Bifurcation of affine maps
- Part 1: Topology of polynomial functions -

Mihai Tibăr

Example ([Broughton-1988])

$f : \mathbb{C}^2 \to \mathbb{C}, \quad f(x, y) = x + x^2 y.$
One checks that the function f has no singular points.
We have $f^{-1}(\varepsilon) = \{y = (\varepsilon - x)/x^2\}$ for $\varepsilon \neq 0$, thus $f^{-1}(\varepsilon) \overset{\text{homeo}}{\simeq} \mathbb{C}^* := \mathbb{C} \setminus \{0\}$
and $f^{-1}(0) = \{x(xy + 1) = 0\}$, thus $f^{-1}(0) \overset{\text{homeo}}{\simeq} \mathbb{C} \sqcup \mathbb{C}^*$.
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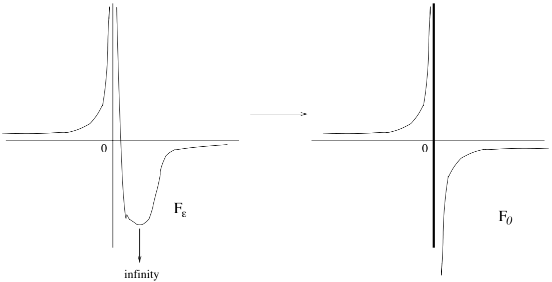

Question: what about the real polynomial $f : \mathbb{C}^2 \to \mathbb{C}$, $f(x,y) = \bar{x} + x^2 y$?

As we have seen in the preceding lectures, the presence of a singularity is the only reason for the non-triviality of the local fibration associated to a function germ g, where its local Milnor fibre has non-trivial reduced homology, while the fibre containing the singularity is contractible. In the global affine setting, we have just seen that the fibres of a polynomial function may not be all homeomorphic *even in the absence of singularities*.

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Let us state this more precisely:

Definition

One says that f *is topologically trivial at* $t_0 \in \mathbb{C}$ if there is a neighbourhood D of $t_0 \in \mathbb{C}$ such that the restriction $f_| : f^{-1}(D) \to D$ is a topologically trivial fibration. If t_0 does not satisfy this property, then we say that t_0 is an *atypical value* and that $f^{-1}(t_0)$ is an *atypical fibre*. We shall denote by $\mathrm{Atyp}\, f$ the set of atypical fibres of f.

In the above example, we have seen that the set $\operatorname{Atyp} f$ contains the value 0 (one may actually show that $\operatorname{Atyp} f = \{0\}$). In general, there is the following inclusion:

$$f(\mathrm{Sing}\, f) \subset \operatorname{Atyp} f,$$

where $\mathrm{Sing}\, f := Z(\mathrm{Jac}\, f)$ denotes the singular locus of f, without any condition on its dimension.

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Since $\mathrm{Sing}\, f$ is an algebraic set, by the Tarski-Seidenberg theorem $f(\mathrm{Sing}\, f)$ is semi-algebraic subset of $\mathbb{C}$, hence finite. It turns out that the set of atypical values $\mathrm{Atyp}\, f$ is also *finite*.

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The proof of the finiteness of the set of atypical values has been sketched by Thom and uses the existence of Whitney stratifications. A complete proof along these lines can be deduced from Verdier's study on Bertini-Sard theorems. Another proof using the resolution of singularities was given by Pham.

It will turn out (in the end of this lecture) that in 2 variables one can give a very precise meaning of *singularities at infinity*. But this is not the case in higher dimensions: the problem of characterising the atypical non-singular values is still open for $n \geq 3$.

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Another evidence of the impact of singularities at infinity is the following famous open problem (a similar conjecture holds in $\mathbb{C}^n$):

Conjecture (Jacobian Conjecture in dimension 2)

Let $f, h \in \mathbb{C}[x, y]$. If $\mathrm{Sing}(f, h) = \emptyset$ then f is equivalent to x modulo an automorphism of $\mathbb{C}^2$.

This conjecture has the following equivalent formulation in terms of singularities at infinity:

Conjecture

If $f : \mathbb{C}^2 \to \mathbb{C}$ has no critical points but has $\boxed{\textit{singularities at infinity}}$ *then, for any polynomial $h : \mathbb{C}^2 \to \mathbb{C}$, the critical locus* $\mathrm{Sing}(f, h)$ *is not empty.*

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Proof.

If the polynomial f has no critical points and no singularities at infinity then f is a trivial fibration (a theorem will come later), and thus all the fibres of f are CW-complexes of dimension ≤ 1 with trivial homotopy groups, hence they are contractible. In this case the Suzuki-Abhyankar-Moh theorem tells that f is linearisable. The case which is still not covered is therefore that of singularities at infinity. □

How topology is changing at infinity

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree d and let $\tilde{f}(x, x_0)$ be the homogenization of f by the new variable x_0. Consider the closure in $\mathbb{P}^n \times \mathbb{C}$ of the graph of f, that is the hypersurface

$$\mathbb{X} := \{(x; x_0), t) \in \mathbb{P}^n \times \mathbb{C} | F := \tilde{f}(x, x_0) - tx_0^d = 0\}.$$

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Let:

$$\tau : \mathbb{X} \to \mathbb{C}$$

be the projection to $\mathbb{C}$, let us denote by H^∞ the hyperplane at infinity $\{x_0 = 0\} \subset \mathbb{P}^n$. Let $\mathbb{X}^\infty := \mathbb{X} \cap (H^\infty \times \mathbb{C})$ be the part at infinity of $\mathbb{X}$. Note that τ is a *proper map*, whereas f is not proper. We denote by $\mathbb{X}_t$ the fibre $\tau^{-1}(t)$, for some $t \in \mathbb{C}$; this is a projective hypersurface in $\mathbb{P}^n$.

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One may identify $\mathbb{C}^n$ to $\mathbb{X} \setminus \mathbb{X}^\infty$ via the canonical map $x \overset{i}{\mapsto} ([x : 1], f(x))$ which fits into the commuting diagram:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{i} & \mathbb{X} \\ {\scriptstyle f} \searrow & & \swarrow {\scriptstyle \tau} \\ & \mathbb{C} & \end{array} \quad . \qquad (1)$$

Exercise. Prove that the singularities of $\mathbb{X}$ are contained in $\mathbb{X}^\infty$, namely $\mathrm{Sing}(\mathbb{X}) = \Sigma \times \mathbb{C}$, where:

$$\Sigma := \{\frac{\partial f_d}{\partial x_1} = \cdots = \frac{\partial f_d}{\partial x_n} = 0,\ f_{d-1} = 0\} \subset H^\infty,$$

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If $\mathrm{Sing}\, \mathbb{X}_t$ denotes the singular locus of the projective hypersurface $\mathbb{X}_t$, then show that $\Sigma = H^\infty \cap \mathrm{Sing}\, \mathbb{X}_t$, $\forall t \in \mathbb{C}$.

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The singularities of f, i.e. the affine set $\mathrm{Sing}\, f := Z(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$, can be identified by the above diagram (1), with the singularities of τ on $\mathbb{X} \setminus \mathbb{X}^\infty$.

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Exercise Prove that $\overline{\mathrm{Sing}\, f} \cap H^\infty \subset \Sigma$, where $\overline{\mathrm{Sing}\, f}$ denotes the closure (analytic or algebraic, the same) of $\mathrm{Sing}\, f$ in $\mathbb{P}^n$. In particular $\dim \mathrm{Sing}\, f \leq 1 + \dim \Sigma$. What happens when $\Sigma = \emptyset$?

In case f has isolated singularities, the quotient algebra:

$$\mathbb{C}[x_1, \ldots, x_n]/\mathrm{Jac}(f)$$

is a finite dimensional complex vector space and its dimension $\mu(f)$ will be called *global Milnor number*.

Exercise Prove that $\mu(f)$ is the total sum of the local Milnor numbers of f at its singular points.

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Remark (The case $\Sigma = \emptyset$)

In this case $\mathrm{Sing}\, \mathbb{X} = \emptyset$, which also implies that f has at most isolated singularities. The restriction of τ to $\mathbb{X} \setminus \mathbb{X}^\infty$ has a singular set equal to $\mathrm{Sing}\, f$, which is a discrete set. Moreover the restriction of τ to $\mathbb{X}^\infty$ is a submersion, since $\mathbb{X}^\infty$ is a product space by the variable t, and τ is also a submersion on the neighbourhood $\mathcal{N} \cap (\mathbb{X} \setminus \mathbb{X}^\infty)$, where $\mathcal{N}$ is the complement of a large enough ball B_M in $\mathbb{C}^n$. Apply now *Ehresmann's theorem* to the manifold $\mathcal{N} \cap (\mathbb{X} \setminus \mathbb{X}^\infty)$ with boundary $\mathcal{N} \cap \mathbb{X}^\infty$ and to the submersive proper map τ. The conclusion is that the restriction $\tau : (\mathcal{N} \cap (\mathbb{X} \setminus \mathbb{X}^\infty), \mathbb{X}^\infty) \to \mathbb{C}$ is a *locally trivial fibration*.

Definition

We say that f is *topologically trivial at infinity at the value* $t_0 \in \mathbb{C}$ if there exists a compact set $K \subset \mathbb{C}^n$ and a disk D_δ centred at t_0 such that the restriction:

$$f_| : (\mathbb{C}^n \setminus K) \cap f^{-1}(D_\delta) \to D_\delta \tag{2}$$

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In the case $\Sigma = \emptyset$ discussed above, we say that thre are no "singularities at infinity", more precisely we have that our polynomial is topologically trivial at infinity at any value $t \in \mathbb{C}$. This implies that $\mathrm{Atyp}\, f = f(\mathrm{Sing}\, f)$.

Case $\dim \Sigma = 0$.

Remark that any non-constant polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ verifies the condition $\dim \Sigma = 0$.

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Unlike the case $\Sigma = \emptyset$ treated before, here we need to replace Ehresmann's theorem by the more powerful Thom Isotopy theorem for spaces endowed with a Whitney stratification.
Here we work with a semi-algebraic Whitney stratification $\mathcal{W}$ of $\mathbb{X}$ having $\mathbb{X} \setminus \mathbb{X}^\infty \simeq \mathbb{C}^n$ as one of the strata. The other strata are lower dimensional and are included in the hyperplane at infinity $\mathbb{X}^\infty$. Under our hypothesis, they are as follows: $\mathbb{X} \setminus \mathbb{X}^\infty$, $\mathbb{X}^\infty \setminus (\Sigma \times \mathbb{C})$, $(\Sigma \times \mathbb{C}) \setminus R$ and R, where R is some finite set of points.
Our proper map τ is therefore submersive on each stratum of $\mathcal{W}$ except at the set $\mathrm{Sing}\, f$ and at the point strata R. The points where the transversality of τ to strata fails, are called *$\mathcal{W}$-singularities*.

The set of all $\mathcal{W}$-singularities is denoted by $\operatorname{Sing}^\infty f$; this depends on the chosen coordinates on $\mathbb{C}^n$. This construction can be done in general for any f, namely endow $\mathbb{X}$ with (the coarsest) Whitney stratification at infinity, and define $\operatorname{Sing}^\infty f$.

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The image $A_{\mathrm{inf}} := \tau(\operatorname{Sing}^\infty f)$ is a finite subset of $\mathbb{C}$ because the restriction of τ to each stratum is a semi-algebraic function, thus its critical values are finite many, again by Tarski-Seidenberg.

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By the Thom Isotopy theorem we get the inclusion $\mathrm{Atyp}\, f \subset A_{\mathrm{inf}} \cup f(\mathrm{Sing}\, f)$, and in particular that $\mathrm{Atyp}\, f$ is a finite set. This proof works in general for any f.

The global bouquet theorem for isolated $\mathcal{W}$-singularities at infinity

Theorem ([ST-1995])

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial with isolated $\mathcal{W}$-singularities at infinity, i.e. $\dim(\mathrm{Sing}^\infty f \cup \mathrm{Sing}\, f) \leq 0$. Then the general fibre of f is homotopy equivalent to a bouquet of spheres of real dimension $n-1$.

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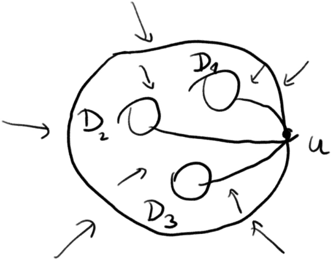
Sketch of proof. Let $c \notin \mathrm{Atyp}\, f$, let D_b denote a small enough disk centred at $b \in \mathrm{Atyp}\, f$. Let $F_V := f^{-1}(V)$ and $\mathbb{X}_V := \tau^{-1}(V)$ for some $V \subset \mathbb{C}$. Let c_b be some point of the boundary ∂D_b. We get like in the proof by Brieskorn of the local bouquet theorem (lecture B1), by deformation retraction and excision, the following splitting:

$$\tilde{H}_i(F_c) = H_{i+1}(\mathbb{C}^n, F_c) = \oplus_{b \in \mathrm{Atyp}\, f} H_{i+1}(F_{D_b}, F_{c_b}).$$

We stick to such a term: for simplicity, let D be one of the discs D_b and let $u \in \partial D$ be fixed. We have, according to [Broughton-1988, Proposition 5.2] the following Lefschetz type duality:

$$H_*(F_D, F_u) \cong H^{2n-*}(\mathbb{X}_D, \mathbb{X}_u).$$



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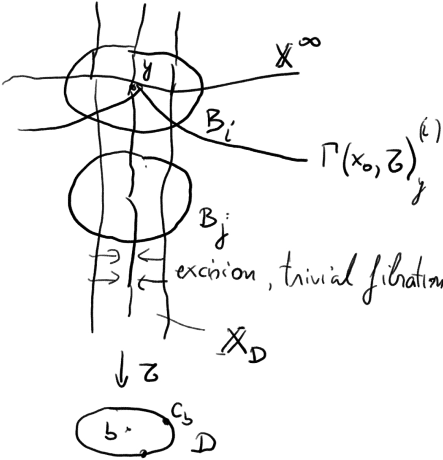

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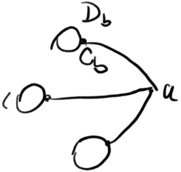

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It remains to prove that $H^*(\mathbb{X}_D, \mathbb{X}_u)$ is concentrated. The stratified singularities of $\tau_{|\mathbb{X}_D}$ are included in $\mathbb{X}_b$; let those be denoted by $a_1, \ldots, a_k$. We may choose a good neighbourhood of a_i, say of the form $B_i \cap \mathbb{X}_D$, where B_i is a small enough closed ball in some local chart and also suppose D small enough such that the restriction $\tau : B_i \cap \mathbb{X}_{D^*} \to D^*$ is a well-defined Milnor-Lê locally trivial fibration.

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Since the fibres are transversal to the semi-algebraic Whitney stratification induced by $\mathcal{W}$ on $\partial B_i \cap \mathbb{X}_D$, there is a fibration on the exterior of the balls:

$$\tau : \mathbb{X}_D \setminus \cup_{i=1,k} B_i \to D$$

This is a trivial fibration since τ is a submersion (no singularities of the map) and it is proper. By an *excision*, we get the isomorphism:

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Definition

We denote by λ_a the number of spheres in the Milnor fibre of the germ $\tau : (\mathbb{X}, a) \to (\mathbb{C}, b)$ and call it *the Milnor-Lê number (at infinity) at a*.

The fibre $B_i \cap \mathbb{X}_u$ is the Milnor fibre of the Milnor-Lê fibration of the map germs $\tau : (\mathbb{X}, a) \to (\mathbb{C}, b)$, and $\mathbb{X}$ is a hypersurface.
The general local Bouquet Theorem says that in this case the Milnor fibre of a function germ with isolated singularity on a hypersurface of dimension n has the reduced homology concentrated in dimension $n - 1$.

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In order to finish the proof of the global affine bouquet theorem, we need to pass from homology to homotopy type, and this is done like in Milnor's proof of the local Bouquet Theorem (see Course B1). □

Notations. $\mu(f)$ = the total Milnor number of f, $\lambda(f) := \sum_{a \in X^\infty} \lambda_a$,
$\mu_{F_b}(f) := \sum_{v \in F_b} \mu_v(f)$, $\lambda_{F_b}(f) := \sum_{a \in X^\infty \cap X_b} \lambda_a$.

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The above proof shows that the relative homology $H_*(F_{D_b}, F_b)$ is concentrated in dimension n and its rank is equal to $\lambda_{F_b}(f) + \mu_{F_b}(f)$. Applying the additive function "Euler characteristic" to the homology exact sequence of the pair (F_{D_b}, F_u), where $u \in \partial D_b$, one gets:

$$(-1)^n(\lambda_{F_b}(f) + \mu_{F_b}(f)) = \chi(F_{D_b}) - \chi(F_u)$$

and one also has $\chi(F_{D_b}) = \chi(F_b)$. We have obtained:

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$\mu_{F_b}(f) := \sum_{v \in F_b} \mu_v(f)$, $\lambda_{F_b}(f) := \sum_{a \in X^\infty \cap X_b} \lambda_a$.

The above proof shows that the relative homology $H_*(F_{D_b}, F_b)$ is concentrated in dimension n and its rank is equal to $\lambda_{F_b}(f) + \mu_{F_b}(f)$. Applying the additive function "Euler characteristic" to the homology exact sequence of the pair (F_{D_b}, F_u), where $u \in \partial D_b$, one gets:

$$(-1)^n(\lambda_{F_b}(f) + \mu_{F_b}(f)) = \chi(F_{D_b}) - \chi(F_u)$$

and one also has $\chi(F_{D_b}) = \chi(F_b)$. We have obtained:

Corollary

Let f be a polynomial with isolated $\mathcal{W}$-singularities at infinity. Then:

1. *The number of spheres in the structure of a general fibre $F_{\rm gen} \overset{\rm ht}{\simeq} \vee S^{n-1}$ is equal to $\mu(f) + \lambda(f)$.*
2. $\chi(F_b) - \chi(F_{\rm gen}) = (-1)^n(\lambda_{F_b}(f) + \mu_{F_b}(f))$. □

Polar number interpretation

Let us consider the map germ:

$$(x_0, \tau) : (\mathbb{X} \cap (\{x_i \neq 0\} \times \mathbb{C}), y) \to (\mathbb{C}^2, 0),$$

at some point $y \in \mathbb{X}^\infty$ in the chart $U_i \times \mathbb{C}$, for $i = 1, \ldots, n$. Consider the polar locus of the map (x_0, τ):

$$\Gamma(x_0, \tau)_y^{(i)} := \mathrm{closure}[\mathrm{Sing}(x_0, \tau)_{|\mathbb{C}^n \cap (U_i \times \mathbb{K})} \setminus \mathrm{Sing}\, \tau_{|\mathbb{C}^n \cap (U_i \times \mathbb{C})}] \subset \mathbb{X},$$

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If f has isolated $\mathcal{W}$-singularities at infinity, then $\Gamma(x_0, \tau)_y^{(i)}$ is a curve or empty, for any $y \in \mathbb{X}^\infty$. Using the "box" neighbourhood like in the local Bouquet Theorem, we obtain:

Corollary

If f has isolated $\mathcal{W}$-singularities at infinity, then the polar number λ_{a_j} is equal to the intersection number $\mathrm{mult}_{a_j}(\mathbb{X}_b, \Gamma(x_0, \tau)_{a_j}^{(j)})$. In particular, the latter does not depend on the chart U_j.

One may then prove the following implication:

$$\Gamma(x_0, \tau)_y^{(i)} \neq \emptyset \Longrightarrow y \text{ is a } \mathcal{W}\text{-singularity at infinity of } f. \qquad (3)$$

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In case of isolated $\mathcal{W}$-singularities one may prove the following equivalence:

Theorem ([ST-1995, Ti-1999, Ti-2007])

Let $y \in \mathbb{X}^\infty$ where τ has a $\mathcal{W}$-singularity at infinity which is at most isolated. Then y is not a singularity at infinity of f if and only if $\Gamma(x_0, \tau)_y^{(i)} = \emptyset$ for all i. □

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The implication "⇐" is proved in 4 steps, and we refer to [ST-1995], [Ti-1999], [Ti-2007] for the details of the proof and for the definition of the notions evoked here: one first shows that if a regular fibre F_b has no atypical points at infinity, then it is t-regular. It turns out that t-regularity implies ρ-regularity. The later allows one to produce a trivialisation of the fibration of the restriction of P to the complement of a large enough ball B intersected by a tube $P^{-1}(D)$, where D is a small enough disk centred at b. Finally, this trivialisation extends to the tube $P^{-1}(D)$.

By chaining together the above corollaries and the last theorem, we obtain:

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Theorem

Let f be a polynomial with isolated $\mathcal{W}$-singularities at infinity. Then $b \in \mathrm{Atyp}\, f$ if and only if $\chi(F_b) \neq \chi(F_{\mathrm{gen}})$.

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Theorem

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This is the generalisation of the following result obtained by Suzuki [Suzuki-1974], and rediscovered by Hà and Lê in 1984:

Theorem ([Suzuki-1974])

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function and let $\lambda \in \mathbb{C} \setminus f(\mathrm{Sing}\, f)$. Then $\lambda \notin \mathrm{Atyp}\, f$ if and only if the Euler characteristic of the fibres $\chi(F_t)$ is constant for t varying in some small neighbourhood of λ.

Milnor number jump interpretation

Consider a non-constant polynomial of two variables $f : \mathbb{C}^2 \to \mathbb{C}$. Let us assume that f has at most isolated singularities (at least on the fibre F_b in the neighbourhood of which we study the variation of topology).

We have seen that in case $\mathrm{Sing}\,\mathbb{X}$ is not empty, it is a finite collection of lines $\bigsqcup_j \{p_j\} \times \mathbb{C}$, where p_j is an isolated singularity of the fibre $\mathbb{X}_t := \tau^{-1}(t)$ for all $t \in \mathbb{C}$.

We denote by $\mu_{p_j}(\mathbb{X}_t)$ its Milnor number. This number is constant along the line $\{p_j\} \times \mathbb{C}$, except for a finite number of *special values* of t. Let then $\mu_{p_j,gen}$ be the generic value of $\mu_{p_j}(\mathbb{X}_t)$ along the line.

At some special value $t = b$, we have: $\mu_{p_j}(\mathbb{X}_b) > \mu_{p_j,gen}$.

The atypical points at infinity of f are then precisely the points (p_j, b) where the Milnor number jumps, namely the following difference is positive[1]:

$$\lambda_{p_j}(b) = \mu_{p_j}(\mathbb{X}_b) - \mu_{p_j,gen} \tag{4}$$

We then have $\mathrm{Atyp}\, f = f(\mathrm{Sing}\, f) \cup \{b \in \mathbb{C} \mid \exists p \in \Sigma, \lambda_p(b) > 0\}$.

[1] see e.g. [ST-1995], and [Ti-2007, Prop. 3.3.6] for a more general statement.

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