Milnor fibrations, part 2

Mihai Tibăr

The bouquet theorem in the singular setting

We present here the third proof of the bouquet theorem, which is based on polar curves and monodromy.
Let $(X, 0)$ denote the germ of a singular complex analytic space, embedded in $(\mathbb{C}^N, 0)$ for some $N > 0$, with $\dim(X, 0) = n$, $n \geq 2$. The existence of Thom-Whitney stratifications on singular analytic spaces allows one to prove local fibration results where the fibres are no more smooth. Historically, the first such result seems to have been the following:

Theorem (Lê D.T.)

Let $g : (X, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ. For any sufficiently small radius $\varepsilon > 0$, and any $0 < \delta \ll \varepsilon$, the restriction:

$$g_| : B_\varepsilon \cap g^{-1}(D_\delta \setminus \{0\}) \to D_\delta \setminus \{0\} \tag{1}$$

is a locally trivial stratified C^0-fibration. The isotopy type of the fibration does not depend on the choice of the radii ε and δ.

The above fibration is called *Milnor-Lê (tube) fibration*, and its fibre is the *Milnor-Lê fibre*.

We shall explain the structure of the *Milnor-Lê fibre* F_g in case g defines an isolated singularity in the stratified sense.

Let $(X, 0)$, with $\dim(X, 0) = n$, $n \geq 2$, be a reduced, irreducible complex analytic germ. We fix some Whitney stratification $\mathcal{S} := \{\mathcal{S}_i\}_{i \in R}$ on $(X, 0)$, and assume that $\{0\}$ is a point-stratum, denoted by $\mathcal{S}_0$.

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One says that g has an *isolated singularity* with respect to the stratification $\mathcal{S}$ iff $\mathrm{Sing}_{\mathcal{S}}\, g = \{0\}$.

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Let $\mathcal{N}$ be a linear normal slice to S_i at $x \in S_i$ (i.e. $\mathcal{N} \subset \mathbb{C}^N$ is the germ of a manifold transversal to S_i and $\mathcal{N} \cap S_i = \{x\}$). For a general linear form $l : \mathcal{N} \to \mathbb{C}$, the restriction $l_{|X\cap\mathcal{N}}$ has an isolated stratified critical point at x, and it has a local Milnor-Lê fibration. One calls *complex link of the stratum S* the Milnor-Lê fibre of the function germ $l_{|X\cap\mathcal{N}}$ at x, namely:

$$\mathbb{C}\mathrm{lk}_X(S) := X \cap \mathcal{N} \cap B_{\varepsilon'}(x) \cap \{l = u\} \quad \text{for} \quad 0 < |u| \ll \varepsilon' \ll 1.$$

Let $S^k(Y)$ denote the k-times repeated suspension of some space Y, where k the complex dimension of Y. By convention, the suspension of the empty set is S^0, the 0-sphere. Let also define $S^0(Y) := Y$.

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Theorem (T, 1994)

Let $g : (X, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with a stratified isolated singularity on a complex analytic germ $(X, 0)$ without 1-dimensional components. Then the Milnor fibre F_g is homotopy equivalent to a bouquet, namely:

$$F_g \overset{\mathrm{ht}}{\simeq} \bigvee_{i \in R} \bigvee_{\# M_i \,\mathrm{times}} S^{k_i}(\mathbb{C}\mathrm{lk}_X(\mathcal{S}_i))$$

where the last wedge is taken $\# M_i$ times, for some integer $\# M_i \geq 0$ which depends on the stratum and will be defined during the proof, with the convention that $\# M_0 = 1$ for the zero-dimensional stratum $\mathcal{S}_0$.

The above theorem excepts the case of curve components of the space germ $(X, 0)$ from the hypotheses because these 1-dimensional components just contribute to the Milnor fibre by discrete points[1]. In particular, the Milnor fibre would not be connected in such a case.

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The complex link of the maximal dimensional stratum is empty, the suspension of the empty set is the 0-sphere, and so its n-th suspension is S^{n-1}. Therefore the maximal dimensional stratum contribution in the Milnor fibre F_g is by a bouquet of spheres S^{n-1}. In particular if $X\setminus\{0\}$ is nonsingular, like in the cases (a) and (e) of the following Corollary, we get this contribution $\vee S^{n-1}$.

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Several remarkable classical results which become particular cases of the above theorem.

Corollary

(a) J. Milnor: *if* $(X,0) = (\mathbb{C}^n, 0)$, *then* $F_g \overset{\mathrm{ht}}{\simeq} \vee_\mu S^{n-1}$.

(b) H. Hamm: *if* $(X,0)$ *is an isolated complete intersection singularity (abbreviated ICIS), then* $F_g \overset{\mathrm{ht}}{\simeq} \vee_\mu S^{n-1}$.

(c) Lê D.T. : *if* $(X,0)$ *is a complete intersection, then* $F_g \overset{\mathrm{ht}}{\simeq} \vee_\mu S^{n-1}$.

(d) Lê D.T.: *if* $(X,0)$ *is an equidimensional analytic germ with* $\mathrm{rhd}\,(X) = \dim_0 X = n$, *then* $F_g \overset{\mathrm{ht}}{\simeq} \vee_\mu S^{n-1}$, *where* $\mathrm{rhd}\,(X)$ *is the rectified homotopical depth.*
The same statement holds for the rectified homological depth $\mathrm{rHd}(X;\mathbb{Q})$ *instead of* rhd[a].

(e) D. Siersm *If* $(X,0)$ *is isolated (i.e.* $X \setminus \{0\}$ *is nonsingular) and* $\dim_0 X \neq 3$, *then* $F_g \overset{\mathrm{ht}}{\simeq} \mathbb{C}\mathrm{lk}_X(\{0\}) \vee \bigvee S^{n-1}$.

[a] idem.

Proof.

(a). The minimal Whitney stratification is trivial (since X is smooth), hence the complex link of the unique stratum is empty, and we can have a certain number of spheres $S^{n-1} = S^n(\emptyset)$ in the wedge.
(b). X has only two strata: $\{0\}$ and $X \setminus \{0\}$. The complex link $\mathbb{C}\mathrm{lk}_X(\{0\})$ turns out to have the homotopy type[a] of a bouquet of spheres S^{n-1}. Besides that, the wedge may also contain spheres $S^{n-1} = S^n(\emptyset)$ from the smooth stratum $X \setminus \{0\}$.
Hamm also proved that a map germ $G : (\mathbb{C}^{n+p-1}, 0) \to (\mathbb{C}^p, 0)$ defining an ICIS has a fibration and its Milnor fibre is homotopy equivalent to a bouquet of spheres S^{n-1}. This result can be reduced to the above by a coordinate change in $\mathbb{C}^p$ such that $X := Z(f_1, \ldots, f_{p-1})$ and $f_p : (X, 0) \to (\mathbb{C}, 0)$ has an isolated singularity.

(d). Lê proved that $\mathrm{rhd}\,(X) = \dim_0 X$ is equivalent to $\mathbb{C}\mathrm{lk}_X(\mathcal{S}_i) \overset{\mathrm{ht}}{\simeq} \vee S^{\mathrm{codim}_{\mathbb{X}} \mathcal{S}_i - 1}$, $\forall i \in R$.
(c). Lê also proved that a complete intersection $(X, 0)$ satisfies $\mathrm{rhd}\,(X) = \dim_0 X$, thus (c) follows from (d).
(e). Like the proof of (b), but here the complex link of the singular stratum $\{0\}$ may not have a particular structure, so this complex link is part of the wedge formula.

□

[a] Exercise. Prove this by induction.

The proof of Theorem 3 will follow from the next handlebody statement.

Theorem

The Milnor fibre F_g is obtained from the complex link $\mathbb{C}\mathrm{lk}_X(\{0\})$ to which one attaches thimbles over local Milnor fibres of stratified Morse singularities, such that the image in $\mathbb{C}\mathrm{lk}_X(\{0\})$ of each such attaching map is contractible within $\mathbb{C}\mathrm{lk}_X(\{0\})$.

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In order to obtain the control over the attaching, one uses a special geometric monodromy which has been introduced by Lê D.T. under the name *carrousel monodromy*. We introduce it below.

We use the following fundamental result of Bertini-Sard type which goes back to Hamm-Lê, Kleiman and Teissier; we send to [Ti2, Theorem 7.1.2] for proofs and for more ample discussions:

Lemma

[Polar Curve Lemma]
There is a Zariski open dense subset $\Omega' := \Omega'_g \subset \check{\mathbb{P}}^{N-1}$ *such that* $\Gamma(l, g)$ *is either a curve germ for all* $l \in \Omega'$*, or is empty for all* $l \in \Omega'$*.*

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The linear forms $l \in \Omega$ have the property that the hyperplane $\{l = 0\}$ is transverse to all strata of $X \setminus \{0\}$, and thus to all strata of $g^{-1}(0) \setminus \{0\}$, in some neighbourhood of 0. The strata of dimension 1 in $\mathcal{S}$ are, by definition, components of the polar curve.

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Saying that l is *generic* with respect to g relatively to the stratification $\mathcal{S}$, means $l \in \Omega$ as in the above Lemma.

The polar neighbourhood

If $l \in \Omega$, then $\Gamma(l, g)$ intersects the fibre $(l, g)^{-1}(0, 0)$ at the origin only. In turn, this implies that the map germ (l, g) is open at the origin of the target, and that there exists a fibration outside the *discriminant*:

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$$\Delta := (l, g)(\Gamma(l, g) \cup \mathrm{Sing}_{\mathcal{S}}\, g), \tag{2}$$

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in the following sense:

Let B_ε denote a Milnor ball for g, that is the intersection of a small enough ball at the origin of the ambient space with a suitable representative of the germ $(X, 0)$. As shown already by Lê, one can use a "box neighbourhood"
$B := B_\varepsilon \cap l^{-1}(D) \cap g^{-1}(D')$ and the map $(l, g) : B \to \mathbb{C}^2$ in order to describe the local Milnor fibration of g and its relation to the Milnor fibration of the slice $g_{|l=0}$.
Here follows the detailed setting.

Let $l \in \Omega_g$. There exist small enough radii $0 < r' \ll r \ll \varepsilon$ such that $\Delta(l, g) \cap \partial \overline{D_r} \times D_{r'} = \emptyset$, and such that the map $(l, g) : B \to D_r \times D_{r'}$, with $B := B_\varepsilon \cap l^{-1}(D_r) \cap g^{-1}(D_{r'})$, restricts to a locally trivial fibration:

$$(l, g)_| : B \setminus (l, g)^{-1}(\Delta) \to (D_r \times D_{r'}) \setminus \Delta. \tag{3}$$

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and moreover, that g induces a locally trivial topological fibration

$$g_| : B \cap g^{-1}(D_{r'} \setminus \{0\}) \to D_{r'} \setminus \{0\}$$

which is homeomorphic to the Milnor fibration of g, and a locally trivial topological fibration

$$g_| : B \cap g^{-1}(D_{r'} \setminus \{0\}) \cap \{l = 0\} \to D_{r'} \setminus \{0\},$$

which is homeomorphic to the Milnor fibration of $g_{|\{l=0\}}$.

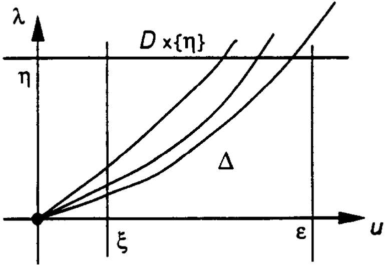
Symmetrically, the following restriction of (3):

$$l_| : B \cap l^{-1}(D_r \setminus \{0\}) \to D_r \setminus \{0\} \quad (4)$$

is homeomorphic to the Milnor fibration of l, which in particular implies the following homeomorphism:

$$F_l \overset{\text{homeo}}{\simeq} B \cap l^{-1}(\xi) \quad (5)$$

for any small enough $|\xi| > 0$.



The carrousel disk

Step 1. Let $S_{r'} := \partial D_{r'}$. We construct a special vector field on $D_r \times S_{r'}$. Namely, there exists an integrable smooth vector field on $D_r \times S_{r'}$ which is a lift of the unit tangent vector field on the circle $S_{r'}$ by the projection

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and such that it is tangent to the circles $\Delta(l, g) \cap (D_r \times S_{r'})$. In addition, one may impose to this vector field to be the unit vector field on the circles $\{0\} \times S_{r'}$ and $\{p\} \times S_{r'}$, $\forall p \in \partial \overline{D_r}$.

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Let us fix some $\eta \in S_{r'}$. The integration of the vector field on $D_r \times S_{r'}$ produces a homeomorphism

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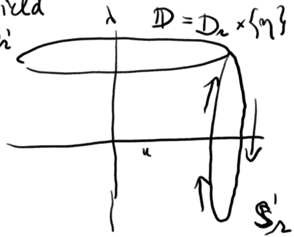
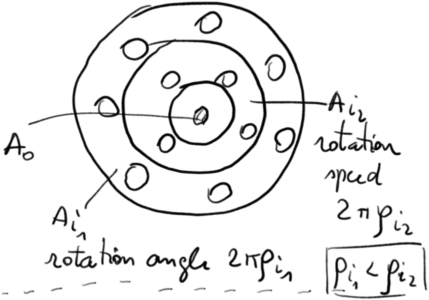
Some point a of the *carrousel disk* $D_r \times \{\eta\}$ has a trajectory inside $D_r \times S_{r'}$ such that, after one turn around $S_{r'}$, it arrives at the point $a' := \mathrm{h}(a) \in D_r \times \{\eta\}$.

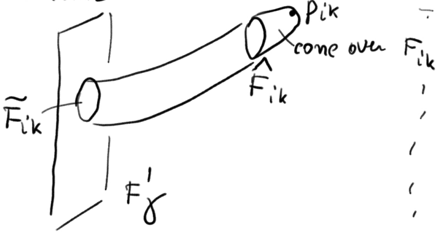
Vector field on $D_\lambda \times D_u$



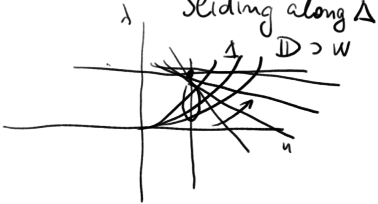
The carousel disk $\mathbb{D}$



Thimble



Sliding along Δ



Step 2. The vector field that we have constructed at Step 1 will be lifted through the map (l, g) to a controlled continuous, more precisely "rugueux" vector field (in Verdier's terminology [Verdier]) on $X \setminus \{0\}$ which lifts our carrousel vector field.

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By the Thom's Second Isotopy Lemma (see e.g. Mather, [GLPW], [Verdier, Theorem 4.14]), one may integrate the latter vector field and obtain a characteristic homeomorphism of the fibration induced by g over $S_{r'}$, and thus a geometric monodromy h_g of the Milnor fibre F_g of g. This is called the *carrousel monodromy*.

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Puiseux expansions of the discriminant $\Delta(l, g)$, and the decomposition of the carrousel disk[2]

The discriminant $\Delta(l, g)$ is a plane curve. By construction, the centre $(0, \eta)$ of the carrousel disk is fixed, and the circle $\partial\overline{D_r} \times \{\eta\}$ is pointwisely fixed too. But each point $a \in \Delta(l, g) \cap D_r \times \{\eta\}$ is moved by the carrousel h around $(0, \eta)$. Its new position h(a) is called *carrousel motion* and depends on the Puiseux parametrizations of the branches of Δ. These Puiseux expansions determine also the motion of the points in the carrousel which are in the neighbourhood of $\Delta(l, g) \cap D_r \times \{\eta\}$.

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The carrousel construction works for holomorphic function germs $g : (X, 0) \to (\mathbb{C}, 0)$, with any singular locus. In case $\mathrm{Sing}_{\mathcal{S}}\, g$ is a positive dimensional set, it is mapped by (l, g) to the u-axis, and in this case we denote by Δ_0 the irreducible component of Δ which coincides to the u-axis. We this write $\Delta'(l, g)$ for the union of the branches of Δ which are not contained in the u-axis.

[2]For more details one may consult [Ti1].

Let $\Delta' = \cup_{i\in I}\Delta_i$, where $I := \{1, \dots, r\}$, be the decomposition into irreducible components. Since the polar curve $\Gamma(l, g)$ projects one-to-one to Δ', this yields a one-to-one correspondence among the components of $\Gamma(l, g) = \cup_{i\in I}\Gamma_i$ and those of Δ'. For any $i \in I$, we then consider a Puiseux parametrisation $(u(t), \lambda(t))$ of Δ_i, namely:

$$u = \sum_{j\geq m_i} c_{i,j} t^j, \quad \lambda = t^{n_i},$$

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where

$$\begin{aligned} m_i &:= \mathrm{mult}_0 \Delta_i \ = \ \mathrm{mult}_0 \Gamma_i \\ n_i &:= \mathrm{mult}_0(\Delta_i, \{\lambda = 0\}) \ = \ \mathrm{mult}_0(\Gamma_i, \{g = 0\}). \end{aligned}$$

where the multiplicity mult_0 is defined in each ambient space, $\mathbb{C}^2$ or $\mathbb{C}^N$, respectively.

Let $\Delta' = \cup_{i \in I} \Delta_i$, where $I := \{1, \ldots, r\}$, be the decomposition into irreducible components. Since the polar curve $\Gamma(l, g)$ projects one-to-one to Δ', this yields a one-to-one correspondence among the components of $\Gamma(l, g) = \cup_{i \in I} \Gamma_i$ and those of Δ'. For any $i \in I$, we then consider a Puiseux parametrisation $(u(t), \lambda(t))$ of Δ_i, namely:

$$u = \sum_{j \geq m_i} c_{i,j} t^j, \quad \lambda = t^{n_i},$$

where

$$\begin{aligned} m_i &:= \mathrm{mult}_0 \Delta_i \;=\; \mathrm{mult}_0 \Gamma_i \\ n_i &:= \mathrm{mult}_0(\Delta_i, \{\lambda = 0\}) \;=\; \mathrm{mult}_0(\Gamma_i, \{g = 0\}). \end{aligned}$$

where the multiplicity mult_0 is defined in each ambient space, $\mathbb{C}^2$ or $\mathbb{C}^N$, respectively.

Denote by $(m_{i,k}, n_{i,k})$, for $1 \leq k \leq g_i$ the k^{th} Puiseux pair of Δ_i, where g_i is the *genus* of the iterated toric knot which is the link of Δ_i.

Denote by $\rho_i := \frac{m_{i,1}}{n_{i,1}} = \frac{m_i}{n_i}$ the *Puiseux ratios* and notice that $\rho_i \leq 1$, since $l \in \Omega_g$ is general.

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Each Δ_i has a Puiseux series expansion with rational exponents, of which we consider here only the essential terms:

$$u = a_{i,1}\lambda^{m_{i,1}/n_{i,1}} + \sum_{l=1}^{l_1} b_{i,1,l}\lambda^{(m_{i,1}+l)/n_{i,1}} + a_{i,2}\lambda^{m_{i,2}/n_{i,1}n_{i,2}} +$$
$$+ \sum_{l=1}^{l_2} b_{i,2,l}\lambda^{(m_{i,2}+l)/n_{i,1}n_{i,2}} + \cdots + a_{i,g_i}\lambda^{m_{i,g_i}/n_{i,1}\cdots n_{i,g_i}} + \sum_{l>0} b_{i,g_i,l}\lambda^{(m_{i,g_i}+l)/n_{i,1}\cdots n_{i,g_i}}$$

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The roots of unity of order $n_i = n_{i,1}\cdots n_{i,g_i}$ act on the coefficients, and each of the resulting expansion is called a *Puiseux-conjugated expansion*.

The curve $C_i: \ u = a_{i,1}\lambda^{m_{i,1}/n_{i,1}}$ is the *first truncation* of Δ_i.
Then C_i intersects the carrousel disk $\mathbb{D} := D_r \times \{\eta\}$ at $n_{i,1}$ points situated on a circle and their carrousel motion is a rotation of angle $2\pi\rho_i$.

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Definition

We consider $n_{i,1}$ disjoint small disks $\delta_{i,j}$, $j \in \{1, \ldots, n_{i,1}\}$, of the same radius, centred at the points $C_i \cap (D_r \times \{\eta\})$, such that each disk contains $n_{i,2} \cdots n_{i,g_i} = n_i/n_{i,1}$ points of the set $\Delta_i \cap (D_r \times \{\eta\})$ such that, if $C_{i_1} = C_{i_2}$, then the corresponding smaller carrousel disks coincide, but if $C_{i_1} \neq C_{i_2}$, then their smaller carrousel disks are totally disjoint.
We say that they are *smaller carrousel disks* of Δ_i.

Annuli, transition zones and rotation speeds

We call *annulus* the difference of two different disks centred at the origin of the carrousel disk $\mathbb{D} := D_r \times \{\eta\}$, and denote by $\mathrm{ex}(A)$ the circle boundary of the largest disk of such an annulus A. To each branch Δ_i there corresponds an annulus A_i of the carrousel disk $D_r \times \{\eta\}$, such that A_i contains $\Delta_i \cap (D_r \times \{\eta\})$, that $A_{i_1} = A_{i_2}$ if and only if $\rho_{i_1} = \rho_{i_2}$, and that different annuli are totally disjoint. The disjoint annuli are ordered according to the radius of their exterior circle ex, in such a way that if $1 > \rho_{i_1} > \rho_{i_2}$ then the radius of $\mathrm{ex}(A_{i_1})$ is smaller than the radius of $\mathrm{ex}(A_{i_2})$.

We shall say that the polar ratio ρ_i is the *rotation speed of* A_i.

We denote by A_{i_0} the annulus corresponding to the polar ratio $\rho = 1$ (and which is the closest to the origin), if this polar ratio exists, and we denote by A_0 an arbitrarily small open disk centred in $(0, \eta)$, not intersecting Δ', and disjoint from all other annuli.

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For any $i \in I$, there are $n_{i,1}$ smaller carrousel disks $\delta_{i,j}$, $j \in \{1, \ldots, n_{i,1}\}$, centred at the $n_{i,1}$ points $C_i \cap (D_r \times \{\eta\})$, and thus contained in A_i. The centres of the disks $\delta_{i,j}$, as well as the points of A_i which are outside any disk $\delta_{i,j}$, have a *carrousel motion* which is, by definition, a rotation of angle $2\pi\rho_i$.

One defines a smooth transition between the angular speeds corresponding to successive annuli. This *transition zone* is a sufficiently thin annulus which we squeeze between A_i and A_{i+1}, in such a way that the collection of annuli and the transition zones defines a partition of the carrousel disk. The disk A_0 is by definition a transition zone and its centre is fixed by the carrousel motion. In each transition zone, the rotation speed depends continuously on the distance to the origin, and it is constant on each circle centred at the origin contained in $D_r \times \{\eta\}$. Altogether, this partition into annuli defines a filtration by disks of the carrousel disk $\mathbb{D}$, which we call *polar filtration*.

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Recursively, each carrousel disk $\delta_{i,j}$ decomposes into annuli which contain *2nd level smaller carrousel disks*, and so on, in a number of g_i steps for each Δ_i. We refer to [Ti1] for this iterated carrousel decomposition and for applications to the computation of the zeta function of the monodromy.

Generalised thimbles via the carrousel

The construction of thimbles is done, starting with a smooth space $\mathcal{B}$ and a C^∞ trivial fibration $\beta : \mathcal{B} \to [0,1]$ with smooth fibre, is classical, see e.g. [AGV1] and [Ebeling1]. Here we have to deal with a singular space, but the definition is analogous.

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Definition

Let $\mathcal{B}$ be a singular space endowed with a Whitney stratification, and let us assume that one can define a stratified C^0 trivial fibration $\beta : \mathcal{B} \to [0, 1]$ by lifting the unit vector field on $[0, 1]$ into a continuous and integrable vector field tangent to the strata of $\mathcal{B}$. Let S be a subset of $\beta^{-1}(1)$ and denote by $T_\beta(S)$ the associated tube, which is homeomorphic to $S \times [0, 1]$.

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The set $T_\beta(S) \cup \mathrm{Cone}(S)$ is called a *(generalised) thimble* on $\beta^{-1}(0)$ along β.

A privileged system of paths in the carrousel disk $\mathbb{D}$

We define a "good" system of paths in the carrousel disk $\mathbb{D}$ by using the carrousel motion. Let $\gamma := (\varepsilon, \eta) \in \partial\mathbb{D}$ (thus $|\varepsilon|$ is the radius of $\mathbb{D}$) and denote $F'_\gamma := B \cap (l, g)^{-1}(\gamma)$. By definition, the carrousel h fixes the boundary $\partial\mathbb{D}$ pointwisely. In case g has an isolated singularity, the restriction $g_| : B \cap (l, g)^{-1}(\{\varepsilon\} \times D_{r'}) \to \{\varepsilon\} \times D_{r'}$ is a trivial fibration, since $\Delta \cap \{\varepsilon\} \times D_{r'} = \emptyset$, and therefore the geometric monodromy h acts trivially on F'_γ.

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Let us fix some simple path $w : [0,1] \to \mathbb{D}$ from γ to the centre $(0, \eta) \in \mathbb{D}$ which avoids any smaller carrousel disk and intersects at only one point any circle centred at the origin of $\mathbb{D}$. For each i, the annulus A_i contains $n_{i,1}$ smaller carrousel disks corresponding to the approximation C_i, and one can order them counter-clockwisely by starting from the path $\mathrm{Im} w$.

We define a connected open subset $W \subset \mathbb{D}$, as follows. If $\rho_i < 1$, we consider an angular sector $V_i \subset \overline{A_i}$ bounded by the path w, such that V_i contains precisely $m_{i,1}$ consecutive order smaller carrousel disks $\delta_{i,j}$ associated to C_i, counted from $\mathrm{Im} w \cap A_i$. In case $A_i = A_j$, we may also consider that the sectors are equal, $V_i = V_j$.
Let then W be the union of all these sectors together with the small disk $\overline{A_0}$.

We now define ordered paths $w_i \subset W$ such that the initial speeds are clockwisely ordered, from γ to the point $d_i := \mathrm{Im} w \cap \mathrm{ex}(\bar{A}_i)$, where $\mathrm{ex}(\bar{A}_i)$ denotes the exterior circle of the annulus $\bar{A}_i$, such that each w_i is therefore a slight alteration of the piece of the path w between γ and d_i.
Let then pick up a point $a_{i,j}$ on the boundary $\bar{\delta}_{i,j}$ of some smaller carrousel disk associated to C_i and define a path $u_{i,j}$ from d_i to $a_{i,j}$ such that $\mathrm{Im} u_{i,j} \subset A_i \cap W$ and that the non-intersecting paths $\{u_{i,j}\}_j$ have clockwisely ordered initial speeds. Then the path $v_{i,j}$ is defined as the composition of $u_{i,j}$ with $w_{i,j}$.

For each smaller carrousel disk $\delta_{i,j} \subset W$ and the point $a_{i,j}$ on its boundary, we reproduce the above construction for the second level smaller carrousel disks in place of the carrousel disk $\mathbb{D}$, and we iterate this procedure the total number of g_i times for each Δ_i. The last g_ith level smaller carrousel disks contain, each of them, a single point of Δ_i.

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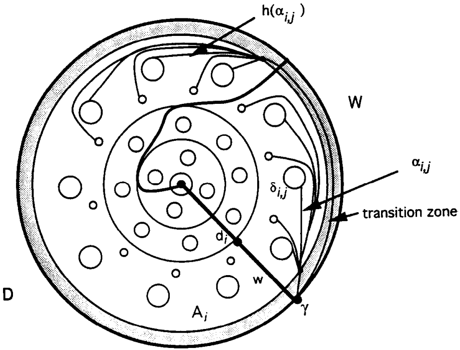
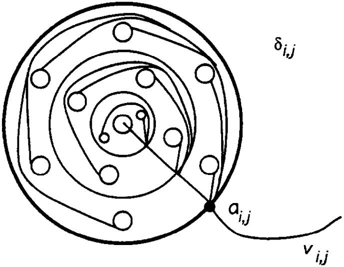
The result of this procedure is that in each 1st level smaller carrousel disk $\delta_{i,j} \subset W$ we have a number of $n_i/n_{i,1}$ non-intersecting paths, each of which connecting the point $a_{i,j}$ on its boundary to some point $b_{i,k}$ very close to some point $b'_{i,k} \in \Delta \cap W$. We finally compose the path $v_{i,j}$ with some path inside $\delta_{i,j}$ as defined just above, and the result is a path $\alpha_{i,k}$ in W connecting γ to the point $b_{i,k} \in \delta_{i,j}$, where $k \in \{1, \ldots, m_i\}$:

$$\alpha_{i,k} : [0,1] \to W, \quad \alpha_{i,k}(0) = \gamma, \quad \alpha_{i,k}(1) = b_{i,k}.$$


h(α_{i,j})
W
α_{i,j}
δ_{i,j}
transition zone
d_i
w
D
A_i
γ


δ_{i,j}
a_{i,j}
v_{i,j}

Thimbles over W, and thimbles over $\mathbb{D} \setminus W$

For each path $\alpha_{i,k}$ from γ to $b_{i,k}$ we may define a generalised thimble $\mathrm{e}_{i,k}$, in the following concrete way.
Let $p_{i,k} \in B \cap (l, g)^{-1}(b'_{i,k})$ be the isolated critical point of the map l restricted to the slice $\{g = \eta\}$, and consider a Milnor ball $B_{i,k}$ at each such point $p_{i,k}$.

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We denote by $\hat{F}_{i,k}$ the local Milnor fibre of $l_{|\{g=\eta\}}$ at $p_{i,k}$ and we identify it with $B_{i,k} \cap (l,g)^{-1}(b_{i,k})$. A fixed trivialisation of the trivial fibration

$$\beta_{i,k} : B \cap (l,g)^{-1}(\alpha_{i,k}([0,1])) \to [0,1] \tag{7}$$

enables one to define a subset $\tilde{F}_{i,k} \subset F'_\gamma = B \cap (l,g)^{-1}(\gamma)$, such that $\tilde{F}_{i,k} \overset{\text{homeo}}{\simeq} \hat{F}_{i,k}$.

Let then

$$\mathrm{e}_{i,k} := T_{\alpha_{i,k}}(\hat{F}_{i,k}) \cup \mathrm{Cone}(\hat{F}_{i,k})$$

be the generalised thimble on $\tilde{F}_{i,k} \subset F'_\gamma$ along $\beta_{i,k}$ as defined at (7).

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We have defined the thimbles in W. Let us define the thimbles outside W, i.e. inside $\mathbb{D} \setminus W$ with help of the carrousel monodromy h. This is based on the following observation:

Lemma

Let $\mathrm{e}_{i,k}$ be a thimble on $\tilde{F}_{i,k}$ along $\beta_{i,k}$. Then $\mathrm{e}'_{i,k} := h(\mathrm{e}_{i,k})$ is a thimble on $\tilde{F}_{i,k}$ along $\beta^h_{i,k} : B \cap (l, g)^{-1}(h \circ \alpha_{i,k}([0,1])) \to [0,1]$.

Let then

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Proof.

Since the carrousel monodromy h is a homeomorphism, we have that $e'_{i,k}$ is homeomorphic to $e_{i,k}$. Moreover, $e'_{i,k}$ is a thimble on $\tilde{F}_{i,k}$, along $\beta^h_{i,k}$ since the restriction $h' : F'_\gamma \to F'_\gamma$ of h is the identity, and therefore $h(\tilde{F}_{i,k}) = \tilde{F}_{i,k}$ pointwisely. □

Let $\{\alpha_{i,k}\}_{i\in I,\ k\in\{1,\ldots,m_i\}}$ be the above defined system of paths within W, and define:

$$r_{i,k} := \begin{cases} \max\{s \mid \mathrm{h}^s(b_{i,k}) \cap W = \emptyset\}, & \text{if } \mathrm{h}(b_{i,k}) \notin W \\ 0\ , \quad \text{if } \mathrm{h}(b_{i,k}) \in W \end{cases}$$

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Let us observe that $\rho_i < 1$ iff $\exists k > 0$ such that $r_{i,k} > 0$.

Definition

For all $i \in I$, $k \in \{1, \dots, m_i\}$, and $s \in \{1, \dots, r_{i,k}\}$, we define the paths

$$\alpha_{i,k,s} := \mathsf{h}^s(\alpha_{i,k}),$$

and if $r_{i,k}$, then we set $\alpha_{i,k,0} := \alpha_{i,k}$.

The set of these paths $\alpha_{i,k,s}$ is a system of non-intersecting paths, each connecting γ with a point near some $\Delta \cap \mathbb{D}$. The paths in $\mathbb{D}\backslash W$ are precisely those paths $\alpha_{i,k,s}$ with $s > 0$.
This "good" system of paths defines our *privileged* system of generalised thimbles.

Third proof of the Bouquet Theorem

We are now in position to prove the following bouquet statement:

Theorem

$$F_g \stackrel{\mathrm{ht}}{\simeq} B \cap (l, g)^{-1}(W) \vee \bigvee_{i \in I} \bigvee_{k \in \{1,\dots,m_i\}} \bigvee_{s \in \{1,\dots,r_{i,k}\}} S(\hat{F}_{i,k}).$$

Proof. The Milnor fibre F_g is homotopy equivalent to the attachment of all the thimbles to the fibre F'_γ, namely it is homotopy equivalent to the union $F'_\gamma \cup_{i,k,s} \mathrm{e}_{i,k,s}$, for $i \in I$, $k \in \{1,\dots,m_i\}$, $s \in \{0,\dots,r_{i,k}\}$. Let us show that we have a good control of the attaching map for the "exterior" thimbles. Namely, by Lemma 9 and the fact that $h_{|F'_\gamma} = \mathrm{id}$, we have the equality:

$$\mathrm{e}_{i,k,s+1} \cap F'_\gamma = \mathrm{e}_{i,k,s} \cap F'_\gamma,$$

which shows that the thimble $\mathrm{e}_{i,k,s+1}$ is attached to F'_γ exactly "at the same place" where the cone $\mathrm{e}_{i,k,s}$ is atached.

It follows that, for fixed $i \in I$ and $k \in \{1, \ldots, m_i\}$ such that $r_{i,k} > 0$, we have the homotopy equivalence:

$$F'_\gamma \cup \mathrm{e}_{i,k,0} \cup \mathrm{e}_{i,k,1} \overset{\mathrm{ht}}{\simeq} (F'_\gamma \cup \mathrm{e}_{i,k,0}) \vee S(\hat{F}_{i,k}),$$

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$$F'_\gamma \cup \bigcup_{s \in \{0,1,\ldots,r_{i,k}\}} \mathsf{e}_{i,k,s} \overset{\mathrm{ht}}{\simeq} (F'_\gamma \cup \mathsf{e}_{i,k,0}) \vee \bigvee_{s \in \{1,\ldots,r_{i,k}\}} S(\hat{F}_{i,k}).$$

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and therefore:

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Finally, by the definition of the zone W, we have that:

$$B \cap (l,g)^{-1}(W) \overset{\text{ht}}{\simeq} F'_\gamma \cup_{i \in I} \cup_{k \in \{1,\ldots,m_i\}} \mathrm{e}_{i,k,0},$$

which ends our proof.

From the preceding Theorem one may deduce the general Bouquet Theorem if we prove the homotopy equivalence

$$B \cap (l, g)^{-1}(W) \overset{\mathrm{ht}}{\simeq} \mathbb{C}\mathrm{lk}_X(\mathcal{S}_0),$$

where $\mathcal{S}_0$ is the zero-dimensional stratum $\{0\}$, i.e. $\mathbb{C}\mathrm{lk}_X(\mathcal{S}_0)$ is the Milnor fibre of our general linear function $l \in \Omega$, that we shall denote it by F_l in the following.

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where $\mathcal{S}_0$ is the zero-dimensional stratum $\{0\}$, i.e. $\mathbb{C}\mathrm{lk}_X(\mathcal{S}_0)$ is the Milnor fibre of our general linear function $l \in \Omega$, that we shall denote it by F_l in the following.

We have by our identifications: $F_l \overset{\text{ht}}{\simeq} B \cap (l, g)^{-1}(\{\xi\} \times D_{r'})$, and we need one more:

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This is a consequence of the procedure called "rabattement dans le diagramme de Cerf" given in [Ti2]. Originally it was introduced by Lê and Perron in the case of a smooth space germ $(X, 0) = (\mathbb{C}^n, 0)$. Their idea is to make the path $\mathrm{Im}\alpha_{i,k}$ slide along a real surface, into a path included in the disk $\{\xi\} \times D_{r'}$.

End of the proofs

We have obtained the bouquet decomposition:

$$F_g \overset{\mathrm{ht}}{\simeq} F_l \vee \bigvee_{i \in I} \bigvee_{k \in \{1,\ldots,m_i\}} \bigvee_{j \in \{1,\ldots,r_{i,k}\}} \mathsf{S}(\hat{F}_{i,k}), \tag{8}$$

where $\hat{F}_{i,k}$ is the Milnor fibre of the stratified Morse singularity of l_α at $p_{i,k}$, and where $p_{i,k}$ denotes some point of the set $\Gamma(l,g) \cap B \cap (l,g)^{-1}(W)$.

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If we denote by $\mathcal{S}_i \subset \mathcal{S}$, where $\mathcal{S} := \{\mathcal{S}_i\}_{i \in R}$, the stratum which contains the point $p_{i,k}$ then, by [Goresky-MacPherson, Main Theorem] we have the homotopy equivalence:

$$\hat{F}_{i,k} \overset{\mathrm{ht}}{\simeq} S^{k_i - 1}(\mathbb{C}\mathrm{lk}_X(\mathcal{S}_i)),$$

where $\mathbb{C}\mathrm{lk}_X(\mathcal{S}_i)$ denotes the complex link of the stratum $\mathcal{S}_i$, and where $k_i := \dim \mathcal{S}_i$.

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We recall that $\mathcal{S}_0$ denotes the stratum $\{0\}$. For each stratum $\mathcal{S}_i \neq \mathcal{S}_0$ (i.e. for any $i \in R\setminus\{0\}$), let $I_i \subset I$ be the subset with the following property: "$j \in I_i$ if and only if $\Gamma_j \subset \mathcal{S}_i$".

We then define a finite set M_i, the number of elements of which is:

$$\#M_i := \sum_{j \in I_i} \sum_{k \in \{1,\dots,m_j\}} r_{j,k} = \sum_{j \in I_i} (n_j - m_j). \tag{9}$$

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In case all the numbers $r_{j,k}$ in the above double sum are zero, then M_i is the empty set. This happens for instance whenever $I_i = \emptyset$. For $i = 0$ we set $\#M_0 = 1$.

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With this last notations, one easily deduces from (8) the wedge decomposition of Theorem 3 from a different partition of the sets of indices. Namely we have the equality:

$$F_I \vee \bigvee_{i\in I} \bigvee_{k\in\{1,\dots,m_i\}} \bigvee_{j\in\{1,\dots,r_{i,k}\}} \mathsf{S}(\hat{F}_{i,k}) = \bigvee_{i\in R} \bigvee_{\#M_i \,\mathrm{times}} S^{k_i}(\mathbb{C}\mathrm{lk}_X(\mathcal{S}_i))$$

where F_I on the left hand side corresponds to $S^0(\mathbb{C}\mathrm{lk}_X(\mathcal{S}_0)) = \mathbb{C}\mathrm{lk}_X(\mathcal{S}_0)$ in the wedge of the right hand side.

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The Handlebody Theorem is also a consequence of the above proofs.

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