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APPROXIMATE SHOCK CURVES FOR NON-CONSERVATIVE HYPERBOLIC SYSTEMS IN ONE SPACE DIMENSION

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Abstract. For non-conservative hyperbolic systems, several definitions of shock waves exist in the literature. In this paper, we propose a new, rather simple one in the case of genuinely non-linear fields. Independently, taking a vanishing viscosity process, we prove the existence of shock curves when one considers viscosities given by a matrix commuting with the matrix which occurs in the original system. This setting generalizes a recent proof given by Bianchini and Bressan. At the end of the paper we prove that both definitions agree to third order near a given left state.

Keywords: Non-conservative hyperbolic systems; shock curves; center manifold

1. Introduction

We consider the strictly hyperbolic system in one space dimension

$$u_t + A(u)u_x = 0, \qquad u(x,t) \in \mathcal{U}, \qquad \forall t > 0,\, x \in \mathbf{R}, \tag{1}$$

where $\mathcal{U}$ is an open convex set of $\mathbf{R}^n$ and A a $n \times n$ strictly hyperbolic matrix with smooth coefficients. Throughout this paper we will note $(\lambda_1(u), \dots, \lambda_n(u))$ the n distinct eigenvalues of $A(u)$, and $(r_1(u), \dots, r_n(u))$ a smooth field of associated right eigenvectors such that $|r_j(u)| = 1$. Systems like (1) are called non-conservative and naturally arise in e.g. multiphase flow models (see [7] for instance). They have been much less studied than systems of conservation laws

$$u_t + f(u)_x = 0, \qquad u(x,t) \in \mathcal{U}, \qquad \forall t > 0,\, x \in \mathbf{R} \tag{2}$$

for which solutions do exist (at least weakly) and for which discontinuities may appear in finite time. (We refer to [14] for the theory concerning these systems and in particular the notions of strict hyperbolicity, genuine non-linearity, shock profiles and Riemann problem.) One of the main reasons is probably that in systems like (1), there is a lack of a natural definition for the non conservative product $A(u)u_x$ when u possesses discontinuities, whereas a term like $f(u)_x$ still possesses a meaning in the sense of distributions.

Nonetheless, systems written in the non-conservative form (1) have been recently studied in two different directions. The first one is given by Dal Maso, LeFloch and Murat in [6] and has been used by Le Floch and Tzavaras [9]. The idea is to define a family of paths

$$\phi : [0,1] \times \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}^n,$$

which satisfies $\phi(0,u^-,u^+) = u^-$ and $\phi(1,u^-,u^+) = u^+$. Then the term $A(u)u_x$ when u is a step function

$$u(x) = \begin{cases} u^- \text{ if } x < 0, \\ u^+ \text{ if } x > 0, \end{cases}$$

is understood as $\left(\int_0^1 A(\phi(\tau,u^-,u^+))\phi'(\tau,u^-,u^+)\,d\tau\right)\delta_0$, where δ_0 stands for the Dirac mass at 0. Furthermore, admissible paths must satisfy the Rankine-Hugoniot like condition

$$\int_0^1 A(\phi(\tau,u^-,u^+))\phi'(\tau,u^-,u^+)\,d\tau = s(u^+ - u^-). \tag{3}$$

It has been remarked [8] that shock waves in that sense are indeed vanishing viscosity limits of traveling-wave solutions to

$$u_t + A(u)u_x = \varepsilon\left(B(u)u_x\right)_x, \qquad \forall (x,t) \in \mathbf{R} \times \mathbf{R}_+, \tag{4}$$

when the viscosity ε tends to 0, provided that the paths ϕ is correctly chosen (defined by the viscous shock profiles, associated to B). Eventually, this notion of admissible paths leads to existence and uniqueness results [10,2] for (1) where the nonconservative product is meant in the previous sense. This theory requires the knowledge of the diffusion matrix B and no canonical paths can be associated with a nonconservative system.

On the other hand, using a regularizing technique, Bianchini and Bressan [3] have yet more recently defined solutions to (1) as unique limits of solutions to the viscous system

$$u_t + A(u)u_x = \varepsilon u_{xx}, \qquad \forall (x,t) \in \mathbf{R} \times \mathbf{R}_+, \tag{5}$$

when the viscosity ε tends to 0. In particular, in this very technical work, they solve the Riemann problem associated to the original system for small values of $|u^+ - u^-|$

$$\begin{cases} u(x,0) = u^\pm, & \text{if } x \gtrless 0, \\ u = \lim_{\varepsilon\to 0} u^\varepsilon, & \text{in } \mathbf{L}^1_{loc}, \qquad \text{where } u^\varepsilon \text{ solves (5).} \end{cases}$$

In this general setting, the authors recover the classical succession of auto-similar i-waves and give a characterization of them. As in the conservative case, with a genuine nonlinear hypothesis on the i-th eigenvalue, these elementary i-waves are shocks or rarefaction waves. We note that this study is limited to the identity diffusion matrix only. In particular, the vanishing viscosity limit of (4) when the viscosity ε tends to 0 is not given by this theory.

Going further, the existence of small amplitude traveling wave solutions of (4) has been proved [12,13] for very general viscosity matrices $B(u)$ and for speeds in a neighborhood of $\lambda_i(u^-)$ provided the i-characteristic field is genuinely nonlinear and with a few more not very restrictive assumptions. Both definitions for a shock coincide provided the paths ϕ describe the shock profiles. Of course, if one wants to compute a shock curve[a] associated to a given left state u^-, which is of particular interest for the design of numerical methods[b], one faces the problem of computing the shock profiles and this may be very difficult.

In the conservative case (2) where $A = d_u f$, it is well known that the shock profiles may depend on $B(u)$ but not the shock curves. Indeed, a shock wave with left and right states (u^-, u^+) and speed σ satisfies the Rankine-Hugoniot condition

$$f(u^+) - f(u^-) = \sigma(u^+ - u^-). \tag{6}$$

Therefore the i-th shock curve starting from u^-: $u^+(\sigma) := \mathcal{S}_i(u^-, \sigma)$ may be parametrized by σ and satisfies (6). Differentiating (6) with respect to σ we get the ordinary differential system

$$\begin{cases} (A(u^+) - \sigma)u^+_\sigma = u^+ - u^-, \\ \quad u^+(\lambda_i(u^-)) = u^-. \end{cases} \tag{7}$$

As remarked by [1], this last characterization of the shock curves does not rely on the fact that the system is in conservative form, and it is tempting to use it as a definition of shock curves in the non conservative case and to set the following definition.

Definition 1. *We call approximate i-shock curve of (1) a non-constant solution to (7) for* σ *in a neighborhood of* $\lambda_i(u^-)$.

We stress that this definition does not make any assumption on any kind of viscosity in the original system nor it gives the freedom to specify any path in the discontinuities. Nonetheless, from the computation above, we immediately recover the exact shock curves when the system is actually conservative.

In this paper, we give several arguments to justify the definition 1. First, we prove that solutions to the ODE (7) do exist (locally in a neighborhood of $\lambda_i(u^-)$). Then, we compare our approximate shock curves with the shock curves found by a

[a]set of states u^+ such that (u^-, u^+) solves (3).

[b]For instance, the knowledge of shock curves is needed for exact Riemann solvers which can be plugged directly into a Godunov scheme.

vanishing viscosity limit process. In that aim, let us introduce the following definition.

Definition 2. *Let $B(v)$ be a positive $n \times n$ matrix which depends smoothly on $v \in \mathbf{R}^n$. We say that u is a vanishing viscosity solution to (1) associated to B if $u := \lim_{\varepsilon \to 0} u^\varepsilon$ in $\mathbf{L}^1_{loc}$ where*

$$u_t^\varepsilon + A(u^\varepsilon)u_x^\varepsilon = \varepsilon(B(u^\varepsilon)u_x^\varepsilon)_x, \qquad \forall (x,t) \in \mathbf{R} \times \mathbf{R}_+. \tag{8}$$

Let us see if we could formally recover (7) in the setting of a general quasilinear hyperbolic system (1) with vanishing viscosity associated to B. The shock curves are defined by the endpoints of shock profiles starting from a given left state u^-. More precisely, we assume that we have a family $(U(\cdot,\sigma))_\sigma$, $\sigma \in (\lambda_i(u^-) - \alpha, \lambda_i(u^-) + \alpha)$ such that $U(\cdot,\sigma)$ solves

$$\begin{cases} (A(U) - \sigma)U_x(x,\sigma) = (B(U)U_x)_x \qquad \forall x \in \mathbf{R}, \\ U(-\infty,\sigma) = u^- \qquad \forall \sigma \in (\lambda_i(u^-) - \alpha, \lambda_i(u^-) + \alpha), \\ U(\cdot, \lambda_i(u^-)) \equiv u^-. \end{cases} \tag{9}$$

As usual, for such a family of profiles, the map $U^\varepsilon(x,t,\sigma) := U\left(\dfrac{x - \sigma t}{\varepsilon}\right)$ solves (8) for all $\varepsilon > 0$. Therefore $U(\cdot,\sigma)$ is a family of i-shock profiles starting from u^- with right states $u^+(\sigma) := \lim_{x \to +\infty} U(x,\sigma)$. Integrating (9) along the profile gives

$$\sigma(u^+ - u^-) = \int_{\mathbf{R}} A(U)U_x dx. \tag{10}$$

Differentiating this equation with respect to σ and integrating by parts yields

$$(A(u^+) - \sigma)u_\sigma^+(\sigma) = u^+ - u^- + \int_{\mathbf{R}} A(U)_x U_\sigma - A(U)_\sigma U_x dx, \tag{11}$$

and we recover (7) up to the default term

$$R(U,\sigma) := \int_{\mathbf{R}} A(U)_x U_\sigma - A(U)_\sigma U_x dx.$$

We hope that $R(U,\sigma)$ is small at least for small jumps. Of course, if A is in fact the differential of a flux f, then $R(U,\sigma)$ clearly vanishes. We also remark that if the shock profiles are all on one fixed curve then R also vanishes. Therefore a natural way of finding the ordinary differential equation (7) is to assume that shock profiles and the shock curve coincide. We may also integrate (7) with respect to σ and obtain

$$\int_{\lambda_i(u^-)}^{\tau} A(u^+(\sigma))u_\sigma^+ \, d\sigma = \int_{\lambda_i(u^-)}^{\tau} \frac{d}{d\sigma}\left(\sigma(u^+(\sigma) - u^-)\right) d\sigma = \tau(u^+(\tau) - u^-), \tag{12}$$

which means that we recover the correct Rankine-Hugoniot condition (3) along such profiles.

In addition to this quantitative difference (compare (7) and (11)), there is also a qualitative difference between the solutions of (7) and the i-shock curves obtained by the vanishing viscosity technique. Indeed, in the construction of an approximate shock curve, we have chosen to fix the left state but it could have been possible also to fix a right state u^+ and to define an approximate shock curve of left states $v(\sigma)$ by

$$\begin{cases} v(\lambda_i(u^+)) = u^+, \\ (A(v(\sigma)) - \sigma)v_\sigma(\sigma) = u^+ - v(\sigma). \end{cases} \tag{13}$$

Now, let $u^- \in \mathcal{U}$ and $u(\sigma)$ be a solution of (7), we pick up a speed σ^+ and note $u^+ := u(\sigma^+)$, then we construct v the approximate i-shock curve of left states satisfying (13). Unfortunately, unlike the conservative case, the equality $v(\sigma^+) = u^-$ does not necessarily hold. This point should be investigated in the near future.

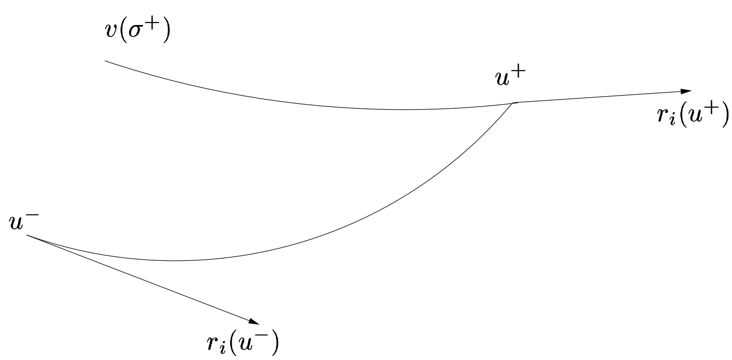

Fig. 1. approximate i-shock curves: shocks starting from u^-, shocks ending at u^+.

The paper is organized as follows. In the next section, we prove the existence and uniqueness of approximate i-shock curves. Section 3 is devoted to the definition of shock curves according to the technique of Bianchini and Bressan. In particular, if no assumption is made about the genuine non-linearity of the i-th characteristic field, we give the setting to generalize their method to viscosities $B(u)$ which commute with $A(u)$. For the sake of simplicity, we give an alternate proof that supposes the i-th characteristic field to be genuinely non-linear. In section 4 we estimate the term $R(\sigma)$ for σ close to the i-th characteristic speed $\lambda_i(u^-)$ and conclude by a comparison between approximate and true i-shock curves.

2. Existence and uniqueness of the approximate $i-$shock curve.

We first study the ordinary differential equation (7). The existence of solutions is not classical because of the degeneracy at $\sigma = \lambda_i(u^-)$. Moreover this equation does not apparently prescribe the first derivative of its solutions around this point. Let us first find which are the possible values of the first derivatives. Without loss of generality, we may suppose $i = 1$, $u^- = 0$ and $\lambda_1(0) = 0$. Then, we decompose the vector u in the basis $\mathcal{B} = (r_1(0), \cdots, r_n(0))$ and write $u = (v, w)$ with $v := u_1$ and $w := (u_2, \cdots, u_n)$. Finally, we write A in the basis $\mathcal{B}$:

$$A(v, w) := \begin{pmatrix} a(v, w) & b(v, w) \\ C(v, w) & D(v, w) \end{pmatrix},$$

where a is a real, b and C are respectively $(n-1)$-line and column vectors and D is a $(n-1) \times (n-1)$-square matrix. In particular, for $\sigma = 0$, we have $a(v, w) = 0$, $b(v, w) = 0$, $C(v, w) = 0$ and

$$D(v, w) = \begin{pmatrix} \lambda_2(0) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n(0) \end{pmatrix}.$$

We stress that D is invertible in the neighborhood of 0. Let us also notice that $r_1 \cdot \nabla\lambda_1(0) = a_v(0)$, therefore if we assume that A is C^1 and that the i-th eigenvalue of A is genuinely non-linear, we have $a_v(0) \neq 0$. In these coordinates $u(\sigma) = (v(\sigma), w(\sigma))$ is a solution of (7) if and only if u is a differentiable function defined in a neighborhood I of 0 such that $v(0) = 0$, $w(0) = 0$ and $\forall \sigma \in I$,

$$\begin{cases} a(v, w)v_\sigma + \langle b(v, w); w_\sigma \rangle = (\sigma v)_\sigma & (14) \\ C(v, w)v_\sigma + D(v, w)w_\sigma = (\sigma w)_\sigma. & (15) \end{cases}$$

Only (14) is degenerate. From (15) and $w(0) = 0$, we get $w_\sigma(0) = 0$ and by Gronwall's lemma $w_\sigma = \mathcal{O}(\sigma v_\sigma) + \mathcal{O}(w)$, with $w = \mathcal{O}\big(\int_0^\sigma \tau v_\sigma(\tau)\big)$. Keeping in mind these estimates, we write $v = v_\sigma(0)\sigma + o(\sigma)$ in (14), and we have

$$(a_v(0)v_\sigma(0) - 1 + o(1))\,\sigma v_\sigma = v + \mathcal{O}(\sigma w). \tag{16}$$

Dividing this equation by σ and letting σ go to 0, we are led to

$$\lim_{\sigma \to 0} v_\sigma(\sigma) = \frac{v_\sigma(0)}{a_v(0)v_\sigma(0) - 1}. \tag{17}$$

Since v_σ has a limit at 0, we deduce that v must be of class C^1 at 0, and (17) reads

$$v_\sigma(0)(a_v(0)v_\sigma(0) - 2) = 0. \tag{18}$$

We now consider the two cases:

Case 1: If $v_\sigma(0) = 0$. Since v_σ is continuous at 0, integrating (14) gives $v = o(\sigma \sup_{|\tau| \le |\sigma|} |v_\sigma(\tau)|)$ and replacing this estimate in (16) yields $\sigma v_\sigma = o(\sigma \sup_{|\tau| \le |\sigma|} |v_\sigma(\tau)|)$. Therefore $v \equiv 0$ and from the non degeneracy of D, $w \equiv 0$.

Case 2: If $v_\sigma(0) \neq 0$, we deduce $v_\sigma(0) = \dfrac{2}{a_v(0)}$.

The first case is the null solution and the second case corresponds to the solution we are looking for. We now show that this solution indeed exists and that there is no other one.

Proposition 3. *Let $A : \mathcal{U} \to \mathbf{R}^{n^2}$ be a C^2 function such that for all $u \in \mathcal{U}$, $A(u)$ is a strictly hyperbolic matrix with eigenvalues $(\lambda_1(u), \cdots, \lambda_n(u))$ and corresponding eigenvectors $(r_1(u), \cdots, r_n(u))$, $|r_i(u)| = 1$. We suppose that for a given $i \in (1, \cdots, n)$ and $u^- \in \mathcal{U}$, $r_i \cdot \nabla\lambda_i(u^-) \neq 0$ (i.e: the i-th eigenvalue is genuinely non-linear) then (7) has a local unique non trivial solution u in the neighborhood of $\lambda_i(u^-)$. Moreover*

$$u - u^- = \frac{2}{r_i \cdot \nabla\lambda_i(u^-)}(\sigma - \lambda_i(u^-)) + o(\sigma - \lambda_i(u^-)).$$

Proof. We suppose again $i = 1$, $u^- = 0$ and $\lambda_1(0) = 0$ and we will use the same notations as above. If the system were conservative (in the form (2)), then u would solve the following system

$$\begin{cases} f^1(v(\sigma), w(\sigma)) = \sigma v(\sigma), & (19) \\ \tilde{f}(v(\sigma), w(\sigma)) = \sigma w(\sigma), & (20) \end{cases}$$

where $v := u^1$, $w := (u^2, \cdots, u^n)$ and $\tilde{f} := (f^2, \cdots, f^n)$. We fix (σ, v) close to $(0,0)$. Since $\mathrm{d}_w\tilde{f}$ is invertible, using the implicit function theorem, we may solve w (20) into $w = W(v, \sigma)$. Now a non trivial solution of (19,20) $(\sigma, v, W(v, \sigma))$ would solve

$$F(v, \sigma) := \frac{f^1(v, W(v, \sigma))}{v} - \sigma = 0 \tag{21}$$

and thanks to the the genuine non linearity hypothesis, one can easily see that $\mathrm{d}_v F$ is invertible. Using again the implicit function theorem, the solution v to (21) may be defined as a function of σ. Hence, in a neighborhood of $(0,0)$, (u, σ) solves (6) with $u \not\equiv 0$ if and only if $u = (v(\sigma), W(v(\sigma), \sigma))$. We will follow the same strategy in the general case. Let δ be a positive real such that $\delta < \dfrac{1}{a_v(0)}$, we define

$$\begin{aligned} E_w &:= \left\{ w \in C^1((-1,1), \mathbf{R}^{n-1})\, ;\, w(0) = 0 \right\}, \\ E_v &:= \left\{ v \in C^1((-1,1), \mathbf{R})\, ;\, v(0) = 0 \right\}, \\ \Omega_\delta &:= \left\{ v \in E_v\, ;\, \left|\left|v_\sigma - v^0_\sigma\right|\right|_\infty < \delta \right\} \text{ where } v^0(\sigma) := \frac{2}{a_v(0)}\sigma. \end{aligned}$$

Let u be a solution of (7) and set $u_\varepsilon = \frac{1}{\varepsilon}u(\varepsilon\sigma)$. Writing the system under the form (14), (15), $u_\varepsilon = (v, w)$ verifies

$$\begin{cases} a(\varepsilon v, \varepsilon w)v_\sigma + \langle b(\varepsilon v, \varepsilon w); w_\sigma \rangle = \varepsilon(\sigma v)_\sigma & (22) \\ (\varepsilon v, \varepsilon w)v_\sigma + D(\varepsilon v, \varepsilon w)w_\sigma = \varepsilon(\sigma w)_\sigma. & (23) \end{cases}$$

Our aim is to solve (22,23) for small ε with (v, w) in $\Omega_\delta \times E_w$. We first define

$$\begin{aligned} W \,:\, \Omega_\delta \times (-\alpha, \alpha) &\longrightarrow E_w \\ (v, \varepsilon) \quad &\longmapsto W(v, \varepsilon) \end{aligned}$$

by $w := W(v,\varepsilon)$ is such that (23) holds, i.e.

$$\begin{cases} w(0) = 0, \\ w_\sigma = (D(\varepsilon v, \varepsilon w) - \varepsilon\sigma)^{-1}(\varepsilon w - C(\varepsilon v, \varepsilon w)v_\sigma). \end{cases}$$

Since D is invertible in the neighborhood of $(0,0)$, W is well defined. Moreover it can be shown (see below) that this is a C^1 function with respect to the C^1 norms. Let us also remark that $W(v,0)=0$. Then, we define

$$F \,:\, \Omega_\delta \times (-\alpha,\alpha) \longrightarrow E_v$$

by

$$F(v,\varepsilon)(\sigma) := \begin{cases} \dfrac{1}{\varepsilon v(\sigma)} \displaystyle\int_0^\sigma (a(\varepsilon v,\, \varepsilon W)v_\sigma + \langle b(\varepsilon v,\, \varepsilon W); W_\sigma\rangle)\; d\tau - \sigma & \text{if } \varepsilon \neq 0, \\ \dfrac{a_v(0,0)v(\sigma)}{2} - \sigma & \text{if } \varepsilon = 0, \end{cases}$$

where we have set $W = W(v,\varepsilon)$ to lighten the notations. We notice that for $v \in \Omega_\delta$ and a fixed small real ε, $(v,\, w := W(v,\varepsilon))$ solves (22) if and only if $F(v,\varepsilon)=0$. We compute

$$F(v_0,0) = 0, \;\; D_vF(v_0,0)(\delta v) = \frac{a_v(0,0)}{2}\delta v$$

and $D_vF(v_0,0)$ is invertible. Since F is a C^1 function, we can apply the implicit function theorem in Banach spaces and there is a C^1 function V and a sufficiently small $\alpha > 0$ so that: $\forall (v,w) \in \Omega_\delta \times E_w$, $\forall \varepsilon \in (-\alpha,\, \alpha)$,

$$(v,w,\varepsilon) \text{ solves } (22,23) \Leftrightarrow (v = V(\varepsilon) \text{ and } w = W(V(\varepsilon),\varepsilon))\,.$$

We conclude that (7) has a unique non locally constant C^1 solution in the neighborhood of 0. Proving that $F : \Omega_\delta \times (-\alpha,\, \alpha) \to E_w$ is C^1 with respect to C^1 norm is tedious but direct and not difficult. This comes from the fact that

$$\begin{aligned} \Omega_\delta &\longrightarrow C^0((-1,1)) \\ v &\longmapsto \left(\sigma \mapsto \frac{v(\sigma)}{\sigma}\right) \end{aligned}$$

is a C^1 function with respect to C^1 norm. □

3. Construction of shock waves with the center manifold technique of Bianchini and Bressan.

In [3] S. Bianchini and A. Bressan solve the Riemann problem with left and right states (u^-,u^+)

$$\begin{cases} \omega(x,t) = \omega(x/t,1),\; \omega(x,0) = u^\pm \text{ if } x \gtrless 0, \\ \omega \text{ is a vanishing viscosity solution of (9) with viscosity matrix } B \equiv \mathrm{Id}. \end{cases} \tag{24}$$

As usual, for each $u^- \in \mathcal{U}$ they construct n independent curves $\Psi_i(s)(u^-)$ such that the Riemann problem with left and right states $(u^-, \Psi_i(s)(u^-))$ is solved with a i-wave. Then, using an implicit function theorem, they find for each right state u^+ sufficiently close to u^-, a sequence $(s_1, \cdots, s_n)$ such that

$$u^+ = \Psi_n(s_n) \circ \cdots \circ \Psi_1(s_1)(u^-).$$

(Here we suppose that the eigenvalues are numbered such that $\lambda_1(u) < \cdots < \lambda_n(u)$.) In the next section, our aim is to estimate the difference between $\Psi_i(s)(u^-)$ and the approximate shock curve that has been defined in the preceding section in the case of genuinely nonlinear fields. In a first step, we generalize the construction made by Bianchini and Bressan to slightly more general viscosity matrices $B(u)$. Indeed, we will assume that $B(u)$ is uniformly invertible and positive, but also that it commutes with $A(u)$ for all u. This additional assumption will be used at many crucial points of the proof. It does not seem to be just a technical assumption that could be removed by adapting the proof. At the end of this section, we estimate the difference between the shock curves and the approximate shock curves for small amplitude shocks. We expect that this result concerning the distance between approximate and exact shock curves is wrong for general viscosity matrices (or at least is less precise).

3.1. *The center manifold*

From now on, we assume $B(u)$ is a $n \times n$ viscosity matrix, smooth in u that satisfies

$$\begin{cases} \exists c > 0, \forall u \in \mathcal{U}, \forall r \in \mathbf{R}^n, < B(u)r; r > \geq C|r|^2, & (25) \\ \forall u \in \mathcal{U}, \ A(u)B(u) = B(u)A(u). & (26) \end{cases}$$

Remark 4. Since the system has been assumed to be strictly hyperbolic, a natural consequence of the fact that $A(u)$ and $B(u)$ commute is that $B(u)$ is also diagonalizable in the same eigenvectors basis than $A(u)$. Indeed from

$$A(u)B(u)r_i(u) = B(u)A(u)r_i(u),$$

we obtain

$$A(u)B(u)r_i(u) = \lambda_i(u)B(u)r_i(u),$$

which means that $B(u)r_i(u) \in \mathbf{R}r_i(u)$.

We recall that the shock profiles satisfy the equation

$$(A(U) - \sigma)U_x = (B(U)U_x)_x, \qquad x \in \mathbf{R}. \tag{27}$$

We want to use the center manifold theorem (see [15]) to select among the solutions of (27) the ones which may remain bounded. As in [3], we rewrite (27) as the first order system

$$\begin{cases} u' = B(u)^{-1}w, \\ w' = (A(u) - \sigma)B(u)^{-1}w, \\ \sigma' = 0. \end{cases} \tag{28}$$

Now, let $u^- \in \mathcal{U}$. We are looking for i-shock profiles passing in the neighborhood of u^-. Of course, the constant shock wave $(u^-, 0, \lambda_i(u^-))$ is an equilibrium point of the system (28). We rewrite (28) as

$$\begin{cases} u' = B(u^-)^{-1}w + \left(B(u)^{-1} - B(u^-)^{-1}\right) w, \\ w' = (A(u^-) - \lambda_i(u^-))B(u^-)^{-1}w \\ \qquad\qquad + \left((A(u) - \sigma)B(u)^{-1} - (A(u^-) - \lambda_i(u^-))B(u^-)^{-1}\right) w, \\ \sigma' = 0, \end{cases}$$

which has the form $X' = MX + f(X)$, with $f(X) = \mathcal{O}(|X - X_0|^2)$ and $X_0 := (u^-, 0, \lambda_i(u^-))$. Applying the center manifold theorem, and calling $\mathcal{N} := \bigcup_k \mathrm{Ker}(M^k)$, there exists a smooth manifold $\mathcal{M}$ and $\Omega \subset \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ a neighborhood of $(u^-, 0, \lambda_i(u^-))$ such that

(1) $\mathcal{M} \subset \Omega$ is tangent to $\mathcal{N}$ at the stationary point X_0,
(2) $\mathcal{M}$ is locally invariant for the system (28),
(3) $\mathcal{M}$ contains all the small trajectories of (28) around $(u^-, 0, \lambda_i(u^-))$.

Here, we have

$$M = \begin{pmatrix} 0 & B(u^-)^{-1} & 0 \\ 0 & (A(u^-) - \lambda_i(u^-))B(u^-)^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

from which we may compute the tangent manifold

$$\mathcal{N} := \left\{(u, w, \sigma);\ w \in \mathbf{R}B(u^-)r_i(u^-)\right\}.$$

We already stress that without (26) the tangent manifold might be more complicated and might not even have the same dimension for instance. From the tangent space $\mathcal{N}$, we may parametrize $\mathcal{M}$ as

$$\mathcal{M} = \{(u, w, \sigma) \in \Omega,\ w = w_i q_i(u, w_i, \sigma)\},$$

where $w_i \in \mathbf{R}$ and q_i is a normalized ($|q_i(u, w_i, \sigma)| = 1$) smooth vector field, which satisfying $q_i(u^-, 0, \lambda_i(u^-)) \in \mathbf{R}B(u^-)r_i(u^-)$. Let us introduce the vectors

$$\tilde{r}_i(u, w_i, \sigma) := B(u)^{-1}q_i(u, w_i, \sigma),$$

we get $\tilde{r}_i(u^-, 0, \lambda_i(u^-)) \in \mathbf{R}r_i(u^-)$. Actually, for u close to u^-, the above construction may be also used to define the center manifold around the point $(u, 0, \lambda_i(u))$ instead of $(u^-, 0, \lambda_i(u^-))$. This leads to $\tilde{r}_i(u, 0, \lambda_i(u)) \in \mathbf{R}r_i(u)$ for all u in a neighborhood of u^-. Now, since we have assumed in (26) that $A(u)$ and $B(u)$ commute for all u in Ω, $A(u)$ and $B(u)$ share common eigenvectors and

$$q_i(u, 0, \lambda_i(u)) \in \mathbf{R}r_i(u). \tag{29}$$

We now show that this relation is not only true for $\sigma = \lambda_i(u)$, but also for any σ. This point is absolutely crucial for the convergence of the method below.

Lemma 5. *For all u in a neighborhood of u^- and all σ in a neighborhood of $\lambda_i(u)$, one has*

$$q_i(u,0,\sigma) \in \mathbf{R}r_i(u). \tag{30}$$

Proof. Let $(u^*,w^*,\sigma) \in \mathcal{M}$, from the center manifold theorem, there exists a curve $\{(u(x),w_i(x),\sigma), x \in (-\alpha,\alpha)\}$ such that $u(0)=u^*$, $w(0)=w^*$ and

$$(B(u)u_x)_x = (A(u)-\sigma)u_x, \qquad x \in (-\alpha,\alpha).$$

Writing $w = w_i q_i(u,w_i,\sigma)$ and $u_x = w_i\tilde{r}_i(u,w_i,\sigma)$, we obtain

$$w_{i,x}q_i(u,w_i,\sigma) + w_i q_{i,x}(u,w_i,\sigma) = (A(u)-\sigma)w_i\tilde{r}_i(u,w_i,\sigma). \tag{31}$$

But, since q_i is normalized, taking the dot product with q_i, we get

$$w_{i,x} = w_i(\tilde{\lambda}_i(u,w_i,\sigma)-\sigma) < \tilde{r}_i(u,w_i,\sigma); q_i(u,w_i,\sigma) >, \tag{32}$$

where we have defined

$$\tilde{\lambda}_i(u,v_i,\sigma) := \frac{< A(u)\tilde{r}_i(u,w_i,\sigma)\ ;\ q_i(u,w_i,\sigma) >}{< \tilde{r}_i(u,w_i,\sigma)\ ;\ q_i(u,w_i,\sigma) >}. \tag{33}$$

Plugging (32) in (31) and dividing by w_i yields

$$q_{i,x} = \Big(A(u) - \tilde{\lambda}_i < \tilde{r}_i; q_i > B(u)\Big)\,\tilde{r}_i - \sigma\,(Id - < \tilde{r}_i; q_i > B(u))\,\tilde{r}_i. \tag{34}$$

But $q_{i,x} = w_i\tilde{r}_i \cdot\nabla_u q_i + w_{i,x}\cdot\nabla_v q_i = \mathcal{O}(w_i)$ and letting w_i go to 0, we are led to

$$\Big(A(u) - \sigma Id + (\sigma - \tilde{\lambda}_i(u,0,\sigma)) < \tilde{r}_i\ ;\ q_i > B(u)\Big)\,\tilde{r}_i(u,0,\sigma) = 0. \tag{35}$$

From the assumption (26), the linear operator in the left-hand side is diagonal in the basis $(r_1(u),\ldots,r_n(u))$. The corresponding eigenvalues are

$$(\nu_k(u,\sigma))_{1\le k\le n} := (\lambda_k(u) - \sigma + (\sigma - \tilde{\lambda}_i(u,0,\sigma)) < \tilde{r}_i\ ;\ q_i >)\mu_k(u))_{1\le k\le n}$$

where $(\mu_k(u))_k$ are the eigenvalues of $B(u)$. For $\sigma = \lambda_i(u)$ we know that $\tilde{r}_i(u,0,\lambda_i(u)) \in \mathbf{R}r_i(u)$ and then $\nu_k(u,\lambda_i(u)) = 0 \Leftrightarrow k = i$. We get from a continuity argument

$$\tilde{r}_i(u,0,\sigma) \in \mathbf{R}r_i(u) \tag{36}$$

as claimed. In particular, we also have

$$q_i(u,0,\sigma) = B(u)^{-1}\tilde{r}_i(u,0,\sigma) \in \mathbf{R}r_i(u) \tag{37}$$

and since $q_i(u,0,\sigma)$ has been normalized, $q_i(u,0,\sigma)$ and therefore $\tilde{r}_i(u,0,\sigma)$ do not depend on σ. Hence, we may rewrite Eq. (35) as

$$\Big(A(u) - \tilde{\lambda}_i(u,0,\sigma) < \tilde{r}_i\ ;\ q_i > B(u)\Big)\,\tilde{r}_i(u,0,\sigma) = 0, \tag{38}$$

where $\tilde{\lambda}_i(u,0,\sigma)$ does not depend on σ. □

Remark 6. In the preceding proof, we have written a characterization of the solutions to (27). Let I be an open subset of $\mathbf{R}$, let $\{(u(x), w(x), \sigma),\ x \in I\} \subset \mathcal{M}$, then u satisfies

$$(B(u)u_x)_x = (A(u) - \sigma)u_x, \qquad x \in I,$$

if and only if (32) holds on I.

3.2. *The fixed point method*

In this section, we generalize the approach of Bianchini and Bressan [3] to our situation $B(u) \neq Id$. However, in a sake of simplicity, we do not give here all the details of the proof explicitly, but rather leave the reader follow exactly the strategy used in [3] in the provided context. On the other hand, in the following section using the stronger hypothesis of genuine non-linearity, we give a more elementary proof to existence of shock profiles which is sufficient for our purpose.

Let us introduce the following notations:

$$\begin{aligned}
\Gamma &:= \left\{\gamma = (u, v, \sigma) \in \mathcal{C}^0([0,1],\, \Omega)\, ;\, u(0) = u^-\right\},\\
\rho(\gamma(t)) &:=< \tilde{r}_i(\gamma(t))\ ;\ q_i(\gamma(t)) >,\\
f_i(t,\gamma) &:= \int_0^t \tilde{\lambda}_i(\gamma(\tau))\rho(\gamma(\tau))d\tau,\\
g_i(t,\gamma) &:= \sup\left\{ \int_0^t \lambda(\tau)\rho(\gamma(\tau))d\tau\ ;\ \lambda \text{ increasing and } \forall 0 \le T \le 1 : \right.\\
&\qquad\qquad \left. \int_0^T \lambda(\tau)\rho(\gamma(\tau))d\tau\ \le\ f_i(T,\gamma)\right\}.
\end{aligned}$$

Remark that from (25) ρ is positive and that in the case $B \equiv Id$, the function $g_i(\cdot,\gamma)$ is just the convex envelop of $f_i(\cdot,\gamma)$. We are now ready to describe the fixed point method of Bianchini and Bressan: let us fix a small positive parameter $s > 0$ and let γ be an element of Γ, then we define $\hat{\gamma}(\gamma, s) = (\hat{u}, \hat{w}_i, \hat{\sigma})$ by

$$\left\{\begin{aligned}
\hat{u}(t) &:= u^- + s\int_0^t \tilde{r}_i(\gamma(\tau))d\tau,\\
\hat{w}_i(t) &:= s\left(f_i(t,\gamma(\cdot)) - g_i(t,\gamma(\cdot))\right),\\
\hat{\sigma}(t) &:= \tfrac{1}{\rho(\gamma(t))}\frac{\mathrm{d}}{\mathrm{d}t}g_i(t,\gamma(\cdot)),
\end{aligned}\right. \tag{39}$$

In the case $B \equiv Id$, Bianchini and Bressan [3] show that there exists a unique $\gamma(\cdot, s)$ in Γ such that $\hat{\gamma}(\gamma(\cdot,s), s) = \gamma(\cdot, s)$. The result is still true in our case, but for shortness and since we are only interested with shock curves we do not give the proof here, the only additional difficulty is the definition of $g_i(\cdot,\gamma)$ which is no more the convex envelop of $f_i(\cdot,\gamma)$.

Assuming the existence and uniqueness of solution to (39) for small s, we define a family $\Psi_i(s)(u^-) := u(1,s)$ of right states joined from the left state u^- by a

i-wave.

$$\omega^s(x,t) = \begin{cases} u^- & \text{if } x/t \leq \sigma(0,s), \\ u(\tau,s) & \text{if } x/t = \sigma(\tau,s), \\ \Psi_i(s)(u^-) & \text{if } x/t \geq \sigma(1,s). \end{cases}$$

To simplify, let us suppose that $\lambda_i(u^-)$ is genuinely non linear and let us formally describe the i-wave. Let $\gamma(t,s) = {}^t(u,v,\sigma)$ be a solution to the fixed point (39) in a neighborhood of ${}^t(u^-,0,\lambda_i(u^-))$, a simple calculation, (30) and (39) yield

$$\frac{d}{dt}\tilde{\lambda}_i(\gamma(t,s))) = sr_i(u^-)\cdot\nabla_u\lambda_i(u^-) + o(s).$$

Now, if the i-field is genuinely nonlinear

$$r_i(u^-)\cdot\nabla_u\lambda_i(u^-) \neq 0,$$

and we have the following alternative :

(a) if $r_i(u^-)\cdot\nabla\lambda_i(u^-) > 0$, then the $t \mapsto \tilde{\lambda}_i(\gamma(t,s))$ is increasing and we have from (39) and (30), $\sigma(t,s) = \lambda_i(u(t,s))$, the function $\omega^s(x,t)$ is therefore a classical rarefaction wave.

(b) if $r_i(u^-)\cdot\nabla\lambda_i(u^-) < 0$, then $t \mapsto \tilde{\lambda}_i(\gamma(t,s))$ is decreasing. Coming back to (39), we find that $\sigma(t,s) = \sigma(s)$ is actually constant in t, and $w_i(t,s) > 0$ for $0 < t < 1$. We then make the change of variable $dt = w_i(t,s)dx$, which is increasing and maps $(0,1)$ onto $\mathbf{R}$. Setting $U(x) := u(t(x),s)$, we have

$$U_x(x) = u_t(t,s)\frac{dt}{dx} = w_i s\tilde{r}_i(U(x), w_i(t(x),s), \sigma(s))$$

from (39). We also have from (39) that

$$w_{i,x} = w_{i,t}\frac{dt}{dx} = sw_i\rho(\tilde{\lambda}_i - \sigma(s))$$

which are the conditions for U to solve

$$(B(U)U_x)_x = (A(U)-\sigma(s))U_x, \qquad x \in \mathbf{R}. \tag{40}$$

In this case, $\omega^s(x,t)$ is a vanishing viscosity limit of the viscous traveling waves of speed $\sigma(s)$, $u_\varepsilon(x,t) := U\left(\dfrac{x}{\varepsilon t}\right)$.

For $s < 0$ small enough in magnitude, the same method may be used replacing sup by inf and $\geq$ by $\leq$ in the definition of g_i.

Remark 7. In the case $B \equiv Id$ studied in [3], the parameter s is the length of the path $\{\omega^s(z,1),\ z \in \mathbf{R}\}$ and $\Psi_i(\cdot)(u^-)$ form a curve of right states reached from u^- by means of i-waves. Without any genuine nonlinearity hypothesis, these i-waves are not necessarily i-shocks or i-rarefaction curves but can be made of a possibly infinite succession of these two kinds of waves and the function $\Psi_i(\cdot)(u^-)$ is merely Lipschitz. In the sequel, we avoid this situation.

3.3. *Shock curves*

Here, we give an alternate proof of existence in the case the i-th characteristic field is moreover genuinely nonlinear. Clearly this result is weaker than the profile construction of Sainsaulieu [12] which has been rephrased using the center manifold technique by Schecter [13]. Nonetheless, we give it here for the sake of completeness and because our assumption that $A(u)$ and $B(u)$ commute will be absolutely crucial in the comparison of exact and approximate shock curves in the next section. In particular, (30) might not hold otherwise. We will come back to that point later.

Therefore, we now choose $u^- \in \Omega$, $i \in \{1, \cdots, n\}$ and make the genuine nonlinearity hypothesis, replacing r_i by $-r_i$ if necessary, we may assume

$$r_i(u^-) \cdot \nabla\lambda_i(u^-) < 0. \tag{41}$$

For data close to $(u^-, 0, \lambda_i(u^-))$, and $s > 0$ small enough, the fixed point $\gamma(t,s)$ satisfies

$$\begin{cases} u(t,s) = u^- + s\displaystyle\int_0^t \tilde{r}_i(\gamma(\tau,s))\mathrm{d}\tau, \\ w_i(t,s) = s\displaystyle\int_0^t \rho(\gamma(\tau,s))(\tilde{\lambda}_i(\gamma(\tau,s)) - \sigma(s))d\tau, \\ \sigma(s) = \dfrac{f_i(1,\gamma(1,s))}{\displaystyle\int_0^1 \rho(\gamma(\tau,s))d\tau}, \end{cases} \tag{42}$$

and describes a i-shock profile. Indeed, if $dt := w_i(t,s)dx$ and $U(x) := u(t(x),s)$, then (27) holds. Moreover we will show in the next section that $\sigma_s(0) = \dfrac{r_i \cdot \nabla\lambda_i(u^-)}{2} \neq 0$, hence, for the parametrization of the profiles $u(\cdot,s)$ one can take σ instead of s. From these observations we introduce the following

Definition 8. *We will call i-shock curve:*

$$\begin{aligned} \mathcal{S}_i(\cdot,u^-) : (\lambda_i(u^-) - \alpha_0\,,\, \lambda_i(u^-) + \alpha_0) &\longrightarrow \mathcal{U} \\ \sigma(s) &\longmapsto u(1,s), \end{aligned}$$

where u, w_i, σ solves (42).

In our case, the situation is easier than the problem without (41) and we prove the existence of such shock curves.

Proposition 9. *Suppose A satisfies (41) for u^- in $\mathcal{U}$, B is smooth, $B(u)$ and $A(u)$ commute for all u in $\mathcal{U}$ and (25) holds for some $c > 0$. Then $\mathcal{S}_i(\cdot,u^-)$ is uniquely defined and the corresponding functions $u(t,s)$, $w_i(t,s)$ and $\sigma(s)$ are smooth.*

Proof. Let k be a positive integer. Suppose that A and thus $\tilde{r}_i$ are C^k. We want to apply an implicit function theorem, let us introduce the space

$$E := \left\{{}^t(u, w_i, \sigma) \in C^k([0,1],\Omega)\ ;\ \sigma \text{ constant}\right\},$$

and the function

$$\begin{aligned} F : (-\alpha,\, \alpha) \times E &\longrightarrow E \\ (s, \gamma) \qquad &\longmapsto \overline{\gamma} = {}^t(\overline{u}, \overline{w_i}, \overline{\sigma}) \end{aligned}$$

where

$$\begin{cases} \overline{u}(t) := u(t,s) - u^- - s\displaystyle\int_0^t \tilde{r}_i(\gamma(\tau))\mathrm{d}\tau, \\ \overline{\sigma} := \quad \sigma \quad - \dfrac{\displaystyle\int_0^1 \rho(\gamma(\tau))\tilde{\lambda}_i(\gamma(\tau))d\tau}{\displaystyle\int_0^1 \rho(\gamma(\tau))d\tau}, \\ \overline{w_i}(t) := \ w_i(t) \ - s\displaystyle\int_0^t \rho(\gamma(\tau))(\tilde{\lambda}_i(\gamma(\tau)) - \sigma)d\tau. \end{cases} \tag{43}$$

We remark that $F \in C^k((-\alpha,\, \alpha) \times E, E)$ and that $F(0, \gamma_0) = 0$ where γ_0 is the constant function $\gamma_0 \equiv {}^t(u^-, 0, \lambda_i(u^-))$. Then we compute

$$\mathrm{D}F_\gamma(\gamma_0) \cdot \begin{pmatrix} \delta u \\ \delta v \\ \delta\sigma \end{pmatrix} = \begin{pmatrix} \delta u \\ \delta v \\ \delta\sigma \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ D_u\overline{\sigma}(\delta u) + D_{w_i}\overline{\sigma}(\delta w_i) + D_\sigma\overline{\sigma}(\delta\sigma) \end{pmatrix},$$

but, since (30) holds for all u, σ, we get that $\rho(u, 0, \sigma)$ and $\tilde{\lambda}_i(u, 0, \sigma)$ are independent from σ and then $D_\sigma\overline{\sigma}(\gamma_0) = 0$. Therefore $\mathrm{D}F_\gamma(\gamma_0)$ is invertible and by the implicit function theorem, we get a unique $s \to u(\cdot, s)$ defined on a neighborhood of 0 such that u is C^k and u solves (42). □

4. Order of the remaining term

We now study $R(\sigma)$ defined in the introduction and show that it is a term of third order in σ. The definition of i-shock profiles of the previous section allows us to derive rigorous estimates of R and then of the difference between the approximate i-shock curve and the i-shock curve of definition 8. More precisely, let $u^- \in \mathcal{U}$ and $i \in \{1, \cdots, n\}$ such that the inequality (41) holds, and let us consider the functions $u(t,s)$, $w_i(t,s)$ and $\sigma(s)$ constructed in the proposition 9. We can define $\tilde{u}(t, \sigma(s)) := u(t,s)$ and

$$U(x, \sigma) := \tilde{u}(\psi(x, \sigma), \sigma), \qquad \forall x \in \mathbf{R},$$

with ψ defined by

$$\begin{aligned} &\psi_x(x, \sigma) = w_i(\psi(x, \sigma(s)), s), \qquad \forall x \in \mathbf{R}, \\ &\psi(0, \sigma) = \frac{1}{2}. \end{aligned}$$

Then $(U(\cdot,\sigma))$ is a smooth family of i-shock profiles for which the formal calculus of the introduction which yields (11) is valid. We have

$$\big(A(\tilde u^+)-\sigma\big)\,\tilde u^+_\sigma(\sigma)=\tilde u^+-u^-+\tilde R(\sigma), \tag{44}$$

with

$$\tilde R(\sigma):=\int_0^1 A(\tilde u)_\tau\tilde u_\sigma - A(\tilde u)_\sigma\tilde u_\tau\,d\tau, \tag{45}$$

and $\tilde u^+(\sigma):=\tilde u(1,\sigma)$.

Proposition 10. *Under the above regularity and genuine nonlinearity hypothesis:*

$$\tilde R(\sigma)=\mathcal{O}(\big|\sigma-\lambda_i(u^-)\big|^3).$$

Proof. Since $\sigma_s\neq 0$, we have to prove that

$$R(s):=\int_0^1 A(u)_\tau u_s - A(u)_s u_\tau\,d\tau=\mathcal{O}(s^3).$$

Since $u(\tau,0)=u^-$, we have $R(0)=0$. Let us note $(\cdot,\,\cdot):\nabla_{u,u}A(u)$ the second order derivative of A at u. We compute

$$\begin{aligned}R_s(s)=&\int_0^1 u_{\tau s}\cdot\nabla_u A(u)u_s - u_s\cdot\nabla_u A(u)u_{\tau s}\,d\tau\\&+\int_0^1 u_\tau\cdot\nabla_u A(u)u_{ss} - u_{ss}\cdot\nabla_u A(u)u_\tau\,d\tau\\&+\int_0^1 (u_\tau,u_s):\nabla_{u,u}A(u)u_s-(u_s,u_s):\nabla_{u,u}A(u)u_\tau\,d\tau.\end{aligned}$$

Since $u_\tau(\tau,0)=0$, we have

$$\begin{aligned}R_{ss}(0)=&\int_0^1 u_{\tau ss}\cdot\nabla_u A(u^-)u_s - u_s\cdot\nabla_u A(u^-)u_{\tau ss}\,d\tau\\&+2\int_0^1 u_{\tau s}\cdot\nabla_u A(u^-)u_{ss} - u_{ss}\cdot\nabla_u A(u^-)u_{\tau s}\,d\tau\\&+2\int_0^1 (u_{\tau s},\,u_s):\nabla_{u,u}A(u^-)u_s-(u_s,\,u_s):\nabla_{u,u}A(u^-)u_{\tau s}\,d\tau.\end{aligned} \tag{46}$$

From the differentiation with respect to s of (42), we get the following equalities (where γ stands for $\gamma(\tau,s)$)

$$u_s(t,s)=s\int_0^t\left[u_s\cdot\nabla_u\tilde r_i(\gamma)+w_{i,s}\cdot\nabla_{w_i}\tilde r_i(\gamma)+\sigma_s\cdot\nabla_\sigma\tilde r_i(\gamma)\right]d\tau+\int_0^t\tilde r_i(\gamma)d\tau,$$

$$\sigma_s(s)=\frac{\displaystyle\int_0^1\Big[u_s\cdot\nabla_u(\rho\tilde\lambda_i)(\gamma)+w_{i,s}\cdot\nabla_{w_i}(\rho\tilde\lambda_i)(\gamma)+\sigma_s\cdot\nabla_\sigma(\rho\tilde\lambda_i)(\gamma)\Big]\,d\tau}{\displaystyle\int_0^1\rho(\gamma)d\tau}$$

$$-\frac{\displaystyle\int_0^1 \rho\tilde{\lambda}_i(\gamma)d\tau}{\left(\displaystyle\int_0^1 \rho(\gamma)d\tau\right)^2}\int_0^1 \left[u_s \cdot\nabla_u \rho(\gamma) + w_{i,s}\cdot\nabla_{w_i}\rho(\gamma) + \sigma_s\cdot\nabla_\sigma\rho(\gamma)\right] d\tau,$$

$$w_{i,s}(t,s) = s\left(\int_0^t \left[u_s \cdot\nabla_u(\rho\tilde{\lambda}_i)(\gamma) + w_{i,s}\cdot\nabla_{w_i}(\rho\tilde{\lambda}_i)(\gamma) + \sigma_s \cdot\nabla_\sigma(\rho\tilde{\lambda}_i)(\gamma)\right] d\tau - t\sigma_s(s)\right)$$
$$+\left(f_i(t,\gamma) - \sigma(s)\int_0^t \rho(\gamma)d\tau\right).$$

For $s = 0$, we compute $w_{i,s}(t,0) = 0$, $u_s(t,0) = tr_i(u^-)$ and we easily deduce that $R_s(0) = 0$. Moreover since we have proved that $q_i(u,0,\sigma) \in \mathbf{R}r_i(u)$, we have $\nabla_\sigma\tilde{\lambda}_i(\gamma_0) = 0$ and $\nabla_\sigma\rho(\gamma_0) = 0$. Therefore, $\sigma_s(0) = \frac{1}{2} r_i \cdot\nabla_u \lambda_i(u^-)$ as claimed earlier. We insist on the fact that the hypothesis that $A(u)$ and $B(u)$ commute is crucial here. In particular, if it was not the case (for example if one considers traveling wave solutions constructed by Sainsaulieu [12] or Schecter [13]) the computation above would be wrong.

Another differentiation of (42) leads to

$$u_{ss}(t,0) = 2\int_0^t u_s \cdot\nabla_u\tilde{r}_i(\gamma(\tau,0)) + v_s \cdot\nabla_v\tilde{r}_i(\gamma(\tau,0)) + \sigma_s \cdot\nabla_\sigma\tilde{r}_i(\gamma(\tau,0))d\tau,$$
$$= t^2 r_i(u^-)\cdot\nabla_u r_i(u^-).$$

Substituting in (46) $u_{ss}(\tau,0) = \tau^2 r_i(u^-)\cdot\nabla_u r_i(u^-)$,$u_{\tau ss}(\tau,0) = 2\tau r_i(u^-)\cdot\nabla_u r_i(u^-)$, $u_s(\tau,0) = \tau r_i(u^-)$ and $u_{\tau s}(\tau,0) = r_i(u^-)$, we get that the two first terms in the right hand side cancel each other and that the last term vanishes. Finally $R_{ss}(0) = 0$. □

We are now able to compare the approximate i-shock curve and the i-shock curve of the definition 8, namely, we prove the following proposition.

Proposition 11. *Let us call $\tilde{u}(\sigma)$ the re-parametrization by σ of the family of i-shock curves found above, and $u(\sigma)$ the approximate i-shock curve of the section 2. We define the vector $\overline{u} := u - \tilde{u}$ whose components in the basis $(r_1(u^-),\cdots, r_n(u^-))$ are denoted $\overline{u} = (\overline{u}_1,,\cdots,\overline{u}_n)$. Then we have*

$$\begin{cases} \overline{u}_i = \mathcal{O}(|\sigma - \lambda_i(u^-)|^3), \\ \overline{u}_j = \mathcal{O}(|\sigma - \lambda_i(u^-)|^4),\ \forall j \neq i. \end{cases} \tag{47}$$

In other words $u(\sigma) = \tilde{u}(\sigma) + \mathcal{O}(|\sigma - \lambda_i(u^-)|^3)$ and there exists a smooth re-parametrization of u: $u^*(\sigma) := u(\psi(\sigma))$ with $\psi_\sigma(\lambda_i(u^-)) = 1$ such that $u^*(\sigma) = \tilde{u}(\sigma) + \mathcal{O}(|\sigma - \lambda_i(u^-)|^4)$.

Proof. Without loss of generality, we may suppose again that $i = 1$, $u^- = 0$ and $\lambda_1(0) = 0$. We substract the equation $(A(\tilde{u}) - \sigma)\,\tilde{u}_\sigma = \tilde{u} + \mathcal{O}(\sigma^3)$ from $(A(u) - \sigma)\,u_\sigma = u$ and get

$$(A(u) - \sigma)\,\overline{u}_\sigma = \overline{u} + (A(\tilde{u}) - A(u))\,\tilde{u}_\sigma + \mathcal{O}(\sigma^3). \tag{48}$$

We use the same decomposition of the vectors in the basis $(r_1(0), \cdots, r_n(0))$ as in the proof of proposition 3. We write $\overline{u} = (v, w)$ with $v := \overline{u}_1$, $w := (\overline{u}_2, \cdots, \overline{u}_n)$ and

$$A(u) := \begin{pmatrix} a(u) & b(u) \\ C(u) & D(u) \end{pmatrix}.$$

Now Eq. (48) reads

$$\begin{pmatrix} (a(u)-\sigma)v_\sigma + < b(u); w_\sigma > \\ C(u)v_\sigma + (D(u)-\sigma)w_\sigma \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix} + (A(\tilde{u}) - A(u))\tilde{u}_\sigma + \mathcal{O}(\sigma^3). \tag{49}$$

On one hand, we have the expansion

$$(A(\tilde{u}) - A(u))\tilde{u}_\sigma = -\overline{u} \cdot \nabla_u A(u)\tilde{u}_\sigma + \underbrace{\mathcal{O}(|\overline{u}|^2)}_{\mathcal{O}(\sigma\overline{u})}.$$

Since $\tilde{u}_\sigma = \dfrac{2r_1(0)}{r_1(0) \cdot \nabla_u \lambda_1(0)} + \mathcal{O}(\sigma)$ and $r_1(0) \cdot \nabla_u a(0) = r_1(0) \cdot \nabla_u \lambda_1(0) r_1(0)$, we get

$$(A(\tilde{u}) - A(u))\tilde{u}_\sigma = \begin{pmatrix} -2v + \mathcal{O}(\sigma v) + \mathcal{O}(w) \\ \mathcal{O}(v) + \mathcal{O}(w) \end{pmatrix}. \tag{50}$$

On the other hand, let us recall that $D(u)$ is invertible and that $a(u) - \sigma = \sigma(1 + \mathcal{O}(\sigma))$, $b(u) = \mathcal{O}(\sigma)$, $C(u) = \mathcal{O}(\sigma)$. Plugging the preceding equalities and (50) in (49), we deduce the estimates

$$v + \sigma v_\sigma = \mathcal{O}(\sigma v) + \mathcal{O}(\sigma^2 v_\sigma) + \mathcal{O}(w) + \mathcal{O}(\sigma w_\sigma) + \mathcal{O}(\sigma^3), \tag{51}$$

$$w_\sigma = \mathcal{O}(v) + \mathcal{O}(\sigma v_\sigma) + \mathcal{O}(w) + \mathcal{O}(\sigma^3). \tag{52}$$

From (52) and the regularity of $\overline{u}$, we deduce $w_\sigma = \mathcal{O}(\sigma v_\sigma) + \mathcal{O}(w) + \mathcal{O}(\sigma^3)$, then by Gronwall's lemma we get $w_\sigma = \mathcal{O}(\sigma v_\sigma) + \mathcal{O}(\sigma^3)$. Now from (51), we get

$$(\sigma v)_\sigma = \mathcal{O}(\sigma^2 v_\sigma) + \mathcal{O}(\sigma^3),$$

which yields $v = \mathcal{O}(\sigma^3)$ and consequently $w = \mathcal{O}(\sigma^4)$. □

This last proposition shows that the approximate i-shock curve is strictly closer to the i-shock curve than any re-parametrization of the i-rarefaction curve $\mathcal{R}_i(u^-)(\sigma)$ defined by

$$\begin{cases} \left(A(\mathcal{R}_i(u^-)(\sigma)) - \sigma\right) \dfrac{d}{d\sigma}\mathcal{R}_i(u^-)(\sigma) = 0, \\ \mathcal{R}_i(u^-)(\lambda_i(u^-)) = u^-. \end{cases}$$

Indeed, in the conservative case, it is well known that $\mathcal{R}_i(u^-)(\lambda_i(u^-) + 2(\sigma - \lambda_i(u^-))) = \mathcal{S}_i(u^-)(\sigma) + \mathcal{O}(|\sigma - \lambda_i(u^-)|^2)$ with in general no better estimates.

5. conclusion

Differentiating the Rankine-Hugoniot condition with respect to the shock speed leads to a definition of shock curves which extends to the non conservative case. On the other hand, defining the shock curves through a vanishing viscosity method in the case the viscosity matrix commutes with the matrix of the system gives a solution which agrees with the preceding one to third order. These results are of prime interest for numerical computations for two reasons. First, the approximate shock curves are simply given by an ODE and are therefore easy to compute. This might be implemented into a Riemann solver in the spirit of Colella-Glaz's solver for Euler equations [5]. Second, the viscosities which arise in the numerical computation of solutions to (1) when one uses a Roe approximate Riemann solver [11] or VFFC method [7] are proportional to $|A(u)|$. This obviously commutes with $A(u)$, and therefore, the shock curves generated by such schemes are very close (third order) to our approximate shock curves. From the above study, this behavior is by no means generic since for general dissipations, the shock curves agree to the approximate shock curves only to second order.

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