Milnor fibrations, part 1

Mihai Tibăr

Foreword

I'll give three different proofs of the bouquet structure of the Milnor fibre, the main interest being that these proofs use a variety of tools which are interesting in themselves.
The first is the original one by Milnor.
The second uses a Morsification and is due to Brieskorn. At the level of homology, it may be extended over a singular space.
The third proof is based on a geometric monodromy and works for any function with isolated singularity on a singular space.

These lectures are extracted from a monograph in preparation. The main part is based on the paper [Ti2]. See also [Ti1] for details.

Introduction

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This means that the Jacobian ideal:

$$\mathrm{Jac}(g) = (\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n})$$

has finite codimension.
Equivalently, that the zero locus $Z(\mathrm{Jac}(g))$ is $\{0\}$. If this zero locus is empty one says that g is *non-singular*.

We have already seen in the previous lectures several results issued from the study of singular functions and maps. One of the analytic invariants attached to g is its *local Milnor number* [Mi]:

$$\mu(g) = \dim_{\mathbb{C}} \mathbb{C}\{x_1, \dots, x_n\}/\mathrm{Jac}(g).$$

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It is possible to compute $\mu(g)$ from a Gröbner base. For instance, such a computation has been one of the original motivations to develop the computer algebra system `Singular`.
Let us point out that, by definition, the Milnor number $\mu(g)$ does not depend on g but on the zero locus $Z(g)$, which is a hypersurface. (More precisely, if we replace g by ug where u is a holomorphic germ and $u(0) \neq 0$ then $Z(ug) = Z(g)$ and $\mu(ug) = \mu(g)$). One speaks about *the Milnor number of a hypersurface germ*.

Possibility of a fibration

One associates to the holomorphic germ g a local fibration, called *Milnor fibration*. Then $\mu(g)$ gets a topological interpretation as the rank of the $(n-1)$th homology group of the Milnor fibre; it is therefore a topological invariant. This is what we will show here in particular.

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More generally, we shall give a geometric description of the monodromy and several of its consequences.

We start by reviewing the main lines of Milnor's proof of his fibration result. We explain two other different methods of proof which enable one to extend this result to a larger class of functions. The general *Bouquet Theorem* based on the controlled attaching via a geometric monodromy is the main part of this lecture.

Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $n \geq 2$, be a non-constant holomorphic germ, and let $B_\varepsilon \subset \mathbb{C}^n$ denote the closed ball of radius $\varepsilon > 0$, $S_\varepsilon^{2n-1} := \partial B_\varepsilon$, and $D_\delta \subset \mathbb{C}$ a disk of radius $\delta > 0$.

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Theorem (Milnor)

For any sufficiently small $\varepsilon > 0$ and any $0 < \delta \ll \varepsilon$, the following restriction:

$$g_| : B_\varepsilon \cap g^{-1}(D_\delta \setminus \{0\}) \to D_\delta \setminus \{0\} \tag{1}$$

is a locally trivial C^∞ fibration. Its fibre is called Milnor fibre.
The isotopy type of the fibration does not depend on the choice of the radii ε and δ.
In case g has an isolated singularity, its Milnor fibre F_g is homotopy equivalent to a bouquet of spheres:

$$F_g \overset{\mathrm{ht}}{\simeq} S^{n-1} \vee \cdots \vee S^{n-1},$$

the number of which is called "Milnor number" and is denoted by $\mu(g)$.

For any g (without any restriction on the singular locus), there is a second fibration:

$$g/\|g\| : S_{\varepsilon}^{2n-1} \setminus g^{-1}(0) \to S^1. \quad (2)$$

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We will give a more precise meaning of the ε and δ in the above statement.

The tube fibration

In the *tube fibration* (1) proved by Milnor [Mi] in case of isolated singularities, i.e. $\mathrm{Sing}\, g = \{0\}$, the name "tube" comes from the fact that this locally trivial fibration retracts to a sub-fibration over some circle S included in $D_\delta \setminus \{0\}$ and centred at 0:

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Milnor proves first that the zero locus $V := \{g = 0\}$ is transversal to all spheres S_ε^{2n-1} provided that the radius $\varepsilon > 0$ is *small enough*. This amounts to showing that there exists $\varepsilon_0 > 0$ such that $V \pitchfork S_\varepsilon^{2n-1}$ for any positive $\varepsilon < \varepsilon_0$.

Question: How to prove this fact?

In particular this defines the *link* $K_g := V \cap S_\varepsilon^{2n-1}$ of the hypersurface singularity as a topological invariant, thus independent of $\varepsilon < \varepsilon_0$.
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Except of $\{g = 0\}$, the nearby fibres of g are all non-singular. This fact follows from the Curve Selection Lemma, see [Mi, §3]. The transversality of V to S_ε insures that S_ε is also transversal to all nearby fibres. More precisely, this means that, for each fixed $0 < \varepsilon \leq \varepsilon_0$, there exists some $\delta_\varepsilon > 0$ such that $g^{-1}(a) \pitchfork S_\varepsilon^{2n-1}$ for any $a \in \mathbb{C}$ with $0 < \|a\| \leq \delta_\varepsilon$.

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This shows that the map (3) is proper submersion. Then *Ehresmann theorem* for manifolds with boundary[1] applies to this situation and yields that (3) is a locally trivial fibration, hence (1) too.

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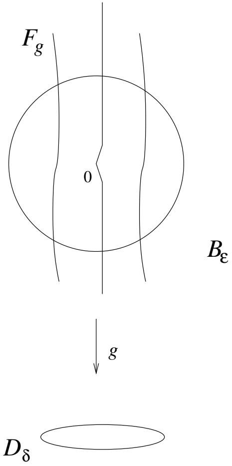
Milnor fibration

From the independency of ε and δ it follows that the total space of the "tube" $B_\varepsilon \cap g^{-1}(D_\delta)$ is contractible.
This is the picture of the "tube fibration" that we see usually.



But why this picture is "cheating", and by far?

The sphere fibration and the bouquet structure

The map (2) holds an important place in the monograph [Mi].
To show that (2) is a locally trivial fibration, Milnor starts by proving that the map $g/\|g\| : B_\varepsilon^{2n} \setminus V \to S^1$ is transversal to all small enough spheres.

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One cannot apply here Ehresmann Theorem because the map (2) is not proper (because of removing the link K_g. Milnor constructs an explicit trivialisation of the map (2) by developing a technically beautiful method which uses the holomorphy of g in an essential way, and which applies to any singular g, not only with isolated singularity. In particular, the existence of a fibration structure in the neighbourhood of the link K_g is shown by constructing a special vector field which yields an "open book structure" with binding K_g.

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It appears that this construction can be replaced by an argument based on the existence of a stratification of V satisfying the Thom $(a)_g$-regularity condition.

Milnor uses the sphere fibration (2) (which is isotopic to the fibration (1)), in order to find the bouquet structure of its fibre F_θ, where θ denotes the angle corresponding to some point on the circle S^1.
His proof consists in the following steps:

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1). F_θ is a CW-complex of dimension $\leq n-1$, using Morse theory for the function $|g|$ on the manifold F_θ.

2). $\tilde{H}^i(\bar{F}_{\theta'}) \simeq \tilde{H}_{2n-2-i}(S_\varepsilon \setminus \bar{F}_{\theta'}) \simeq \tilde{H}_{2n-2-i}(F_\theta)$, for any $\theta' \neq \theta$, where the first isomorphism is by the Alexander duality, and the second is due to the homeomorphism $S_\varepsilon \setminus \bar{F}_{\theta'} \simeq F_\theta \times (S^1 \setminus \theta')$ wich is a consequence of the fibration (2).

3). F_θ is $(n-2)$-connected. Indeed, $H_i(F_\theta) = 0$ for $i \geq n$ by step 1, and $\tilde{H}_i(F_\theta) = 0$ for $i \leq n-2$ by step 2.
Therefore the reduced homology of F_θ is concentrated in dimension $n-1$. It remains to show that F_θ is simply connected for $n \geq 3$ (since for $n = 2$ the fibre is a Riemann surface with holes, hence simply connected). This is done by Milnor by using the Morse function $|g|$ on the manifold $S_\varepsilon \setminus N(K_\varepsilon)$, where $N(K_\varepsilon)$ denotes some small tubular neighbourhood of the link $K_\varepsilon := g^{-1}(0) \cap S_\varepsilon$. Then apply Hurewicz theorem.

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4). Hurewicz theorem also yields $\pi_{n-1}(F_\theta) \simeq H_{n-1}(F_\theta)$, which is a free abelian group. One may further construct a map $S^{n-1} \vee \cdots \vee S^{n-1} \to F_\theta$ which induces an isomorphism at the level of homology groups. By Whitehead's theorem, this is a homotopy equivalence.

Definition (Milnor number)

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In particular, the Milnor fibre F_θ is diffeomorphic to the Milnor fibre F_g of the tube fibration (3) and of the fibration (1), they have the same homology and the same homotopy type.

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By integrating a lift of the unit vector field on the directly oriented circle S, one obtains a *geometric monodromy*. In case g has *isolated singularity*, this can be done such that it is the identity on the boundary ∂F_g, due to the triviality of the fibration

$$g_| : S_\varepsilon^{2n-1} \cap g^{-1}(D_\delta) \to D_\delta \tag{4}$$

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$$h_\gamma : (F_g, \partial F_g) \to (F_g, \partial F_g).$$

which induces the *algebraic monodromy*:

$$h_* : \widetilde{H}_i(F_g, \mathbb{Z}) \to \widetilde{H}_i(F_g, \mathbb{Z}) \tag{5}$$

and, since $h_{|\partial F_g} = \mathrm{id}$ by (4), the *variation map* of h:

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and similarly in co-homology, where the only non-trivial groups are in dimension $i = n - 1$ in case of isolated singularity.

Details about these objects can be found e.g. in the monographs by Arnold, Gusein-Zade and Varchenko, Ebeling, and many other sources. In case g has non-isolated singularities, there is no more triviality at the whole boundary but one can still construct geometric variation maps (see Siersma's paper in Topology 1991).

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Still in this general case of non-isolated singularity, the famous *monodromy theorem* asserts that the algebraic monodromy (5) has only roots of unity as eigenvalues (one says that the monodromy is quasi-unipotent). This was conjectured by Milnor and proved by Grothendieck, Landman, Clemens, Brieskorn, and a bunch of other authors, by different means. There are also several important refinements and extensions of this theorem in different other categories.

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The study of the monodromy engendered a lot of interesting results, with special emphasis in case of curves and surfaces, essentially in two ways: using a blow-up of the singular locus of the function g (e.g. Deligne, A'Campo, Brieskorn, Steenbrink, etc) or by constructing geometric monodromies (A'Campo, Gabrielov, Ebeling, Lê D.T., Perron, Siersma, etc).

Bouquet theorem via Morsification

We present a method due to Brieskorn and applies to holomorphic function germs g with isolated singularity at the origin. One fixes a Milnor ball B_ε and considers the following deformation:

$$g_t = g + tl$$

where l is a linear function, and $t \in \mathbb{C}$. For l general enough, and for t close enough to 0, the deformation g_t has only complex Morse points (thus isolated) and with different values. We say that g_t is a *Morsification of* g.

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During the deformation, the fibres of g_t keep transverse to the boundary S_ε for all t close enough to 0. As the Milnor tube $B_\varepsilon \cap g^{-1}(D_\delta)$ is contractible, the deformed tube $B_\varepsilon \cap g_t^{-1}(D_\delta)$ is contractible too, since homotopy equivalent to the original one. By the same transversality reason, the nonsingular fibres of the deformed tube are homotopy equivalent to the Milnor fibre of g.

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In case of a complex *Morse singularity* (also denoted by A_1), the homology of $(B_j \cap g_t^{-1}(D_j), B_j \cap g_t^{-1}(\gamma_j))$ is $\mathbb{Z}$ in dimension n, and 0 in any other dimension.

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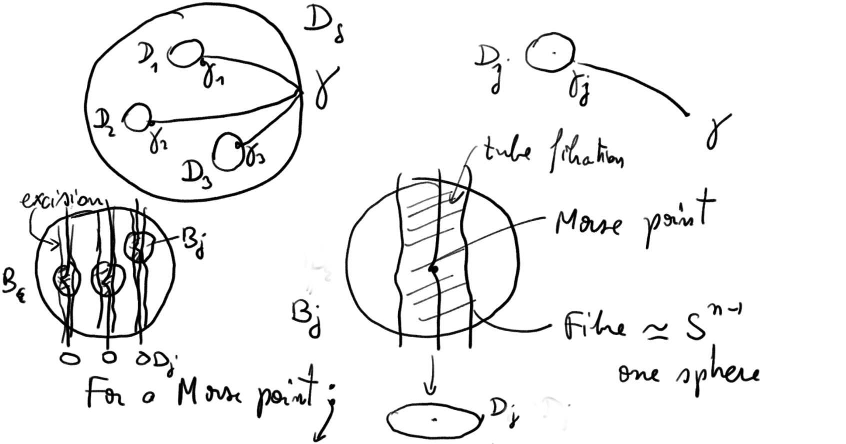
We get that $H_*(B_\varepsilon \cap g^{-1}(D_\delta), B_\varepsilon \cap g^{-1}(\gamma))$ is concentrated in degree n, and:

$$H_n(B_\varepsilon \cap g^{-1}(D_\delta), B_\varepsilon \cap g^{-1}(\gamma)) \simeq \mathbb{Z}^\mu,$$

where μ is the number of Morse points in the Morsification of g.

Mayer-Vietoris, and direct sum splitting of relative homology



$H_i(B_j \cap g_t^{-1}(D_j), B_j \cap g_t^{-1}(\gamma_j)) \simeq \mathbb{Z}$ if $i = n-1$ and $\simeq 0$ otherwise

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This proof shows in particular that the number of spheres in the bouquet is equal to the number of Morse points in some Morsification of g. In particular this number does not depend on the chosen Morsification.

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Exercise

Compute the index of the gradient vector field $\mathrm{grad} g$ at the origin and relate it to the number of Morse points in a Morsification of g. (Source: Milnor's topology book – for how to compute the index and for this relation).
Then compare this index to the Milnor number defined algebraically as the codimension of the Jacobian ideal.
Deduce that all these numbers are equal.

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