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A characteristic of gyroisometries in Möbius gyrovector spaces

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Abstract

Hugo Steinhaus (1966a, b) has asked whether inside each acute angled triangle there is a point from which perpendiculars to the sides divide the triangle into three parts of equal areas. In this paper, we prove that $f : \mathbb{D} \to \mathbb{D}$ is a gyroisometry (hyperbolic isometry) if, and only if it is a continuous mapping that preserves the partition of a gyrotriangle (hyperbolic triangle) asked by Hugo Steinhaus.

Keywords: Gyrogroups, Möbius Gyrovector Spaces, Gyroisometry.

MSC: 51B10, 30F45, 20N05, 51M09, 51M10, 51M25.

1 Introduction

A Möbius transformation $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a mapping of the form $w = (az+b)/(cz+d)$ satisfying $ad - bc \neq 0$, where $a, b, c, d \in \mathbb{C}$. The set of all Möbius transformations Möb$(\mathbb{C} \cup \{\infty\})$ is a group with respect to the composition and any $f \in$ Möb$(\mathbb{C} \cup \{\infty\})$ is conformal, i.e. it preserves angles. Let us define

$$\Omega = \{S \subset \mathbb{C} \cup \{\infty\} : \text{ S is an Euclidean circle or a Euclidean line } \cup \infty\}.$$

It is well known that if $C \in \Omega$ and $f \in$ Möb$(\mathbb{C} \cup \{\infty\})$, then $f(C) \in \Omega$. There are well-known elementary proofs that if f is a continuous injective map of the extended complex plane $\mathbb{C} \cup \{\infty\}$ that maps circles into circles, then f is Möbius. In addition to this a map is Möbius if, and only if it preserves cross ratios. In function theory, it is known that a function f is Möbius if, and only if the Schwarzian derivative of f vanishes when $f'(z) \neq 0$. Using this differential criterion, H. Haruki and T.M. Rassias[2] proved that if f is meromorphic and preserves Apollonius quadrilaterals, then f is Möbius.

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[2]Haruki and Rassias, 1998, "A new characteristic of Möbius transformations by use of Apollonius quadrilaterals".

The Möbius invariant property is also naturally related to hyperbolic geometry. For instance, in the hyperbolic plane, Möbius transformations can be characterized by Lambert (and Saccheri) quadrilaterals, i.e., a continuous bijection which maps Lambert quadrilaterals to Lambert quadrilaterals (or Saccheri quadrilaterals to Saccheri quadrilaterals) must be Möbius, see Yang and Fang (2006a), Yang and Fang (2006b). Moreover, in literature there are many characterizations of Möbius transformations by using of triangular domains[3], regular hyperbolic polygons[4], hyperbolic regular star polygons[5], polygons having type A[6], and others.

2 Möbius transformations of the Disc, Möbius Addition and Möbius Gyrovector Spaces

Let us denote the complex open unit disc (centered at origin) in $\mathbb{C}$ by $\mathbb{D}$ and z_0 be an element of $\mathbb{C}$. Clearly the mapping

$$f(z) = e^{i\theta}\frac{z_0 + z}{1 + \overline{z_0}z}, \qquad \theta \in \mathbb{R}$$

is a Möbius transformation satisfying $f(\mathbb{D}) = \mathbb{D}$. L. Ahlfors[7] proved that the most general Möbius transformation of $\mathbb{D}$ is given by the polar decomposition

$$z \to e^{i\theta}\frac{z_0 + z}{1 + \overline{z_0}z} = e^{i\theta}(z_0 \oplus z).$$

It induces the Möbius addition "$\oplus$" in the disc, allowing the Möbius transformation of the disc to be viewed as Möbius *left gyrotranslation*

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0}z}$$

followed by rotation. Here $\theta \in \mathbb{R}$, $z_0 \in \mathbb{D}$ and Möbius substraction "$\ominus$" is defined by $a \ominus z = a \oplus (-z)$. Clearly $z \ominus z = 0$ and $\ominus z = -z$.The groupoid $(\mathbb{D}, \oplus)$ is not a group since the groupoid operation "$\oplus$" is not associative. In addition to this the commutative property does not hold. However, the groupoid $(\mathbb{D}, \oplus)$ has a group-like structure.

The breakdown of commutativity in Möbius addition is "repaired" by the introduction of gyration,

$$gyr : \mathbb{D} \times \mathbb{D} \to Aut(\mathbb{D}, \oplus)$$

[3]Li and Wang, 2009, "A new characterization for isometries by triangles".

[4]Demirel and Seyrantepe, 2011.

[5]Demirel, 2013, "A characterization of Möbius transformations by use of hyperbolic regular star polygons".

[6]Liu, 2006, "A new characteristic of Möbius transformations by use of polygons having type A".

[7]Ahlfors, 1978, *Complex analysis: An introduction to the theory of analytic functions of one complex variable*.

given by the equation

$$gyr\,[a,b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b} \tag{1}$$

where $Aut(\mathbb{D},\oplus)$ is the automorphism group of the groupoid $(\mathbb{D},\oplus)$. Therefore, the *gyrocommutative law* of Möbius addition $\oplus$ follows from the definition of gyration in (1),

$$a \oplus b = gyr\,[a,b]\,(b \oplus a). \tag{2}$$

Coincidentally, the gyration $gyr[a,b]$ that repairs the breakdown of the commutative law of $\oplus$ in (2), repairs the breakdown of the associative law of $\oplus$ as well, giving rise to the respective *left* and *right gyroassociative laws*

$$\begin{aligned} a \oplus (b \oplus c) &= (a \oplus b) \oplus gyr\,[a,b]\,c \\ (a \oplus b) \oplus c &= a \oplus (b \oplus gyr\,[b,a]\,c) \end{aligned}$$

for all $a,b,c \in \mathbb{D}$.

Definition 1 – A groupoid $(\mathbb{G},\oplus)$ is a gyrogroup if its binary operation satisfies the following axioms

($\mathbb{G}$1) For each $a \in G$, there is an element $0 \in G$ such that $0 \oplus a = a$.

($\mathbb{G}$2) For each $a \in G$, there is an element $b \in G$ such that $b \oplus a = 0$.

($\mathbb{G}$3) For all $a,b \in G$, there is an automorphism $gyr[a,b] \in Aut(G,\oplus)$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a,b]c$$

($\mathbb{G}$4) For all $a,b \in G$, $gyr\,[a,b] = gyr\,[a \oplus b, b]$.

Additionally, if the binary operation "$\oplus$" obeys the *gyrocommutative law* $\mathbb{G}$5, then $(\mathbb{G},\oplus)$ is called a gyrocommutative gyrogroup.

($\mathbb{G}$5) For all $a,b \in G$, $a \oplus b = gyr\,[a,b]\,(b \oplus a)$.

Clearly, with these properties, one can now readily check that the Möbius complex disc groupoid $(\mathbb{D},\oplus)$ is a gyrocommutative gyrogroup. We refer readers to Ungar (2001, 2008) for more details about gyrogroups.

Identifying complex numbers of the complex plane $\mathbb{C}$ with vectors of the Euclidean plane $\mathbb{R}^2$ in the usual way:

$$\mathbb{C} \ni u = u_1 + iu_2 = (u_1, u_2) = \mathbf{u} \in \mathbb{R}^2.$$

Then the equations

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \mathrm{Re}(\overline{u}v) \\ \|\mathbf{u}\| &= |u|. \end{aligned} \tag{3}$$

give the inner product and the norm in $\mathbb{R}^2$, so that Möbius addition in the disc $\mathbb{D}$ of $\mathbb{C}$ becomes Möbius addition in the disc $\mathbb{R}_1^2 = \left\{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\| < 1\right\}$ of $\mathbb{R}^2$. In fact we get from (3)

$$\begin{aligned} u \oplus v &= \frac{u+v}{1+\bar{u}v} \\ &= \frac{(1+u\bar{v})(u+v)}{(1+\bar{u}v)(1+u\bar{v})} \\ &= \frac{\left(1+\bar{u}v+u\bar{v}+|v|^2\right)u+\left(1-|u|^2\right)v}{1+\bar{u}v+u\bar{v}+|u|^2\,|v|^2} \\ &= \frac{\left(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2\right)\mathbf{u}+\left(1-\|\mathbf{u}\|^2\right)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2} \\ &= \mathbf{u}\oplus\mathbf{v} \end{aligned} \tag{4}$$

for all $u, v \in \mathbb{D}$ and all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_1^2$.

Let $\mathbb{V} = (\mathbb{V}, +, \cdot)$ be any inner-product space and

$$\mathbb{V}_s = \{v \in \mathbb{V} : \|v\| < s\}$$

be the open ball of $\mathbb{V}$ with radius $s > 0$. Möbius addition in $\mathbb{V}_s$ is motivated by (4) and it is given by the equation

$$\mathbf{u}\oplus\mathbf{v} = \frac{\left(1+\left(2/s^2\right)\mathbf{u}\cdot\mathbf{v}+\left(1/s^2\right)\|\mathbf{v}\|^2\right)\mathbf{u}+\left(1-\left(1/s^2\right)\|\mathbf{u}\|^2\right)\mathbf{v}}{1+(2/s^2)\,\mathbf{u}\cdot\mathbf{v}+\left(1/s^4\right)\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2} \tag{5}$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$. Without loss of generality, we may assume that $s = 1$ in (5). However we prefer to keep s as a free positive parameter in order to exhibit the results that in the limit as $s \to \infty$, the ball $\mathbb{V}_s$ expands to the whole of its real inner product space $\mathbb{V}$, and Möbius addition $\oplus$ reduces to vector addition $+$ in $\mathbb{V}$, i.e.,

$$\lim_{s\longrightarrow\infty} \mathbf{u}\oplus\mathbf{v} = \mathbf{u}+\mathbf{v}$$

and

$$\lim_{s\to\infty} \mathbb{V}_s = \mathbb{V}.$$

Möbius scalar multiplication "$\otimes$" is given by the equation

$$r\otimes\mathbf{v} = s\tanh\left(r\tanh^{-1}\|\mathbf{v}\|/s\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{6}$$

where $r \in \mathbb{R}$, $\mathbf{u},\mathbf{v} \in \mathbb{V}_s$, $\mathbf{v} \neq 0$ and $r \otimes 0 = 0$. Möbius scalar multiplication possesses the following properties:

(P1) $n \otimes \mathbf{v} = \mathbf{v} \oplus \mathbf{v} \oplus \cdots \oplus \mathbf{v}$, ($n$-term)

(P2) $(r_1 + r_2) \otimes \mathbf{v} = r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}$ scalar distribute law

(P3) $(r_1 r_2) \otimes \mathbf{v} = r_1 \otimes (r_2 \otimes \mathbf{v})$ scalar associative law

(P4) $r \otimes (r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}) = r \otimes (r_1 \otimes \mathbf{v}) \oplus r \otimes (r_2 \otimes \mathbf{v})$ monodistribute law

(P5) $\|r \otimes \mathbf{v}\| = |r| \otimes \|\mathbf{v}\|$ homogenity property

(P6) $\frac{|r| \otimes \mathbf{v}}{\|r \otimes \mathbf{v}\|} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ scaling property

(P7) $\mathrm{gyr}[\mathbf{a},\mathbf{b}](r \otimes \mathbf{v}) = r \otimes \mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v}$ gyroautomorphism property

(P8) $1 \otimes \mathbf{v} = \mathbf{v}$ multiplicative unit property

Definition 2 (Möbius gyrovector spaces) – Let $(\mathbb{V}_s,\oplus)$ be a Möbius gyrogroup equipped with scalar multiplication $\otimes$. The triple $(\mathbb{V}_s,\oplus,\otimes)$ is called a Möbius gyrovector space.

Definition 3 – The Möbius gyrodistance between the points $\mathbf{A},\mathbf{B}$ in Möbius gyrovector space $(\mathbb{V}_s,\oplus)$ is given by the equation

$$d(\mathbf{A},\mathbf{B}) = \|\mathbf{A} \ominus \mathbf{B}\|.$$

The Möbius gyrodistance function, in general gyrodistance function, gives rise to a gyrotriangle inequality which involves a gyroaddition $\oplus$. In contrast, the familiar hyperbolic distance function in the literature is designed so as to give rise to a triangle inequality which involves the addition +. The connection between the gyrodistance function and the standard hyperbolic distance function is described in Ungar (1999).

Definition 4 – A map $\phi : (\mathbb{V}_s,\oplus) \to (\mathbb{V}_s,\oplus)$ is a gyroisometry of $(\mathbb{V}_s,\oplus)$ if it preserves the gyrodistance between any two points of $(\mathbb{V}_s,\oplus)$, that is, if

$$d(\phi(\mathbf{A}),\phi(\mathbf{B})) = d(\mathbf{A},\mathbf{B})$$

for all $\mathbf{A},\mathbf{B} \in \mathbb{V}_s$, see Ungar (2005).

A.A. Ungar[8] proved that a map ϕ defined from Einstein gyrovector space $(\mathbb{R}^n_s,\oplus)$ to itself is a gyroisometry if and only if the map ϕ is of the form

$$\phi(\mathbf{X}) = A \oplus R(\mathbf{X})$$

where $R \in O(n)$ is an $n \times n$ orthogonal matrix $A = \phi(0) \in \mathbb{R}^n_s$, 0 being the origin of $\mathbb{R}^n_s$. This theorem is also valid in Möbius gyrovector space.

[8] Ungar, 2014, *An introduction to hyperbolic barycentric coordinates and their applications.*

3 Möbius Gyroline and Möbius Gyrotriangle

In full analogy with straight lines in the standard vector space approach to Euclidean geometry, a Möbius gyroline (briefly a gyroline) passing through the point P and has a directional vector $\mathbf{u}$ in the ball $\mathbb{V}_1$, is represented by

$$\alpha(t) = P \oplus (\mathbf{u} \otimes t).$$

For more details about gyrovectors, we refer to Ungar (2005). A gyroline passing through the points K and L is represented by

$$\alpha_{KL}(t) = K \oplus (\ominus K \oplus L) \otimes t$$

as expected, in full analogy with Euclidean geometry.

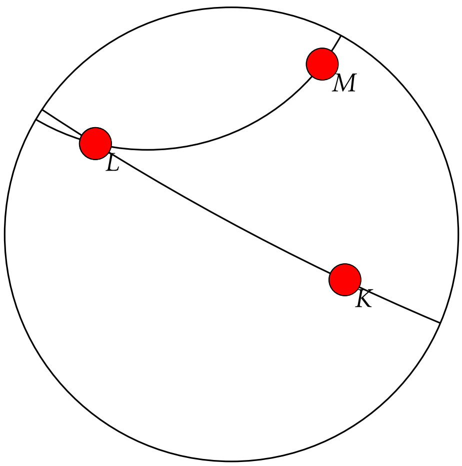

Figure 1 – A gyroline passing through the points K and L is a circular arc that intersect the disc $\mathbb{D}$ orthogonally. The gyrolines passing through the center of the disc are also correspond to chords of the disc.

A Möbius gyrotriangle ΔKLM (briefly a gyrotriangle) in the ball $\mathbb{V}_1$ is shown in *Fig. 2.* It has vertices $K, L, M \in \mathbb{V}_1$, sides $\mathbf{k}, \mathbf{l}, \mathbf{m} \in \mathbb{V}_1$ and side gyrolengths $-1 < k, l, m < 1$,

$$\mathbf{a} = \ominus L \oplus M, \qquad a = \|\mathbf{a}\|$$
$$\mathbf{b} = \ominus M \oplus K, \qquad b = \|\mathbf{b}\|$$
$$\mathbf{c} = \ominus K \oplus L, \qquad c = \|\mathbf{c}\|$$

The following equations allow us to find the gyroangle measures α, β and γ of the gyroangles at the vertices of the gyrotriangle ΔKLM:

$$cos\alpha = \frac{\ominus K \oplus L}{\|\ominus K \oplus L\|} \cdot \frac{\ominus K \oplus M}{\|\ominus K \oplus M\|}$$
$$cos\beta = \frac{\ominus L \oplus K}{\|\ominus L \oplus K\|} \cdot \frac{\ominus L \oplus M}{\|\ominus L \oplus M\|}$$
$$cos\gamma = \frac{\ominus M \oplus K}{\|\ominus M \oplus K\|} \cdot \frac{\ominus M \oplus L}{\|\ominus M \oplus L\|}$$

A most important advantage of studying hyperbolic geometry is the fact that the gyrotriangle gyroangles determine uniquely its side gyrolengths as follows:

Theorem 1 – *Let ΔKLM be gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_1, \oplus, \otimes)$ with vertices K, L, M, corresponding gyroangles α, β, γ and side gyrolengths k, l, m, as shown in* Fig. 2. *Then the following equations hold:*

$$k^2 = \frac{\cos\alpha + \cos(\beta+\gamma)}{\cos\alpha + \cos(\beta-\gamma)}$$
$$l^2 = \frac{\cos\beta + \cos(\alpha+\gamma)}{\cos\beta + \cos(\alpha-\gamma)}$$
$$m^2 = \frac{\cos\gamma + \cos(\alpha+\beta)}{\cos\gamma + \cos(\alpha-\beta)}$$

For more details, we refer to Ungar (2005).

The gyroarea $\Delta(ABC)$ of gyrotriangle ABC is given by

$$\Delta(ABC) = \frac{1}{2}\tan\frac{\delta}{2}$$

where δ is called the defect of gyrotriangle ABC defined by $\delta = \pi - (\alpha+\beta+\gamma)$, see Ungar (2005). Similarly the gyroarea $\Delta(ABCD)$ of gyroquadrilateral $ABCD$ with $\angle DAB = \alpha_1, \angle ABC = \alpha_2, \angle BCD = \alpha_3, \angle CDA = \alpha_4$ is given by

$$\Delta(ABCD) = \frac{1}{2}\tan\frac{\delta}{2}$$

where δ is called the defect of gyroquadrilateral $ABCD$ defined by $\delta = 2\pi - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$.

4 A Characteristic of Gyroisometries in Möbius Gyrovector Spaces

Theorem 2 – *(Kuratowski-Steinhaus[9]) Let $T \subseteq R^2$ be a bounded measurable set, and let $|T|$ be the measure of T. Let $\theta_1, \theta_2, \theta_3$ be the angles determined by three rays emanating*

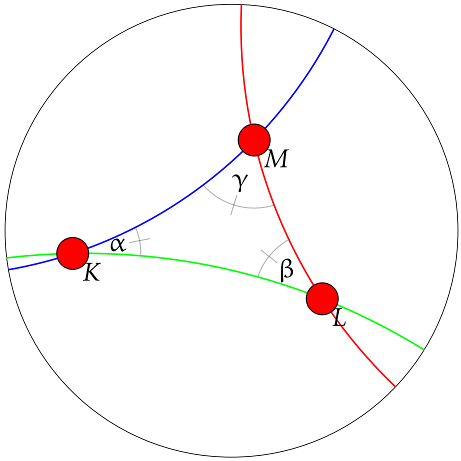

Figure 2 – A Möbius gyrotriangle in the unit disc $\mathbb{D}$.

from a point, and let $\theta_1 < \pi, \theta_2 < \pi, \theta_3 < \pi$. Let r_1, r_2, r_3 be nonnegative numbers such that $r_1 + r_2 + r_3 = |T|$ Then there exists a translation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $|f(T) \cap \theta_1| = r_1, |f(T) \cap \theta_2| = r_2, |f(T) \cap \theta_3| = r_3$.

H. Steinhaus asked whether inside each acute angled triangle there is a point from which perpendiculars to the sides divide the triangle into three parts with equal areas, see Steinhaus (1966a,b). For the solution of this problem, we refer to Tyszka (2007).

Naturally, one may wonder whether the solution of this problem exists in hyperbolic geometry. In Demirel (2018), O. Demirel solved this problem in the Poincaré disc model of hyperbolic geometry. Now, we try to get a characteristic of gyroisometries by use of the partition of a gyrotriangle asked by Hugo Steinhaus.

Example 1 – Let ABC be an equilateral gyrotriangle in Möbius gyrovector space $(\mathbb{D}, \oplus, \otimes)$ with vertices A, B, C satisfying $\angle ABC = \angle BCA = \angle CAB = \alpha$ and $|A \ominus B| = |B \ominus C| = |C \ominus A| = p$. Let us denote the gyromidpoints of the segments AB, AC, BC by M_{AB}, M_{AC}, M_{BC}, respectively and D be the gyrocentroid of ABC. Since $\angle DM_{AB}B = \angle DM_{BC}C = \angle DM_{CA}A = \frac{\pi}{2}$, then it is clear that

$$\Delta(AM_{AB}DM_{AC}) = \Delta(BM_{BC}DM_{AB}) = \Delta(CM_{AC}DM_{BC}).$$

[9]Kuratowski and Steinhaus, 1985, *Une application géométrique du théoréme de Brouwer sur les points invariants*.

Notice that if ABC is an acute angled isosceles gyrotriangle satisfying $d(A,B) = d(A,C)$, then one can easily see that there exists a point on $[A,H]$, where H is the midpoint of B and C, such that

$$\Delta(AM_{AB}DM_{AC}) = \Delta(BM_{BC}DM_{AB}) = \Delta(CM_{AC}DM_{BC})$$

holds.

Throughout the paper, we denote by X' the image of X under f, by $[P,Q]$ the geodesic segment between points P and Q, by PQ the gyroline through points P and Q. If we say f preserves the Steinhaus partition of gyrotriangles, this means that for all acute angled gyrotriangles ABC in $(\mathbb{D},\oplus,\otimes)$, if P divides ABC into three parts of equal gyroareas by the perpendiculars drawn from P to the sides of ABC satisfying

$$\Delta(AM_1PM_3) = \Delta(BM_2PM_1) = \Delta(CM_3PM_2),$$

then P' divides $A'B'C'$ into three parts of equal gyroareas by the perpendiculars drawn from P' to the sides of $A'B'C'$ satisfying

$$\Delta(A'M_1'P'M_3') = \Delta(B'M_2'P'M_1') = \Delta(C'M_3'P'M_2').$$

Notice that the points P and P' must be interior points of the gyrotriangles ABC and $A'B'C'$, respectively.

Naturally, one may wonder whether the solution of the Steinhaus problem for an arbitrary acute angled hyperbolic triangle exists? For the affirmative answer of this question we refer to Demirel (2018).

Lemma 1 – *Let $f : \mathbb{D} \to \mathbb{D}$ be a mapping which preserves the Steinhaus partition of gyrotriangles, then f is injective.*

Proof. Let us take two distinct points K, L in $\mathbb{D}$. Then there exists a point M in $\mathbb{D}$ such that KLM is an equilateral gyrotriangle. By Example 1 on the preceding page, the gyrocentroid of KLM, say P, divides KLM into three parts of equal gyroareas. Therefore, P' divides $K'L'M'$ into three parts of equal gyroareas which implies $K' \neq L'$. Therefore, f is injective. □

Lemma 2 – *Let $f : \mathbb{D} \to \mathbb{D}$ be a mapping which preserves the Steinhaus partition of gyrotriangles. If K, L, M_1 are three gyrocollinear points in $\mathbb{D}$ such that $d(K,M_1) = d(M_1,L)$ then the points K', L', M_1' are gyrocollinear.*

Proof. Let K and L be two distinct points in $\mathbb{D}$ and denote the gyromidpoint of these points by M_1. Firstly, there exists a point S in $\mathbb{D}$ such that KLS is an equilateral gyrotriangle. Let C be the gyrocentroid of KLS and M_2, M_3 be the gyromidpoints of $[L,S],[S,K]$, respectively. By Example 1 on the preceding page, we have

$$\Delta(KM_1CM_3) = \Delta(LM_2CM_1) = \Delta(SM_3CM_2),$$

and by the property of f, we get

$$\Delta(K'M_1'C'M_3') = \Delta(L'M_2'C'M_1') = \Delta(S'M_3'C'M_2').$$

Hence we obtain $\angle C'M_1'L' = \angle C'M_2'S' = \angle C'M_3'K' = \frac{\pi}{2}$, which implies that K', M_1', L' are gyrocollinear points. □

Lemma 3 – *Let $f : \mathbb{D} \to \mathbb{D}$ be a mapping which preserves the Steinhaus partition of gyrotriangles, then f preserves the right gyroangles.*

Proof. Let l_1 and l_2 be two gyrolines in $\mathbb{D}$ such that l_1 meets l_2 perpendicularly. Denote the common point of these gyrolines by M_1. Let K and L be two points on l_1 such that M_1 is the gyromidpoint of K and L. Then, there exists a point on l_2, say S, such that KLS is an equilateral gyrotriangle. As in Example 1 on p. 114, the gyrocentroid of KLS, say C, divides KLS into three parts of equal gyroareas. Hence, by the property of f, we get that C' divides $K'L'S'$ into three parts of equal gyroareas. By Lemma 2 on the previous page we get that $\angle S'M_1'K' = \angle S'M_1'L' = \frac{\pi}{2}$. This ends the proof. □

Lemma 4 – *Let $f : \mathbb{D} \to \mathbb{D}$ be a mapping which preserves the Steinhaus partition of gyrotriangles, then f preserves the gyrolines.*

Proof. Let l_1 and l_2 be two gyrolines in $\mathbb{D}$ such that l_1 meets l_2 perpendicularly. Denote the common point of these gyrolines by M. Let A and B two points on l_1 such that M is the gyromidpoint of A and B. Denote the common points of l_2 with $\partial(\mathbb{D})$ by C, D where $\partial(\mathbb{D})$ is the boundary of $\mathbb{D}$. Clearly there exists a point K on $[C, M]$ such that ABK is a right gyrotriangle with $\angle BKA = \frac{\pi}{2}$. For each point X_i on $[C, K]$ for all $i \in I \subset \mathbb{R}$, it is easy to see that the gyrotriangle BX_iA is an isosceles gyrotriangle. By Demirel (2018) the gyrotriangle BX_iA has a Steinhaus partition for appropriate points Y_i, Z_i, W_i such that

$$\Delta(AMY_iZ_i) = \Delta(BW_iY_iM) = \Delta(W_iX_iZ_iY_i)$$

where $Y_i \in [M, X_i], Z_i \in [A, X_i], W_i \in [B, X_i]$ for all $i \in I \subset \mathbb{R}$. By hypothesis there exists a Steinhaus partition of the gyrotriangle $B'X_i'A'$ such that

$$\Delta(A'M'Y_i'Z_i') = \Delta(B'W_i'Y_i'M') = \Delta(W_i'X_i'Z_i'Y_i')$$

holds. By Lemma 3, $[M', Y_i']$ meets $[A', B']$ perpendicularly for all $i \in I \subset \mathbb{R}$. This implies that the points Y_i' for $i \in I \subset \mathbb{R}$ are gyrocollinear. When the points A and B are sufficiently close to point M, then the points Y_i are close enough to point M. Finally considering the point D as well as point C one can easily see that the image of l_2 must be a gyroline. □

The proof of the following results is clear from Lemma 4. So we omit it.

Result 1 – *Let $f : \mathbb{D} \to \mathbb{D}$ be a mapping which preserves the Steinhaus partition of gyrotriangles, then f preserves the isosceles gyrotriangles.*

Result 2 – *Let $f : \mathbb{D} \to \mathbb{D}$ be a mapping which preserves the Steinhaus partition of gyrotriangles, then f preserves the equilateral gyrotriangles.*

Theorem 3 – *Let $f : \mathbb{D} \to \mathbb{D}$ be a continuous mapping which preserves the Steinhaus partition of gyrotriangles, then f is a gyroisometry.*

Proof. Let K and L be two distinct points in $\mathbb{D}$. Now construct a sequence consists of equilateral gyrotriangles A_iKA_{i+1} such that $\angle A_iKA_{i+1} = \frac{2\pi}{k}, (1 \le i \le k, k \in \mathbb{Z})$. Clearly we get $A_{k+1} = A_1$. Since f preserves all equilateral gyrotriangles by *Result 4.8* we get that the gyrotriangles A_iKA_{i+1} must be equilateral for all $1 \le i \le k$ and $A'_1 = A'_{i+1}$. It is easy to see that $\angle A'_iK'A'_{i+1} = \frac{2\pi}{k}$ holds for all $1 \le i \le k$. Clearly f preserves $\frac{m\pi}{k}$-valued angles at the vertex K, where m, k are integers. Because of the fact that f is a continuous mapping and the set of rational numbers is dense in $\mathbb{R}$, it follows that f preserves all angles at the vertex K. Therefore, by Theorem 1 on p. 113, we get $d_H(K, L) = d_H(K', L')$. □

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