Abstract

We show how the polar degree of an arbitrarily singular projective hypersurface can be decomposed as a sum of non-negative numbers which represent local vanishing cycles.

Polar degree of projective hypersurfaces - Part 4 -

Mihai Tibăr

Polar degree and vanishing cycles

Lat us remember from Part 1 the statement of Huh's breakthrough result in [Huh, Theorem 2 and its Proof]:

Theorem (Huh, 2013)

Let $V \subset \mathbb{P}^n$ be a hypersurface with isolated singularities. For any general hyperplane $\mathcal{H}_p$ passing through some singular point $p \in \mathrm{Sing}(V)$, such that V is not a cone of apex p, one has:

$$\mathrm{pol}(V) = \mu_p^{\langle n-2 \rangle}(V) + \mathrm{rank}\ H_n(\mathbb{P}^n \setminus V, (\mathbb{P}^n \setminus V) \cap \mathcal{H}_p) \quad (1)$$

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The first term is the local invariant $\mu_p^{\langle n-2\rangle}(V) \geq 0$ counting a certain type of vanishing cycles, but the second term rank $H_n(\mathbb{P}^n \setminus V, (\mathbb{P}^n \setminus V) \cap \mathcal{H}_p)$ is a global invariant.

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There is the following *challenge*:

Express $\mathrm{pol}(V)$ as a sum of non-negative local invariants.

Admissible hyperplanes

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is some non-constant homogeneous polynomial function, and $V := \{f = 0\} \subset \mathbb{P}^n$ be endowed with a Whitney stratification $\mathcal{W}$. Let also $\hat{\ell} : \mathbb{C}^{n+1} \to \mathbb{C}$ be a linear function defining a hyperplane $H \in \mathbb{C}^{n+1}$ and let $\mathcal{H} \subset \mathbb{P}^n$ denote its corresponding projective hyperplane.

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Definition

We say that the affine hyperplane $H \subset \mathbb{C}^{n+1}$ through 0 (or that the projective hyperplane $\mathcal{H} \subset \mathbb{P}^n$) is *admissible for f* if:

(i) $\mathcal{H}$ is transversal to all strata of $\mathcal{W}$ except at finitely many points.

(ii) the polar locus $\Gamma(\hat{\ell}, f) \subset \mathbb{C}^{n+1}$ is either of dimension 1, or it is empty.

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By a linear change of coordinates, we may and will assume that $p = [1; 0; \cdots ; 0]$.

Let us remind the definition of the polar locus:

$$\Gamma(\hat{\ell}, f) := \overline{\left\{ x \in \mathbb{C}^{n+1} \mid \mathrm{rank} \begin{bmatrix} \frac{\partial f}{\partial x_0}(x) & \frac{\partial f}{\partial x_1}(x) & \cdots & \frac{\partial f}{\partial x_n}(x) \\ 0 & a_1 & \cdots & a_n \end{bmatrix} < 2 \right\} \setminus \{f = 0\}} \tag{2}$$

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Remark on generic hyperplanes

The set of **admissible hyperplanes** contains by definition the set of *generic hyperplanes* $\mathcal{H}$ *relative to* V, namely hyperplanes which are transversal to all strata of the stratification $\mathbb{P}\mathcal{W}$ of V, since in this case:

- the non-transversality locus is empty, thus condition (i) is fulfilled,
- the polar locus $\Gamma(\hat{\ell}, f)$ is 1-dimensional or empty by the Generic Polar Curve Lemma, thus condition (ii) is fulfilled too.

And we remind (from Part 1 of the lecture) that in this case we have the equality:

$$\mathrm{mult}_0 \Gamma(\hat{\ell}, f) = \mathrm{pol}(V).$$

Here we do not work with transversal hyperplanes, but with admissible ones. For them, we have the following fundamental result telling that the hyperplanes admissible for f at some singular point p, even if they are non-generic, they have a genericity property:

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Theorem (Constrained polar curve theorem, [Siersma-Tibăr])

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$, $n \geq 2$, be a homogeneous polynomial with $\dim \operatorname{Sing} f > 0$. Let $p \in \operatorname{Sing} V$ such that $V := \{f = 0\} \subset \mathbb{P}^n$ is not a cone of apex p.
Then there is a Zariski open dense subset $\hat{\Omega}_p$ of the set of hyperplanes through p such that the polar locus $\Gamma(\hat{\ell}, f) \subset \mathbb{C}^{n+1}$ is either a curve for all $\hat{\ell} \in \hat{\Omega}_p$, or it is empty for all $\hat{\ell} \in \hat{\Omega}_p$.

The local Milnor-Lê number $\alpha_p(V, \mathcal{H})$.

In some affine chart $\mathbb{C}^n \subset \mathbb{P}^n$ containing $p \in V$, let us consider a linear function $\ell : \mathbb{C}^n \to \mathbb{C}$ such that $\ell(p) = 0$, and let $H_s := \{\ell = s\}$ for $s \in \mathbb{C}$, where $H_0 := H$. Let $\mathcal{H} \in \mathbb{P}^n$ be the projective closure of H. It contains the point p.

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We assume that $\mathcal{H}$ is transversal to all the strata of the stratification $\mathcal{W}$ of V in the neighbourhood of p, except at the point p itself. This is equivalent to saying that the restriction of the function ℓ to some small neighbourhood B_ε of p in $\mathbb{P}^n$ has a stratified isolated singularity at p with respect to $\mathcal{W}$. Consequently, it's local Milnor-Lê fibre $B_\varepsilon \cap (V \cap H_s)$, for some s close enough to 0, has the homotopy type of a bouquet of spheres[1] of dimension $n - 2$.

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If $\mathcal{H}_{\rm gen}$ is a general hyperplane through p, then $\alpha_p(V, \mathcal{H}_{\rm gen})$ is the Milnor number of the *complex link* of V at p. We will denote it by $\alpha_p(V)$.

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By its definition, the integer $\alpha_p(V, \mathcal{H})$ is non-negative. It depends only on the *reduced* structure of V at p, and on the chosen hyperplane $\mathcal{H}$.

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Let $f_p = 0$ be a local equation of the reduced hypersurface germ (V, p). Then $\alpha_p(V, \mathcal{H})$ equals the *polar multiplicity of* f_p *with respect to* ℓ *at* p, namely:

$$\alpha_p(V, \mathcal{H}) = \mathrm{mult}_p(H_0, \Gamma(\ell, f_p)), \tag{3}$$

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This polar intersection multiplicity might be higher than the *generic polar number* $\mathrm{mult}_p \Gamma(\ell_{\mathrm{gen}}, f_p) = \alpha_p(V)$.

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Exercice. Prove that $\alpha_p(V) = 0$ except of finitely many points $p \in V$, for any projective hypersurface V.

Lemma

Let $\mathcal{H}$ be admissible for f. Then $\alpha_p(V, \mathcal{H}) > 0 \implies V \not\pitchfork_p \mathcal{H}$.

Proof.

Our claim is equivalent to the following:
$V \pitchfork_p \mathcal{H} \implies \Gamma_p(\ell, f_p) = \emptyset$ and $V \not\pitchfork_p \mathcal{H} \implies \dim_p \Gamma_p(\ell, f_p) \leq 1$.
If p is a point of stratified transversal intersection $V \pitchfork_p \mathcal{H}$, then the polar locus $\Gamma_p(\ell, f_p)$ is empty as a direct consequence of its definition.
Let now $V \not\pitchfork_p \mathcal{H}$. Since p is an isolated non-transversality, the polar locus $\Gamma_p(\ell, f_p)$ intersects $B_p \cap V$ at most at p, for some small enough ball B_p centred at p. Thus $\dim \Gamma_p(\ell, f_p) \leq 1$. □

The above lemma shows in particular that $\alpha(V, \mathcal{H})$ is a well-defined non-negative integer.

Definition

Let $\mathcal{H}$ be an admissible hyperplane for V. We define:

$$\alpha(V, \mathcal{H}) := \sum_{p \in V \cap \mathcal{H}} \alpha_p(V, \mathcal{H}). \tag{4}$$

Local vanishing cycles of a polynomial $\mathbb{C}^n \to \mathbb{C}$

By some linear change of coordinates, one may assume that the admissible hyperplane $\mathcal{H}$ has equation $x_n = 0$. We consider it as the hyperplane at infinity for the coordinate system on $\mathbb{C}^n = \mathbb{P}^n \setminus \mathcal{H}$.
We then consider the polynomial:

$$P_{\mathcal{H}} : \mathbb{C}^n \to \mathbb{C}, \quad P_{\mathcal{H}}(x_0, \ldots, x_{n-1}) := f(x_0, \ldots, x_{n-1}, 1).$$

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Exercice. Show that if $\mathcal{H}$ is admissible then $\deg P_{\mathcal{H}} = \deg f = d$.

Let $\mathbb{X} := \{f(x_0, \ldots, x_n) - t x_n^d = 0\} \subset \mathbb{P}^n \times \mathbb{C}$. Let

$$\tau : \mathbb{X} \to \mathbb{C}$$

be the projection on the second factor, and let us denote by $\mathbb{X}_t := \tau^{-1}(t)$ its fibres. The set $\mathbb{X}$ is precisely the closure in $\mathbb{P}^n \times \mathbb{C}$ of the graph of $P_{\mathcal{H}}$ and

$$\mathbb{X}^\infty := \mathbb{X} \cap (\mathcal{H} \times \mathbb{C}) = (V \cap \mathcal{H}) \times \mathbb{C}$$

is the divisor at infinity.

The non-isolated singular locus of the fibre $\mathbb{X}_0 = V$ (if there is any) intersects the hyperplane at infinity $\mathcal{H}$. We are interested in another type of singularities, the so-called *singularities at infinity* of the fibres $\mathbb{X}_t$ for $t \neq 0$.

Definition (Partial Thom stratification at infinity, see e.g. [Ti3])

A locally finite stratification of $\mathbb{X}^\infty$ such that each stratum is Thom (a_{x_n})-regular with respect to the smooth stratum $\mathbb{X} \setminus \mathbb{X}^\infty$ is called a *∂-Thom stratification at infinity*. This is independent on the affine chart.

Definition (t-singularities at infinity, [Ti3])

Let $\mathcal{G}$ be a ∂-Thom stratification at infinity of $\mathbb{X}$, and let $\eta \in \mathbb{X}^\infty$. If the map $\tau : \mathbb{X} \to \mathbb{C}$ is transversal to the stratification $\mathcal{G}$ at η then we say that $P_{\mathcal{H}}$ is *t-regular at infinity* at this point. Otherwise we say that $P_{\mathcal{H}}$ has a *t-singularity at infinity* at η.

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We say that η is an *isolated t-singularity* at infinity of $P_\mathcal{H}$ if the map $\tau : \mathbb{X} \to \mathbb{C}$ has an isolated non-transversality at η with respect to $\mathcal{G}$, and if moreover the map τ has no other singularity on $\mathbb{X} \setminus \mathbb{X}^\infty$ in the neighbourhood of η.

Theorem ([Siersma-Tibăr])

Let $V := \{f = 0\} \subset \mathbb{P}^n$. *If the hyperplane* $\mathcal{H} = \{x_n = 0\}$ *is admissible for* f, *then the polynomial* $P_{\mathcal{H}}$ *has, outside* $\overline{P_{\mathcal{H}}^{-1}(0)}$, *only isolated t-singularities and only isolated affine singularities. The set of these singular points is finite.*

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Let

$$\beta^{\mathrm{aff}}(V, \mathcal{H}) := \sum_{v \in (\operatorname{Sing} P_{\mathcal{H}}) \setminus V} \mu_v(P_{\mathcal{H}})$$

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At each point $(q, t) \in \mathbb{X}^{\infty}$ which is an isolated t-singularity at infinity, one may define a number of "vanishing cycles at infinity" $\lambda(q, t)$ (see [Ti3]). Let then

$$\beta^{\infty}(V, \mathcal{H}) := \sum_{t \neq 0, q \in V \cap \mathcal{H}} \lambda(q, t)$$

denote the sum of the numbers of vanishing cycles at infinity $\lambda(q, t)$ of the isolated t-singularities at infinity outside the fibre $\mathbb{X}_0 = V$.

We may then define the non-negative finite integer :

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The quantisation theorem for the polar degree may be stated as follows:

Theorem ([Siersma-Tibăr])

Let $V := \{f = 0\} \subset \mathbb{P}^n$ be a projective hypersurface and let $\mathcal{H}$ be an admissible hyperplane for V. Then:

$$\mathrm{pol}(V) = \alpha(V,\mathcal{H}) + \beta(V,\mathcal{H}). \tag{6}$$

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This formula is a sum of non-negative numbers representing local vanishing cycles. It yields lower bounds for the polar degree, actually any of the terms is a lower bound. This recovers in particular Huh's lower bound.

Proof of the theorem

As a general fact, the bifurcation set $\mathcal{B}_P \subset \mathbb{C}$ of any polynomial function $P : \mathbb{C}^n \to \mathbb{C}$ is finite. Let then $D_0 \subset \mathbb{C}$ be some disk such that $\mathcal{B}_{P_{\mathcal{H}}} \cap D_0 = \{0\}$. By using that $P_{\mathcal{H}}$ has isolated singularities outside the fibre over 0, including at infinity, it follows that the relative homology $H_*(\mathbb{C}^n, P_{\mathcal{H}}^{-1}(D_0))$ is concentrated in dimension n. Moreover:
the top Betti number $b_{n-1}(P_{\mathcal{H}}^{-1}(D_0)) = b_n(\mathbb{C}^n, P_{\mathcal{H}}^{-1}(D_0))$ is precisely $\beta(V, \mathcal{H})$.

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Since the tube $P_{\mathcal{H}}^{-1}(D_0)$ and the fibre $P_{\mathcal{H}}^{-1}(0)$ have the same Euler characteristic, we have:

$$(-1)^n \mathrm{rank}\ H_n(\mathbb{C}^n, P_{\mathcal{H}}^{-1}(D_0)) = \chi(\mathbb{C}^n, P_{\mathcal{H}}^{-1}(D_0)) = 1 - \chi(P_{\mathcal{H}}^{-1}(D_0)) = 1 - \chi(V \setminus \mathcal{H}).$$

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Exercice. Why $\chi(P_{\mathcal{H}}^{-1}(D_0)) = \chi(P_{\mathcal{H}}^{-1}(0))$?

Consider the germ of a pencil $\mathcal{P}_\delta$ of hyperplanes of $\mathbb{P}^n$ which contains our admissible hyperplane $\mathcal{H}$, parametrised by an arbitrarily small disk $\delta \subset \mathbb{C} \subset \mathbb{P}^1$ centred at 0, where $\pi : \mathcal{P}_\delta \backslash A \to \delta$ is the projection to the parameter, such that $\pi(\mathcal{H}) = 0$.

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We require that $\mathcal{P}_\delta$ is generic with respect to V, in the sense that the base locus A of this pencil $\mathcal{P}_\delta$ (which is of dimension $n-2$) is transversal to the Whitney stratification $\mathcal{W}$ of $V \subset \mathbb{P}^n$, and more precisely transversal to the induced stratification $\mathcal{W}_H$ on the slice $V \cap \mathcal{H}$. The choice of the axis A covers a Zariski-open subset of all hyperplane slices of $V \cap \mathcal{H}$.

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The general member $\mathcal{H}_{\mathrm{gen}}$ of this pencil germ is a general hyperplane with respect to V. By definition $\boxed{4}$ of the polar degree, we therefore have:

$$(-1)^n \operatorname{pol}(V) = 1 - \chi(V \setminus \mathcal{H}_{\mathrm{gen}}).$$

Taking the difference, we obtain:

$$
\begin{aligned}
(-1)^n[\mathrm{pol}(V) - \mathrm{rank}\ H_n(\mathbb{C}^n, P_{\mathcal{H}}^{-1}(D_0))] = \\
= \chi(V \setminus \mathcal{H}) - \chi(V \setminus \mathcal{H}_{\mathrm{gen}}) = \chi(V \cap \mathcal{H}_{\mathrm{gen}}) - \chi(V \cap \mathcal{H}).
\end{aligned}
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Since the axis A of the pencil $\mathcal{P}_\delta$ is stratified-transversal to $\mathcal{W}$ and the stratified singularities of the pencil $\mathcal{P}_\delta$ outside A are precisely the set of points of non-transversality $\mathrm{Sing}_{\mathcal{W}}(V \cap \mathcal{H})$, it follows that the variation of the topology of the pencil $\mathcal{P}_\delta$ at its fibre $\mathcal{H}$ is localisable, by excision, at the points $q \in \mathrm{Sing}_{\mathcal{W}}(V \cap \mathcal{H})$.

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In homology, this variation is concentrated in dimension $n-1$, and its contribution is the number $\alpha_q(V, \mathcal{H})$ defined at (6).

$$\chi(V \cap \mathcal{H}_{\text{gen}}) - \chi(V \cap \mathcal{H}) = - \sum_{q \in \mathrm{Sing}_{\mathcal{W}}(V \cap \mathcal{H})} \chi(B_q \cap V \cap \mathcal{P}_\delta, B_q \cap V \cap \mathcal{H}_{\text{gen}})$$

$$= (-1)^n \sum_{q \in \mathrm{Sing}_{\mathcal{W}}(V \cap \mathcal{H})} \alpha_q(V, \mathcal{H})$$

for some small enough balls B_q at $q \in V \cap \mathcal{H}$

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From this we obtain:

$$\beta(V, \mathcal{H}) = \text{rank } H_n(\mathbb{C}^n, P_{\mathcal{H}}^{-1}(D_0)) = \mathrm{pol}(V) - \alpha(V, \mathcal{H})$$

which ends the proof of our formula.

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