Real Fibrations
Part 1: Setting up most general theorems

Mihai Tibăr

Stating a meaningful local fibration theorem

Let $G : (\mathbb{K}^m, 0) \to (\mathbb{K}^p, 0)$ be an analytic map germ, where $m \geq p \geq 1$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In case $p = 1$ and $\mathbb{K} = \mathbb{C}$, we have seen in Lecture B1 that holomorphic function germs have a local Milnor fibration. Whenever $\mathbb{K} = \mathbb{R}$, removing the origin disconnects the real line and, by similar arguments, one obtains a Milnor fibration $G : (\mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ which is a trivial fibration over each of the two connected components of the set germ $(\mathbb{R} \setminus \{0\}, 0)$.

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We thus focus here on the setting $p \geq 2$. As we have seen in the statement of Milnor's Theorem for the function germs, the Milnor fibration makes sense only if it is independent on the choices of ε and δ. In case of map germs, this is still an important issue.

Question: What is the reason of this independency in case $p = 1$?

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Without this condition one cannot speak about "Milnor fibration". Some part of the recent literature does not care about this condition whatsoever. We avoid here this false track by taking into account carefully "the independence on ε and δ". It actually turns out that this is a difficult problem which leaves many open questions[1].

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Example 1. $F : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$, $(x, y, z) \mapsto (x^2 - y^2 z, y)$. The singular locus of F is included in the central fibre $F^{-1}(0, 0)$.
The image of F is open in 0, namely we have the equality of set germs $(\mathrm{Im} F, 0) = (\mathbb{C}^2, 0)$. Nevertheless this map germ F does not have a locally trivial fibration over the set germ $(\mathbb{C}^2 \setminus \{0\}, 0)$[2], since we cannot apply Ehresmann Theorem.

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Example 2. $F : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$, $F(x, y) = (x, xy)$. The image $F(B_\varepsilon)$ of the ball B_ε centred at 0 depends heavily on its radius $\varepsilon > 0$. Here, not only that the image $F(B_\varepsilon)$ does not contain a neighbourhood of the origin, but it turns out that the image of F is not even well-defined as a set germ.

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We therefore have to answer the following question in case $p \geq 2$:

What analytic map germs define local fibrations?
In other words, what kind of local fibration and under what conditions.

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Map germs having germ image sets

In the second example, the image of the map germs G is not well-defined as a set germ. Let us define this condition more carefully. We state this in case $\mathbb{K} = \mathbb{R}$ but one may replace everywhere by $\mathbb{C}$.

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For two subsets $A, A' \subset \mathbb{R}^p$ containing the origin, one has the equality of set germs $(A, 0) = (A', 0)$ if and only if there exists some open ball $B_\varepsilon \subset \mathbb{R}^p$ centred at 0 and of radius $\varepsilon > 0$ such that $A \cap B_\varepsilon = A' \cap B_\varepsilon$.

Definition

Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 0$, be a continuous map germ. We say that the image $G(K)$ of a set $K \subset \mathbb{R}^m$ containing 0 is a *well-defined set germ* at $0 \in \mathbb{R}^p$ if for any small enough open balls $B_\varepsilon, B_{\varepsilon'}$ centred at 0, with $\varepsilon, \varepsilon' > 0$, we have the equality of germs $(G(B_\varepsilon \cap K), 0) = (G(B_{\varepsilon'} \cap K), 0)$.

Nice map germs

Whenever both images $\mathrm{Im} G$ and $G(\mathrm{Sing}\, G)$ are well-defined as set germs, one says that G is a *nice map germ*, abbreviated NMG. (The first does not imply the second!)

Exercise. The product of two mixed NMG is a NMG? Check the following example: $(1 + z_1)z_2$ and $\bar{z}_2$.

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Proposition

Let $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n \geq p$, be a holomorphic map germ. If the fibre $F^{-1}(0)$ has dimension $n - p$ then $(\mathrm{Im}F, 0) = (\mathbb{C}^p, 0)$. If moreover $\mathrm{Sing}\, F \cap F^{-1}(0) = \{0\}$ then F is a NMG.

Question. Does the second condition in the above Proposition implies the first?

Proof.

Let $B_\varepsilon \subset \mathbb{C}^n$ be some open neighbourhood of the origin where the holomorphic map F is defined. Let $Z \subset B_\varepsilon$ be a general complex p-plane through 0. Then 0 is an isolated point of $Z \cap F^{-1}(0)$. By a classical result (folklore), for any small enough open neighbourhood U_ε of 0 in Z, the induced map $F_{|U_\varepsilon} : U_\varepsilon \to \mathbb{C}^p$ is finite-to-one. By the Open Mapping Theorem of Grauert and Remmert, this implies that $F(U_\varepsilon)$ is open. This shows the equality of germs $(\mathrm{Im} F, 0) = (\mathbb{C}^p, 0)$.

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In case $\mathrm{Sing}\, F \cap F^{-1}(0) = \{0\}$, the set germ $(F^{-1}(0), 0)$ is an *isolated complete intersection singularity*, abbreviated ICIS[a] and it has been proved[b] that $(F(\mathrm{Sing}\, F), 0)$ is a hypersurface germ.

□

[a] This object has been studied in many papers, and will be a topic in this School (2022 part of it).

[b] See Looijenga's book.

A real counterpart of the above result is the following:

Proposition ([JT1])

Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 0$ *be an analytic map germ.*
If $\mathrm{Sing}\, G \cap G^{-1}(0) = \{0\}$, *then* G *is a NMG.*

I do not prove this here, but only remark that in case $\dim G^{-1}(0) > 0$, one has:

Lemma

If $\mathrm{Sing}\, G \cap G^{-1}(0) \subsetneq G^{-1}(0)$ *then* $(\mathrm{Im}\, G, 0) = (\mathbb{R}^p, 0)$.

Proof of the Lemma. Let $q \in G^{-1}(0) \setminus \mathrm{Sing}\, G$, which is nonempty by hypothesis. Then G is a submersion on some small open neighbourhood N_q of q, thus the restriction $G_{|N_q}$ is an open map, and therefore $\mathrm{Im}\, G$ contains some open neighbourhood of the origin of the target. □

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If a non-constant analytic map germ $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 1$ *is a NMG, then its discriminant* $\mathrm{Disc}\, G := G(\mathsf{Sing}\, G)$ *is a well-defined closed subanalytic set germ.*

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Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant analytic map germ, $m \geq p \geq 1$. Let $U \subset \mathbb{R}^m$ be a connected manifold, and let

$$M(G_{|U}) := \left\{ x \in U \mid \rho_{|U} \not\pitchfork_x G_{|U} \right\}$$

be the set of ρ*-nonregular points* of $G_{|U}$, or *the Milnor set of* $G_{|U}$, where $\rho := \| \cdot \|$ denotes here the Euclidean distance function, and $\rho_{|U}$ is its restriction to U.
It turns out from the definition that $M(G_{|U})$ is real analytic.
In the following we will actually consider the germ at 0 of $M(G_{|U})$.

By definition $M(G_{|U})$ coincides with the singular set $\mathrm{Sing}(\rho, G)_{|U}$ defined in its turn as the set of points $x \in U$ such that either $x \in \mathrm{Sing}(G_{|U})$, or $x \notin \mathrm{Sing}(G_{|U})$ and $\mathrm{rank}_x(\rho_{|U}, G_{|U}) = \mathrm{rank}_x(G_{|U})$.

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Definition (The Milnor set in the stratified setting)

Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, with $m \geq p > 1$, be a non-constant analytic map germ, and let $\mathbb{W}$ be a finite semi-analytic Whitney (a)-stratification of G at 0. Let $W_\alpha \in \mathbb{W}$ denote the germ of some stratum, and let $M(G_{|W_\alpha})$ be the Milnor set of $G_{|W_\alpha}$. One calls

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$M(G)$ is closed because $\mathbb{W}$ is a Whitney (a)-stratification.

Tame map germs

Definition

Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, with $m \geq p > 1$, be a non-constant analytic map germ. We say that G *is tame* if the following inclusion of set germs holds:

$$\overline{M(G) \setminus G^{-1}(0)} \cap G^{-1}(0) \subset \{0\}. \tag{1}$$

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It follows from the definition that if G is tame then the closure of the strata of $\mathcal{W}$ of dimensions $\leq p$ intersect $G^{-1}(0)$ only at $\{0\}$. A particular case of tame maps is: G with $G^{-1}(0) = \{0\}$ as a set germ at 0.

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Exercise. Check that Sabbah's example considered in the beginning is not tame.

"Tame" seems to be not bad

Theorem

Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$*, with* $m \geq p > 1$*, be a non-constant analytic map germ. If* G *is tame then:*

1. G *is a NMG at the origin.*
2. *the image* $G(W_\alpha)$ *is a well-defined set germ at the origin, for any stratum* $W_\alpha \in \mathbb{W}$*.*

Remark

Let us point out two very particular cases where $\mathrm{Im}\, G$ is well-defined as a set germ:

(i). Sing $G \subset G^{-1}(0)$ and G satisfies condition (1).
It was shown by Massey that $(\mathrm{Im}\, G, 0) = (\mathbb{R}^p, 0)$.

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Regular stratification of the tame map germ G

We show that "tame map germs" is a category where we can prove the existence of a local singular fibration. We focus on the general case $\dim \mathrm{Disc}\, G > 0$.

Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant analytic map germ, $m > p > 1$, and let $\mathbb{W}$ be the Whitney (b)-regular stratification at 0 defined above.
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By using the classical stratification theory, there exists a germ of a finite subanalytic stratification $\mathbb{S}$ of the target such that $\mathrm{Disc}\, G$ is a union of strata, and that G is a stratified submersion relative to the couple of stratifications $(\mathbb{W}, \mathbb{S})$, meaning that the image by G of a stratum $W_\alpha \in \mathbb{W}$ is a single stratum $S_\beta \in \mathbb{S}$, and that the restriction $G_| : W_\alpha \to S_\beta$ is a submersion.

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Exercise. Try to find a "regular stratification" in case of the second example given in the beginning. Not possible? Why not?

Definition (Singular Milnor tube fibration)

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We say that G has a *singular Milnor tube fibration* relative to $(\mathbb{W}, \mathbb{S})$ if for any small enough $\varepsilon > 0$ there exists $0 < \eta \ll \varepsilon$ such that the restriction:

$$G_{|} : B^m_\varepsilon \cap G^{-1}(B^p_\eta \backslash \{0\}) \to B^p_\eta \backslash \{0\} \quad (2)$$

is a stratified locally trivial fibration which is independent, up to stratified homeomorphisms, of the choices of ε and η.

What means more precisely "independent, up to stratified homeomorphisms, of the choices of ε and η"?
When replacing ε by some $\varepsilon' < \varepsilon$, and η by some small enough $\eta' < \eta$, then the fibration (2) and its analogous fibration for ε' and η' should have the same stratified image in the smaller ball $B^p_{\eta'} \setminus \{0\}$, and the fibrations should be stratified diffeomorphic over this ball. This property is based on the fact that the image of G is well-defined as a stratified set germ, which amounts to the assumed existence of the regular stratification $(\mathbb{W}, \mathbb{S})$.

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The non-empty fibres are those over some connected stratum $S_\beta \subset \mathrm{Im} G$ of $\mathbb{S}$. Each such fibre is a singular stratified set, namely it is the union of its intersections with all strata $W_\alpha \subset G^{-1}(S_\beta)$.

Singular stratified fibration theorem ("tame" seems just suitable!)

Theorem

Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 1$, *be a non-constant analytic map germ. If* G *is tame, then* G *has a singular Milnor tube fibration* (2).

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Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 1$, be a non-constant analytic map germ. If G is tame, then G has a singular Milnor tube fibration (2).

Proof. Let us fix a regular stratification $(\mathbb{W}, \mathbb{S})$. By definition, the restriction of G to any stratum $W_\alpha \in \mathbb{W}$ is nonsingular and of constant rank.

Let us first consider the strata W_α such that the fibres of the restriction $G_{|W_\alpha}$ are of dimension > 0, equivalently $\operatorname{corank}(G_{|W_\alpha}) \geq 1$. Condition (1) implies the existence of $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, there exists η, $0 < \eta \ll \varepsilon$, such that, for any stratum $S_\beta \in \mathbb{S}$, $S_\beta \subset G(W_\alpha)$, the restriction map

$$G_| : W_\alpha \cap \overline{B^m_\varepsilon} \cap G^{-1}(S_\beta \cap B^p_\eta \setminus \{0\}) \to S_\beta \cap B^p_\eta \setminus \{0\} \tag{3}$$

is a submersion on a manifold with boundary.

Indeed, since the sphere S_ε^{m-1} is transversal to all the finitely many strata of the Whitney stratification $\mathbb{W}$ at $0 \in \mathbb{R}^n$, it follows that the intersection $S_\varepsilon^{m-1} \cap \mathbb{W}$ is a Whitney stratification $\mathbb{W}_{S,\varepsilon}$ of S_ε^{m-1}. Condition (1) tells that the restriction of the map G to any stratum of $\mathbb{W}$ is stratified-transversal to the corresponding stratum of the stratification $\mathbb{W}_{S,\varepsilon}$, for any $0 < \varepsilon < \varepsilon_0$.

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As for the strata W_α such that the fibres of the restriction $G_{|W_\alpha}$ are of dimension 0, we have seen that they belong to $M(G)$, by definition. Thus, since these strata do not intersect the sphere boundary of the source in (3), the restriction of G to such a stratum is proper.

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We refer to [JT1] for the relation between "tame" and the Thom regularity, for more other details, and for several examples illustrating the notions that we have seen here.

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