Algebraic topology computations and Representation theory of GL_n

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Plan :

I. Functors in algebraic topology.

II. Functors in representation theory of GL_n

III. Thm : LF and ΘF coincide.

IV. Some applications.

V. The homology of EML spaces.

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1. How are they related ?
2. Which ones are isomorphic ?
3. Can we describe them from simpler functors ?

   Ex : $H_{10}K(A, 3) = A \otimes A/3A$ (A free abelian)

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Normalized Chain functor :

$\mathcal{N} : s(\mathbb{k}\text{-mod}) \to \mathrm{Ch}_{\geq 0}(\mathbb{k}\text{-mod})$

is an equivalence of categories, with explicit inverse

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Free (or projective) resolution of M, shifted

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Free simplicial $\mathbb{k}$-mod, with homology $M[n]$

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Chain complex of $\mathbb{k}$-mod

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3. Thm [Dold] $H_*(SP^dX)$ computed from $H_*(X)$ and L_iS^d.

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1. $GR(A) := \mathbb{Z}[A]$. Then $L_*GR(A;n) = H_*K(A,n)$.
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I. Functors in algebraic topology (3)

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  But there remains work : for ex, still not clear what $L_iS^d(A,n)$ are !

II. Functors in representation theory of GL_n (1)

1. Functors $\rightarrow$ representations of $GL_n(\mathbb{k})$

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Forget that ρ is a morphism of algebraic group schemes :

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Hence

- $(\Bbbk^n, \rho_{\mathrm{Frob}}) \neq (\Bbbk^n, \mathrm{Id})$ in $GL_{n,\Bbbk}$-mod
- $\mathcal{U}(\Bbbk^n, \rho_{\mathrm{Frob}}) = \mathcal{U}(\Bbbk^n, \mathrm{Id})$ in $GL_n(\Bbbk)$-mod

II. Functors in representation theory of GL_n (3)

3. Strict polynomial functors

$$\begin{array}{ccc} & & GL_{n,\Bbbk}\text{-mod} \\ & & \downarrow \mathcal{U} \\ \mathcal{F}_{\Bbbk} & \xrightarrow{\quad \mathrm{ev} \quad} & GL_n(\Bbbk)\text{-mod} \end{array}$$

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Ex : S^d, $\otimes^d$, $\mathcal{L}^d$ are canonically strict polynomial functors, homogeneous of degree d.

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Def : Strict polynomial functor = pair $(F, \{F_{M,N}\})$ (with $F \in \mathcal{F}_{\Bbbk}$)
Ex : S^d, $\otimes^d$, $\mathcal{L}^d$ (homogeneous of degree d).

II. Functors in representation theory of GL_n (4)

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- Strict polynomial functors are interesting because :

Thm : [Friedlander-Suslin] If $n \geq d$, iso :

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Allows explicit $\mathrm{Ext}_{GL_{n,\Bbbk}}$-computations !

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II. Functors in representation theory of GL_n (5)

Algebraic topology	Representations of group scheme $GL_{n,\mathbb{k}}$
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4. Ringel Duality Θ

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4. Ringel Duality Θ

- [Ringel, 91] repres. finite dim. algebras
- [Donkin, 93] the $GL_{n,\Bbbk}$-mod case
- [Chałupnik, 2008] translated to $\mathcal{P}_{d,\Bbbk}$

Ringel duality linked with theory of tilting modules

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II. Functors in representation theory of GL_n (6)

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$\Theta(\Gamma^d) =$ **big complex !**

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Different insights on computations.
Well-known computations on the right hand side are unknown on the left hand side (and vice versa).

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The proof

Easy. Ingredients :

1. Classical computation : $L_* S^d(M,1) \simeq \Lambda^d(M)[d]$
   equivalent forms :
   - Koszul dual of $S(M)$ is $\Lambda(M)$,
   - $\mathrm{Ext}^*_{S(M)}(\Bbbk, \Bbbk) = \Lambda(M[1])$,
   - $\Omega\mathbb{C}P^\infty \simeq S^1$,
2. Standard simplicial algebra (EZ-thm...)
3. Elementary homological algebra in $\mathcal{P}_{d,\Bbbk}$ (Yoneda lemma).

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A) Plethysm problem.

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Rk : no injective resolution of $S^n \circ \otimes^d$ is known,
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Ex : $\mathrm{Ext}^*_{\mathcal{P}_{nd,\Bbbk}}(\Lambda^{nd}, F \circ S^d) = 0$

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Work in progress with. L. Breen and R. Mikhailov.

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