension : $H_*(pt, R) = H_0(pt, R) = R$.

Cup product

A singular n-simplex in a space X is a continuous map $\Delta^n \to X$. Singular homology defines the n^{th} homology group of X as a quotient of the group $C_n(X)$ of n-chains. Here, we use the dual cohomology.
One can define a cup product on cochains by its value on a singular simplex :

$$C^k(X) \times C^l(X) \to C^{k+l}(X)$$

$$(\psi, \phi) \mapsto \psi \smile \phi$$

$\psi \smile \phi : \sigma \mapsto \psi(\sigma \circ L_k)\phi(\sigma \circ R_l)$

Cup product

This product is both associative and distributive, and has a unit as long as R has one. It induces a graded ring structure on $H^*(X,R) = \bigoplus_n H^n(X,R)$.
For example, considering spheres or projective spaces :

$$H^*(S^n, R) = R[\alpha]/(\alpha)^2$$

$$H^*(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$$

$$H^*(\mathbb{CP}^n, R) = R[\beta]/(\beta^{n+1})$$

$$H^*(\mathbb{HP}^n, R) = R[\gamma]/(\gamma^{n+1})$$

Homotopy groups

Given a space with base point (X, x_0), its n^{th} homotopy group is defined by $\pi_n(X, x_0) := [(I^n, \partial I^n); (X, x_0)]$ equipped with the operation induced by path concatenation.

For any two pointed topological spaces (X, x_0) and (Z, z_0), let $\Omega X := Cont_\bullet(S^1, X)$ and
$\Sigma : X \mapsto I \times X/((0,x) \sim (0,y); (1,x) \sim (1,y); (t, x_0) \sim (s, x_0))$.
Then

$$[\Sigma X, Z] \simeq [X, \Omega Z]$$

Corollary : $\forall n > 1, \pi_n(X, x_0)$ is abelian.

Homotopy groups

A continuous map $f : X \to Y$ is a weak equivalence if it induces an isomorphism $\pi_n(X) \simeq \pi_n(Y)$ for any n.
A continuous map $f : X \to Y$ is a homotopy equivalence if there is a continuous map $g : Y \to X$ such that $f \circ g = id_X$ up to homotopy and $g \circ f = id_Y$ up to homotopy.

Cell complexes

Starting with the discrete 0-skeleton X^0, one can inductively construct every n-skeleton of a cell complexe X

$$X^0 \subseteq X^1 \subseteq \dots \subseteq X^n \subseteq \dots$$

$$\phi_\alpha : S^{n-1} \to X^{n-1}$$

$$X^n = X^{n-1} \sqcup_\alpha D^n_\alpha / (x \sim \phi_\alpha(x))$$

$$X = \cup_n X^n$$

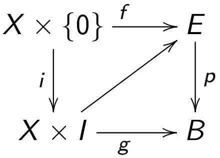
(all with the colimit topology)

Cell complexes

Given any topological space X, there is a functorial construction giving a cell complexe $\tilde{X}$ and a weak equivalence $\tilde{X} \to X$. Furthermore, any weak equivalence between two cell complexes is actually a homotopy equivalence. (Whitehead's theorem)

Fibration

A surjective map $p : E \to B$ is a (Hurewicz) fibration if it has the Homotopy Lifting Property for any topological space X : any commutative square can be completed as follows



It is a Serre fibration if this is true for all spheres.

Fibration

Theorem : If $p : E \to B$ is a Serre fibration, for $e_0 \in E$ let $b_0 = p(e_0)$, and $F = p^{-1}(b_0)$. Then, if one sees b_0 and e_0 as base points, the following sequence is exact :

$$... \longrightarrow \pi_n(F, e_0) \longrightarrow \pi_n(E, e_0) \longrightarrow \pi_n(B, b_0) \longrightarrow \pi_{n-1}(F, e_0)...$$

Fibration

One can associate to any continuous map $f : A \to B$ a topological space $E_f := A \times_f B^I = \{(x, \beta) | \beta(0) = f(x)\}$, so that one can write f as the composition of a homotopy equivalence and a fibration. The long exact sequence induced is the long exact sequence of the map f.

Suspension

A topological space X is said to be n-connected if $\forall k \leq n$, $\pi_k(X) = 0$.
Theorem (Freudenthal) :If X is an n-connected cell complex, then Σ induces an isomorphism $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ for $i < 2n+1$. Corollarily, ΣX is $(n+1)$-connected, and we obtain a stationnary sequence, which limit is denoted $\pi_i^S(X)$.

Suspension

We will denote $\pi_i^S = \pi_i^S(S^0)$.
Theorem (Serre) : $\forall i > 0$, π_i^S is finite.
$\pi_*^S = \bigoplus \pi_i^S$ equipped with composition is a graduated ring.

The Hurewicz homomorphism $h_n : \pi_n(X) \to H_n(X, R)$ is given by $[f] \mapsto H_n(f)(\Theta)$.
Theorem : If X is $(n-1)$-connected, $(n > 2)$, then h_n is an isomorphism. Furthermore, $H_1(X)$ is the abelianization of $\pi_1(X)$.

Hopf's invariant

The Hopf invariant of a map $f : S^{2n+1} \to S^{n+1}$ is defined by the cup product structure on $X = S^{n+1} \cup_f D^{2n+2}$: if a and b respectively generate the $(n+1)^{st}$ and $(2n+2)^{nd}$ cohomology group of X, then $a \smile a = H(f)b$.
It is a group homomorphism $\pi_{2n+1}(S^{n+1}) \to \mathbb{Z}$, an isomorphism for $n = 1$, and the zero map for n even.