Polar degree of projective hypersurfaces
- Part 1 -

Mihai Tibăr

Equivalent definitions, and a couple of conjectures

Polar degree and topology

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This has been proved by Dimca and Papadima [**?**, 2003].

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$\boxed{2.}$ $\mathrm{pol}(V) = \mathrm{mult}_0 \Gamma(\ell, f)$, the multiplicity of the polar locus of the map $(l, f) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^2, 0)$.

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The polar curve $\Gamma(\ell, f)$ is here a homogeneous subset of $\mathbb{C}^{n+1} \Rightarrow$ union of lines, thus $\mathrm{mult}_0\Gamma(\ell, f) =$ the number of these lines. This is also the number of points $x \in V \cap \{l \neq 0\}$ such that $\mathrm{grad} f(x) \in \mathbb{P}^n$ coincides with $\ell \in \mathbb{P}^n$, and this number is equal to $\#(\mathrm{grad} f)^{-1}(\ell)$.

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Exercice. Explain the role of the genericity of ℓ in the definition 2.

$\boxed{3.}$ $\mathrm{pol}(V) = \mathrm{rank}\ H_{n-1}(\mathbb{C}\mathrm{lk}_{\{f=0\}}(\{0\}))$
where $\mathbb{C}\mathrm{lk}_{\{f=0\}}(\{0\})$ denotes the *complex link*[1] of the stratum $\{0\}$ of the hypersurface $\{f = 0\} \subset \mathbb{C}^{n+1}$. This is the local Milnor fibre of the function $\ell_| : (\{f = 0\}, 0) \to (\mathbb{C}, 0)$. See next page $\longrightarrow$

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But already the definition $\boxed{3.}$ shows the same thing.

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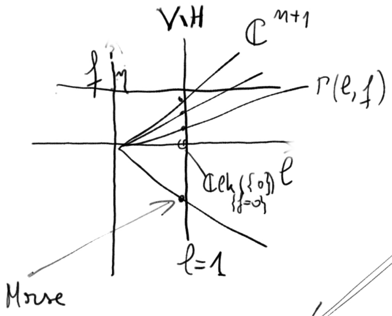

l is general.

$\mathbb{C}^{n+1} \supset \{l=1\} \simeq \mathbb{C}^n$ contractible

$\{l=1\} \overset{ht}{\simeq} \mathbb{C}lk_{\{f=0\}}(\{0\}) \cup$ n-cells

Morse singular points of the function f on $\{l=1\} \simeq \mathbb{C}^n$.

$\mathbb{C}lk_{\{f=0\}}(\{0\}) \overset{ht}{\simeq} \vee S^{n-1}$ by the Bouquet theorem

If V has isolated singularities only:

Morse point $\longleftrightarrow$ line of $\Gamma(l,f)$

$\{l=1\} \overset{ht}{\simeq} \underbrace{(\{f=\eta\} \cap \{l=1\})} \cup$ n-cells $\longrightarrow$ singular points on V

$\{f=\eta\} \cap \{l=0\}$ = Milnor fibre of f on $\{l=0\} \simeq \mathbb{C}^n$.

$f|_{l=0} : x_1^d + \dots + x_n^d \longrightarrow \mu = (d-1)^n$

Hypersurfaces with at most isolated singularities

Dimca and Papadima proved the formula:

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The smooth hypersurface $V_{n,d}$ defined by the equation

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realises the *maximum* polar number $\mathrm{pol}(V_{n,d}) = (d-1)^n$ for fixed n, in this category of "hypersurfaces with at most isolated singularities".

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What are the hypersurfaces with $\mathrm{pol}(V) = 1$, called "homaloidal"?
The above formula shows in particular that the smooth quadratic hypersurface $V_{n,2}$ is the only smooth V which is homaloidal.

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Proposition (Dimca and Papadima, 2003)

The relative homology $H_*(D(f), D(f) \cap H)$ is concentrated in dimension n, and

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$\rightarrow$ Yet this does not help for bounding $\mathrm{pol}(V)$ from below.

Huh's extension

Let V have isolated singularities. Let $\mu_p^{\langle n-2 \rangle}$ be the Milnor number of a hyperplane section[2] $H_p \cap V$ through p.

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If V is not a cone of vertex p, then:

Theorem (Huh, 2014)

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This extends the preceding Dimca-Papadima result to hyperplanes H_p passing through a singular point $p \in \text{Sing } V$.

The proof relies on *the theory of slicing by pencils with singularities in the axis* from [**?**, **?**, **?**].

* If V is a cone of vertex p then pol $V = 0$.

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Theorem (Huh)

A projective hypersurface $V \subset \mathbb{P}^n$ with only isolated singularities and $\mathrm{pol}(V) = 1$ is one of the following, after a linear change of homogeneous coordinates:

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(iii) $(n = 2, d = 3)$ *the union of a smooth conic and one of its tangents:*

$$f = x_0(x_1^2 + x_0 x_2) = 0, \qquad (A_3).$$

This list contains the homaloidal plane curves found by Dolgachev.

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