An Initial-and-Boundary Value Problem for the KP-II equation on a strip and on the half plane

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1. Introduction

In order to describe the evolution of one-dimensional long waves of small amplitude propagating on the surface of water, Boussinesq and later D. J. Korteweg and G. de Vries derived the Korteweg-de Vries equation. The pure Cauchy problem on the whole line for this equation reads as follows

$$\text{(KdV)} \quad : \quad \begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & \text{for} \quad x \in \mathbf{R},\ t \geq 0, \\ u(x,0) = u_0(x) & \text{for} \quad x \in \mathbf{R}. \end{cases}$$

A very large number of mathematical publications are related to the well-posedness of (KdV). Let us just cite some of the most significant [1], [7], [15], [23].

A two dimensional model was given by B. B. Kadomtsev and V. I. Petviashvili when they investigated in [14] the transverse stability of the solitary waves of (KdV). The associated pure initial-value problem on the whole space is

$$\text{(KP)} \quad : \quad \begin{cases} u_t + u_{xxx} + \varepsilon v_y + uu_x = 0 & \text{in} \quad \mathbf{R}^2 \times [0,T), \\ v_x = u_y & \text{in} \quad \mathbf{R}^2 \times [0,T), \\ u(x,y,0) = u_0(x,y) & \text{in} \quad \mathbf{R}^2. \end{cases}$$

This equation is usually called KP-I when $\varepsilon = -1$ and KP-II when $\varepsilon = +1$. The pure Cauchy problem for (KP) was and is still extensively studied. First, S. Ukai proved in [25] a local result for an initial data in some H^s with $s \geq 3$ valid in both case $\varepsilon = -1$ or $\varepsilon = +1$. More specifically for the KP-II equation, the reader can find in a work of J. Bourgain [8], the proof of the global well-posedness for initial data in $L^2(\mathbf{R}^2)$. For less regular initial data we refer to [13], [21], [22] and [24]. Concerning the well-posedness of the KP-I equation, we refer to [11, 18, 19].

Let $L \in]0,+\infty]$ and $\Lambda_L = (0,L)$. A one dimensional model that describes the waves propagating in a channel is the following initial-and-boundary value problem for the KdV equation (see [5]),

$$\mathcal{T}(\Lambda_L) \quad : \quad \begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & \text{for} \quad x \in \Lambda_L,\, t \geq 0, \\ u(x,0) = u_0(x) & \text{for} \quad x \in \Lambda_L, \\ u(0,t) = g(t) & \text{for} \quad t \geq 0, \end{cases}$$

where u_0 is the initial data and g is a time dependent Dirichlet boundary condition.

The first works concerning $\mathcal{T}(\Lambda_L)$ are due to J. Bona and R. Winther in the special case of the quarter plane, that is when $L = +\infty$. In [5], these authors show that the $\mathcal{T}(\mathbf{R}_+)$ system possesses a solution u in $L^\infty_{loc}(\mathbf{R}_+, H^4(\mathbf{R}_+))$ for given $u_0 \in H^4(\mathbf{R}_+)$ and $g \in H^2_{loc}(\mathbf{R}_+)$ such that the compatibility relations $u_0(0) = g(0)$ and $g'(0) + \partial_x u_0(0) + u_0(0)\partial_x u_0(0) + \partial_x^3 u_0(0) = 0$ hold. In [6] the same authors prove that the solution depends continuously on both u_0 and g.

Later, in [9] T. Colin and J. M. Ghidaglia attack the case of a finite spatial domain $\Lambda_L = (0,L)$ with a Dirichlet condition $u(0,t) = g(t)$ at the left boundary and $(u_x(L,t) = h(t),\, u_{xx}(L,t) = k(t))$ at the right boundary. The authors show that this problem is locally well-posed in $H^1(0,L)$. Precisely, for an initial data u_0 in $H^1(0,L)$ there exist $T_L > 0$ and a unique solution u^L in $L^\infty([0,T_L], H^1(0,L)) \cap C^0([0,T_L], L^2(0,L))$. Moreover, the problem becomes globally well-posed (that is $T_L = +\infty$) if the H^1-norm of u_0 and g are small enough. The two works above are based on a long-wave type regularization inspired by [1] and an energy method using strongly the smoothing effects of the KdV equation.

More recently, T. Colin and M. Gisclon (see [10]) proved that these smoothing effects do not depend on L when $\Lambda_L = (0,L)$. Precisely, the authors prove that for a familly of initial data $u_0^L \in L^2((0,L))$ such that

$$\sup_{L\geq 1} \int_0^L |u_0^L(x)|^2(1+x^2) < \infty,$$

and such that u_0^L tends to u_0 strongly in $L^2_{loc}(\mathbf{R}_+)$, for all $T > 0$, if L is large enough, the solution u^L of $\mathcal{T}((0,L))$ is defined on $[0,T]$ and tends to u strongly in $L^p([0,T], L^2_{loc}(\mathbf{R}_+))$ for $1 \leq p < +\infty$ where u is the solution of $\mathcal{T}(\mathbf{R}_+^*)$ with initial data u_0. This method gives in addition the well-posedness of the quarter-plane problem in the weighted space $L^2((1+x^2)dx)$.

A very recent work [2] by J. Bona, S. M. Sun and B.-Y. Zhang, improves these results concerning $\mathcal{T}(\mathbf{R}_+^*)$ by using techniques introduced by

J. Bourgain for the study of nonlinear dispersive wave equations. Problem $\mathcal{T}(\mathbf{R}_+^*)$ is proved to be locally well-posed for u_0 in $H^s(\mathbf{R}_+)$, $s > 3/4$ and g in $H_{loc}^{\frac{s+1}{3}}(\mathbf{R}_+)$. Furthermore, global well-posedness is obtained for u_0 in $H^s(\mathbf{R}_+)$ and g in $H_{loc}^{\frac{3s+7}{12}}(\mathbf{R}_+)$ when $1 \leq s \leq 3$, and for u_0 in $H^s(\mathbf{R}_+)$ and g in $H_{loc}^{\frac{s+1}{3}}(\mathbf{R}_+)$ when $s \geq 3$. In addition the dependence of the solution on the initial data and the boundary condition is proved to be analytic.
The same authors study in [4] problem $\mathcal{T}([0,1])$ and show its global well-posedness in $H^s(0,1)$ for $s \geq 0$. Their other work [3] is devoted to the study of a damped version of $\mathcal{T}(\mathbf{R}_+)$.

In a very nice work [12], J. E. Colliander and C. E. Kenig study the initial-boundary value problem for the generalized K-dV equation on a half-line by adapting the initial value methods of C. E. Kenig, G. Ponce and L. Vega [15] and J. Bourgain [7]. The idea is to replace the initial-boundary value problem by a forced initial value problem. They obtain local and global well-posedness in some H^s.

In order to take into account weak transverse effects as weak perturbations of the wave in the transverse direction, the KP equation must be considered. Let $\mathbf{G}$ denote either the line $\mathbf{R}$ or the circle $\mathbf{S}^1$ and let $\Omega_L := \{(x,y)\,,\, 0 < x < L \text{ and } y \in \mathbf{G}\}$ for a positive L or $L = +\infty$. The initial-and-boundary value problem for KP-II reads

$$\mathcal{P}(\Omega_L^T) \quad : \quad \begin{cases} u_t + u_{xxx} + \partial_x^{-1} u_{yy} + u u_x = 0 & \text{in} \quad \Omega_L \times [0,T), \\ u(x,y,0) = u_0(x,y) & \text{in} \quad \Omega_L, \\ u(0,y,t) = g(y,t) & \text{in} \quad \mathbf{G} \times [0,T), \end{cases}$$

where T is a positive time, u_0 is the initial data and g is a time dependent Dirichlet boundary condition. For u in $C_0^\infty(\overline{\Omega_L})$, the operator ∂_x^{-1} is defined as follows

$$\partial_x^{-1} u(x,y) := -\int_x^{+\infty} u(s,y) ds. \tag{1}$$

In this paper, we show that $\mathcal{P}(\Omega_L^T)$ possesses solutions. In order to close the system, we have to add a set of right boundary conditions. In this paper, we have chosen:

$$< BC_L >: \ \left(\partial_x^{-1} u(L,y,t) = 0, u(L,y,t) = 0, u_x(L,y,t) = 0, \text{ in } \mathbf{G} \times [0,T)\right).$$

We use an energy method in some weighted spaces, very close to that of T. Colin and M. Gisclon [10] for K-dV. Here, the regularizing effects of the KP-II equation are crucial. First, we attack the non homogeneous linear problem associated to $\mathcal{P}(\Omega_L^T)$ with $< BC_L >$ as right boundary conditions and a homogeneous Dirichlet left boundary condition. Precisely, we prove the global well-posedness of this linear problem in the weighted space $\mathcal{H}_L := L^2_{e^x}(\Omega_L)$. Then, by a fixed point method we show that local weak solutions of the $\mathcal{P}(\Omega_L^T)$ initial-and-boundary value problem exist for initial data in $\mathcal{F}_L := \left\{u \in L^2_{(1+x)^2}(\Omega_L),\, u_y \in L^2_{(1+x)^2}(\Omega_L)\right\}$ and boundary data g in a set containing the space $C^0([0,T], H^3(\mathbf{R}_y)) \cap C^1([0,T], H^1(\mathbf{R}_y))$. We use linear energy estimates and anisotropic Sobolev estimates for the nonlinear terms. The solutions are obtained as limit of functions satisfying the right boundary conditions described above. Moreover, we show continuous dependence with respect to the initial and boundary data. If $L = +\infty$, the solution is unique. In the case $L < \infty$, we do not have any uniqueness result, the main obstacle is that we do not know whether the solutions are smooth enough so that the right boundary conditions $< BC_L >$ make sense. The spaces in which the non-linear problem is solved are defined at the beginning of section 4 and the main results are stated in section 5. We point out that our method can be performed with other sets of right boundary data (see Remark 5.2).

Remark 1.1. *In this paper, all the results are stated for domains* $[0, L] \times \mathbf{R}$ *(i.e.* $\mathbf{G} = \mathbf{R}$*). However, they remain valid with only minor changes in the proofs for periodic strips* $[0, L] \times \mathbf{S}^1$.

The sequel is organized as follows. The two next sections deal with the linear problem. The notations and spaces are set in section 2 and the non-homogeneous linear problem with homogeneous boundary conditions is solved in section 3. Moreover useful linear *a priori* estimates are stated at the end of this section. The sections 4, 5 deal with the non-linear problem. In Section 4, we describe our method, introduce new spaces and prove some linear and non-linear estimates in these spaces. The existence of local solutions to the problem $\mathcal{P}(\Omega_L^T)$ for initial data in $\mathcal{F}_L$ is proved in section 5.

2. Notations

We recall that $\Omega_L := \{(x, y)\,,\, 0 < x < L \text{ and } y \in \mathbf{R}\}$ and $\Omega := \Omega_\infty$ denotes the half-plane $\mathbf{R} \times \mathbf{R}_+$. The notation $\|\cdot\|_{L^p(\omega)}$ will be used for the norms of classical Lebesgue spaces on $\omega \subset \mathbf{R}^2$. Let us define

$$L^2_\rho(\Omega_L) := \left\{u \in L^2(\Omega_L),\ \int_{\Omega_L} u^2(x, y)\rho(x)dxdy < \infty\right\},$$

endowed with the scalar product

$$(u,v)_{L^2_\rho(\Omega_L)} := \int_{\Omega_L} u(x,y)v(x,y)\rho(x)dxdy,$$

where $\rho(x)$ is a nonnegative weight. Moreover, for any non negative integer k, we define the Hilbert space

$$X^k_\rho(\Omega_L) := \left\{u \in L^2_\rho(\Omega_L),\ \partial_x^j u \in L^2_\rho(\Omega_L),\ \text{for } j = 0,\dots,k\right\},$$

equipped with the norm

$$|u|^2_{X^k_\rho(\Omega_L)} := \sum_{j=0}^{k} |\partial_x^j u|^2_{L^2_\rho(\Omega_L)}.$$

The linear problem will be studied for data in $X^0_{e^x}(\Omega_L)$. For this reason, we set

$$\mathcal{H}_L := X^0_{e^x}(\Omega_L).$$

The following Proposition gives a meaning to $\partial_x^{-1}u$, for $u \in \mathcal{H}_L$.

Proposition 2.1. *Let $k \geq 0$, $l \geq 0$ and a weight ρ such that $\rho(x) \geq Cx^2$ for some constant C (e. g. $\rho(x) = e^x$, $(1+x)^p$ $p \geq 2$). The linear application* (1) *extends to a unique continuous map from $L^2_\rho(\Omega)$ into $L^2(\Omega)$.*

Proof. Let $u \in C^\infty_c(\overline{\Omega})$ and $\phi := \partial_x^{-1}u$ defined by (1). Integrating by parts and using Cauchy-Schwarz inequality yields $\int_\Omega \phi^2 dxdy = -2\int_\Omega x\phi\phi_x dxdy \leq \left(\int_\Omega x^2u^2dxdy\right)^{\frac{1}{2}}\left(\int_\Omega \phi^2dxdy\right)^{\frac{1}{2}}$ and thus $\int_\Omega \phi^2dxdy \lesssim \int_\Omega u^2\,\rho(x)dxdy$. We conclude by a density argument. □

In the case $L < +\infty$, we extend by zero u which belongs to $\mathcal{H}_L$ as a function on Ω and thus we may define again $\partial_x^{-1}u$ by (1). An important fact is that this choice implies that the boundary condition $\partial_x^{-1}u(L,\cdot) \equiv 0$ is satisfied.

Let us define a space of regular functions with suitable decreasing properties,

$$\mathcal{Y}_L := \left\{f \in \mathcal{S}_L,\ \mathcal{F}_y(f) \in C^\infty_c(\overline{\Omega_L})\right\},$$

where $\mathcal{S}_L := \mathcal{S}(\overline{\Omega_L})$ is a Schwartz space and $\mathcal{F}_y(f)$ denotes the Fourier transform with respect to the y direction. Let us remark that it is classical to prove that $\mathcal{Y}_L$ is a dense subset of $\mathcal{H}_L$ for every positive L. We can add the boundary conditions to these spaces and define

$$\mathcal{Y}_{0,L} := \{u \in \mathcal{Y}_L,\ u(0,\cdot) \equiv 0, \ \text{ and } \ < BC_L >\},$$

$$\mathcal{S}_{0,L} := \{u \in \mathcal{S}_L,\ u(0,\cdot) \equiv 0, \ \text{ and } \ < BC_L >\}.$$

In the particular case $L = +\infty$, the conditions on the right boundary relaxes:

$$\mathcal{S}_\infty = \mathcal{S}_{0,\infty}, \quad \text{and} \quad \mathcal{Y}_\infty = \mathcal{Y}_{0,\infty}.$$

For $I \subset \mathbf{R}$ and E a Banach space, $C^{0,1/2}(I, E)$ will denote the Hölder space of continuous functions f from I into E such that there exists $C > 0$ with

$$|f(x) - f(y)|_E \ \leq \ C\sqrt{|x-y|}, \qquad \forall\, x, y \in I.$$

In all the paper, the notation $A \lesssim B$ means that there exists a universal constant $c \geq 0$ such that $A \leq cB$.

3. The linear problem with homogeneous boundary conditions

In this section, we will use standard evolution equation theory (see [20]) for the study of the non-homogeneous linear problem associated to $\mathcal{P}(\Omega_L)$ namely,

$$\mathcal{L}(\Omega_L) \quad : \quad \begin{cases} \dfrac{du}{dt} + \mathcal{A}u = f, & \text{in} \quad \Omega_L \times [0,T), \\ u(x,y,0) = u_0(x,y) & \text{in} \quad \Omega_L, \\ u(0,y,t) = 0 & \text{in} \quad \mathbf{R} \times [0,T), \end{cases}$$

where $\mathcal{A}$ is the linear operator defined on $C_0^\infty(\overline{\Omega_L})$ given by

$$\mathcal{A}u := u_{xxx} + \partial_x^{-1} u_{yy}.$$

We will define an unbounded operator A_L on $\mathcal{H}_L$ from $\mathcal{A}$ and the boundary conditions $< BC_L >$. Let us begin by the following,

Lemma 3.1. *There exists $L_0 > 0$ such that for $L_0 \leq L \leq +\infty$, and every f in $\mathcal{Y}_L$ there exists a unique u in $\mathcal{S}_{0,L}$ satisfying,*

$$2u + \mathcal{A}u = f \qquad \text{in } \Omega_L, \tag{2}$$

We set $R_L(f) := u$.

Proof. Let us denote by

$$F(x,\eta) := \mathcal{F}_y(f)(x,\eta) = \int_{\mathbf{R}} e^{-iy\eta} f(x,y)dy,$$
$$U(x,\eta) := \mathcal{F}_y(u)(x,\eta) = \int_{\mathbf{R}} e^{-iy\eta} u(x,y)dy,$$

the Fourier transform of f and u in the y direction. Equation (2) reads

$$U_{xxx} + 2U - \eta^2 \partial_x^{-1} U = F \qquad (x,\eta) \in \Omega_L \times \mathbf{R}.$$

Let us fix η in $\mathbf{R}$, differentiating the last equation with respect to x, we get the following ordinary differential system

(3) $$\mathcal{U}_x = S\mathcal{U} + B, \quad \text{in} \quad \Omega_L \times \mathbf{R},$$

where $\mathcal{U}(x,\eta) = \begin{pmatrix} U \\ U_x \\ U_{xx} \\ U_{xxx} \end{pmatrix}$, $B(x,\eta) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ F_x \end{pmatrix}$, and

$$S(\eta) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \eta^2 & -2 & 0 & 0 \end{pmatrix}.$$

Let $\{V_k(\eta),\ k = 1, \dots, 4\}$ the basis of eigenvectors of S associated to the family of eigenvalues $\{\lambda_k(\eta),\ k = 1, \dots, 4\}$. It is well known that $V_k(\eta) = \big(1, \ \lambda_k(\eta), \ \lambda_k^2(\eta), \ \lambda_k^3(\eta)\big)^t$. A direct computation gives the following properties,

- $\lambda_k(\eta)$ is an even smooth function of η, for $k = 1, \dots, 4$,
- $|\lambda_k(\eta)| \approx |\eta|^{1/2}$, when $\eta \to \pm\infty$,
- $\lambda_1(\eta)$ and $\lambda_2(\eta)$ are real functions satisfying $\lambda_1(\eta) \le -2^{\frac{1}{2}}$ and $\lambda_2(\eta) \ge 0$ for all η,
- $\lambda_4 = \overline{\lambda_3}$,
- Finally, writing $\lambda_3(\eta) = r(\eta)\exp(i\theta(\eta))$, we have $r(\eta) \ge 2^{\frac{1}{2}}$, and $\pi/3 \le \theta(\eta) < \pi/2$, $\forall \eta \in \mathbf{R}$.

Thus, a general solution of (3) has the form,

$$(4)\ \ \mathcal{U}(x,\eta)=\sum_{k=1}^{4}\alpha_k(\eta)\exp(\lambda_k(\eta)x)V_k(\eta)$$
$$+\sum_{k=1}^{4}\left(\int_0^x \exp(\lambda_k(\eta)(x-s)) < V_k^*(\eta), B(s,\eta) > ds\right)V_k(\eta),$$

where $\alpha_k(\eta)$ are functions depending only on η, and $\{V_k^*(\eta),\ k=1,\dots,4\}$ is the dual basis of $\{V_k(\eta),\ k=1,\dots,4\}$.

In the special case $L=+\infty$, we can choose

$$\alpha_k=-\int_0^{+\infty}\exp(-\lambda_k(\eta)s) < V_k^*(\eta), B(s,\eta) > ds,$$

for $k=2,3,4$ in order to obtain only bounded solutions. The constant α_1 suffices to fix the boundary conditions $U(0,\eta)=0$ because the first component of $V_1(\eta)$ does not vanish for any value of η. Precisely, we take $\alpha_1=\frac{1}{V_1^1}\left(-V_2^1\alpha_2-V_3^1\alpha_3-V_4^1\alpha_4\right)$ where V_k^j denotes the j-th component of V_k. Clearly, $\mathcal{U}$ belongs to $C^\infty(\overline{\Omega})$, furthermore $\mathcal{U}(0,\cdot)=0$, for $|\eta|$ sufficiently large $\mathcal{U}(\cdot,\eta)=0$, and for x large enough $\mathcal{U}(x,\eta)=\alpha_1(\eta)\exp(\lambda_1(\eta)x)V_1(\eta)$. Thus $u=\mathcal{F}_y^{-1}(\mathcal{U})$ belongs to $\mathcal{S}_\infty$.

In the case $L<+\infty$, we consider the linear map

$$\Xi^L(\eta)\ :\ (\alpha_k(\eta))_{1\le k\le 4}\mapsto\Big(\mathcal{U}(0,\eta),(\mathcal{U}_{xxx}+2\mathcal{U})(L,\eta),\mathcal{U}(L,\eta),\mathcal{U}_x(L,\eta)\Big).$$

Using the fact that $\lambda_1(\eta)$ is negative and that all the other eigenvalues have a non-negative real part, a direct computation gives,

$$\Im(\det\left(\Xi^L(\eta)\right))\le -2e^{-\lambda_1 L}\Im(\lambda_3)(\lambda_1^3+2\Re(\lambda_3)|\lambda_3|^2)+\mathcal{O}(\sum_{k=1}^{4}|\lambda_k|^4),$$

uniformly with respect to η. Thus, for L large enough, $\det\left(\Xi^L(\eta)\right)$ does not vanish and we may conclude that the functions α_1, α_2, α_3 and α_4 are uniquely defined for given conditions on $U(0,\cdot)$, $(U_{xxx}+2U)(L,\cdot)$, $U(L,\cdot)$ and $U_x(L,\cdot)$. Since $< BC_L >$ is equivalent to

$$(U(0,\eta)=0,(U_{xxx}+2U)(L,\eta)=\eta^2F(L,\eta),U(L,\eta)=0,U_x(L,\eta)=0),$$

we conclude that α_1, α_2, α_3 and α_4 can be uniquely chosen in order to fix the boundary conditions $< BC_L >$. □

We have the following *a priori* estimates,

Lemma 3.2. *Let $L \in [L_0, +\infty]$ (L_0 is given by Lemma 3.1). Let f in $\mathcal{Y}_L$, then the following estimate hold:*

$$|R_L(f)|_{\mathcal{H}_L} + |R_L(f)_x|_{\mathcal{H}_L} \lesssim |f|_{\mathcal{H}_L}. \tag{5}$$

Proof. We begin by the case $L < +\infty$. Let us set $u = R_L(f) \in \mathcal{S}_{0,L}$, we multiply (2) by $e^x u$, and we integrate over Ω_L to obtain,

$$\begin{aligned} &\frac{3}{2}\int_{\Omega_L} u_x^2\, e^x dxdy + \frac{1}{2}\int_{\Omega_L} (\partial_x^{-1}u_y)^2\, e^x dxdy + \frac{3}{2}\int_{\Omega_L} u^2\, e^x dxdy \\ &+\frac{1}{2}\int_{\mathbf{R}} \Big[\big(-u_x^2 - (\partial_x^{-1}u_y)^2 + u^2 + 2u_{xx}u - 2u_x u\big)\, e^x \Big]_{x=0}^{x=L} dy \\ &= \int_\Omega f u\, e^x dxdy \end{aligned} \tag{6}$$

The boundary condition $< BC_L >$ together with the condition $u(0,\cdot) \equiv 0$ implies that the three first boundary terms of (6) are non negative and that the two last are zero. Thus, Young inequality yields

$$\begin{aligned} &\frac{3}{2}\int_{\Omega_L} u_x^2\, e^x dxdy + \frac{1}{2}\int_{\Omega_L} (\partial_x^{-1}u_y)^2\, e^x dxdy + \int_{\Omega_L} u^2\, e^x dxdy \\ &+\frac{1}{2}\int_{\mathbf{R}} \big(u_x^2(0) + (\partial_x^{-1}u_y)^2(0) + u^2(L)\, e^L\big)\, dy \\ &\le \frac{1}{2}\int_\Omega f^2\, e^x dxdy, \end{aligned} \tag{7}$$

then,

$$\frac{3}{2}\int_{\Omega_L} u_x^2\, e^x dxdy + \int_{\Omega_L} u^2\, e^x dxdy \le \frac{1}{2}\int_\Omega f^2\, e^x dxdy. \tag{8}$$

Finally, we obtain (5).
In the case $L = +\infty$, the fact that u belongs to $\mathcal{S}_\infty$ simplifies the proof because every right boundary term vanishes. □

The estimate (5) of Lemma 3.2 allows us to extend R_L as a bounded operator on $\mathcal{H}_L$, and thus we can set $D(A_L) := R_L(\mathcal{H}_L)$. Clearly, $(\mathcal{A} + 2)(R_L(f)) = f$ in $\mathcal{D}'(\Omega_L)$ for $f \in \mathcal{H}_L$ and R_L is one to one on $D(A_L)$. Now we define the closed unbounded operator A_L by $A_L(u) := f - 2u$ for $u \in D(A_L)$ and $f \in \mathcal{H}_L$ such that $R_L(f) = u$. We equip $D(A_L)$ with the graph norm $|\cdot|_{D(A_L)} := |\mathcal{A}\cdot|_{\mathcal{H}_L} + |\cdot|_{\mathcal{H}_L}$, estimate (5) shows that $D(A_L)$ is continuously embedded in $X^1_{e^x}(\Omega_L)$.

Proposition 3.1.

(i) *The space* $D(A_L)$ *is a dense subspace of* $(\mathcal{H}_L, |\cdot|_{\mathcal{H}_L})$.

(ii) $\mathcal{S}_{0,L}$ *is dense in* $D(A_L)$ *for the graph norm.*

Proof. We show that $\mathcal{Y}_{0,L} \subset D(A_L)$. Let v in $\mathcal{Y}_{0,L}$, then $f := (\mathcal{A}+2)v$ belongs to $\mathcal{Y}_L$. We set $\overline{v} := R_L(f)$, and $u := v - \overline{v}$. Since u is smooth and satisfies the appropriate boundary conditions, the fact that $(\mathcal{A}+2)u = 0$ tells us that (7) is satisfied with $f = 0$. Thus $v = R_L(f)$ belongs to $D(A_L)$ and (i) follows.

R_L is a bi-continuous map from $\mathcal{H}_L$ onto $D(A_L)$, and $\mathcal{Y}_L$ is a dense subspace of $\mathcal{H}_L$, moreover, since $R_L(\mathcal{Y}_L) \subset \mathcal{S}_{0,L}$ we get (ii). $\square$

Corollary 3.1. *We have the following continuous embeddings*

$$D(A_L) \hookrightarrow X^1_{e^x}(\Omega_L) \hookrightarrow C^{0,\frac{1}{2}}([0,L]_x, L^2(\mathbf{R}_y)).$$

Moreover, if u *belongs to* $D(A_L)$, u *satisfies the boundary conditions* $u(0,\cdot) \equiv 0$ *and* $u(L,\cdot) \equiv 0$ *in* $L^2(\mathbf{R})$.

Proof. Let u in $\mathcal{S}_L$, (5) yields

$$|u|_{L^2_{e^x}} + |u_x|_{L^2_{e^x}} \lesssim |u|_{L^2_{e^x}} + |\mathcal{A}u|_{L^2_{e^x}} \lesssim |u|_{D(A_L)}$$

and the embedding follows from the density of $\mathcal{S}_L$ in $D(A_L)$. Since the desired boundary conditions are satisfied by the elements of $\mathcal{S}_L$ and are continuous with respect to the $C^{0,\frac{1}{2}}([0,L]_x, L^2(\mathbf{R}_y))$-norm, they are also satisfied by the elements of $D(A_L)$. $\square$

Proposition 3.2. *The operator* $A_L + 2$ *is positive on* $\mathcal{H}_L$.

Proof. Let u in $\mathcal{S}_{0,L}$, applying the linear estimate derived in the proof of Lemma 3.2, we have

(9)

$$\begin{aligned}(A_L u, u)_{L^2_{e^x}} =& \frac{3}{2}\int_{\Omega_L} u_x^2\, e^x dxdy + \frac{1}{2}\int_{\Omega_L} (\partial_x^{-1} u_y)^2\, e^x dxdy - \frac{1}{2}\int_{\Omega_L} u^2\, e^x dxdy \\ &+ \frac{1}{2}\int_{\mathbf{R}} \left(u_x^2(0) + (\partial_x^{-1} u_y)^2(0) + u^2(L)e^L\right) dy\end{aligned}$$

thus $(A_L u + 2u, u)_{L^2_{e^x}} \geq 0$. We conclude by a density argument due to Proposition 3.1 that $A_L + 2$ is positive on $\mathcal{H}_L$. $\square$

Now using standard evolution equation theory, we can claim the well-posedness of $\mathcal{L}(\Omega_L)$ in $\mathcal{H}_L$,

Theorem 3.1. *Let there be given $T > 0$ and $L \in [L_0, +\infty]$. Let $f \in C^1([0,T], \mathcal{H}_L)$ and $u_0 \in D(A_L)$, then problem $\mathcal{L}(\Omega_L)$ possesses a unique solution u in $C^0([0,T], D(A_L)) \cap C^1([0,T], \mathcal{H}_L)$.*

In section 5, the following linear estimate will be usefull.

Proposition 3.3. *For $p \in \mathbb{N}$, $L \in (L_0, +\infty]$ and for every v in $D(A_L)$, we have*

$$(A_L v, v)_{L^2_{x^p}} \geq \frac{p}{2} \left[\int_{\Omega_L} \left(3v_x^2 + (\partial_x^{-1} v_y)^2 \right) x^{p-1} dxdy - (p-1)(p-2) |v|^2_{L^2_{x^{p-3}}} \right].$$

Proof. Let v in $\mathcal{S}_{0,L}$, integrations by parts lead to

$$(10) \qquad (A_L v, v)_{L^2_{x^p}} =$$
$$\frac{p}{2} \left[\int_{\Omega_L} \left(3v_x^2 + (\partial_x^{-1} v_y)^2 \right) x^{p-1} dxdy - (p-1)(p-2) \int_{\Omega_L} v^2 \, x^{p-3} dxdy \right]$$
$$+ \frac{1}{2} \int_{\mathbf{R}} \left(v_x^2(0) 0^p + (\partial_x^{-1} v_y(0))^2 0^p + p(p-1) v^2(L) L^{p-2} \right) dy$$
$$\geq \frac{p}{2} \left[\int_{\Omega_L} \left(3v_x^2 + (\partial_x^{-1} v_y)^2 \right) x^{p-1} dxdy - (p-1)(p-2) \int_{\Omega_L} v^2 \, x^{p-3} dxdy \right],$$

and the Proposition is obtained by density. ☐

4. The non-linear problem, notations and density arguments

Let $L_0 < L \leq \infty$ and $T > 0$. From now on $\Omega_L^T := \Omega_L \times (0,T)$. We would like to solve the following problem: find a function u in $\mathcal{D}'(\Omega_L^T)$ such that

$$\mathcal{P}_0(\Omega_L^T) \quad : \quad \begin{cases} u_t + \mathcal{A}u = -uu_x, \text{ in } & \mathcal{D}'(\Omega_L^T), \\ u(x,y,0) = u_0(x,y), & \text{in} \quad \overline{\Omega_L}, \\ u(0,y,t) = g(y,t), & \text{in} \quad \mathbf{R} \times (0,T), \\ u \text{ satisfies } < BC_L >, & \end{cases}$$

where u_0 and g are initial and boundary conditions. First we will transform $\mathcal{P}_0(\Omega_L^T)$ in order to deal with homogeneous boundary conditions on the left boundary. Let ψ in $C_0^\infty(\mathbf{R}_+)$ such that $\text{supp}(\psi) \subset [0, L_0)$ and $\psi(0) = 1$. We define $\phi := \phi(g)$ and $\Sigma := \Sigma(g)$ in the following way

$$\phi(x,y,t) = g(y,t) \cdot \psi(x), \qquad \Sigma = -(\phi_t + \phi_{xxx} + \partial_x^{-1} \phi_{yy} + \phi \phi_x).$$

Finally, we set $u = w + \phi(g)$ then $\mathcal{P}_0(\Omega_L^T)$ is formally equivalent to:

$$\mathcal{Q}(\Omega_L^T) \ : \ \begin{cases} w_t + \mathcal{A}w = -\Sigma(g) - \partial_x(w\phi(g)) - \frac{1}{2}\partial_x(w^2), & \text{in} \quad \Omega_L^T, \\ w(x,y,0) = u_0(x,y) - \phi(g)(x,y,0), & \text{in} \quad \Omega_L, \\ w(0,y,t) = 0, & \text{in} \quad \mathbf{R}\times(0,T), \\ w \text{ satisfies } <BC_L>. \end{cases}$$

In section 5, we will solve $\mathcal{Q}(\Omega_L^T)$ using a fixed point method for initial data in $(\mathcal{F}_L, |\cdot|_{\mathcal{F}_L})$, where this space is defined by

$$\begin{aligned} \mathcal{F}_L := & \quad \left\{ u \in L^2_{(1+x)^2}(\Omega_L),\, u_y \in L^2_{(1+x)^2}(\Omega_L) \right\}, \\ |u|^2_{\mathcal{F}_L} := & \quad |u|^2_{L^2_{(1+x)^2}(\Omega_L)} + |u_y|^2_{L^2_{(1+x)^2}(\Omega_L)}. \end{aligned}$$

From Proposition 2.1 we know that for each u in $\mathcal{F}$, the operator ∂_x^{-1} defines a unique $\partial_x^{-1}u := \phi \in L^2(\Omega)$ such that $u = \phi_x$.

Let us now introduce the space in which this method is performed,

$$\mathcal{F}_L^T := \left\{ w \in C^0([0,T], \mathcal{F}_L),\, \int_{\Omega_L^T} (w_x^2 + w_{xy}^2)(1+x)dxdydt < +\infty \right\},$$

equipped with the norm

$$|w|^2_{\mathcal{F}_L^T} := \sup_{[0,T]} \left(\int_{\Omega_L} (w^2 + w_y^2)(1+x)^2 dxdy \right) + \int_{\Omega_L^T} (w_x^2 + w_{xy}^2)(1+x)dxdydt.$$

The initial and boundary conditions will belong to the following space.

$$\mathcal{B}_L^T := \left\{ (u_0, g) \in \mathcal{F}_L \times C^0([0,T], H^1(\mathbf{R})),\ |(u_0,g)|_{\mathcal{B}_L^T} < \infty \right\},$$

whose norm is defined by

$$\begin{aligned} |(u_0,g)|_{\mathcal{B}_L^T} := & |u_0|_{\mathcal{F}_L} + \sup_{[0,T]} \left(\int_{\mathbf{R}} g^2 + g_y^2 \right)^{\frac{1}{2}} + \left(\int_{\mathbf{R}\times[0,T]} g^2 + g_y^2 \right)^{\frac{1}{2}} \\ & + \int_0^T \left(\int_{\mathbf{R}} g_t^2 + g_{yt}^2 + g_{yy}^2 + g_{yyy}^2 \right)^{\frac{1}{2}} + \left(\int_0^T \left(\int_{\mathbf{R}} g^4 + g_y^4 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}. \end{aligned}$$

In fact for a fixed T,

$$\mathcal{B}_L^T = \mathcal{F}_L \times \left(L^1([0,T], H^3(\mathbf{R})) \cap W^{1,1}([0,T], H^1(\mathbf{R})) \right).$$

In addition, we will use

$$\mathcal{I}_L^T := L^1([0,T],\mathcal{F}_L).$$

equipped with the norm

$$|\Sigma|_{\mathcal{I}_L^T} := \int_0^T \left(\int_{\Omega_L} \Sigma^2(1+x)^2 + \Sigma_y^2(1+x)^2\right)^{\frac{1}{2}},$$

The space $(\mathcal{B}_L^T, |\cdot|_{\mathcal{B}_L^T})$ has been chosen in order to satisfy the next Proposition.

Proposition 4.1. *Let $\gamma = (u_0, g)$ and $\overline{\gamma} = (\overline{u_0}, \overline{g})$ be in $\mathcal{B}_L^T$, then*

$$|u_0 - \phi(g)(\cdot,\cdot,0)|_{\mathcal{F}_L} \lesssim |\gamma|_{\mathcal{B}_L^T}, \quad |\phi(g)|_{\mathcal{F}_L^T} \lesssim |\gamma|_{\mathcal{B}_L^T}, \quad |\Sigma(g)|_{\mathcal{I}_L^T} \lesssim |\gamma|_{\mathcal{B}_L^T} + |\gamma|^2_{\mathcal{B}_L^T},$$
$$|\Sigma(g) - \Sigma(\overline{g})|_{\mathcal{I}_L^T} \lesssim (1 + |\gamma + \overline{\gamma}|_{\mathcal{B}_L^T})|\gamma - \overline{\gamma}|_{\mathcal{B}_L^T}.$$

The constants involved in the $\lesssim$ do not depend on T or L, they only depend on ψ.

Proof. These estimates come from the fact that ψ belongs to $C_0^\infty(\mathbf{R}_+)$ and the choice of $|\cdot|_{\mathcal{B}_L^T}$. □

We will need some density results stated here. First we introduce the space $\mathcal{S}_{0,L}^T$ of elements $(\overline{w}, (u_0, g)) \in \mathcal{F}_L^T \times \mathcal{B}_L^T$ such that for $j = 0, 1$,

$$\partial_y^j(u_0 - \phi(g))(\cdot,\cdot,0) \in D(A_L),$$
$$\partial_y^j\Sigma(g), \partial_y^j\partial_x(\overline{w}^2), \partial_y^j\partial_x(\overline{w}\phi(g)) \in C^1([0,T],\mathcal{H}_L).$$

We claim that

Proposition 4.2. *The space $\mathcal{S}_{0,L}^T$ is dense in $(\mathcal{F}_L^T \times \mathcal{B}_L^T, |\cdot|_{\mathcal{F}_L^T \times \mathcal{B}_L^T})$.*

Proof. In virtue of Proposition 3.1, it is not difficult to see that

$$\left\{(\overline{w}, u_0, g)\,;\, \overline{w} \in C_0^\infty(\Omega_L^T),\, \mathcal{F}_y(g) \in C_0^\infty([0,T]\times\mathbf{R}),\, u_0 \in \mathcal{S}_{0,L}\right\} \subset \mathcal{S}_{0,L}^T,$$

and the Proposition is obtained by classical density results. □

Let us describe the fixed point method performed in the next section. Let $(\overline{w}, (u_0, g))$ in $\mathcal{S}_{0,L}^T$, we will set $K_L^T(\overline{w}, u_0, g) := w$, where $w \in C^1([0,T],\mathcal{H}_L)$ is the solution given by Theorem 3.1 of

$$\mathcal{Q}(\Omega_L^T)(\overline{w}, u_0, g) \;:\; \begin{cases} w_t + A_L w = -\Sigma(g) - \partial_x(\overline{w}\phi(g) - \frac{1}{2}\overline{w}^2) & \text{in} \quad \Omega_L^T, \\ w(x,y,0) = u_0(x,y) - \phi(g)(x,y,0) & \text{in} \quad \Omega_L. \end{cases}$$

In fact w, w_y belong to $C^1([0,T],\mathcal{H}_L)\cap C^0([0,T],D(A_L))$. We will show that K_L^T is continuous from $(\mathcal{S}_{0,L}^T,|\cdot|_{\mathcal{F}_L^T\times\mathcal{B}_L^T})$ into $(\mathcal{F}_L^T,|\cdot|_{\mathcal{F}_L^T})$, so, by the density argument of Proposition 4.2, K_L^T will extend continuously as a map from $\mathcal{F}_L^T\times\mathcal{B}_L^T$ into $\mathcal{F}_L^T$.

Definition 4.1. *Let (u_0,g) in $\mathcal{B}_L^T$ and u in $\mathcal{F}_L^T$, we will write "u solves $\mathcal{P}_0(\Omega_L^T)$" if $u=w+\phi(g)$ where $w=K_T^L(w,u_0,g)$.*

This definition is an abuse of notation since the boundary conditions $< BC_L >$ have no meaning in $\mathcal{F}_L^T$ but

Proposition 4.3. *Provided the map K_L^T is continuous. A solution u of $\mathcal{P}_0(\Omega_L^T)$ in the sense of Definition 4.1 for a given data (u_0,g) in $\mathcal{B}_L^T$ satisfies*

(i) $u_t+u_{xxx}+\partial_x^{-1}u_{yy}+uu_x=0,\qquad$ *in* $\mathcal{D}'(\Omega_L^T)$,
(ii) $u(x,y,0)=u_0(x,y)$, *for almost every* (x,y) *in* Ω_L,
(iii) $u(0,y,t)=g(y,t)$, *for almost every* (y,t) *in* $\mathbf{R}\times[0,T]$,
(iv) $u(L,y,t)=0$, *for almost every* (y,t) *in* $\mathbf{R}\times[0,T]$.

Proof. Let $(\overline{w},u_0,g)\in\mathcal{S}_{0,L}^T$ and $w=K_L^T(\overline{w},u_0,g)$, then from Theorem 3.1, w belongs to $C([0,T],D(A_L))$ and

(11)
$$w_t+w_{xxx}+\partial_x^{-1}w_{yy}=-\Sigma(g)-\partial_x(\overline{w}\phi(g))-\frac{1}{2}\partial_x(\overline{w}^2),\quad \text{in } \mathcal{D}'(\Omega_L^T),$$
(12)
$$w(x,y,0)=u_0(x,y)-\phi(g)(x,y),\quad \text{for almost every } (x,y) \text{ in } \Omega_L.$$

Moreover, from Corollary 3.1,

$$w(0,y,t)=0,\quad \text{for almost every } (y,t) \text{ in } \mathbf{R}\times[0,T], \tag{13}$$
$$w(L,y,t)=0,\quad \text{for almost every } (y,t) \text{ in } \mathbf{R}\times[0,T]. \tag{14}$$

Now, let $(\overline{w},u_0,g)$ in $\mathcal{F}_L^T\times\mathcal{B}_L^T$ and $w=K_L^T(\overline{w},u_0,g)$, from Proposition 4.2, there exists a sequence $(\overline{w}^k,u_0^k,g^k)_k$ in $\mathcal{S}_{0,L}^T$ converging to $(\overline{w},u_0,g)$ in $\mathcal{F}_L^T\times\mathcal{B}_L^T$. From Proposition 4.1 and the fact that $\mathcal{F}_L^T$ is continuously embedded in $L^2(\Omega_L^T)$, we get

$$\Sigma(g^k)\underset{k\to+\infty}{\longrightarrow}\Sigma(g),\quad \text{in } L^1([0,T],L^2(\Omega_L)),$$

and

$$\phi(g^k)\overline{w}^k\underset{k\to+\infty}{\longrightarrow}\phi(g)\overline{w},\quad (\overline{w}^k)^2\underset{k\to+\infty}{\longrightarrow}\overline{w}^2\quad \text{in } L^1(\Omega_L^T).$$

We set $w^k = K_L^T(\overline{w}^k, u_0^k, g^k)$. From the continuity of K_L^T we get that $w^k \underset{k\to+\infty}{\longrightarrow} w$ in $\mathcal{F}_L^T$ and thus, $\partial_x^{-1} w^k \underset{k\to+\infty}{\longrightarrow} \partial_x^{-1} w$ in $C([0,T], L^2(\Omega_L^T))$. Collecting the preceding convergences, we get that $w, \overline{w}$ and g satisfy (11). Now, it suffices to set $w = \overline{w}$ and substitute $w = u - \phi(g)$ in (11) to get (i).

Since $\mathcal{F}_L^T$ is continuously embedded in $C([0,T], L^2(\Omega_L))$, we get that w satisfies (12) and $u = w + \phi(g)$ satisfies (ii). Now, since $\mathcal{F}_L^T$ is continuously embedded in $C^{0,\frac{1}{2}}([0,L]_x, L^2(\mathbf{R}_y \times [0,T]_t))$ we deduce (iii) and (iv) from (13) and (14). □

The next proposition shows that Definition 4.1 does not depend on the choice of ψ.

Proposition 4.4. *Let us fix $L_0 < L \leq \infty$ and $T > 0$. Let ψ_1 and ψ_2 in $C_0^\infty([0, L_0))$ such that $\psi_1(0) = \psi_2(0) = 1$ and let us note*

$$\phi_1 = \phi, \quad \Sigma_1 = \Sigma, \quad K_1 = K_L^T, \quad \text{constructed with } \psi := \psi_1,$$
$$\phi_2 = \phi, \quad \Sigma_2 = \Sigma, \quad K_2 = K_L^T, \quad \text{constructed with } \psi := \psi_2.$$

Provided K_1 and K_2 are continuous from $\mathcal{F}_L^T \times \mathcal{B}_L^T$ into $\mathcal{F}_L^T$, then for every (u_0, g) in $\mathcal{B}_L^T$ and every $\overline{u}$ in $\mathcal{F}_L^T$

$$\phi_1(g) + K_1(\overline{u} - \phi_1(g), u_0, g) = \phi_2(g) + K_2(\overline{u} - \phi_2(g), u_0, g).$$

Proof. Let $L, T, K_1, K_2, \phi_1, \phi_2, \Sigma_1, \Sigma_2$ as above. From the continuity of K_1 and K_2 and a density argument, it is sufficient to prove the result for $\overline{u}$ in $C([0,T], \mathcal{H}_L)$, u_0 in $D(A_L)$ and g in $\mathcal{F}_y^{-1}(C_0^\infty(\mathbf{R} \times [0,T]))$. Let $(\overline{u}, u_0, g)$ satisfying these assumptions, we set $w_i = K_i(\overline{u} - \phi_i(g), u_0, g)$ for $i = 1, 2$. From Theorem 3.1, we have for $i = 1, 2$, w_i belongs to $C([0,T], D(A_L))$ and

$$w_{i,t} + A_L w_i = -\Sigma_i(g) - \frac{1}{2}\partial_x \left(\overline{u}^2 - {\phi_i}^2(g)\right).$$

Subtracting these two equalities, we get

$$\partial_t(w_1 - w_2) + A_L(w_1 - w_2) = -\Sigma_1(g) + \Sigma_2(g) - \frac{1}{2}\partial_x \left({\phi_1}^2(g) - {\phi_2}^2(g)\right).$$

Now let $u_i = w_i + \phi_i(g)$ for $i = 1, 2$, from our assumptions on g, $u_1 - u_2 = w_1 - w_2 + \phi_1(g) - \phi_2(g)$ belongs to $C([0,T], D(A_L))$, substituting in the preceding equality, and using the definition of Σ_1 and Σ_2, we get

$$\partial_t(u_1 - u_2) + A_L(u_1 - u_2) = 0.$$

Moreover $(u_1 - u_2)(\cdot, \cdot, 0) = 0$ and from the uniqueness result of Theorem 3.1, we have $u_1 \equiv u_2$. □

To end this section, we present in the following Proposition the estimates we will use to treat the nonlinear part of $\mathcal{Q}(\Omega_L^T)$.

Proposition 4.5. *Let v, w in $L^2(\Omega_L)$, such that v_y and w_x belong to $L^2(\Omega_L)$ and w satisfies one of the boundary condition $w(0,\cdot) \equiv 0$ or $w(L,\cdot) \equiv 0$ in $L^2(\mathbf{R})$.*

(i) *If moreover $v, v_y, w \in L^2_{(1+x)^2}(\Omega_L)$, then*

$$\int_{\Omega_L} v^2 w^2 (1+x)^3 \leq 2 \left(\int_{\Omega_L} v^2 (1+x)^2 \right)^{\frac{1}{2}} \left(\int_{\Omega_L} v_y^2 (1+x)^2 \right)^{\frac{1}{2}} \times \left[\int_{\Omega_L} w^2 + 2 \left(\int_{\Omega_L} w^2 (1+x)^2 \right)^{\frac{1}{2}} \left(\int_{\Omega_L} w_x^2 \right)^{\frac{1}{2}} \right].$$

(ii) *If moreover $v, v_y \in L^2_{1+x}(\Omega_L)$ and $w \in L^2_{(1+x)^2}(\Omega_L)$, then*

$$\int_{\Omega_L} v^2 w^2 (1+x)^2 \leq 2 \left(\int_{\Omega_L} v^2 (1+x) \right)^{\frac{1}{2}} \left(\int_{\Omega_L} v_y^2 (1+x) \right)^{\frac{1}{2}} \times \left[\int_{\Omega_L} w^2 + 2 \left(\int_{\Omega_L} w^2 (1+x)^2 \right)^{\frac{1}{2}} \left(\int_{\Omega_L} w_x^2 \right)^{\frac{1}{2}} \right].$$

Proof. Let V, W in $L^2(\Omega_L)$ such that V_y and W_x belong also to $L^2(\Omega_L)$ and W satisfies one of the boundary conditions $W(0,\cdot) = 0$ or $W(L,\cdot) = 0$ in $L^2(\mathbf{R})$. We estimate the L^2-norm of the product $V \cdot W$ by splitting the integration in the x and y directions,

$$\int_{\Omega_L} V^2 W^2 \leq \int_{[0,L]} \left\| V^2(x,\cdot) \right\|_{L^\infty(\mathbf{R})} dx \times \int_{\mathbf{R}} \left\| W^2(\cdot,y) \right\|_{L^\infty([0,L])} dy.$$

Now, since $\int_{\Omega_L} V_y^2$ is finite, we get for almost every (x,y) in Ω_L,

$$V^2(x,y) = 2 \int_{-\infty}^{y} V(x,z) V_y(x,z) dz \leq 2 \int_{\mathbf{R}} |V(x,z) V_y(x,z)| dz,$$

integrating along $[0,L]$, we get $\int_{[0,L]} \left\| V^2(x,\cdot) \right\|_{L^\infty(\mathbf{R})} dx \leq 2 \int_{\Omega_L} |V V_y|$.
For the second integral, we can write

$$W(x,y)^2 = 2 \int_0^x W_x(z,y) W(z,y) dz, \qquad \forall y \in \mathbf{R}, \text{ a.e.}$$

if have $W(0,\cdot)\equiv 0$ and

$$W(x,y)^2=-2\int_x^L W_x(z,y)W(z,y)dz,\qquad \forall y\in\mathbf{R},\ \text{a.e.}$$

if $W(L,\cdot)\equiv 0$ thus integrating on$\mathbf{R}$, both cases lead to $\int_{\mathbf{R}}\left\|W^2(\cdot,y)\right\|_{L^\infty([0,L])}dy\le 2\int_{\Omega_L}|WW_x|$, and finally

$$\int_{\Omega_L}V^2W^2\le 4\int_{\Omega_L}|VV_y|\times\int_{\Omega_L}|WW_x|\,.$$

Now, let v, w satisfying the assumptions of the Lemma, we prove (i), (resp. (ii)) by using this last inequality with $V(x,y)=(1+x)v(x,y)$ and $W(x,y)=\sqrt{1+x}\,w(x,y)$ (resp. $V(x,y)=\sqrt{1+x}\,v(x,y)$ and $W(x,y)=\sqrt{1+x}\,w(x,y)$) and the Cauchy-Schwarz inequality. □

Remark 4.1. *For domains $[0,L]\times\mathbf{S}^1$ (see Remark 1.1), the preceding estimates are not valid. In fact, for V in $L^2([0,L],\mathbf{S}^1)$, we have*

$$\int_0^L\left(\sup_{y\in\mathbf{S}^1}V^2(x,y)\right)dx\lesssim\left(\int_0^L\int_{\mathbf{S}^1}|VV_y|dxdy\right)+\int_0^L\int_{\mathbf{S}^1}V^2dxdy,$$

where the last term do not appear in the preceding proof. This difference introduce harmless changes in the following proofs.

5. The nonlinear boundary value problem in $\mathcal{F}_L$

In this section, we present the existence for the Cauchy problem $\mathcal{P}_0(\Omega_L^T)$ in the space $\mathcal{F}_L$.

We first state the key Lemma of our proof

Lemma 5.1. *Let $L_0<L\le\infty$ and $T>0$. Let there be given Σ in $\mathcal{I}_L^T$, and f, w in $\mathcal{F}_L^T$ such that Σ, Σ_y, $\partial_x(fw)$ and $\partial_x(fw)_y$ belong to $C^1([0,T],\mathcal{H}_L)$ and u_0 in $\mathcal{F}_L$ such that u_0 and $u_{0,y}$ belong to $D(A_L)$. Let u be the solution of*

$$\left\{\begin{array}{lll} u_t+A_Lu=\Sigma+\partial_x(fw) & in & \Omega_L\times[0,T),\\ u(x,y,0)=u_0(x,y) & in & \Omega_L,\end{array}\right.\tag{15}$$

given by Theorem 3.1. We recall that $u\in C^0([0,T],D(A_L))\cap C^1([0,T],\mathcal{H}_L)$ and $u_y\in C^0([0,T],D(A_L))\cap C^1([0,T],\mathcal{H}_L)$.

Then

(i)

$$|u|^2_{\mathcal{F}_L^T} \lesssim |u_0|^2_{\mathcal{F}_L} + T^{\frac{1}{4}}(1+T^{\frac{1}{2}})|f|_{\mathcal{F}_L^T}|w|_{\mathcal{F}_L^T}|u|_{\mathcal{F}_L^T} + |\Sigma|_{\mathcal{I}_L^T}\,|u|_{\mathcal{F}_L^T}.$$

(ii)

$$\int_0^T \int_{\Omega_L} \left((\partial_x^{-1}u_y)^2 + (\partial_x^{-1}u_{yy})^2\right)(1+x)dxdydt$$
$$\lesssim |u_0|^2_{\mathcal{F}_L} + T^{\frac{1}{4}}(1+T^{\frac{1}{2}})|f|_{\mathcal{F}_L^T}|w|_{\mathcal{F}_L^T}|u|_{\mathcal{F}_L^T} + |\Sigma|_{\mathcal{I}_L^T}\,|u|_{\mathcal{F}_L^T}.$$

All the constants involved in the $\lesssim$ signs do not depend on L and T.

Proof. Multiplying equation (15) by $(1+x)^2u$, integrating over Ω_L, and using Proposition 3.3 with $v=u$ and $p=0,1,2$, we get

$$\frac{1}{2}\partial_t \int_{\Omega_L} u^2(1+x)^2 + 3\int_{\Omega_L}(1+x)u_x^2 + \int_{\Omega_L}(1+x)(\partial_x^{-1}u_y)^2$$
$$\leq \int_{\Omega_L}(fw)_x u(1+x)^2 + \int_{\Omega_L}\Sigma u(1+x)^2\,.$$

Expanding the derivative $(fw)_x$ in the right hand side, integrating over $[0,T]$, and using Cauchy-Schwarz inequality, we get

$$\tfrac{1}{2}\int_{\Omega_L} u^2(T)(1+x)^2 - \frac{1}{2}\int_{\Omega_L} u_0^2(1+x)^2 + \int_{\Omega_L^T}(1+x)\left(3u_x^2 + (\partial_x^{-1}u_y)^2\right)$$
$$\leq \int_0^T I_1(w(t))\times I_2(u(t),f(t))dt + \int_0^T I_1(f(t))\times I_2(u(t),w(t))dt + I_3, \tag{16}$$

where $\quad I_1(w) := \left(\int_{\Omega_L} w_x^2(1+x)\right)^{\frac{1}{2}},\quad I_2(u,f) := \left(\int_{\Omega_L} u^2f^2(1+x)^3\right)^{\frac{1}{2}},$

and $\quad I_3 := \left(\int_0^T\left(\int_{\Omega_L}\Sigma^2(1+x)^2\right)^{\frac{1}{2}}\right)\left(\sup_{[0,T]}\int_{\Omega_L} u^2(1+x)^2\right)^{\frac{1}{2}}.$

Obviously, $I_3 \leq |\Sigma|_{\mathcal{I}_L^T}\,|u|_{\mathcal{F}_L^T}$. Since u belongs to $C^0([0,T],D(A_L))$, from Corollary 3.1, we have $u(0,\cdot,t)=0$ in $L^2(\mathbf{R})$ for every $t\in[0,T]$. Moreover, the estimate of Proposition 3.3 with $v:=u$ tells us that $u_x(\cdot,\cdot,t)$ belongs to $L^2_{(1+x)}(\Omega_L)$ for every $t\in[0,T]$. Thus, for every t in $[0,T]$, we can apply Proposition 4.5 (i) with $v:=f(t)$ et $w:=u(t)$ to estimate $I_2(u(t)f(t))$.

Then we get

$$I_2(u(t), f(t)) \leq 2 \left(\sup_{[0,T]} \int_{\Omega_L} f^2(1+x)^2 \right)^{\frac{1}{4}} \left(\sup_{[0,T]} \int_{\Omega_L} f_y^2(1+x)^2 \right)^{\frac{1}{4}}$$
$$\times \left[\left(\sup_{[0,T]} \int_{\Omega_L} u^2 \right)^{\frac{1}{2}} + 2 \left(\sup_{[0,T]} \int_{\Omega_L} u^2(1+x) \right)^{\frac{1}{4}} \left(\int_{\Omega_L} u_x^2(t) \right)^{\frac{1}{4}} \right].$$

Multiplying this last inequality by $I_1(w(t))$, integrating over $[0, T]$ and using the Hölder inequality $\int_0^T h^{1/2}(t)dt \leq T^{1/2} \left(\int_0^T h(t)dt \right)^{1/2}$, we get

$$\int_0^T I_1(w(t)) \times I_2(u(t), f(t))dt \lesssim \left(\sup_{[0,T]} \int_{\Omega_L} f^2(1+x)^2 \right)^{\frac{1}{4}}$$
$$\times \left(\sup_{[0,T]} \int_{\Omega_L} f_y^2(1+x)^2 \right)^{\frac{1}{4}} \left(\sup_{[0,T]} \int_{\Omega_L} u^2 \right)^{\frac{1}{2}} \int_0^T \left(\int_{\Omega_L} w_x^2(1+x) \right)^{\frac{1}{2}} dt$$
$$+ \left(\sup_{[0,T]} \int_{\Omega_L} f^2(1+x)^2 \right)^{\frac{1}{4}} \left(\sup_{[0,T]} \int_{\Omega_L} f_y^2(1+x)^2 \right)^{\frac{1}{4}} \left(\sup_{[0,T]} \int_{\Omega_L} u^2(1+x)^2 \right)^{\frac{1}{4}}$$
$$\times \int_0^T \left(\int_{\Omega_L} u_x^2 \right)^{\frac{1}{4}} \left(\int_{\Omega_L} w_x^2(1+x) \right)^{\frac{1}{2}} dt,$$
$$\leq T^{\frac{1}{2}} |f|_{\mathcal{F}_L^T} |u|_{\mathcal{F}_L^T} |w|_{\mathcal{F}_L^T} + T^{\frac{1}{4}} |f|_{\mathcal{F}_L^T} |u|_{\mathcal{F}_L^T} |w|_{\mathcal{F}_L^T}.$$

The same estimate holds with the same proof for $\int_0^T I_1(f(t)) \times I_2(u(t), w(t))dt$, thus from (16), we have

$$\frac{1}{2} \int_{\Omega_L} u^2(T)(1+x)^2 - \frac{1}{2} \int_{\Omega_L} u_0^2(1+x)^2 + \int_{\Omega_L^T} (1+x) \left(3u_x^2 + (\partial_x^{-1} u_y)^2 \right)$$
$$\lesssim T|u|^2_{\mathcal{F}_L^T} + T^{\frac{1}{4}}(1 + T^{\frac{1}{4}})|f|_{\mathcal{F}_L^T} |w|_{\mathcal{F}_L^T} |u|_{\mathcal{F}_L^T} + |\Sigma|_{\mathcal{I}_L^T} |u|_{\mathcal{F}_L^T}. \tag{17}$$

Now we estimate u_y and u_{xy} following the same line. Multiplying equation $(15)_y$ by $(1+x)^2 u_y$, integrating over Ω_L, and using Proposition 3.3

with $v = u_y$ and $p = 0, 1$, we get

$$\frac{1}{2}\partial_t \int_{\Omega_L} u_y^2(1+x)^2 + 3\int_{\Omega_L} u_{xy}^2(1+x) + \int_{\Omega_L} (\partial_x^{-1}u_{yy})^2(1+x)$$
$$\leq \int_{\Omega_L} f_{xy}wu_y(1+x)^2 + \int_{\Omega_L} f_x w_y u_y(1+x)^2 + \int_{\Omega_L} f w_{xy} u_y (1+x)^2$$
$$+ \int_{\Omega_L} f w_{xy} u_y(1+x)^2 + \int_{\Omega_L} f_y w_x u_y (1+x)^2 + \int_{\Omega_L} \Sigma_y u_y (1+x)^2,$$

Using Cauchy-Schwarz inequality and integrating over $[0,T]$, we obtain

$$\frac{1}{2}\int_{\Omega_L} u_y^2(T)(1+x)^2 - \frac{1}{2}\int_{\Omega_L} u_{0,y}^2(1+x)^2 + \int_{\Omega_L^T} \left(3u_{xy}^2 + (\partial_x^{-1}u_{yy})^2\right)(1+x)$$
$$\leq \int_0^T I_4(w(t)) \times I_5(f(t), u(t))dt + \int_0^T I_6(w(t)) \times I_7(f(t), u(t))dt \tag{18}$$
$$+ \int_0^T I_4(f(t)) \times I_5(w(t), u(t))dt + \int_0^T I_6(f(t)) \times I_7(w(t), u(t))dt + I_8,$$

where

$$I_4(w) := \left(\int_{\Omega_L} w_{xy}^2(1+x)\right)^{\frac{1}{2}}, \qquad I_5(f,u) := \left(\int_{\Omega_L} f^2u_y^2(1+x)^3\right)^{\frac{1}{2}},$$
$$I_6(w) := \left(\int_{\Omega_L} w_y^2(1+x)^2\right)^{\frac{1}{2}}, \qquad I_7(f,u) := \left(\int_{\Omega_L} f_x^2u_y^2(1+x)^2\right)^{\frac{1}{2}},$$
$$I_8 := \left(\int_0^T \left(\int_{\Omega_L} \Sigma_y^2(1+x)^2\right)^{\frac{1}{2}}\right)\left(\sup_{[0,T]}\left(\int_{\Omega} u_y^2(1+x)^2\right)^{\frac{1}{2}}\right).$$

We will bound the left hand side with $T^{\frac{1}{4}}(1+T^{\frac{1}{2}})|f|_{\mathcal{F}_L^T}|w|_{\mathcal{F}_L^T}|u|_{\mathcal{F}_L^T}+|\Sigma|_{\mathcal{I}_L^T}|u|_{\mathcal{F}_L^T}$. First, we have $I_8 \leq |\Sigma|_{\mathcal{I}_L^T}|u|_{\mathcal{F}_L^T}$. The third and the fourth term in the the right hand side of (18) are of same kind of the first and the second ones. Only $\int_0^T I_4(w(t)) \times I_5(f(t), u(t))dt$ and $\int_0^T I_6(w(t)) \times I_7(f(t), u(t))dt$ remain. Using the estimate of Proposition 3.3 with $v := u_y$, we get $u_{xy}(\cdot,\cdot,t)$ in $L^2(\Omega_L)$ for every t in $[0,T]$. Moreover, since f belongs to $\mathcal{F}_L^T$, we can use

Proposition 4.5 (ii) with $v := f(t)$ and $w := u_y(t)$. We have for t in $[0,T]$,

$$I_5(f(t),u(t)) \lesssim \left(\sup_{[0,T]}\int_{\Omega_L} f^2(1+x)^2\right)^{\frac14}\left(\sup_{[0,T]}\int_{\Omega_L} f_y^2(1+x)^2\right)^{\frac14}$$
$$\times\left[\left(\sup_{[0,T]}\int_{\Omega_L} u_y^2\right)^{\frac12}+\left(\sup_{[0,T]}\int_{\Omega_L} u_y^2(1+x)^2\right)^{\frac14}\left(\int_{\Omega_L} u_{xy}^2(t)\right)^{\frac14}\right].$$

Thus multiplying by $I_4(w(t))$, integrating over $[0,T]$ and using Hölder inequality, we obtain

$$\int_0^T I_4(w(t))\times I_5(f(t),u(t))dt \lesssim |f|_{\mathcal{F}_L^T}|u|_{\mathcal{F}_L^T}\int_0^T\left(\int_{\Omega_L} w_{xy}^2(1+x)\right)^{\frac12}dt$$
$$+|f|_{\mathcal{F}_L^T}|u|^{\frac12}_{\mathcal{F}_L^T}\int_0^T\left(\int_{\Omega_L} u_{xy}^2\right)^{\frac14}\left(\int_{\Omega_L} w_{xy}^2(1+x)\right)^{\frac12}dt$$
$$\lesssim T^{\frac14}(1+T^{\frac14})|f|_{\mathcal{F}_L^T}|u|_{\mathcal{F}_L^T}|w|_{\mathcal{F}_L^T}. \tag{19}$$

Finally using Proposition 4.5 (ii) with $v := f_x(t)$ and $w := u_y(t)$, we have for every t in $[0,T]$,

$$I_7(f(t),u(t)) \lesssim \left(\sup_{[0,T]}\int_{\Omega_L} f_x^2(1+x)\right)^{\frac14}\left(\sup_{[0,T]}\int_{\Omega_L} f_{xy}^2(1+x)\right)^{\frac14}$$
$$\times\left[\left(\sup_{[0,T]}\int_{\Omega_L} u_y^2\right)^{\frac12}+\left(\sup_{[0,T]}\int_{\Omega_L} u_y^2(1+x)^2\right)^{\frac14}\left(\int_{\Omega_L} u_{xy}^2(t)\right)^{\frac14}\right].$$

Multiplying by $I_6(w(t))$, integrating over $[0,T]$ and using again Hölder inequality, we obtain

$$\int_0^T I_6(w(t))\times I_7(f(t),u(t))$$
$$\lesssim |w|_{\mathcal{F}_L^T}|u|_{\mathcal{F}_L^T}|f|^{\frac12}_{\mathcal{F}_L^T}\int_0^T\left(\int_{\Omega_L} f_x^2(1+x)\right)^{\frac14}dt$$
$$+|w|_{\mathcal{F}_L^T}|u|^{\frac12}_{\mathcal{F}_L^T}|f|^{\frac12}_{\mathcal{F}_L^T}\int_0^T\left(\int_{\Omega_L} f_x^2(1+x)\right)^{\frac14}\left(\int_{\Omega_L} u_{xy}^2\right)^{\frac14}dt$$
$$\lesssim T^{\frac12}(1+T^{\frac14})|f|_{\mathcal{F}_L^T}|u|_{\mathcal{F}_L^T}|w|_{\mathcal{F}_L^T}. \tag{20}$$

From (18), (19), (20), we get

$$\frac{1}{2}\int_{\Omega_L} u_y^2(T)(1+x)^2 - \frac{1}{2}\int_{\Omega_L} u_{0,y}^2(1+x)^2 + \int_{\Omega_L^T} \left(3u_{xy}^2 + (\partial_x^{-1}u_{yy})^2\right)(1+x)$$

$$\lesssim T^{\frac{1}{4}}(1+T^{\frac{1}{2}})|f|_{\mathcal{F}_L^T}|u|_{\mathcal{F}_L^T}|w|_{\mathcal{F}_L^T} + |\Sigma|_{\mathcal{I}_L^T}|u|_{\mathcal{F}_L^T}. \tag{21}$$

Summing (17), (21), the Lemma is proved. □

We state the main result of this section.

Theorem 5.1. *Let L in $(L_0, +\infty]$ and $T^* > 0$,*

(i) *For every $\gamma := (u_0, g)$ in $\mathcal{B}_L^{T^*}$, there exists $0 < T \leq T^*$, and $u := \mathcal{V}^L(u_0, \gamma)$ in $\mathcal{F}_L^T$ solving $\mathcal{P}_0(\Omega_L^T)(u_0, g)$ in the sense of Definition 4.1. Moreover, $T = T(|\gamma|_{\mathcal{B}_L^{T^*}})$ is a non increasing function of $|\gamma|_{\mathcal{B}_L^{T^*}}$ and $\lim_{|\gamma|_{\mathcal{B}_L^{T^*}} \to 0} T(|\gamma|_{\mathcal{B}_L^{T^*}}) = +\infty$.*

(ii) *In particular, the conclusions of Proposition 4.3 hold for u.*

(iii) *For every $r_1 > 0$, there exists $T = T(r_1)$ such that the map*

$$\mathcal{V}^L : B_{\mathcal{B}_L^T}(0, r_1) \longrightarrow \mathcal{F}_L^T \qquad \textit{is uniformly continuous.}$$

(iv) *Finally, the application*

$$\begin{aligned} B_{\mathcal{B}_L^T}(0, r_1) &\longrightarrow \left[L^2\left([0,T], L^2_{1+x}(\Omega_L)\right)\right]^2, \\ \gamma &\longmapsto (\partial_x^{-1}u_y, \partial_x^{-1}u_{yy}) \qquad \textit{is uniformly continuous.} \end{aligned}$$

Remark 5.1. *The time T does not depend on L, moreover the maps $\mathcal{V}^L$ are uniformly continuous, uniformly in $L > L_0$.*

Proof. Let us fix $L_0 < L \leq \infty$ and $r_1 > 0$. Let $T \in (0, T^*]$ and $r > 0$ to be fixed later. For every $\gamma := (u_0, g)$ in $\mathcal{B}_L^T$ and every $\overline{w}$ in $\mathcal{F}_L^T$ such that $|\gamma|_{\mathcal{B}_L^T} \leq r_1$ and $|\overline{w}|_{\mathcal{F}_L^T} \leq r$, we define $w := K_L^T(\overline{w}, \gamma)$ the element of $\mathcal{F}_L^T$ solving

$$\mathcal{Q}(\Omega_L^T)(\overline{w}, \gamma) \; : \; \begin{cases} w_t + A_L w = \Sigma(\gamma) - \partial_x(\overline{w}(\phi(\gamma) + \overline{w}/2)) & \text{in} \quad \Omega_L^T, \\ w(x,y,0) = u_0(x,y) - \phi(\gamma)(x,y,0) & \text{in} \quad \Omega_L, \\ w(0,y,t) = 0 & \text{in} \quad \mathbf{R}\times[0,T), \end{cases}$$

We postpone the density argument which allows us to define K_L^T to the end of the proof. Applying (i) of Lemma 5.1 to problem $\mathcal{Q}(\Omega_L^T)(\overline{w}, \gamma)$, there exists $C_0 > 0$ such that,

$$|w|^2_{\mathcal{F}_L^T} \le C_0\Big[|u_0 - \phi(\gamma)(\cdot,\cdot,0)|^2_{\mathcal{F}_L} + |\Sigma|_{\mathcal{I}_L^T}|w|_{\mathcal{F}_L^T} + T^{\frac{1}{4}}(1+T^{\frac{1}{2}})\left(|\overline{w}|_{\mathcal{F}_L^T} + |\phi(\gamma)|_{\mathcal{F}_L^T}\right)|\overline{w}|_{\mathcal{F}_L^T}|w|_{\mathcal{F}_L^T}\Big], \tag{22}$$

and

$$\int_{\Omega_L^T}\left((\partial_x^{-1}w_y)^2 + (\partial_x^{-1}w_{yy})^2\right)(1+x) \le C_0\Big[|u_0 - \phi(\gamma)(\cdot,\cdot,0)|^2_{\mathcal{F}_L} + |\Sigma|_{\mathcal{I}_L^T}|w|_{\mathcal{F}_L^T} + T^{\frac{1}{4}}(1+T^{\frac{1}{2}})\left(|\overline{w}|_{\mathcal{F}_L^T} + |\phi(\gamma)|_{\mathcal{F}_L^T}\right)|\overline{w}|_{\mathcal{F}_L^T}|w|_{\mathcal{F}_L^T}\Big]. \tag{23}$$

The later will be useful for the proof of (iv). We focus on (22). Using the estimates of Proposition 4.1, there exists C depending only on the choice of ψ such that,

$$\begin{aligned}
&|w|^2_{\mathcal{F}_L^T}\\
&\le C\left[|\gamma|^2_{\mathcal{B}_L^T} + \left((|\gamma|_{\mathcal{B}_L^T} + |\gamma|^2_{\mathcal{B}_L^T}) + (T^{\frac{1}{4}} + T^{\frac{1}{2}})(|\gamma|_{\mathcal{B}_L^T} + |\overline{w}|_{\mathcal{F}_L^T})|\overline{w}|_{\mathcal{F}_L^T}\right)|w|_{\mathcal{F}_L^T}\right],\\
&\le Cr_1^2 + C\Big[(1+r_1)r_1 + T^{\frac{1}{4}}(1+T^{\frac{1}{2}})(r+r_1)r\Big]|w|_{\mathcal{F}_L^T}.
\end{aligned}$$

We choose $r = r(r_1)$ such that $Cr_1^2 \le r^2/3$ and $C(1+r_1)r_1 \le r/3$. Then for T small enough, say $T < T_0 = T_0(r_1)$ where $CT_0^{\frac{1}{4}}(1+T_0^{\frac{1}{4}})(r+r_1) \le 1/3$, $K^T_{1,L}$ maps $B_{\mathcal{F}_L^T}(0,r) \times B_{\mathcal{B}_L^T}(0,r_1)$ into $B_{\mathcal{F}_L^T}(0,r)$.

We fix $T \le T_0(r)$, let us show that $K_L^T(\cdot,\gamma)$ is a uniform contraction with respect to γ in $B_{\mathcal{B}_L^T}(0,r_1)$. Let γ in $B_{\mathcal{B}_L^T}(0,r_1)$, let $\overline{w}_1, \overline{w}_2$ which belong to $B_{\mathcal{F}_L^T}(0,r)$ and let $w_1 := K_L^T(\overline{w}_1,\gamma)$, $w_2 := K_L^T(\overline{w}_2,\gamma)$. The function $w := w_1 - w_2$ solves

$$\begin{cases}
w_t + A_L w = -\partial_x\left[\overline{w}\times(\phi(\gamma) + (\overline{w}_1+\overline{w}_2)/2)\right], & \text{in} \quad \Omega_L\times[0,T],\\
w(x,y,0) = 0, & \text{in} \quad \Omega_L,\\
w(0,y,t) = 0, & \text{in} \quad \mathbf{R}\times[0,T),
\end{cases}$$

where $\overline{w} := \overline{w}_1 - \overline{w}_2$.
From Lemma 5.1 and Proposition 4.1, we deduce

$$\begin{aligned}
|w|_{\mathcal{F}_L^T} &\le CT_0^{\frac{1}{4}}(1+T_0^{\frac{1}{4}})\left[|\phi(\gamma)|_{\mathcal{F}_L^T} + \left|\frac{\overline{w}_1+\overline{w}_2}{2}\right|_{\mathcal{F}_L^T}\right]|\overline{w}|_{\mathcal{F}_L^T},\\
&\le 2CT_0^{\frac{1}{4}}(1+T_0^{\frac{1}{4}})(r_1+r)|\overline{w}|_{\mathcal{F}_L^T} \le \frac{2}{3}|\overline{w}|_{\mathcal{F}_L^T}.
\end{aligned}$$

Thus $\left|K_L^T(\overline{w}_1,\gamma) - K_L^T(\overline{w}_1,\gamma)\right|_{\mathcal{F}_L^T} \leq 1/2|\overline{w}_1 - \overline{w}_2|_{\mathcal{F}_L^T}$ and the the part (i) of the theorem is achieved by the Picard fixed point theorem and Proposition 4.3 (i). Let $\gamma_1 = (u_{0,1}, g_1)$, $\gamma_2 = (u_{0,2}, g_2)$ be two elements of $B_{\mathcal{B}_L^T}(0, r_1)$ and $\overline{w}_1$, $\overline{w}_2$ two elements of $B_{\mathcal{F}_L^T}(0,r)$. Let $w_1 := K^L(\gamma_1, \overline{w}_1)$, $w_2 = K^L(\gamma_2, \overline{w}_2)$, then the difference $w := w_1 - w_2$, satisfies

$$\begin{cases} w_t + A_L w = -\Sigma - \partial_x \left(\overline{w}\dfrac{\overline{w}_1 + \overline{w}_2}{2}\right) - \partial_x\left(\overline{w}_2\phi\right) - \partial_x\left(\overline{w}\phi_1\right), & \text{in} \quad \Omega_L^T, \\ w(x,y,0) = u_0(x,y) - \phi(x,y,0), & \text{in } \Omega_L, \\ w(0,y,t) = 0, & \text{in } \mathbf{R}\times[0,T), \end{cases}$$

where $\overline{w} := \overline{w}_1 - \overline{w}_2$, $\phi := \phi(\gamma_1) - \phi(\gamma_2)$, $u_0 := u_{1,0} - u_{2,0}$ and $\Sigma := \Sigma(\gamma_1) - \Sigma(\gamma_2)$.
Applying again Lemma 5.1 to this problem, and Proposition 4.1, there exists constants $C_2 = C_2(r_1)$ and $C_3 = C_3(T_0, r_1, r)$ such that

$$|w|^2_{\mathcal{F}_L^T} \leq C_2|\gamma|_{\mathcal{B}_L^T} + C_3\left(|\gamma|_{\mathcal{B}_L^T} + |\overline{w}|_{\mathcal{F}_L^T}\right)|w|_{\mathcal{F}_L^T}, \tag{24}$$

$$\int_{\Omega_L^T} (\partial_x^{-1}w_y)^2(1+x)^2 + (\partial_x^{-1}w_{yy})^2 \leq C_2|\gamma|_{\mathcal{B}_L^T} + C_3\left(|\gamma|_{\mathcal{B}_L^T} + |\overline{w}|_{\mathcal{F}_L^T}\right)|w|_{\mathcal{F}_L^T}. \tag{25}$$

From (24), the application K_L^T is locally uniformly continuous and since K_L^T is a uniform contraction, we get (iii). Now, using (25) and (23), we get (iv).

In fact the Theorem (3.1) gives solutions to $\mathcal{Q}(\Omega_L^T)(\overline{w},\gamma)$ for $(\overline{w},\gamma)$ in $\mathcal{S}_{0,L}^T$. Using Proposition 4.2 and the estimate (24), we can continuously extend the definition of K_L^T to a map from $B_{\mathcal{F}_L^T}(0,r) \times B_{\mathcal{B}_L^T}(0,r_1)$ into $B_{\mathcal{F}_L^T}(0,r)$. Thus the construction of K_L^T is justified. □

Remark 5.2. *In* [17], *we treat the same problem, with the right boundary conditions*

$$\partial_x^{-1}u(L,y,t) = 0,\ u_x(L,y,t) = 0,\ u_{xx}(L,y,t) = 0, \quad \textit{in } \mathbf{R}\times[0,T),$$

instead of $< BC_L >$. *We obtain a result analogous to Theorem 5.1. Moreover, in this case, we have stronger smoothing effects and for a solution* $u \in \mathcal{F}_L^T$, *there exists* $0 < T^* \leq T$ *such that:*

$$\sup_{[0,T^*]}\left(t\int_{\Omega_L} u_x^2(1+x)dxdy\right) + \int_{\Omega_L^{T^*}} tu_{xx}^2 dxdydt < \infty.$$

With the regularity obtained in Theorem 5.1, we do not know whether the solution is unique in $\mathcal{F}_L^T$ for $L < \infty$, but for $L = +\infty$ the right boundary conditions $< BC_L >$ relax and we have,

Theorem 5.2. *Let $T > 0$ and u, $\overline{u}$ in $\mathcal{F}_\infty^T$ solving*

$$u_t + u_{xxx} + \partial_x^{-1} u_{yy} + \partial_x \left(\frac{u^2}{2} \right) = 0, \qquad \textit{in } \mathcal{D}'((0,+\infty) \times \mathbf{R} \times (0,T)),$$

$$\overline{u}_t + \overline{u}_{xxx} + \partial_x^{-1} \overline{u}_{yy} + \partial_x \left(\frac{\overline{u}^2}{2} \right) = 0, \qquad \textit{in } \mathcal{D}'((0,+\infty) \times \mathbf{R} \times (0,T)).$$

If $u(0,\cdot,\cdot) \equiv \overline{u}(0,\cdot,\cdot)$ in $L^2(\mathbf{R} \times (0,T))$ and $u(\cdot,\cdot,0) \equiv \overline{u}(\cdot,\cdot,0)$ in $L^2(\mathbf{R}_+ \times \mathbf{R})$, then $u = \overline{u}$.

Proof. Let T, u, $\overline{u}$ as above, we set $w = u - \overline{u}$ and $\overline{w} = (u + \overline{u})/2$. From the boundary condition $w(0,\cdot,\cdot) \equiv 0$ in $L^2(\mathbf{R} \times (0,T))$, w is in

$$H_0^1(\mathbf{R}_{+,x}, L^2(\mathbf{R}_y \times (0,T)) \cap L^2_{loc}\left((0,T)_t, H_0^1(\mathbf{R}_{+,x} \times \mathbf{R}_y)\right).$$

Moreover w satisfies the initial condition $w(\cdot,\cdot,0) \equiv 0$ in $L^2(\mathbf{R}_+ \times \mathbf{R})$ and solves

$$(26)\quad w_t + w_{xxx} + \partial_x^{-1} w_{yy} = -\partial_x(\overline{w} w), \quad \text{in } \mathcal{D}'((0,+\infty) \times \mathbf{R} \times (0,T)).$$

Let $0 < t < T$. From the proof of Lemma 5.1, the product $(1+x)^2 \partial_x(\overline{w} w) w$ belong to $L^1(\mathbf{R}_+ \times \mathbf{R} \times (0,t))$ and there exists a universal positive constant C such that the following estimate holds

$$(27)\quad \int_0^t \int_{\mathbf{R}_+ \times \mathbf{R}} \partial_x(\overline{w}) w (1+x)^2 dxdydt \le C t^{\frac{1}{4}} (1 + t\frac{1}{2}) |\overline{w}|_{\mathcal{F}_\infty^t} |w|^2_{\mathcal{F}_\infty^t}.$$

Let ϕ in $C_0^\infty((0,+\infty) \times \mathbf{R} \times (0,t))$, multiplying $\phi_t + \mathcal{A}\phi$ by $(1+x)^2\phi$, integrating on $\mathbf{R}_+ \times \mathbf{R} \times (0,t)$ and integrating by parts as in the proof of Proposition 3.3, we have

$$\int_0^t \int_{\mathbf{R}_+ \times \mathbf{R}} (\phi_t + \mathcal{A}\phi)(1+x)^2 \phi dxdydt = \frac{1}{2} \int_{\mathbf{R}_+ \times \mathbf{R}} \phi^2(x,y,t)(1+x)^2 dxdy$$

$$(28)\qquad + \int_0^t \int_{\mathbf{R}_+ \times \mathbf{R}} \left(3\phi_x^2 + (\partial_x^{-1}\phi_y)^2\right)(1+x) dxdydt.$$

Multiplying (26) by $(1+x)^2 w$ and integrating on $\mathbf{R}_+ \times \mathbf{R} \times (0,t)$, we get using (27)

$$\int_0^t \int_{\mathbf{R}_+\times\mathbf{R}} \left(w_t + w_{xxx} + \partial_x^{-1} w_{yy}\right) w(1+x)^2$$
$$= -\int_0^t \int_{\mathbf{R}_+\times\mathbf{R}} \partial_x(\overline{w}w)w(1+x)^2 dxdydt$$
$$\leq Ct^{\frac{1}{4}}(1+t\frac{1}{2})|\overline{w}|_{\mathcal{F}^t_\infty}|w|^2_{\mathcal{F}^t_\infty}. \tag{29}$$

Letting ϕ tends to w in $\mathcal{F}^T_L$ in (28) and plugging the result in (29), we get

$$\frac{1}{2}\int_{\mathbf{R}_+\times\mathbf{R}} w^2(1+x)^2 dxdy + 3\int_0^t \int_{\mathbf{R}_+\times\mathbf{R}} w_x^2(1+x)^2 dxdydt$$
$$\leq Ct^{\frac{1}{4}}(1+t\frac{1}{2})|\overline{w}|_{\mathcal{F}^t_\infty}|w|^2_{\mathcal{F}^t_\infty}.$$

Thus, for every $0<t<T$, $|\overline{w}|^2_{\mathcal{F}^t_\infty} \leq 2Ct^{\frac{1}{4}}|\overline{w}|^2_{\mathcal{F}^t_\infty}|w|^2_{\mathcal{F}^t_\infty}$ and we conclude that $w \equiv 0$ on $\mathbf{R}_+ \times \mathbf{R} \times (0,T)$. □

Corollary 5.1. *Let $T^* > 0$, (u_0, g) in $\mathcal{B}^{T^*}_\infty$ and let $u := \mathcal{V}^\infty(u_0, g)$, $T := T(|u_0, g|_{\mathcal{B}^{T^*}_\infty})$, then u is the unique solution to $\mathcal{P}_0(\mathbf{R}_+ \times \mathbf{R})(u_0, g)$ in $\mathcal{F}^T_\infty$. In particular $\mathcal{V}^\infty_1(u_0, g) = \mathcal{V}^\infty_2(u_0, g)$.*

For the numerical simulation of these solutions, it will be convenient to work in a finite domain (see also Remark 1.1). The following result will be useful.

Proposition 5.1. *Let $T^* > 0$, and for $L_0 < L \leq +\infty$, let (u_0^L, g^L) in $\mathcal{B}^{T^*}_\infty$. We suppose that*

$$(u_0^L, g^L) \underset{L\to+\infty}{\longrightarrow} (u_0^\infty, g^\infty), \text{ in } \mathcal{B}^{T^*}_\infty.$$

Let $r_1 := \liminf_{L\to\infty} |(u_0^L, g)|_{\mathcal{B}^{T^*}_\infty}$ and $T := T(r_1)$. We set $u^L := \mathcal{V}^L(u_0^L, g)$ for $L < L_0 \leq \infty$, then for every $\bar{L} > 0$ and every $0 < \bar{T} < T$,

$$u^L \underset{L\to+\infty}{\rightharpoonup} u^\infty, \text{ weakly in } \mathcal{F}^{\bar{T}}_{\bar{L}}.$$

Proof. For $L_0 < L \leq \infty$, let u_0^L, g^L and u^L as above, let $0 < \bar{T} < T$, from Theorem 5.1, the family (u^L) is bounded in the separable Hilbert space $\mathcal{F}^{\bar{T}}_{\bar{L}}$, uniformly in $\bar{L} > 0$, thus using a diagonal argument, there exist a subsequence (L_k) converging to $+\infty$ and an element $\bar{u}$ in $\mathcal{F}_{\infty}^{\bar{T}}$ such that

$$u^{L_k} \underset{k\to+\infty}{\rightharpoonup} \bar{u}, \text{ weakly in } \mathcal{F}_{\bar{L}}^{\bar{T}},$$

for every $\bar{L} > 0$. Thus $\left(u^{L_k}\right)$, $\left(\left(u^{L_k}\right)^2\right)$ converge to $\bar{u}$, $\bar{u}^2$ in $\mathcal{D}'(\mathbf{R}_+ \times \mathbf{R})$ and $\bar{u}$ satisfies

$$\bar{u}_t + \bar{u}_{xxx} + \partial_x^{-1}\bar{u}_{yy} + \partial_x\left(\frac{\bar{u}^2}{2}\right), \text{ in } \mathcal{D}'(\mathbf{R}_+ \times \mathbf{R}).$$

Moreover for every $L > L_0$, u^L satisfies the initial and boundary conditions $u^L(\cdot,\cdot,0) = u_0^L$ in $L^2(\mathbf{R}_+ \times \mathbf{R})$ and $u^L(0,\cdot,\cdot) = g^L$ in $L^2(\mathbf{R}_+ \times (0,\bar{T}))$, then $\bar{u}(\cdot,\cdot,0) = u_0^{\infty}$ and $\bar{u}(0,\cdot,\cdot) = g^{\infty}$ and from Corollary 5.1 $\bar{u} = u^{\infty}$. From the uniqueness of the limit, we have the convergence of the whole family (u^L). □

6. Concluding remarks

We point out that our method does not lead to global solutions. In order to obtain a global result, the initial-and-boundary value problem must be solved in L^2. Thus it would be relevant to use the method introduced by J. Bourgain [8], [7] for the study of nonlinear dispersive wave equations as J.E. Colliander and C.E. Kenig in [12] or by J.L. Bona, S.M. Sun and B.-Y. Zang in [2, 4] for the K-dV equation.

A work in progress [16] concerns the numerical problem related to our study. Precisely, numerical simulations for the initial-and-boundary value KP-II problem in $[0, L] \times \mathbb{S}^1$ will be performed. This describes the evolution of waterwaves created by a wavemaker in a channel.

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