TWO FURTHER PROBABILISTIC APPLICATIONS OF BESSEL FUNCTIONS

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Abstract. We revisit two classical formulas for the Bessel function of the first kind, due to von Lommel and Weber-Schafheitlin, in a probabilistic setting. The von Lommel formula exhibits a family of solutions to the van Dantzig problem involving the generalized semi-circular distributions and the first hitting times of a Bessel process with positive parameter, whereas the Weber-Schafheitlin formula allows one to construct non-trivial moments of Gamma type having a signed spectral measure. Along the way, we observe that the Weber-Schafheitlin formula is a simple consequence of the von Lommel formula, the Fresnel integral and the Selberg integral.

1. Two classical formulas for the Bessel function

The Bessel function of the first kind

\[ J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha + 1)} \binom{\alpha}{1} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\alpha}}{n! \Gamma(n+\alpha+1)} \]

defined for all \( \alpha \in \mathbb{R} \) and \( z \in \mathbb{C}/\mathbb{R}^- \), is one of the most important special functions of mathematical physics. Its connections with probability are well-known, numerous and manifold. In this note, we shall consider two applications of these functions in a context related to infinite divisibility. The fact that Bessel functions appear in such matters is classical - see [16] and all the references therein. However, it seems that our following observations have so far passed unnoticed. We shall rely on two formulas, which date back to the fundamental analysis of Bessel functions performed during the second half of the XIX-th century. Their self-contained proof is given in the Appendix for the convenience of the reader.

The first formula is the computation of a Fourier transform, which will be used in Section 2.

**Lemma A** (von Lommel). For every \( \alpha > -1/2 \) and \( z \in \mathbb{C}/\mathbb{R}^- \), one has

\[ \frac{1}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \left( \frac{z}{2} \right)^\alpha \int_{-1}^{1} e^{it z}(1-t^2)^{\alpha-1/2} dt = J_\alpha(z). \]

The second formula is the computation of a Mellin transform, which will be applied in Section 3. Let us mention that this formula has suffered several generalizations over the years - see e.g. [3, 18, 10] and the references therein. The latter, however, do not seem to have direct applications in our setting.

**Lemma B** (Weber-Schafheitlin). For every \( \alpha > -1/2 \) and \( s \in (0, \alpha + 1/2) \), one has

\[ \int_0^\infty z^{-2s} J_\alpha^2(z) \, dz = \frac{\Gamma(s) \Gamma(\alpha + 1/2 - s)}{2\sqrt{\pi} \Gamma(1/2 + s) \Gamma(\alpha + 1/2 + s)}. \]

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2. A family of solutions to van Dantzig’s problem

The van Dantzig problem asks whether, for a given characteristic function \( f \) on the line, the function \( t \mapsto 1/f(it) \) is also a characteristic function. When this is the case, one says that \( (f(t), 1/f(it)) \) is a van Dantzig pair. Whereas a general solution to this problem does not seem to exist, several examples have been given in [14]. See also Section 3 in [12], which uses the different terminology associated pair. The prototype is the standard normal distribution with the van Dantzig pair \((e^{-t^2/2}, e^{-t^2/2})\), which is called self-reciprocal because its two components are equal. Another example is the uniform distribution on \((-1, 1)\) whose associated van Dantzig pair is

\[
\left( \frac{\sin t}{t}, \frac{t}{\sinh t} \right),
\]

the second component being a characteristic function by the Euler product formula

\[
\frac{t}{\sinh t} = \prod_{n \geq 1} \left( 1 + \frac{t^2}{n^2 \pi^2} \right)^{-1} = \mathbb{E} \left[ \exp \left\{ \frac{it}{\pi} \sum_{n \geq 1} \frac{X_n}{n} \right\} \right],
\]

where \( \{X_n, n \geq 1\} \) is an infinite sample of the Laplace distribution with density \( e^{-|x|/2} \) on \( \mathbb{R} \) and characteristic function

\[
\mathbb{E}[e^{itX_1}] = \frac{1}{1 + t^2}.
\]

Above, the a.s. convergence of the series is ensured by Kolmogorov’s one-series theorem.

In [17], an analogy has been given between van Dantzig pairs and infinitely divisible (ID) Wald couples. More precisely, the main result of [17] states that for every integrable Lévy measure \( \nu \) on \( \mathbb{R}^- \) there exist a real, integrable, centered and ID random variable \( X \) having Lévy measure \( \nu \), and a positive ID random variable \( T \) such that

\[
\mathbb{E}[e^{\lambda X}] \mathbb{E}[e^{-\lambda^2 T/2}] = 1, \quad \lambda \geq 0.
\]

By Brownian subordination - see Proposition I.8.1 in [17], this transforms into

\[
\mathbb{E}[e^{i\lambda Y}] = \frac{1}{\mathbb{E}[e^{\lambda X}]}
\]

for every \( \lambda \geq 0 \) with \( Y = B_T \), and we see that the characteristic functions of \( X \) and \( Y \) would form a van Dantzig pair if the equality could be extended to all \( \lambda < 0 \). However, the fact that \( Y \) is ID and the corollary p.117 in [14] show that this is possible only if \( Y \) is Gaussian, that is in the degenerate case when \( T \) is constant.

In this section, we will produce a family of van Dantzig pairs constructed from a couple \((X, T)\) where \( T \) is ID and \( X \) is not ID. On the one hand, we consider the power semicircle distribution with density

\[
h_\alpha(x) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} (1 - x^2)^{\alpha - 1/2} \text{1}_{(-1,1)}(x),
\]

where \( \alpha > -1/2 \) is the index parameter. Up to affine transformation, this law can be viewed as an extension of the arcsine, uniform and semicircle distributions which correspond to \( \alpha = 0, \alpha = 1/2 \) and \( \alpha = 1 \) respectively. It has been studied in [2] as a non ID factor of the standard Gaussian distribution.
See also the references therein for other aspects of this distribution. We set

\[ \hat{h}_\alpha(t) = \int_{\mathbb{R}} e^{itx} h_\alpha(x) \, dx \]

for the characteristic function of the density \( h_\alpha \). On the other hand, we consider the Bessel process of dimension \( d = 2\alpha + 1 > 0 \) starting from zero, which we recall to be the unique strong solution to the SDE

\[
\begin{align*}
\left\{ 
&dX_t = \frac{\alpha}{X_t} \, dt + dB_t \\
&X_0 = 0
\end{align*}
\]

where \( \{B_t, t \geq 0\} \) is a standard Brownian motion, and we denote by \( T_\alpha = \inf\{t > 0, X_t = 1\} \) its first hitting time of one. It is well-known - see [16] - that this random variable is ID and has finite exponential moments. Finally, we set \( Y_\alpha = \hat{B}_{T_\alpha} \) where \( \{\hat{B}_t, t \geq 0\} \) is an independent standard Brownian motion. By subordination, this random variable is ID, integrable and centered.

**Proposition 1.** For every \( \alpha > -1/2 \), the function \( 1/\hat{h}_\alpha(it) \) is the characteristic function of \( Y_\alpha \). In particular, \( (\hat{h}_\alpha(t), 1/\hat{h}_\alpha(it)) \) is a van Dantzig pair.

**Proof.** By Lemma A, one has

\[ \hat{h}_\alpha(t) = \frac{\Gamma(\alpha + 1)}{(t/2)^\alpha} J_\alpha(t), \quad t > 0. \]

From the Hadamard factorization - see (4.14.3) and (4.14.4) in [1], we obtain

\[ \hat{h}_\alpha(z) = \prod_{n \geq 1} \left( 1 - \frac{z^2}{j_\alpha^2,n} \right), \quad z \in \mathbb{C}, \]

where \( 0 < j_{\alpha,1} < j_{\alpha,2} < \ldots \) are the positive zeroes of \( J_\alpha \) and the product is absolutely convergent on every compact set of \( \mathbb{C} \). This implies

\[ \frac{1}{\hat{h}_\alpha(it)} = \prod_{n \geq 1} \left( 1 + \frac{t^2}{j_{\alpha,n}^2} \right)^{-1}. \]

On the other hand, it follows from Formula (3.8) in [11] that

\[
\mathbb{E}[e^{-\lambda T_\alpha}] = \frac{(\sqrt{2\lambda})^\alpha}{2^\alpha \Gamma(1 + \alpha) I_\alpha(\sqrt{2\lambda})} = \frac{(i\sqrt{2\lambda})^\alpha}{2^\alpha \Gamma(1 + \alpha) J_\alpha(i\sqrt{2\lambda})} = \prod_{n \geq 1} \left( 1 + \frac{2\lambda}{j_{\alpha,n}^2} \right)^{-1}
\]

for every \( \lambda > 0 \), where \( I_\alpha \) is the modified Bessel function of the first kind and we have used the first equality in (4.12.2) of [1] for the second equality. The Brownian subordination argument of Proposition I.8.1 in [17] concludes the proof. \( \square \)

**Remark.** (a) The case \( \alpha = 1/2 \) corresponds to the aforementioned pair

\[ \left( \frac{\sin t}{t}, \frac{t}{\sinh t} \right), \]

recalling e.g. from (4.6.3) in [1] that \( j_{1/2,n} = n\pi \) for all \( n \geq 1 \). The case \( \alpha = 0 \) is also explicitly mentioned in [14] as an example pertaining to Theorem 5 therein - notice that this theorem actually covers the whole range \( \alpha \in (-1/2, 1/2) \). In general, one has \( \hat{h}_\alpha \in \mathcal{D}_1 \) for all \( \alpha > -1/2 \) with the notation of [14], and our pairs can hence be viewed as further explicit examples of van Dantzig pairs corresponding to \( \mathcal{D}_1 \). The case \( \alpha = 1 \) is worth mentioning because it shows that the semicircle characteristic function
belongs to a van Dantzig pair, as does the Gaussian characteristic function.

(b) Following the notation of [14], the characteristic function

\[ \hat{g}_\alpha(t) = \frac{\hat{h}_\alpha(t)}{\hat{I}_\alpha(t)} = J_\alpha(t) I_\alpha(t) \]

is self-reciprocal, in other words one has \( \hat{g}_\alpha(t)\hat{g}_\alpha(it) = 1 \). Observe that \( \hat{g}_\alpha(t) \) does not correspond to some ID distribution, by Theorem 3 in [14].

3. A family of moments of Gamma type

3.1. The general setting. Introduce the notation

\[ (a)_s = (a_1)_s \times \cdots \times (a_n)_s = \prod_{i=1}^{n} \frac{\Gamma(a_i + s)}{\Gamma(a_i)} \]

for \( a = \{a_1, \ldots, a_n\} \) with \( 0 < a_1 \leq \ldots \leq a_n \) and \( s > -a_1 \), where we have used the standard definition of the Pochhammer symbol \( (x)_s \). When \( n = 0 \) that is \( a \) is empty, we set \( \min(a) = \infty \) and \( (a)_s = 1 \).

In [4], the authors introduce the notation

\[ D \left[ \begin{array}{cccc} a & b \\ c & d \end{array} \right] \]

for the distribution of the positive random variable \( X \), if it exists, such that

\[ \mathbb{E}[X^s] = \frac{(a)_s(b)_{-s}}{(c)_s(d)_{-s}} \]

for every \( s \in (-\min(a), \min(b)) \), where \( a, b, c \) and \( d \) are four finite and possibly empty sets of positive numbers. A challenging open problem is to characterize the existence of such distributions in terms of the four sets \( a, b, c \) and \( d \).

When they exist, these distributions are unique by inversion of the Mellin transform on the non-trivial strip \((-\min(a), \min(b))\). From the point of view of special functions, the Mellin transform is also formerly invertible in a Meijer \( G \)-function. More precisely, one has by definition

\[ \int_0^\infty f(x) x^{s-1} \, dx = \frac{(a)_s(b)_{-s}}{(c)_s(d)_{-s}} \]

for every \( s \in (-\min(a), \min(b)) \), where \( f \) is the Meijer \( G \)-function

\[ f(x) = G_{m,n}^{m,n} \left( x^{-1} \middle| \begin{array}{c} A \\ B \end{array} \right) \]

with \( A = \{1 - a_n, \ldots, 1 - a_1, d_1, \ldots, d_p\} \) and \( B = \{b_1, \ldots, b_m, 1 - c_q, \ldots, 1 - c_1\} \) finite and possibly empty sets. We refer to Chapter 16 in [15] for more detail on the Meijer \( G \)-functions. One should remember that the above \( f \) is a function strictly speaking on \((0, 1) \cup (1, \infty)\) only, and that there might be a singularity at one. For example, if \( a = (1, 3), c = (2, 2) \) and \( b = d = \emptyset \), an easy computation mentioned in [4] shows that the corresponding random variable \( X \) exists with distribution

\[ \mathbb{P}[X \in dx] = \frac{1}{2} (\delta_1(dx) + 1_{(0,1)} dx). \]

The solution to the above existence problem amounts to characterizing the non-negativity of \( f \) on \((0, 1) \cup (1, \infty)\) and of the singularity at one. To this end, simulations of Meijer \( G \)-functions can be
performed by the Wolfram package [20], which is however not very robust when the amount of parameters becomes large or in the presence of the singularity.

3.2. The case \( b = \emptyset \). Before stating our contribution to this problem in the next paragraph, let us give an account on the case \( b = \emptyset \), which has been studied in [4, 7, 8, 9] from various points of view. It is clear that the condition \( d = \emptyset \) is then necessary for the existence of \( X \). Moreover, considering the random variable \( X^{-1} \), the problem is equivalent to the case \( a = c = \emptyset \). Setting \( n = \sharp \{ a \} \) and \( p = \sharp \{ c \} \), we discard the obvious case \( p = 0 \) where \( X \) always exists and is distributed as the independent product

\[
X \overset{d}{=} \Gamma_{a_1} \times \cdots \times \Gamma_{a_n}.
\]

Here and throughout, \( \Gamma_t \) denotes a standard Gamma random variable with parameter \( t > 0 \) and we make the convention that an empty product is one. When \( p \geq 1 \), the following necessary conditions for the existence of \( X \) are obtained in [4]:

\[
\begin{cases}
  p \leq n, \\
  a_1 \leq c_1, \\
  a_1 + \cdots + a_p \leq c_1 + \cdots + c_p.
\end{cases}
\]

This set of conditions is easily seen to be sufficient for \( p = 1 \) or \( n = 2 \). For \( p = 1 \) one has indeed

\[
X \overset{d}{=} B_{a_1,c_1-a_1} \times \Gamma_{a_2} \times \cdots \times \Gamma_{a_n}
\]

where \( B_{a,b} \) denotes, here and throughout, a standard Beta random variable with parameter \( a, b > 0 \) and we make the conventions \( B_{0,b} = 0 \) for all \( b > 0 \) and \( B_{a,0} = 1 \) for all \( a > 0 \). It is interesting to mention that for \( n = 2 \), an example of such independent Beta-Gamma products is the square of a Brownian supremum area, which is thoroughly studied in [8] - see Theorem 1.6 therein. To show the sufficiency of (3) in the remaining case \( p = n = 2 \), we first write down the general formula

\[
\frac{(a)_s}{(c)_s} = \exp - \left\{ \int_0^\infty (1 - e^{-sx}) \left( \frac{\varphi_a(x) - \varphi_c(x)}{x(1 - e^{-x})} \right) dx \right\}
\]

for all \( p = n \) with the notations

\[
\varphi_a(x) = e^{-a_1 x} + \cdots + e^{-a_n x} \quad \text{and} \quad \varphi_c(x) = e^{-c_1 x} + \cdots + e^{-c_n x},
\]

which follows e.g. from Theorem 1.6.2 (ii) in [1]. By the Lévy-Khintchine formula we see that \( X \) exists, and has support \([0, 1]\) as the exponential of some negative ID random variable, as soon as

\[
\varphi_a \geq \varphi_c \quad \text{on} \quad \mathbb{R}^+.
\]

This condition is then easily checked for \( p = n = 2 \), \( a_1 \leq c_1 \) and \( a_1 + a_2 \leq c_1 + c_2 \). In the case \( a_1 + a_2 = c_1 + c_2 \) there is a singularity at one which is given by

\[
\mathbb{P}[X = 1] = \lim_{s \to \infty} \mathbb{E}[X^s] = \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)} \leq 1,
\]

and the density of \( X \) on \((0, 1)\) can be computed in terms of a hypergeometric function \( {}_2F_1 \) - see Theorem 6.2 in [7]. In the case \( a_1 + a_2 < c_1 + c_2 \) the random variable \( X \) is absolutely continuous, with a density on \((0, 1)\) which is also given in terms of a hypergeometric function \( {}_2F_1 \) - see Formula (5.1) in [7].
We next observe that the set of conditions (3) is not sufficient for \( p = n = 3 \), as the following counterexample shows:

\[
\begin{align*}
& a_1 = 2, a_2 = 16/5, a_3 = 17/5, \\
& c_1 = 11/5, c_2 = 12/5, c_3 = 4.
\end{align*}
\]

Indeed, if there existed a corresponding random variable \( X \), one would have

\[
\mathbb{P}[X = 1] = \lim_{s \to \infty} \mathbb{E}[X^s] = \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} = \frac{150}{132} > 1,
\]

which implies that the density of \( X \) on \((0,1)\) would have to take negative values. Even in the simpler case \( p = n \), characterizing the existence of \( X \) does not seem easy in general. An interesting open question is whether the sufficient condition (5) is also necessary, which would mean

\[
X \text{ exists } \iff \log X \text{ is ID}.
\]

This open question is easily seen to be equivalent to Conjecture 1 in [9]. We also refer to the whole Section 2 in [9] for a set of conditions on the sets \( a \) and \( c \) ensuring (5). The necessity of (5), with the notation \( \varphi_c(x) = e^{-c_1 x} + \cdots + e^{-c_p x} \), can also be asked for \( p < n \). Observe that the necessity of (5) would also imply the equivalence (6) in view of the more general formula

\[
\frac{a}{(c)_s} = \exp \left\{ (\psi(a) - \psi(c)) s + \int_{-\infty}^0 (e^{sx} - 1 - sx) \left( \frac{\varphi_a(|x|) - \varphi_c(|x|)}{|x|(1 - e^{-|x|})} \right) dx \right\}
\]

which follows from a combination of Theorem 1.6.1 (ii) and Theorem 1.6.2 (ii) in [1], where \( \psi \) is the digamma function and we have used the notations

\[
\psi(a) = \psi(a_1) + \cdots + \psi(a_n) \quad \text{and} \quad \psi(c) = \psi(c_1) + \cdots + \psi(c_p).
\]

Notice finally that for \( p < n \), the random variable \( X \) has unbounded support since \( (\mathbb{E}[X^s])^{1/s} \to \infty \) as \( s \to \infty \), whereas its support is bounded in \([0,1]\) for \( p = n \) since then \( (\mathbb{E}[X^s])^{1/s} \to 1 \).

### 3.3. A family of solutions for \( a \neq \emptyset \) and \( b \neq \emptyset \)

We now come back to the general setting of Paragraph 3.1, and we use the notations therein. By the same arguments as in [4], the following conditions are easily seen to be necessary:

\[
\begin{align*}
& p + q \leq n + m, \\
& \min(a) \leq \min(c), \\
& \min(b) \leq \min(d).
\end{align*}
\]

Moreover, in the case \( p = q = 1 \) these conditions are also sufficient, with

\[
X = \frac{B_{a_1,c_1-a_1} \times \Gamma_{a_2} \times \cdots \times \Gamma_{a_n}}{B_{b_1,d_1-b_1} \times \Gamma_{b_2} \times \cdots \times \Gamma_{b_n}}.
\]

The following proposition shows however that the condition \( p \leq n \) which was necessary in the setting of the previous paragraph, is in general not necessary anymore.

**Proposition 2.** For every \( a, b, c, d > 0 \), the distribution

\[
D \left[ \begin{array}{c} a \\ (c, d) \end{array} \right] = D \left[ \begin{array}{c} a \\ (d, c) \end{array} \right]
\]

(a) **exists** if \( c + d \geq 3a + b + 1/2 \) and \( \min(c,d) \geq \min(2a+b,a+1/2) \).

(b) **does not exist** if \( c + d < 3a + b + 1/2 \) or \( \min(c,d) \leq a \).
Proof. We begin with the existence result and we first consider the case $c = 2a + b$ and $d = a + 1/2$. Introducing the function

$$f(x) = \frac{\sqrt{\pi} \Gamma(2a+b) \Gamma(a+1/2)}{\Gamma(a) \Gamma(b)} x^{a-3/2} J_{a+b-1/2}^2(x^{-1/2})$$

which is non-negative on $(0, \infty)$ (and vanishes an infinite number of times), we get from Lemma B and a change of variable

$$\int_0^\infty x^s f(x) \, dx = \frac{2\sqrt{\pi} \Gamma(2a+b) \Gamma(a+1/2)}{\Gamma(a) \Gamma(b)} \int_0^\infty x^{-2a-2s} J_{a+b-1/2}^2(x) \, dx$$

for every $s \in (-a, b)$. We next consider the case $c + d = 3a + b + 1/2$ and $\min(c, d) > \min(2a + b, a + 1/2)$. It follows from the case $p = n = 2, a_1 < c_1$ and $a_1 + a_2 = c_1 + c_2$ in the previous paragraph that the distribution

$$D \begin{pmatrix} (2a + b, a + 1/2) & - \\ (c, d) & - \end{pmatrix}$$

exists and, by the previous case, so does

$$D \begin{pmatrix} a & b \\ (c, d) & - \end{pmatrix} = D \begin{pmatrix} a & b \\ (2a + b, a + 1/2) & - \end{pmatrix} \odot D \begin{pmatrix} (2a + b, a + 1/2) & - \\ (c, d) & - \end{pmatrix},$$

where we have used the notation $\odot$, the concatenation rule and the simplification rule in [4] p.1046. The remaining case in the proof of (a) is $c + d > 3a + b + 1/2$ and $\min(c, d) \geq \min(2a + b, a + 1/2)$, which is an easy consequence of the previous case since, setting $m = \min(c, d)$ and $M = \max(c, d)$, the distribution

$$D \begin{pmatrix} 3a + b + 1/2 - m & - \\ M & - \end{pmatrix}$$

exists from the case $p = n = 1$ and $a_1 < c_1$ in the previous paragraph, and one has

$$D \begin{pmatrix} a & b \\ (c, d) & - \end{pmatrix} = D \begin{pmatrix} a & b \\ (m, 3a + b + 1/2 - m) & - \end{pmatrix} \odot D \begin{pmatrix} 3a + b + 1/2 - m & - \\ M & - \end{pmatrix}.$$  

We now proceed to the non-existence result, which is easy for $m = \min(c, d) \leq a$ : if $m = a$, one has

$$D \begin{pmatrix} a & b \\ (c, d) & - \end{pmatrix} = D \begin{pmatrix} - & b \\ M & - \end{pmatrix}$$

and it is clear that such a distribution cannot exist, whereas if $m < a$ the Mellin transform

$$s \mapsto \frac{(a)_s (b)_{-s}}{(c)_s (d)_s}$$

vanishes at $s = -m$ inside the domain of convergence $(-a, b)$, which is impossible for a probability distribution. To handle the remaining case $c + d < 3a + b + 1/2$, we shall use the following identification

$$G_{1,3}^{1,1} \left( x \bigg| \begin{array}{c} 1 - a \\ 0, 1 - b, 1 - c \end{array} \right) = \frac{\Gamma(a)}{\Gamma(b) \Gamma(c)} \, _2F_1 \left( \begin{array}{c} a \\ b, c \end{array} ; -x \right)$$

for every $a, b, c, x > 0$. This particular case of Formula 16.18.1 in [15] will also be useful in the sequel. By (2) and Formula 16.19.2 in [15] with $\mu = -b$, we deduce

$$(8) \quad D \begin{pmatrix} a & b \\ (c, d) & - \end{pmatrix} \quad \text{exists} \iff \quad _2F_1 \left( \begin{array}{c} a + b \\ c + b, d + b \end{array} ; -x \right) \geq 0 \quad \text{for all } x \geq 0.$$
We can now appeal to the asymptotic behaviour for the generalized hypergeometric functions, to be found e.g. in [15]. Applying Formula 16.11.8 therein with \( q = \kappa = 2 \) and \( \nu = a-b-c-d+1/2 > -2(a+b) \), we get

\[
{\binom{a+b}{c+b+2}\binom{-x}{2}} = \frac{x^{\nu/2}}{\sqrt{\pi}} \cos(\sqrt{x + \nu \pi / 2}) + o(x^{\nu/2}) \quad \text{as } x \to \infty,
\]

which shows that the function on the left-hand side takes negative values on \((0, \infty)\). \(\square\)

**Remark.** (a) For fixed \( a, b > 0 \), the existence set defined in (a) is convex with extremal points \((2a + b, a + 1/2)\) and \((a + 1/2, 2a + b)\). The main argument for (a) is the analysis on these extremal points, which is a consequence of the Weber-Schafheitlin formula.

(b) As expected, the leading term in the asymptotic expansion of the hypergeometric function \( {\binom{a+b}{c+b+2}\binom{-x}{2}} \) becomes positive when \( c + d > 3a + b + 1/2 \) viz. \( \nu < -2(a+b) \). It can be shown from Formula 16.11.8 in [15] and a finer analysis that the leading term takes negative values when \( c + d = 3a + b + 1/2 \) and \( \min(c, d) < \min(2a + b, a + 1/2) \). We omit details.

(c) Setting \( X_{a,b}^s \) for the random variable corresponding to the extremal distribution

\[
D \left[ \frac{a}{(2a + b, a + 1/2)} - \right],
\]

the same argument leading to (7) implies

\[
\mathbb{E}[X_{a,b}^s] = \exp \left\{ (\psi(a) - \psi(b, c, d)) s + \int_{-\infty}^0 (e^{sx} - 1 - sx) \left( \frac{\varphi_a(|x|) - \varphi_{c,d}(|x|)}{|x|(1 - e^{-|x|})} \right) dx \right\}
\]

\[
+ \int_0^\infty (e^{sx} - 1 - sx) \left( \frac{\varphi_b(x)}{x(1 - e^{-x})} \right) dx \}
\]

for every \( s \in (-a, b) \). In the recent terminology of [13], this means that \( \log X_{a,b} \) is quasi-infinitely divisible (QID) with quasi-Lévy measure having density

\[
\frac{1}{|x|(1 - e^{-|x|})} \left( (\varphi_a(|x|) - \varphi_{c,d}(|x|))1_{\{x<0\}} + \varphi_b(x)1_{\{x>0\}} \right).
\]

This density, which is negative in an interval \((a_*, 0)\) with \( a_* < 0 \), vanishes at \( a_* \), and is positive otherwise, tends to \(-\infty\) and is not integrable at \( 0^- \). This implies that \( \log X_{a,b} \) is not ID and that (6) fails. It would be interesting to construct more general families of QID distributions with the help of the Gamma function.

Let us now mention a close connection between Proposition 2 and the recent paper [5]. From the definition of a Mellin transform, for every \( a, b, c, d > 0 \) one has

\[
D \left[ \frac{a}{(c,d)} - \right] \text{ exists } \iff \ D \left[ \frac{a+t}{(c+t,d+t)} - \right] \text{ exists for every } t \in (-a,b),
\]

so that the equivalence (8) can be rewritten

\[
{\binom{a}{b}\binom{-x}{c}} \geq 0 \quad \text{for all } x \geq 0 \iff \ D \left[ \frac{(1-\lambda)a}{(b-\lambda a, c-\lambda a)} - \right] \text{ exists for every } \lambda \in (0, 1).
\]

It is then easy to see that Theorem 6.1 in [5] is actually equivalent to our Proposition 2. Notice that our proof is considerably simpler than all the arguments involved in [5]. This connection with \( {\binom{a}{b}\binom{-x}{c}} \)
Proposition 2 shows that the argument for the second case in Part (a) of Proposition 2 shows clearly that

\[ f_{a,b}(c) \leq d \leq c. \]

Moreover,

(a) for \( u \in [(3a + b)/2 + 1/4, \max(2a + b, a + 1/2)] \), one has \( f_{a,b}(u) = 3a + b + 1/2 - u \).

(b) for \( u > \max(2a + b, a + 1/2) \), one has \( f_{a,b}(u) \in ]a, a + (a + b)/2(u - a)] \).

In particular, one has \( f_{a,b}(u) \to a \) as \( u \to \infty \).

Proof. Fix \( a, b > 0 \) and consider

\[ D_{a,b} = \{ (c, d) \in (a, \infty) \times (a, \infty) \text{ such that } D \left[ \begin{array}{c} a \\ (c, d) \end{array} \right] \text{ exists} \}. \]

It is clear from the definition that the set \( D_{a,b} \) is closed and symmetric with respect to the line \( \{ c = d \} \). Proposition 2 shows that \( D_{t,a,b} = D_{a,b} \cap \{ c + d = t \} \) is non-empty if and only if \( t \geq 3a + b + 1/2 \). The argument for the second case in Part (a) of Proposition 2 shows clearly that \( D_{t,a,b} \) is a closed segment \([(x_t, y_t), (y_t, x_t)] \) with \( x_t \in (a, m) \) for every \( t \geq 3a + b + 1/2 \). Besides, the easy fact that

\[ (c, d) \in D_{a,b} \implies (c, d + s) \in D_{a,b} \text{ for every } s \geq 0 \]

which was used in the argument for the third case in Part (a) of Proposition 2, implies that the function \( t \mapsto x_t \) is non-increasing and that the function \( t \mapsto y_t \) is increasing. Finally, both functions are clearly continuous by the closedness of \( D_{a,b} \). Consider now the mapping

\[ f_{a,b}(u) = \begin{cases} 3a + b + 1/2 - u & \text{if } u \in [(3a + b)/2 + 1/4, \max(2a + b, a + 1/2)] \\ x(y^{-1}(u)) & \text{if } u > \max(2a + b, a + 1/2), \end{cases} \]

which is continuous and non-increasing on \([(3a + b)/2 + 1/4, \infty)\). Putting everything together with Proposition 2 implies the required equivalence (9), and it is clear that (a) is fulfilled. To show (b) and conclude the proof, it suffices to combine (8) and Theorem 4.2 in [6].

\[ \square \]

From the above discussion, the latter result can also be expressed in terms of the hypergeometric function \( _1F_2 \). More precisely, it follows from (8) that

\[ _1F_2 \left[ \begin{array}{c} 2a \\ b \\ c \end{array} ; -x \right] \geq 0 \text{ for all } x \geq 0 \iff (b - a, c - a) \in D_{a,a} \]

for every \( a, b, c > 0 \). This functional representation for the diagram of non-negativity of \( _1F_2 \) on the negative half-line seems unnoticed. Following the introduction in [6], one can also rephrase a problem of Askey and Szegö in the following way: for every \( a, b > 0 \) one has

\[ \int_0^x t^{b-a}J_{a+b-1}(t) \, dt \geq 0 \text{ for all } x \geq 0 \iff (a, 1) \in D_{b,b} \]
Perhaps can this probabilistic reformulation of an old and famous problem on Bessel functions of the first kind - see also Chapter 7.6 in [1] for a discussion - be useful. We would like to finish this paper with the following open problem, which is natural in view of Remark (a).

**Conjecture.** For every $a, b > 0$, the set $D_{a,b}$ is convex.

By Proposition 3, this conjecture amounts to the convexity of the function $f_{a,b}$. This would imply that the latter function is also decreasing, in other words that the non-increasing function $t \mapsto x_t$ which was introduced during the proof of Proposition 3 is actually decreasing. But we were not able to prove this.

**APPENDIX**

**A.1. Proof of Lemma A.** Expanding $e^{itz}$ as a series, switching the sum and the integral, cancelling the odd terms and changing the variable, the right-hand side transforms into

$$
\frac{1}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \left( \frac{z}{2} \right)^\alpha \sum_{n \geq 0} (-1)^n \frac{z^{2n}}{(2n)!} \int_0^1 t^{n-1/2}(1-t)^{\alpha-1/2} \, dt = \left( \frac{z}{2} \right)^\alpha \sum_{n \geq 0} (-1)^n \frac{\Gamma(n + 1/2) z^{2n}}{\sqrt{\pi} (2n)! \Gamma(n + \alpha + 1)}
$$

as required.

**A.2. Proof of Lemma B.** The original proof of the Weber-Schafheitlin formula is the consequence of a more general result which involves a quadratic transformation of some hypergeometric function. See [19] p.402 for details and also Exercises 4.14 and 4.15 in [1] for a more modern presentation of this result. We follow here another, apparently unnoticed and overall simpler argument relying on Lemma A, the Fresnel integral and the Selberg integral. For every $\alpha > -1/2$ and $s \in (0, \alpha + 1/2)$, Lemma A implies first

$$
\int_0^\infty z^{-2s} J_\alpha^2(z) \, dz = \frac{1}{\pi 4^a \Gamma(\alpha + 1/2)^2} \int_0^\infty z^{2\alpha-2s} \left( \int_{[-1,1]^2} \cos(z(t+u))((1-t^2)(1-u^2))^{\alpha-1/2} \, dt \, du \right) \, dz.
$$

Supposing next $s > \alpha$, this transforms into

$$
\int_0^\infty z^{-2s} J_\alpha^2(z) \, dz = \frac{1}{\pi 4^a \Gamma(\alpha + 1/2)^2} \int_{[-1,1]^2} \left( (1-t^2)(1-u^2) \right)^{\alpha-1/2} \left( \int_0^\infty z^{2\alpha-2s} \cos(z(t+u)) \, dz \right) \, dt \, du
$$

$$
= \frac{1}{4^{\alpha+1/2} \Gamma(\alpha + 1/2) \Gamma(2s - 2\alpha) \cos(\pi(s-\alpha))} \times \int_{[-1,1]^2} ((1-t^2)(1-u^2))^{\alpha-1/2} |t-u|^{2s-2\alpha-1} \, dt \, du
$$

where in the second equality we have used the Fresnel integral - see e.g. Exercise 1.19 in [1] - and the change of variable $u \mapsto -u$. On the other hand, one has

$$
4^{\alpha+1/2} \Gamma(2s - 2\alpha) \cos(\pi(s-\alpha)) = \frac{4^{\alpha+1/2} \Gamma(2s - 2\alpha)}{\Gamma(s - \alpha + 1/2) \Gamma(s + 1/2)} = \frac{4^s \sqrt{\pi} \Gamma(s-\alpha)}{\Gamma(\alpha + 1/2 - s)}
$$
by the complement and multiplication formulas for the Gamma function. Therefore,

\[
\int_0^\infty z^{-2s} J_\alpha^2(z) \, dz = \frac{\Gamma(\alpha + 1/2 - s)}{4^s \sqrt{\pi} \Gamma(\alpha + 1/2)^2 \Gamma(s - \alpha)} \int_{[-1,1]^2} ((1 - t^2)(1 - u^2))^{\alpha - 1/2} |t - u|^{2s-2\alpha-1} \, dt \, du
\]

\[
= \frac{4^{\alpha-1/2} \Gamma(\alpha + 1/2 - s)}{\sqrt{\pi} \Gamma(\alpha + 1/2)^2 \Gamma(s - \alpha)} \int_{[0,1]^2} (t(1-t)u(1-u))^{\alpha - 1/2} |t - u|^{2s-2\alpha-1} \, dt \, du
\]

\[
= \frac{4^\alpha \Gamma(s)^2 \Gamma(2s - 2\alpha) \Gamma(\alpha + 1/2 - s)}{\Gamma(s) \Gamma(\alpha + 1/2 - s) \Gamma(s - \alpha + 1/2) \Gamma(\alpha + 1/2 + s) \Gamma(2s - 2\alpha) \Gamma(\alpha + 1/2 + s)}
\]

for every \( s \in (\max(0, \alpha), \alpha + 1/2) \), where in the third equality we have used the Selberg integral - see e.g. Theorem 8.1.1 in [1] - and the fourth equality follows from Gauss’ multiplication formula - see e.g. Theorem 1.5.1 in [1]. By analytic continuation, the formula remains true for every \( s \in (0, \alpha + 1/2) \).

\[\square\]

References


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