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# A highly anisotropic nonlinear elasticity model for vesicles II. Derivation of the thin bilayer bending theory

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**Abstract** We study the thin-shell limit of the nonlinear elasticity model for vesicles introduced in part I. We consider vesicles of width $2\varepsilon \downarrow 0$ with elastic energy of order $\varepsilon^3$. In this regime, we show that the limit model is a bending theory for generalized hypersurfaces — namely, codimension 1 oriented varifolds without boundary. Up to a positive factor, the limit functional is the Willmore energy. In the language of $\Gamma$-convergence, we establish a compactness result, a lower bound result and the matching upper bound in the smooth case.

**Keywords** Calculus of Variation · $\Gamma$-convergence · Willmore functional · Rigidity estimates · Non-linear elasticity · Lipid bilayers

**Mathematics Subject Classification (2000)** 49Q10 · 49Q15 · 74B20 · 74K25 · 74K25

## 1 Introduction

In this article, we study the behavior as $\varepsilon$ goes to 0 of the nonlinear elasticity model for vesicle membranes with finite thickness introduced in [12]. More precisely, we perform a $\Gamma$-limit analysis of the family of functionals $\mathscr{F}/\varepsilon^3$ where $\varepsilon$ is the half-thickness of the membrane. Before stating the main results, we recall the model, set some notation and introduce complementary assumptions. However, this second part strongly depends on the first part of the paper.

### 1.1 An Eulerian nonlinear elasticity model

Let us fix an integer $d \geq 2$. Given $\varepsilon > 0$, a membrane of thickness $2\varepsilon$ in $\mathbf{R}^d$ is modeled by a bounded open set $\Omega \subset \mathbf{R}^d$ and two mappings $\tau \in L^2(\mathbf{R}^d, \mathbf{R}^d)$ and $\sigma \in L^2(\Omega, \mathbf{R}^d)$. These objects are subjected to a set of constraints: first, we assume that $\tau$ is a gradient vector field, more precisely, there exists $t \in W^{1,2}_{loc}(\mathbf{R}^d) \cap C(\mathbf{R}^d, [-\varepsilon, \varepsilon])$ such that $\tau = \nabla t$. Moreover, we assume

$$\Omega = \{y \in \mathbf{R}^d \;:\; |t|(y) < \varepsilon\}.$$

To prevent membranes from escaping to infinity, we fix a large radius $R > 0$ and enforce

$$|y| > R \quad \Longrightarrow \quad t(y) = +\varepsilon.$$

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Eventually, we assume that $\nabla\cdot\sigma = 0$ in $\mathscr{D}'(\Omega)$. Outside the set $\Omega$, we extend $\sigma$ by 0,

$$\sigma(y) \ := \ 0 \quad \text{for every } y \in \mathbf{R}^d \setminus \Omega.$$

We denote by $\mathscr{A}_\varepsilon(R)$ the set of triplets $(\sigma,\tau,\Omega)$ satisfying the above hypotheses and by $\mathscr{A}_\varepsilon$ the union $\cup_{R\uparrow\infty}\mathscr{A}_\varepsilon(R)$.
The material density at some point $x\in\Omega$ is defined as $\sigma(x)\cdot\tau(x)$ and the total quantity of material is,

$$\mathscr{Q}(\sigma,\tau) \ := \ \int_{\mathbf{R}^d} \sigma\cdot\tau.$$

The parameter $2\varepsilon$ should represent the thickness of the vesicle layer, hence the natural definition for the area of the membrane is $\mathscr{Q}(\sigma,\tau)/2\varepsilon$. Given a radius $R>0$ and a $(d-1)$-volume $S>0$, we set

$$\mathscr{A}_\varepsilon(R,S) \ := \ \{(\sigma,\tau,\Omega)\in\mathscr{A}_\varepsilon(R) \ : \ \mathscr{Q}(\sigma,\tau) = 2S\varepsilon\}.$$

The elastic energy associated to a configuration $a=(\sigma,\tau,\Omega)\in\mathscr{A}_\varepsilon$ has the form

$$\mathscr{F}(a) \ := \int_\Omega f(\sigma(y),\tau(y))\,dy,$$

where $f\in C(\mathbf{R}^d\times\mathbf{R}^d,\mathbf{R}_+)$ depends on the material. In our context, the stored-energy functions $f$ of interest vanish on the sphere

$$\mathbb{S}^{d-1} \ := \ \left\{(e,e) : e\in S^{d-1}\right\} \ \subset \mathbf{R}^d\times\mathbf{R}^d,$$

that is

$$f(\mathbb{S}^{d-1}) = \{0\}. \tag{1.1}$$

For the lower bound part of the $\Gamma$-limit analysis, we also require that $f$ does not degenerate with respect to this constraint: we assume that the infimum of $f/d(\cdot,\mathbb{S}^{d-1})^2$ over $\mathbf{R}^d\times\mathbf{R}^d\setminus\mathbb{S}^{d-1}$ is positive. Equivalently, we assume

$$f \ \geq \ \kappa f_0, \qquad \text{for some constant } \kappa>0, \tag{1.2}$$

with $f_0(u,v) \ := \ |u-v|^2+(|u|-1)^2+(|v|-1)^2$, for every $u,v\in\mathbf{R}^d$.
The energy functional associated to this particular function is denoted by

$$\mathscr{F}_0(\sigma,\tau,\Omega) \ := \int_\Omega f_0(\sigma(y),\tau(y))\,dy.$$

We study the $\Gamma$-limit as $\varepsilon$ tends to 0 of the energy $\mathscr{F}/\varepsilon^3$ defined on the set $\mathscr{A}_\varepsilon(R,S)$. For this we consider families $\{a_\varepsilon\}_{\varepsilon\in(0,1]}$ (or sequences $(a_{\varepsilon_k})$ with $\varepsilon_k\downarrow 0$) of triplets $a_\varepsilon=(\sigma_\varepsilon,\nabla t_\varepsilon,\Omega_\varepsilon)\in\mathscr{A}_\varepsilon(R,S)$ with energy of order of $\varepsilon^3$:

$$\sup_\varepsilon \frac{\mathscr{F}(a_\varepsilon)}{\varepsilon^3} \ < \ \infty. \tag{1.3}$$

## 1.2 Topological hypotheses

In the proofs, we use a uniform equicontinuity hypothesis which in general does not follow from (1.3) and might be only a technical assumption.

**Hypothesis 1** *There exists a modulus of continuity $\omega$ (i.e $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ is a non-decreasing concave continuous function such that $\omega(0) = 0$) such that*

$$\text{for every } \varepsilon \in (0,1], \quad t_{(\varepsilon)} : y \in \mathbf{R}^d \mapsto \frac{t_\varepsilon}{\varepsilon}(\varepsilon y) \text{ is } \omega\text{-continuous.}$$

The energy bound $\mathscr{F}_0(\sigma, \nabla t, \Omega) < \infty$ is not sufficient for $t$ being continuous. However, in the cases $d = 2$ and $d = 3$, if the stored energy function satisfies

$$f(\sigma,\tau) \;\geq\; \kappa'(|\tau| - 1)^p$$

for some $p > d$ and $\kappa' > 0$, then Hypothesis 1 is the consequence of the energy bound (1.3). Indeed, in this case,

$$\int_{(1/\varepsilon)\Omega_\varepsilon} (|\nabla t_{(\varepsilon)}| - 1)^p \;=\; \varepsilon^{3-d} \left( \frac{1}{\varepsilon^3} \int_{\Omega_\varepsilon} (|\nabla t_\varepsilon| - 1)^p \right) \;\leq\; \left( \sup_\varepsilon \frac{\mathscr{F}(a_\varepsilon)}{\varepsilon^3} \right) \frac{\varepsilon^{3-d}}{\kappa'}.$$

Consequently, the family $t_{(\varepsilon)}$ has uniformly bounded gradients in $L^p(\mathbf{R}^d)$ and, by Morrey embedding theorem, is uniformly equi-Hölder-continuous with Hölder exponent $1 - d/p$.

In the physical case $d = 3$ we can improve the compactness result under the following assumption which prescribes the genus of the membrane.

**Hypothesis 2** *For every $\varepsilon \in (0,1]$, the open set $\{y \;:\; |t_\varepsilon|(y) < 1/10\}$ is connected. Moreover, there exists $g_0 \geq 0$ such that for every $\varepsilon \in (0,1]$:*
*if $\Gamma$ is a compact subset of a smooth embedded surface $\Gamma' \subset \mathbf{R}^3$ and if $\Phi : \Gamma \times [-1,1] \to \mathbf{R}^3$ is a smooth mapping satisfying*

$$\Phi(\Gamma \times \{\pm 1\}) \;\subset\; \left\{ y \in \mathbf{R}^3 \;:\; \pm t_\varepsilon(y) > 1/10 \right\};$$

*then $\Gamma$ is homeomorphic to a closed subset of the $g_0$-torus.*

## 1.3 Compactness

Let us fix $R, S > 0$ and consider a family $\{a_\varepsilon\}_{0<\varepsilon\leq 1}$, $a_\varepsilon = (\sigma_\varepsilon, \tau_\varepsilon, \Omega_\varepsilon) \in \mathscr{A}_\varepsilon(R,S)$, satisfying Hypothesis 1 and

$$E_0 \;:=\; \sup_{0<\varepsilon\leq 1} \frac{\mathscr{F}_0(a_\varepsilon)}{\varepsilon^3} \;<\; +\infty. \tag{1.4}$$

Our strategy is to approximate the membrane described by the data $a_\varepsilon$ by a smooth hypersurface $\Sigma_\varepsilon = \partial O_\varepsilon$ where $O_\varepsilon$ is an open subset of $B_R$ which is close in $L^1$ to $[t_\varepsilon \equiv -\varepsilon]$ — see Section 2, Proposition 2.1. We obtain uniform bounds on the $(d-1)$-volume of $\Sigma_\varepsilon$ and on the Wilmore energy $\mathscr{W}(\Sigma_\varepsilon)$. Sets of finite perimeter seem reasonable limit objects for the family $\{O_\varepsilon\}$ as up to extraction, $(O_\varepsilon)$ converges towards a set with finite perimeter $O_0$. Unfortunately, we may loose large pieces of membrane in the limit process: two (or more) pieces of the hypersurface $\Sigma_\varepsilon$ may coincide at the limit $\varepsilon \downarrow 0$, leading to $\mathscr{H}^{d-1}(\partial O_0) < S$. Moreover, if we consider the behavior of the Willmore energy, the limit surface $\Sigma_0 = \partial O_0$ may not have square integrable mean curvature, as cusps arise on the boundary of the cancelling pieces of hypersurface (see Figure 1.1). To keep track of these phenomena and prevent cancellation, we have to take into account multiplicity.

We do this by considering hypersurfaces as $(d-1)$-dimensional varifolds. Varifolds have been introduced as a generalization of manifolds by Almgren [2] for the study of Plateau's problem, (see also Allard [1] and the reference book by Simon [15]). More precisely, here we consider the set of *oriented* $(d-1)$-varifolds as introduced by Hutchinson [10].

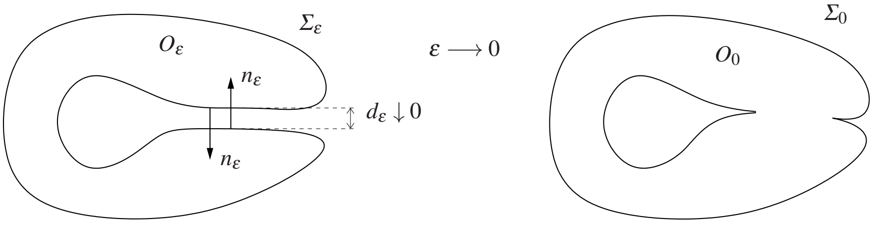


**Fig. 1.1** Cancellation of boundaries with opposite orientations.

**Definition 1.1**

a) The space of oriented $(d-1)$-varifolds in $\mathbf{R}^d$ is the topological dual of $C_c(\mathbf{R}^d \times S^{d-1})$, that is the space $\mathscr{M}(\mathbf{R}^d \times S^{d-1})$ of Radon measures on $\mathbf{R}^d \times S^{d-1}$ endowed with the weak star topology of Radon measures.

b) To any oriented $(d-1)$-varifold $\mathscr{V} \in \mathscr{M}(\mathbf{R}^d \times S^{d-1})$, we associate a distribution $\Lambda\mathscr{V} \in \mathscr{D}'(\mathbf{R}^d, \mathbf{R}^d)$ defined by

$$\langle \Lambda\mathscr{V}; \psi \rangle \;:=\; \langle \mathscr{V}\,; (y,n) \mapsto \psi(y)\cdot n \rangle\,, \quad \text{ for } \psi \in \mathscr{D}(\mathbf{R}^d, \mathbf{R}^d).$$

c) Given a smooth hypersurface $\Sigma$ oriented by $\nu$, we define the oriented $(d-1)$-varifold $\mathscr{V} = \mathscr{V}(\Sigma, \nu)$ by

$$\langle \mathscr{V}; \varphi \rangle \;:=\; \int_\Sigma \varphi(x, \nu(x))\, d\mathscr{H}^{d-1}(x), \quad \text{ for every } \varphi \in C_c(\mathbf{R}^d \times S^{d-1}).$$

*Remark 1.1*

a) Usually, oriented $k$-varifolds are defined as the Radon measures over $\mathbf{R}^d \times G^o(k,d)$, where $G^o(k,d)$ denotes the Grassmannian of oriented $k$-subspaces of $\mathbf{R}^d$. Here, we consider varifolds with co-dimension 1 and we can identify any $(d-1)$-dimensional oriented subspace of $\mathbf{R}^d$ with its positively oriented unit normal. This defines a smooth diffeomorphism $G^o(d-1,d) \overset{\sim}{\to} S^{d-1}$.

b) Similarly, in the literature (see e.g. [10]), one associates to any oriented $k$-varifold a $k$-current $\mathscr{C}\mathscr{V}$. Here, we choose not to treat currents explicitly. We identify simple $(d-1)$ vectors of the form $(-1)^{i+1} e^1 \wedge \cdots \wedge e^{i-1} \wedge e^{i+1} \wedge \cdots \wedge e^d$ with $e_i$ . By duality, this identifies the $(d-1)$-current $\mathscr{C}\mathscr{V}$ with $\Lambda\mathscr{V}$. In fact, in the present paper the boundary of $\mathscr{C}\mathscr{V}$ always vanishes. With our notation, this amounts to saying that $\Lambda\mathscr{V}$ is curl free ($\partial_i[\Lambda\mathscr{V}]_j = \partial_j[\Lambda\mathscr{V}]_i$ for $1 \le i,j \le d$) or equivalently, thanks to Poincaré conditions, that the distribution $\Lambda\mathscr{V}$ is a gradient.

In general the distribution $\Lambda\mathscr{V}$ carries strictly less information than the oriented varifold $\mathscr{V}$. For instance, if $\Sigma \neq \emptyset$ is a smooth compact hypersurface oriented by $\nu$ then $\mathscr{V} := \mathscr{V}(\Sigma,\nu) + \mathscr{V}(\Sigma,-\nu)$ does not vanish but $\Lambda\mathscr{V} \equiv 0$.

c) When $\Sigma$ is the boundary of a smooth and bounded open set $O \subset \mathbf{R}^d$, with outward unit normal $\nu$, then $\Lambda[\mathscr{V}(\Sigma,\nu)] = -\nabla \mathbf{1}_O$.

Our compactness result concerns varifolds constructed from elements of $\mathscr{A}_\varepsilon$.

**Definition 1.2** Let us fix a cut-off function $\chi^\star \in C_c^\infty(1/2,2)$ satisfying $\chi^\star(1)=1$. For $\varepsilon\in(0,1]$, we associate to any element $a=(\sigma,\nabla t,\Omega)$ of $\mathscr{A}_\varepsilon$ the oriented varifold,

$$\langle \mathscr{V}_\varepsilon^\star(a);\varphi\rangle \;:=\; \frac{1}{2\varepsilon}\int_\Omega \chi^\star(|\nabla t|(y))\varphi\left(y,\frac{\nabla t}{|\nabla t|}(y)\right)dy, \quad \forall\varphi\in C(\mathbf{R}^d\times S^{d-1}).$$

**Definition 1.3 (Limit sets)**
a) The limit set $\mathscr{A}_0(R,S)$ in our $\Gamma$-convergence analysis is a set of oriented $(d-1)$-varifolds. Namely, $\mathscr{V}_0\in\mathscr{M}(\mathbf{R}^d\times S^{d-1})$ belongs to $\mathscr{A}_0(R,S)$ if there exists a sequence of smooth open sets $(O_k)_{k\ge 1}\subset B_R$ with outward unit normals $\nu_k$ and boundaries $\Sigma_k$ such that

$$\begin{aligned} &\sup_k \mathscr{W}(\Sigma_k) \;<\; \infty, \qquad\qquad \mathscr{H}^{d-1}(\Sigma_k) \overset{k\uparrow\infty}{\longrightarrow} S,\\ &\text{and} \qquad \mathscr{V}(\Sigma_k,\nu_k) \overset{k\uparrow\infty}{\longrightarrow} \mathscr{V}_0 \;\text{ as Radon measures.}\end{aligned} \tag{1.5}$$

b) In the case $d=3$, we also introduce the subset $\mathscr{A}_{00}(R,S,g_0)$ where $g_0\in\mathbf{N}$ is the genus introduced in Hypothesis 2.
We say that $\mathscr{V}_0\in\mathscr{A}_{00}(R,S,g_0)$ if there exists a sequence of smooth open sets $(O_k)\subset B_R$ such that (1.5) holds and moreover $\Sigma_k$ is connected and $\mathrm{genus}(\Sigma_k)\le g_0$ for every $k\ge 0$.
In this case the second fundamental form $II_{\Sigma_k}=\mathrm{div}_{\Sigma_k}\nu_k$ on $\Sigma_k$ satisfies the identity

$$|II_{\Sigma_k}|^2 \;=\; |h_{\Sigma_k}|^2-2K_k,$$

where $K_k=\det II_{\Sigma_k}$ denotes the Gaussian curvature of $\Sigma_k$. Thanks to the Gauss-Bonnet formula, we have

$$\int_{\Sigma_k}|II_{\Sigma_k}|^2\,d\mathscr{H}^{d-1} \;=\; \mathscr{W}(\Sigma_k)-2\int_{\Sigma_k}K_k\,d\mathscr{H}^{d-1} \;\le\; \mathscr{W}(\Sigma_k)+8\pi(g_0-1).$$

Consequently, if $\mathscr{V}_0\in\mathscr{A}_{00}(R,S,g_0)$, the second fundamental forms of the surfaces $(\Sigma_k)$ in (1.5) are uniformly square integrable .

We postpone further definitions and statements about varifolds to Section 4. Let us say however that the elements of $\mathscr{A}_0(R,S)$ are oriented integer rectifiable $(d-1)$-varifolds (Definition 4.1) which admit a $L^2$-generalized mean curvature (Definition 4.2). The definition of the Willmore energy extends to these objects as a lower semi-continuous functional on $\mathscr{A}_0(R,S)$. If $d=3$, the elements of $\mathscr{A}_{00}(R,S,g_0)$ admit furthermore a $L^2$-generalized second fundamental form.

**Theorem 1.1**
*Let $R,S>0$ and let $\{a_\varepsilon\}_{0<\varepsilon\le 1}$ be a family of configurations $a_\varepsilon=(\sigma_\varepsilon,\nabla t_\varepsilon,\Omega_\varepsilon)\in\mathscr{A}_\varepsilon(R,S)$ satisfying the energy bound* (1.4) *and Hypothesis 1. Then:*

*a) There exists a non negative oriented $(d-1)$-varifold $\mathscr{V}_0$ with total mass $S$ such that, up to extraction,*

$$\mathscr{V}_\varepsilon^\star(a_\varepsilon) \overset{\varepsilon\downarrow 0}{\longrightarrow} \mathscr{V}_0 \;\text{ as Radon measures.}$$

*b) There exists a set of finite perimeter $O_0\subset B_R$, such that*

$$\nabla\mathbf{1}_{O_0}=-\Lambda\mathscr{V}_0, \quad \text{and} \quad T_\varepsilon:=\frac{\varepsilon-t_\varepsilon}{2\varepsilon} \overset{\varepsilon\downarrow 0}{\longrightarrow} \mathbf{1}_{O_0} \;\text{ weakly in } BV(\mathbf{R}^d).$$

*c) Moreover, $\mathscr{V}_0\in\mathscr{A}_0(R,S)$ (hence $\mathscr{V}_0$ has finite generalized Willmore energy).*

*d) If $d=3$ and if Hypothesis 2 holds, then $\mathscr{V}_0\in\mathscr{A}_{00}(R,S,g_0)$.*

*Remark 1.2*
*i.* As a consequence of (b), if the interior domain $M_\varepsilon = [t_\varepsilon \equiv -\varepsilon]$ has prescribed volume or satisfies $\mathscr{H}^d(M_\varepsilon) \to V$, then $\mathscr{H}^d(O_0) = V$.

*ii.* The varifold $\mathscr{V}_0$ can be described by means of a $(d-1)$-rectifiable set $\Sigma_0$ and multiplicity functions $\theta_0^\pm$. The Willmore enrgy $\mathscr{W}(\mathscr{V}_0)$ has an explicit expression in terms of the generalized curvature of $\Sigma_0$ and $\theta_0^\pm$ (see Definition 4.1, Definition 4.2 and formula (4.8)).

## 1.4 Lower bound

For the lower bound, we make further assumptions on the stored energy function. Namely, we assume that

$$f \text{ is of class } C^2 \text{ in some neighborhood } \mathscr{N} \text{ of } \mathbb{S}^{d-1} \text{ in } \mathbf{R}^d \times \mathbf{R}^d, \tag{1.6}$$

and is isotropic in this neighborhood, that is

$$f(Q\sigma, Q\tau) \;=\; f(\sigma,\tau) \qquad \forall Q \in \mathrm{SO}(d), \forall(\sigma,\tau) \in \mathscr{N}. \tag{1.7}$$

This assumption parallels the frame indifference hypothesis in nonlinear elasticity.

**Theorem 1.2** *Let $R, S > 0$ and let $(a_{\varepsilon_k})_{\varepsilon_k \downarrow 0}$ with $a_{\varepsilon_k} \in \mathscr{A}_{\varepsilon_k}(R,S)$ be a sequence of configurations satisfying Hypothesis 1. Assume that there exists a $(d-1)$-varifold $\mathscr{V}_0$ such that $\mathscr{V}_{\varepsilon_k}(a_{\varepsilon_k}) \to \mathscr{V}_0$ as Radon measures.*
*Then, for every $f \in C(\mathbf{R}^d \times \mathbf{R}^d, \mathbf{R}_+)$ satisfying* (1.1)*,* (1.2)*,* (1.6) *and* (1.7)*, we have*

$$c_0(f)\mathscr{W}(\mathscr{V}_0) \;\le\; \liminf_{\varepsilon_k \downarrow 0} \frac{\mathscr{F}(a_{\varepsilon_k})}{\varepsilon_k^3},$$

*where $c_0(f) > 0$ only depends on the Hessian matrix $D^2 f$ on $\mathbb{S}^{d-1}$. Namely,*

$$c_0(f) \;:=\; \frac{\det L}{3 L_{2,2}} \qquad \textit{with} \qquad L \;:=\; \begin{pmatrix} \dfrac{\partial^2 f}{\partial {\sigma_d}^2} & \dfrac{\partial^2 f}{\partial \sigma_d \partial \tau_d} \\[2ex] \dfrac{\partial^2 f}{\partial \sigma_d \partial \tau_d} & \dfrac{\partial^2 f}{\partial {\tau_d}^2} \end{pmatrix} (e_d, e_d).$$

## 1.5 Upper bound in the smooth case

**Theorem 1.3** *Let $f \in C(\mathbf{R}^d \times \mathbf{R}^d, \mathbf{R}_+)$ satisfying* (1.1)*,* (1.6) *and* (1.7) *and let $O_0 \subset \mathbf{R}^d$ be a smooth bounded open subset with boundary $\Sigma_0$ and outward unit normal $\nu_0$. Let us define*

$$R \;:=\; \sup\{|y| : y \in O_0\} + 1 \quad \textit{and} \quad S \;:=\; \mathscr{H}^{d-1}(\Sigma_0).$$

*Then there exists a family $\{a_\varepsilon\}_{0<\varepsilon\le 1}$, $a_\varepsilon \in \mathscr{A}_\varepsilon(R,S)$, such that*

$$\mathscr{V}_\varepsilon^\star(a_\varepsilon) \overset{\varepsilon\downarrow 0}{\longrightarrow} \mathscr{V}(\Sigma_0,\nu_0) \textit{ as Radon measures and } c_0(f)\mathscr{W}(\Sigma_0) \;=\; \lim_{\varepsilon\downarrow 0} \frac{\mathscr{F}(a_\varepsilon)}{\varepsilon^3}.$$

*Moreover, noting $a_\varepsilon = \{\sigma_\varepsilon, \nabla t_\varepsilon, \Omega_\varepsilon\}$, the open sets $\Omega_\varepsilon$ and the vector fields $\sigma_\varepsilon$ and $\nabla t_\varepsilon$ are uniformly smooth. In particular Hypothesis 1 holds.*

To complete the $\Gamma$-limit analysis, we should extend this reconstruction result to any data $\mathscr{V}_0 \in \mathscr{A}_0(R,S)$. Namely, we should prove that given $\mathscr{V}_0 \in \mathscr{A}_0(R,S)$ there exists a family $\{a_\varepsilon\}_{0<\varepsilon\leq 1}$, with $a_\varepsilon \in \mathscr{A}_\varepsilon(R,S)$ such that $\mathscr{V}_\varepsilon^\star(a_\varepsilon) \to \mathscr{V}_0$ and $\mathscr{F}(a_\varepsilon)/\varepsilon^3 \to \mathscr{W}(\Sigma_0)$ as $\varepsilon \downarrow 0$. A natural way for this would be to answer positively the following question.

*Question 1.1* Let $\mathscr{V}_0 \in \mathscr{A}_0(R,S)$. Does there exist a sequence of smooth open sets $(O_k) \subset B_R$ with outward normal $\nu_k$ and boundary $\Gamma_k$ such that (1.5) holds and furthermore,

$$\mathscr{W}(\Sigma_k) \stackrel{k\uparrow\infty}{\longrightarrow} \mathscr{W}(\mathscr{V}_0) \ ?$$

As described in Section 2.7, the elements of $\mathscr{A}_0$ are oriented rectifiable $(d-1)$-varifolds with a generalized mean curvature in $L^2$. To our knowledge the best regularity result established for varifolds with a generalized mean curvature is due to A. Menne [11] who established their $C^2$-rectifiabilty (see also R. Schätzle [14] and references therein). In this context, giving a positive answer to Question 1.1 seems a difficult task.

In the case $d=3$ and under Hypothesis 2, the limit set is reduced to $\mathscr{A}_{00}(R,S)$ (Theorem 1.1.d). The elements of $\mathscr{A}_{00}$ are limit of compact surfaces $(\Sigma_k)$ with uniformly bounded second fundamental forms in $L^2$. These objects are more regular than in the general setting. For instance, a theorem of J. Fu [7] which improves earlier results by T. Toro [17, 18] implies that every $\Sigma_k$ is locally the image of a bi-Lipschitz mapping defined on a subset of $\mathbf{R}^2$. Furthermore, the number of bi-Lipschitz mappings required to cover $\Sigma_k$ and their Lipschitz constants are uniformly bounded with respect to $k$. In view of this result, we believe that giving a positive answer to Question 1.1 for $\mathscr{V}_0 \in \mathscr{A}_{00}(R,S,g_0)$ is a more tractable issue.

## 1.6 Notation

Throughout the paper, the letter $C$ denotes a non negative constant which is either a universal constant or only depends on the dimension $d$. For constants which also depend on other parameters, $\alpha_1,\cdots,\alpha_k$, we write $C(\alpha_1,\cdots,\alpha_k)$. As usual, the values of these constants may change from line to line. For constants which depend on the data introduced in the hypotheses (the dimension $d$, the limiting radius $R$, the prescribed $(d-1)$-volume $S$, the modulus of continuity $\omega$, the cut off function $\chi^\star$ or the energy upper bound $E_0$) but not on $\varepsilon$, we use the short hand $C_{\not\varepsilon}$ and we write $C_{\not\varepsilon}(\alpha_1,\cdots,\alpha_k)$ for constants also depending on other parameters.

We write $B_r(y)$ to denote the open ball in $\mathbf{R}^d$ with center $y$ and radius $r>0$ or simply $B_r$ for $B_r(0)$.

The $k$-dimensional hausdorff measure of a set $E \subset \mathbf{R}^d$ is denoted by $\mathscr{H}^k(E)$.

We often define the set of elements satisfying a property $\mathscr{P}$ by $[\mathscr{P}]$. For instance $[t=\varepsilon]$ is the set $\{y \in \mathbf{R}^d : t(y)=\varepsilon\}$.

Most of the time, we use $y$ or $z$ to denote a generic element of $\mathbf{R}^d$ whereas $x$ is always a point on a hypersurface.

For $e \in S^{d-1}$, $\pi_e$ denotes the orthogonal projection on the space $e^\perp = \{y \in \mathbf{R}^{d-1} : y\cdot e = 0\}$, that is $\pi_e(y) = y-(y\cdot e)e$.

We identify $e_d^\perp$ with $\mathbf{R}^{d-1}$ and for $y \in \mathbf{R}^d$, we write $y'=(y_1,\cdots,y_{d-1})=\pi_{e_d}y$, so that $y=(y',y_d)$.

Some objects introduced along the proofs are used in different and sometimes distant parts of the paper. These objects are singled out by means of a superscript star: $\omega^\star$, $U^\star$, *etc*. We have already met the cut-off function $\chi^\star$ in Definition 1.2.

We use the prefix "I" to refer to a result of the first part of this article. For instance, Theorem 2.1 in [12] is refered as Theorem I.2.1 .

### 1.7 Outline of the paper

In Section 2, we establish the compactness result Theorem 1.1. The main part of this section is devoted to the proof of Proposition 2.1 which contains the relevant constructions and estimates. In particular, we use the rigidity estimates of [12] to show that we can approximate the varifolds $\{\mathscr{V}^\star(a_\varepsilon)\}_\varepsilon$ by hypersurfaces $\{\Sigma_\varepsilon\}_\varepsilon$ with uniformly bounded Willmore energy.
In Section 2.7, we establish Theorem 1.1 as a consequence of Proposition 2.1 and of Allard's compactness theorem for integer rectifiable varifolds applied to the family $\{\Sigma_\varepsilon\}$.

In Section 3, we build a recovery sequence in the smooth case, proving Theorem 1.3. We also describe there the general form of the Hessian matrix of our anisotropic stored energy functions on the set $\mathbb{S}^{d-1}$.

In Section 4 we introduce further material concerning varifolds. The aim of this section is to provide a better understanding of the limit set $\mathscr{A}_0(R,S)$ and to define the Willmore energy of generalized hypersurfaces.

Section 5 is devoted to the proof of the lower bound of Theorem 1.2. In Section 5.1, we introduce an approximate mean curvature at the $\varepsilon$-level. At some point $x \in \Sigma_\varepsilon$, this approximate mean curvature only depends on the restriction of $\sigma_\varepsilon$ to the ball $B_{\sqrt{2}\varepsilon}(x)$. We show that this approximate curvature is indeed an approximation of the mean curvature on $\Sigma_\varepsilon$ in a weak sense. Again, the rigidity estimates are crucial in this step.
In Section 5.2 we pass to the limit $\varepsilon \downarrow 0$ using lower semi-continuity of the Willmore energy. This reduces the lower bound problem to a relatively easy local optimization problem: minimize the local energy under prescribed approximate mean curvature.

In the last section, we discuss the hypotheses and indicate possible generalizations of our results. In particular, we consider the case of a material with spontaneous curvature $\mu \neq 0$.

## 2 Compactness, part 1. Approximation by smooth hypersurfaces

Let $R,S>0$ and let us consider a family $\{a_\varepsilon\}_{0<\varepsilon\leq 1}$ satisfying the hypotheses of Theorem 2.1. In this section, for every $\varepsilon$, we build a smooth open set $O_\varepsilon$ with boundary $\Sigma_\varepsilon$ and outward unit normal $\nu_\varepsilon$ such that $O_\varepsilon$ is close in $L^1$ to $[t_\varepsilon \equiv -\varepsilon]$ and $\mathscr{V}(\Sigma_\varepsilon,\nu_\varepsilon)$ is close to $\mathscr{V}'(a_\varepsilon)$. The relevant uniform bounds and properties of these families are collected in Proposition 2.1 below. We then deduce Theorem 1.1 from the proposition in Section 2.7.

**Proposition 2.1**
*a) For every $\varepsilon \in (0,1]$,*

$$\left|\frac{1}{2\varepsilon}\mathscr{H}^d(\Omega_\varepsilon)-S\right|+\left|\frac{1}{2\varepsilon}\int_{\mathbf{R}^d}|\nabla t_\varepsilon|-S\right|+|\langle\mathscr{V}^\star_\varepsilon(a_\varepsilon)\,;1\rangle-S|\ \leq\ C_{\not g}\varepsilon.$$

*b) For every $\varepsilon\in(0,1]$, there exists a smooth bounded open set $O_\varepsilon\subset B_{R+C\varepsilon}$ with outward unit normal $\nu_\varepsilon$, such that, with the notation $\Sigma_\varepsilon := \partial O_\varepsilon$ and $M_\varepsilon := \{y\in\mathbf{R}^d\ :\ t_\varepsilon(y)=-\varepsilon\}$, we have*

$$\|\mathbf{1}_{O_\varepsilon}-\mathbf{1}_{M_\varepsilon}\|_{L^1}\ \leq\ C_{\not g}\varepsilon^{d-1},\qquad \mathscr{H}^{d-1}(\Sigma_\varepsilon)\ \leq\ C_{\not g},\qquad \mathscr{W}(\Sigma_\varepsilon)\ \leq\ C_{\not g}.$$

*c) For every $\varphi\in C(\mathbf{R}^d\times\mathbf{R}^d)$ such that $\sup_{y,v}|\varphi(y,v)|/(1+|v|^2)<\infty$, we have*

$$\left|\frac{1}{2\varepsilon}\int_{\Omega_\varepsilon}\varphi(y,\nabla t_\varepsilon(y))\,dy-\int_{\Sigma_\varepsilon}\varphi(x,\nu_\varepsilon(x))\,d\mathscr{H}^{d-1}(x)\right|\ \stackrel{\varepsilon\downarrow 0}{\longrightarrow}\ 0.$$

*d) If moreover, $d=3$ and Hypothesis 2 holds, then we may assume that $\Sigma_\varepsilon$ is connected and has genus $g_\varepsilon\leq g_0$.*

Except in the proof of part (c) of the proposition (Section 2.6), the parameter $\varepsilon \in (0,1]$ is fixed. In this case, it is convenient to rescale the domain by a factor $1/\varepsilon$ by setting $\Omega_{(\varepsilon)} := \varepsilon^{-1}\Omega_\varepsilon$. We also define the rescaled data:

$$t_{(\varepsilon)}(y) := \varepsilon^{-1} t_\varepsilon(\varepsilon y), \quad \tau_{(\varepsilon)}(y) := \nabla t_\varepsilon(\varepsilon y), \quad \sigma_{(\varepsilon)}(y) := \sigma_\varepsilon(\varepsilon y) \quad \text{for } y \in \mathbf{R}^d.$$

With this notation, the function $t_{(\varepsilon)} \in C(\mathbf{R}^d, [-1,1])$ is $\omega$-continuous, we have $\nabla t_{(\varepsilon)} = \tau_{(\varepsilon)}$ and $\Omega_{(\varepsilon)} = \{y \in \mathbf{R}^d \,:\, |t_{(\varepsilon)}|(y) < 1\}$. The vector field $\sigma_{(\varepsilon)}$ is divergence free in $\Omega_{(\varepsilon)}$ and vanishes in $\mathbf{R}^d \setminus \Omega_{(\varepsilon)}$. We also have,

$$\begin{aligned} \mathscr{Q}(\sigma_{(\varepsilon)}, \nabla t_{(\varepsilon)}) \;&=\; \int_{\mathbf{R}^d} \sigma_{(\varepsilon)} \cdot \nabla t_{(\varepsilon)} \;=\; 2S/\varepsilon^{d-1}, \\ \text{and} \qquad \mathscr{F}_0(\sigma_{(\varepsilon)}, \nabla t_{(\varepsilon)}, \Omega_{(\varepsilon)}) \;&\le\; E_0/\varepsilon^{d-3}. \end{aligned} \tag{2.1}$$

Part (a) is established in Section 2.1. In Section 2.2, we introduce the harmonic extension $u_{(\varepsilon)}$ of $t_{(\varepsilon)}$ in a subset of $\Omega_{(\varepsilon)}$ which contains $[|t_{(\varepsilon)}| < 4/5]$. The level sets of $u_{(\varepsilon)}$ are candidates for the hypersurface $\Sigma_{(\varepsilon)}$.
In Section 2.3 we state and prove some technical lemmas which follow from the weak rigidity estimates of [12]. These lemmas are designed for selecting the "good" points where the data is sufficiently close to a zero energy limit state, as described in Theorem I.1.1. We also state there all the consequences of the strong rigidity estimates which are relevant to our purpose. Eventually, we bound the volume of "bad" regions. All these results are also used in the proof of the lower bound in Section 5.
In Section 2.4, we build the set $O_{(\varepsilon)}$ (and therefore the hypersurface $\Sigma_{(\varepsilon)}$) and we prove the estimates of part (b). The special case (d) is treated in Section 2.5. Eventually, the convergence result (c) is established in Section 2.6.

To lighten notation, we drop all the subscripts $(\varepsilon)$ and write $t$ for $t_{(\varepsilon)}$, $\sigma$ for $\sigma_{(\varepsilon)}$, $\Omega$ for $\Omega_{(\varepsilon)}$, *etc*. We come back to the unambiguous notation at the beginning of Section 2.6 when considering the limit $\varepsilon \downarrow 0$.

## 2.1 Proof of Proposition 2.1. a

For $-1 \le \alpha_- < \alpha_+ \le 1$, we set

$$\Omega_{\alpha_-}^{\alpha_+} \;:=\; \left\{ y \in \mathbf{R}^d \,:\, \alpha_- < t(y) < \alpha_+ \right\} \;\subset\; \Omega.$$

**Lemma 2.1** *Let $-1 \le \alpha_- < \alpha_+ \le 1$, we have*

$$\int_{\Omega_{\alpha_-}^{\alpha_+}} \nabla t \cdot \sigma \;=\; (\alpha_+ - \alpha_-) S/\varepsilon^{d-1}. \tag{2.2}$$

*Moreover,*

$$\left| \mathscr{H}^d(\Omega_{\alpha_-}^{\alpha_+}) - \frac{(\alpha_+ - \alpha_-)S}{\varepsilon^{d-1}} \right| \;\le\; \frac{5}{2}\mathscr{F}_0(\sigma, \nabla t, \Omega_{\alpha_-}^{\alpha_+}) + 2\sqrt{\frac{S\mathscr{F}_0(\sigma, \nabla t, \Omega_{\alpha_-}^{\alpha_+})}{\varepsilon^{d-1}}}, \tag{2.3}$$

*and similarly,*

$$\left| \int_{\Omega_{\alpha_-}^{\alpha_+}} |\nabla t| - \frac{(\alpha_+ - \alpha_-)S}{\varepsilon^{d-1}} \right| \;\le\; 4\mathscr{F}_0(\sigma, \nabla t, \Omega_{\alpha_-}^{\alpha_+}) + 4\sqrt{\frac{S\mathscr{F}_0(\sigma, \nabla t, \Omega_{\alpha_-}^{\alpha_+})}{\varepsilon^{d-1}}}. \tag{2.4}$$

Before proving the lemma let us show that it implies Proposition 2.1.a. Applying (2.3) and (2.4) with $\alpha_\pm = \pm 1$, unscaling and taking into account (2.1), we obtain

$$\left|\frac{1}{2\varepsilon}\mathscr{H}^d(\Omega_\varepsilon) - S\right| + \left|\frac{1}{2\varepsilon}\int_{\mathbf{R}^d} |\nabla t_\varepsilon| - S\right| \le C_{\not g}\varepsilon. \tag{2.5}$$

Next, we have

$$\langle \mathscr{V}_\varepsilon^\star(a_\varepsilon)\,;1\rangle = \frac{1}{2\varepsilon}\int_{\Omega_\varepsilon} \chi^\star(|\nabla t_\varepsilon|(y))\,dy.$$

Since $|\chi^\star(\tau) - 1| \le C||\tau| - 1|$, we get

$$\begin{aligned} |\langle \mathscr{V}_\varepsilon^\star(a_\varepsilon)\,;1\rangle - S| \quad &\le \quad \frac{1}{2\varepsilon}\int_{\Omega_\varepsilon} ||\nabla t_\varepsilon| - 1| + \left|\frac{1}{2\varepsilon}\mathscr{H}^d(\Omega_\varepsilon) - S\right| \\ &\overset{(1.4),(2.5)}{\le} \sqrt{\mathscr{H}^d(\Omega_\varepsilon)/2\varepsilon}\sqrt{E_0}\,\varepsilon + C_{\not g}\varepsilon \overset{(2.5)}{\le} C_{\not g}\varepsilon. \end{aligned} \tag{2.6}$$

Proposition 2.1.a follows from (2.5),(2.6).

*Proof (of Lemma 2.1)* Let us first establish that for every smooth open set $O$ such that

$$\{x \in \mathbf{R}^d : t(x) > -1\} \subset O \subset \{x \in \mathbf{R}^d : t(x) < 1\},$$

we have

$$\int_{\partial O} \sigma \cdot \nu = S/\varepsilon^{d-1}, \tag{2.7}$$

where $\nu$ denotes the outward normal on $\partial O$ and $\sigma \cdot \nu$ is well defined in $H^{-1/2}(\partial O)$ as the trace on $\partial O$ of the normal component of $\sigma \in H_{\mathrm{div}}(\Omega) = \{\sigma' \in L^2(\Omega, \mathbf{R}^d) : \operatorname{div}\sigma' \in L^2(\Omega)\}$.
Notice that the identity

$$\int_O \nabla\varphi \cdot \sigma = \int_{\partial O} \varphi\sigma \cdot \nu, \tag{2.8}$$

is valid for $\varphi \in C^\infty(\overline{\Omega})$ such that $\varphi \equiv 0$ in the neighborhood of $\{t \equiv -1\}$. In order to extend this formula to the case $\varphi = t + 1$, we introduce for $s \in (0,1)$, the truncated function $t_s$ defined by

$$t_s(y) := \begin{cases} t(y) & \text{if } |t(y)| \le 1 - s, \\ \pm(1-s) & \text{if } \pm t(y) > 1 - s. \end{cases}$$

By continuity of $t$, the function $\varphi_s := t_s + (1-s) \in W^{1,2}(\Omega)$ vanishes in the neighborhood of $\{t = -1\}$. Consequently, $\varphi_s$ belongs to the closure in $W^{1,2}$ of $\{\varphi \in C^\infty(\overline{\Omega}) \,:\, \operatorname{supp}\varphi \cap \{t \le -1\} = \emptyset\}$ and (2.8) is valid with $\varphi = \varphi_s$.

$$\int_O \nabla t_s \cdot \sigma = \int_{\partial O} (t_s + (1-s))\sigma \cdot \nu.$$

Letting $s \downarrow 0$, since $t_s \to t$ in $W^{1,2}_{loc}(\mathbf{R}^d)$, we obtain,

$$\int_O \nabla t \cdot \sigma = \int_{\partial O} (t+1)\sigma \cdot \nu.$$

Similarly, integrating $\nabla(t_s - (1-s)) \cdot \sigma$ on $\Omega \setminus \overline{O}$ and passing to the limit $s \downarrow 0$, we also get,

$$\int_{\Omega\setminus\overline{O}} \nabla t \cdot \sigma = \int_{\partial O} (-t+1)\sigma \cdot \nu.$$

Summing these identities, we obtain $2\int_{\partial O}\sigma\cdot\nu=\int_\Omega\nabla t\cdot\sigma$. By hypothesis, the value of the latter is $2S/\varepsilon^{d-1}$, so (2.7) holds true.

1/ Let us establish (2.2). By hypothesis, this identity is true for $(\alpha_-,\alpha_+)=(-1,1)$. Let us first assume $-1<\alpha_-<\alpha_+<1$ and let $O$ be a smooth bounded open set such that

$$\{t\le\alpha_+\}\subset O\subset\{t<1\}.$$

For instance, we may slightly mollify $t$ and invoke Sard theorem to define $O$ as a smooth sublevel set of the smooth approximation of $t$. Now let us introduce the truncated function,

$$\widetilde{t}(y) := \begin{cases} t(y) & \text{if } \alpha_-\le t(y)\le\alpha_+,\\ \alpha_\pm & \text{if } \pm t(y)>\alpha_\pm.\end{cases}$$

As above, (2.8) is valid with $\varphi=\widetilde{t}-\alpha_-$ and we have,

$$\int_{\Omega_{\alpha_-}^{\alpha_+}}\nabla t\cdot\,\sigma\;=\;\int_{\Omega_{\alpha_-}^{\alpha_+}}\nabla\widetilde{t}\cdot\,\sigma\;=\;(\alpha_+-\alpha_-)\int_{\partial O}\sigma\cdot\nu\;\overset{(2.7)}{=}\;(\alpha_+-\alpha_-)\,S/\varepsilon^{d-1}.$$

Hence, identity (2.2) holds in the case $-1<\alpha_-<\alpha_+<1$. The remaining cases follow by continuity of the integral.

2/ We are ready to establish (2.3). Let $-1\le\alpha_-<\alpha_+\le1$. By (2.2), the left hand side of (2.3) is bounded by $\displaystyle\int_{\Omega_{\alpha_-}^{\alpha_+}}|1-\nabla t\cdot\sigma|$.
To estimate this integral, we write

$$\begin{aligned}|1-\nabla t\cdot\sigma| &= \left|(1-|\nabla t|^2)+(1-|\sigma|^2)+|\nabla t-\sigma|^2\right|/2\\ &\le\;|1-|\nabla t||+|1-|\sigma||+\left[(1-|\nabla t|)^2+(1-|\sigma|)^2+|\nabla t-\sigma|^2\right]/2.\end{aligned}$$

Integrating on $\Omega_{\alpha_-}^{\alpha_+}$ and using the Cauchy-Scwharz inequality, we obtain,

$$\left|X^2-(\alpha^+-\alpha^-)S/\varepsilon^{d-1}\right|\;\le\;\sqrt{2F_0}\,X+F_0/2,$$

with the notation:

$$F_0\;:=\;\mathscr{F}_0(\sigma,\nabla t,\Omega_{\alpha_-}^{\alpha_+}),\qquad X\;:=\;\sqrt{\mathscr{H}^d(\Omega_{\alpha_-}^{\alpha_+})}.$$

In particular, $X^2-\sqrt{2F_0}X\le2S/\varepsilon^2+F_0/2$, and $X\le(\sqrt{2F_0}+\sqrt{8S/\varepsilon^2+2F_0})/2$. Substituting this in the right hand side of the above estimate, we obtain,

$$\left|X^2-\frac{(\alpha^+-\alpha^-)S}{\varepsilon^{d-1}}\right|\;\le\;(3/2)F_0+\sqrt{F_0^2+4F_0S/\varepsilon^2}\;\le\;(5/2)F_0+2\sqrt{F_0S/\varepsilon^2},$$

that is (2.3).

Similarly, the left hand side of (2.4) is bounded by $\displaystyle\int_{\Omega_{\alpha_-}^{\alpha_+}}||\nabla t|-\nabla t\cdot\sigma|$.
Writing $||\nabla t|-\nabla t\cdot\sigma|\le|1-\nabla t\cdot\sigma|+||\nabla t|-1|$, we get $\int_{\Omega_{\alpha_-}^{\alpha_+}}||\nabla t|-\nabla t\cdot\sigma|\le\sqrt{5F_0}X+F_0/2$ which yields (2.4). □

2.2 Construction of a harmonic extension of $t_\varepsilon$. Definition of $\Sigma^s$

Recall that $\omega$ is the modulus of continuity of Hypothesis 1. Let $\delta > 0$ be the largest number such that

$$\delta \le 1/4, \qquad \omega(\delta) \le 1/10. \tag{2.9}$$

Then we define

$$\Omega'_{(\varepsilon)} := \left\{ y \in \mathbf{R}^d \,:\, |t_{(\varepsilon)}|(y) < 9/10 \right\}, \qquad F'_{(\varepsilon)} := \mathbf{R}^d \setminus \Omega'_{(\varepsilon)},$$

and

$$O_{(\varepsilon),\delta} := \left\{ y \in \mathbf{R}^d \,:\, d(y, F'_{(\varepsilon)}) > \delta \right\}, \qquad F_{(\varepsilon),\delta} := \overline{\bigcup_{y \in F'_{(\varepsilon)}} B_\delta(y)} = \mathbf{R}^d \setminus O_{(\varepsilon),\delta}.$$

Notice that $\overline{[|t| < 4/5]} \subset O_\delta$ or equivalently, $t \ge 4/5$ in $F_\delta$. More precisely, for $y \in F'$, we have $t \ge 4/5$ on $\overline{B_\delta}(y)$ if $t(y) \ge 9/10$ and $t \le -4/5$ on $\overline{B_\delta}(y)$ if $t(y) \le -9/10$.

We introduce the harmonic extension $u_{(\varepsilon)}$ of $t_{(\varepsilon)}$ in $O_{(\varepsilon),\delta}$. Its level sets are good candidates for the hypersurface $\Sigma_{(\varepsilon)} = (1/\varepsilon)\Sigma_\varepsilon$ of Proposition 2.1.b.

**Definition 2.1** We set $u_\varepsilon(y) = \varepsilon u_{(\varepsilon)}(y/\varepsilon)$, where $u_{(\varepsilon)}$ is defined as

$$u_{(\varepsilon)} := \operatorname{argmin} \left\{ \int_{\mathbf{R}^d} |\nabla \varphi|^2 \,:\, \varphi \in W^{1,2}_{loc}(\mathbf{R}^d),\, \varphi \equiv t_{(\varepsilon)} \text{ a.e on } F_{(\varepsilon),\delta} \right\}.$$

*Remark 2.1*
a) The definition of $u_{(\varepsilon)}$ is valid since the feasible domain of the minimization problem contains at least $t_{(\varepsilon)}$.
b) We do not define $u_{(\varepsilon)}$ as the harmonic extension of $t_{(\varepsilon)}$ in $\Omega_{(\varepsilon)}$ for two reasons.
• First, in the sequel, we need $u_\varepsilon$ to be equal to $t_\varepsilon$ in a large part of $\Omega_\varepsilon$. Thanks to the definition of $\Omega'_{(\varepsilon)}$, this property holds true in the set $[9/10 < |t_{(\varepsilon)}| < 1]$. When we will apply the rigidity estimates of [12] to $u_{(\varepsilon)}$ and $t_{(\varepsilon)}$ in domains intersecting $[9/10 < |t_{(\varepsilon)}| < 1]$, this will allow us to use the same averaged normal direction for both vector fields $\nabla u_\varepsilon$, $\nabla t_\varepsilon$.
• We also need $u_{(\varepsilon)}$ to be uniformly equicontinuous (independently of $\varepsilon$). For this, we define the harmonic extensions $u_\varepsilon$ in domains satisfying uniformly the exterior ball property. This is the reason for the introduction of the sets $O_{(\varepsilon),\delta}$ which have this property with balls with radius $\delta$.

**Definition 2.2**
a) For $-1/2 < s < 1/2$, we set $\Sigma_\varepsilon^{\varepsilon s} := \varepsilon \Sigma^s_{(\varepsilon)}$ where $\Sigma^s_{(\varepsilon)}$ is the level set

$$\Sigma^s_{(\varepsilon)} := \{ x \in O_{(\varepsilon),\delta} \,:\, u_{(\varepsilon)}(x) = s \}.$$

b) For every $z \in O_{(\varepsilon),\delta}$, we set

$$n_{(\varepsilon)}(z) := \begin{cases} \dfrac{\nabla u_{(\varepsilon)}}{|\nabla u_{(\varepsilon)}|}(z) & \text{if } \nabla u_{(\varepsilon)}(z) \neq 0, \\ e_d & \text{if } \nabla u_{(\varepsilon)}(z) = 0. \end{cases}$$

*Remark 2.2*
Since $u_{(\varepsilon)}$ is harmonic in the neighborhood of $\Sigma^s_{(\varepsilon)}$, this set is an analytic surface with unit normal $n_{(\varepsilon)}$ in the neighborhood of $\{x \in \Sigma^s_{(\varepsilon)} : \nabla u_{(\varepsilon)}(x) \neq 0\}$.

In the sequel, we drop again the subscripts $(\varepsilon)$: we note $u$ for $u_{(\varepsilon)}$, $O_\delta$ for $O_{(\varepsilon),\delta}$, $n$ for $n_{(\varepsilon)}$ and $\Sigma^s$ for $\Sigma^s_{(\varepsilon)}$, *etc*. We first establish that $\nabla u$ is close to $\nabla t$.

**Lemma 2.2** *We have*

$$\int_{\mathbf{R}^d} |\nabla u - \sigma|^2 \;\le\; E_0 \qquad \textit{and} \qquad \int_{\mathbf{R}^d} |\nabla u - \nabla t|^2 \;\le\; 4E_0.$$

*Proof* Since $\sigma$ is divergence free in $O_\delta$, we easily check that $u$ minimizes the functional

$$J(\varphi) \;:=\; \int_{\mathbf{R}^d} |\nabla \varphi - \sigma|^2.$$

in the set $T_\delta := \left\{ \varphi \in W^{1,2}_{loc}(\mathbf{R}^d) \;:\; \varphi \equiv t \text{ in } F_\delta \right\}$. Since $t \in T_\delta$, we have by (2.1),

$$\int_{\mathbf{R}^d} |\nabla u - \sigma|^2 \;=\; J(u) \;\le\; J(t) \;\le\; E_0.$$

The second estimates then follows from (2.1) and the triangular inequality. □

Let us notice that, by construction, $O_\delta \subset B_R$ satisfies the exterior ball property with balls of radii $\delta$. As a consequence, we can use the Perron method to obtain the existence of a function $\widetilde{u} \in C(\overline{O_\delta})$ which is harmonic in $O_\delta$ and satisfies $\widetilde{u} \equiv t$ on $\partial O_\delta$ (see [8], Theorem 2.14 and below). By the maximum principle, we have $\widetilde{u} = u$, hence $u$ is continuous on $\overline{O_\delta}$. Thereafter we also need $u$ to be uniformly continuous with a modulus of continuity that does not depend on $\varepsilon$. We did not find the relevant precise statements in our favorite textbooks for this last result. For the sake of completeness, we provide a proof.

**Lemma 2.3** *There exists a modulus of continuity $\omega^\star$ only depending on $\omega$, $\delta$ and $d$ such that $u$ is $\omega^\star$-continuous on $\mathbf{R}^d$.*

*Proof* We start by regularizing the boundary data $t$ and the domain. For $0 < \eta \le \delta/2$, we set $t'_\eta := t \star \rho_\eta$ where $\rho_\eta = \eta^{-d}\rho(\cdot/\eta)$ is a standard non-negative mollifier, compactly supported in $B_\eta$. We also set $O'_{\delta,\eta} := O_\delta + B_\eta \supset O_\delta$.
Writing $t'_\eta(y) - t'_\eta(z) = \int [t(y-x) - t(z-x)] d\rho_\eta(x) \le \int \omega(y-z) d\rho_\eta = \omega(y-z)$, we see that the functions $\{t'_\eta\}_{0<\eta<\delta/2}$ are $\omega$-continuous. They are also uniformly bounded in $W^{1,2}(B_{R/\varepsilon})$ and satisfy $-1 \le t'_\eta \le 1$.
By construction, the domain $O'_{\delta,\eta}$ satisfies the exterior ball condition for balls with radii $\delta/2$ and the interior ball condition for balls with radii $\eta$. In particular $\partial O'_{\delta,\eta}$ has $C^{1,1}$ regularity and $O'_{\delta,\eta}$ is a bounded Lipschitz domain.
Let $u'_\eta$ be the variational harmonic extension of $t'_\eta$ in $O'_{\delta,\eta}$, defined as

$$u'_\eta \;:=\; \operatorname{argmin} \left\{ \int_{B_{R/\varepsilon}} |\nabla \varphi|^2 \,:\, \varphi \in W^{1,2}_{loc}(\mathbf{R}^d),\, \varphi \equiv t'_\eta \text{ a.e on } \mathbf{R}^d \setminus O'_{\delta,\eta} \right\}.$$

By classical elliptic regularity theory, $u'_\eta$ is of class $C^1$ in $\overline{O'_{\delta,\eta}}$ and $u'_\eta \equiv t'_\eta$ on $\partial O_{\delta,\rho}$. We claim that

$$\exists \text{ a modulus of continuity } \omega' = \omega'(\omega,\delta,d) \text{ such that } u'_\eta \in C_{\omega'}(\overline{O'_{\delta,\eta}}). \tag{2.10}$$

Assuming the claim, there exists a function $u' \in W^{1,2}(B_{R/\varepsilon})$ such that, up to extraction, $u'_\eta \to u'$ as $\eta \downarrow 0$. Moreover, $u'$ is harmonic in $O_\delta$, is $\omega'$-continuous in $\overline{O_\delta}$ and $u' \equiv t$ in $\mathbf{R}^d \setminus \overline{O_\delta}$. Hence

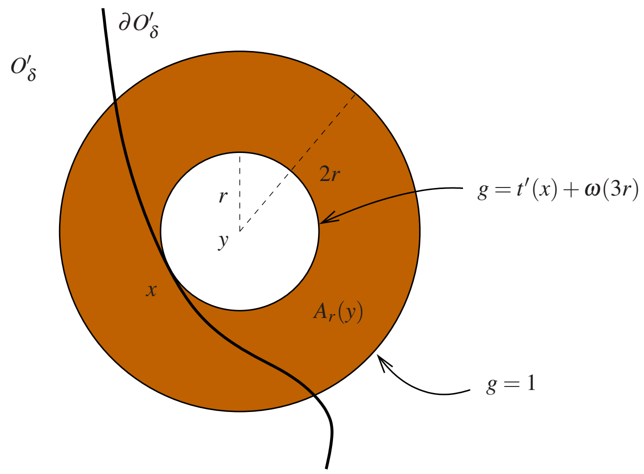


**Fig. 2.1** The barrier function $g$.

$u = u'$ and $u$ is $\omega'$-continuous in $\overline{O_\delta}$. Since $u = t$ is $\omega$-continuous in $\mathbf{R}^d \setminus O_\delta$, we conclude that $u$ is $(\omega + \omega')$-continuous in $\mathbf{R}^d$ and the lemma is proved.

Let us now establish (2.10). We fix $\eta \in (0, \delta/2]$ and drop the subscripts $\eta$.

*Step 1 (interior estimate)* First, by the maximum principle, we have in any case $|u'(y) - u'(z)| \leq 2$. Now, let $0 < r \leq 1/2$ and $B_r(x) \subset O'_\delta$. By harmonic regularity we have $|\nabla u| \leq C/r$ in $B_{r/2}(x)$. So $|u'(y) - u'(z)| \leq C\sqrt{|y-z|}$ for every $y, z \in B_{r^2}(x)$.

From this estimate, we conclude that we only have to establish that for any $x \in \partial O'_\delta$, $u'(z) \to u'(x)$ as $z \in O'_\delta \to x$ with a convergence rate that only depends on $\omega$, $\delta$ and $d$.

*Step 2 (continuity up to the boundary)* Let us fix $x \in \partial O'_\delta$ and let $r \in (0, \delta/2]$. By hypothesis, there exists a ball $B_r(y) \subset \mathbf{R}^d \setminus \overline{O'_\delta}$ such that $x \in \partial B_r(y)$.
As a barrier, we consider the harmonic function $g$ defined on the annulus $A_r(y) := B_{2r}(y) \setminus \overline{B_r}(y)$ satisfying the boundary conditions $g \equiv t'(x) + \omega(3r)$ on $\partial B_r(y)$ and $g \equiv 1$ on $\partial B_{2r}(y)$ (see Figure 2.1).
We claim that $g \geq u'$ in $A_r(y) \cap O'_\delta$. Indeed, $g - u'$ is harmonic on this set, so by the maximum principle it is sufficient to check the inequality on the boundary of $A_r(y) \cap O'_\delta$. This boundary splits into:

$$\partial(A_r(y) \cap O'_\delta) \;=\; (\partial B_{2r}(y) \cap \overline{O'_\delta}) \cup (\overline{A_r(y)} \cap \partial O'_\delta).$$

On the first part, we have $g - u' = 1 - u' \geq 0$. On the second part $u' = t'$ on $\partial O'_\delta$, so, $g(z) - u'(z) = t'(x) + \omega(3r) - t'(z) \geq \omega(3r) - \omega(|x - z|) \geq 0$ since $|x - z| \leq 3r$ and $t'$ is $\omega$-continuous. Hence, $g \geq u'$ in $A_r(y) \cap O'_\delta$.
An explicit computation gives

$$g(z) \;=\; t'(x) + \omega(3r) + (1 - t'(x) - \omega(3r))\frac{G(1) - G(|z-y|/r)}{G(1) - G(2)},$$

where $z \mapsto G(|z|)$ denotes a radially symmetric decreasing multiple of the fundamental solution of the Laplacian ($G(s) = -\ln s$ if $d = 2$ and $G(s) = s^{2-d}$ for $d \geq 3$). Hence, for every $x \in \partial O'_\delta$, $0 < r < \delta/2$ and $z \in O'_\delta \cap B_r(x)$, we have

$$u'(z) - t'(x) \;\leq\; g(z) - t'(x) \;\leq\; \omega(3r) + 2\frac{G(1) - G(|z-y|/r)}{G(1) - G(2)}.$$

Substituting $-t'$ for $t$ and $-u'$ for $u$, we obtain the corresponding result with opposite signs. We conclude that for every $x \in \partial O'_\delta$, $0 < r < \delta/2$ and $z \in O'_\delta \cap B_r(x)$,

$$|u'(z) - t'(x)| \le \omega(3r) + 2\frac{G(1) - G(|z-y|/r)}{G(1) - G(2)}.$$

In particular, for $x \in \partial O'_\delta$ and $z \in O'_\delta$, such that $|z - x| \le \delta^2$, we can choose $r = \sqrt{|x-z|}$. In this case,

$$\frac{|z-y|}{r} \le \frac{|x-y|}{r} + \frac{|z-x|}{r} = 1 + \sqrt{|z-x|}.$$

This leads to,

$$|u'(z) - t'(x)| \le \omega(3\sqrt{|x-z|}) + 2\frac{G(1) - G(1+\sqrt{|z-x|})}{G(1) - G(2)} \overset{|z-x|\downarrow 0}{\longrightarrow} 0.$$

This establishes the claim (2.10). □

Recall that, in $F_\delta$, we have $|t| \ge 4/5$, so as a consequence of Lemma 2.3, the level sets $\{\Sigma^s\}_{-1/2<s<1/2}$ lie at a positive distance from $F_\delta$.

**Corollary 2.1** *For $-1/2 < s < 1/2$, we have*

$$d(\Sigma^s, F_\delta) \ge \delta^\star, \qquad \textit{with } \delta^\star := \max\{r : \omega^\star(r) \le 2/5\}.$$

*In particular, $\delta^\star$ only depends on $\omega$ and $d$.*

## 2.3 Good cylinders. Bad balls

We use here the weak rigidity inequalities of [12] (Theorems I.2.2 ) and I.1.1) to show that $t_\varepsilon$ and $u_\varepsilon$ are close to some affine function in the neighborhood of points with small local energy. These results allow us to select the *good* points where it is possible to carry out the computations and derived the main estimates leading to the compactness (and lower bound) results.

**Definition 2.3** The local energy in an open set $O \subset \mathbf{R}^d$ is defined as

$$\mathscr{E}(O) := \int_{O\cap\Omega} f_0(\sigma, \nabla t) + |\nabla u - \nabla t|^2.$$

When $O$ is the open ball $B_\lambda(y) \subset \mathbf{R}^d$, we use the short hand,

$$\mathscr{E}_\lambda(y) := \mathscr{E}(B_\lambda(y)) = \int_{\Omega\cap B_\lambda(y)} f_0(\sigma, \nabla t) + |\nabla u - \nabla t|^2.$$

We first show that if $z \in \Omega$ is such that $\mathscr{E}_3(z)$ is small enough, then $\Omega - z$ contains a cylinder of the following form.

**Definition 2.4** For $\lambda > 0$ and $\bar{n} \in S^{d-1}$, $D'_\lambda(\bar{n})$ denotes the $(d-1)$-ball,

$$\{y' \in \mathbf{R}^d : |y'| < \lambda,\ y' \cdot \bar{n} = 0\} = B_\lambda \cap \bar{n}^\perp.$$

For $\lambda > 1$, $\xi \in [0,1)$ and $\bar{n} \in S^{d-1}$, $D^\xi_\lambda(\bar{n})$ denotes the finite cylinder

$$\{y' + s\bar{n} : y' \in D'_\lambda(\bar{n}),\ |s| < 1 - \xi\}.$$

Using rotation invariance, we often consider the case $\bar{n} = e_d$, for which we simply write $D'_\lambda$ for $D'_\lambda(e_d)$ and $D^\xi_\lambda$ for

$$D^\xi_\lambda(e_d) = \{y \in \mathbf{R}^d : |y_d| < 1 - \xi\,,\ \sum_{i<d} y_i^2 < \lambda^2\}.$$

**Lemma 2.4** *Let $\xi \in (0,1/2)$, and $\eta > 0$. There exists $\beta = \beta_1(\omega,\xi,\eta)$ such that if $z \in \mathbf{R}^d$ satisfies $|u(z)| \le 1/2$ and $\mathscr{E}_3(z) \le \beta$, then, $|\nabla u(z)| \ge 1/2$ and using the notation*

$$z_0 := z - u(z)n(z), \qquad D := z_0 + D_1^\xi(n(z)), \qquad \varphi(y) = u(z) + (y-z)\cdot n(z),$$

*we have,*

*(a) $\overline{D} \subset \Omega$ (see Figure 2.2) and*

$$\|t-\varphi\|_{L^\infty(\overline{D})} + \|u-\varphi\|_{L^\infty(\overline{D})} + \|\nabla t - \nabla\varphi\|_{L^2(D)} + \|\sigma - \nabla\varphi\|_{L^2(D)} \le \eta;$$

*(b) there exists an analytic mapping $\Psi : D_1'(n(z)) \times (-1/2,1/2) \to \mathbf{R}$, such that $\|\Psi\|_{C^2} \le \eta$ and for every $s \in (-1/2,1/2)$,*

$$\Sigma^s \cap D = [z_0 + s\,n(z)] + \left\{ y' + \Psi(y',s)n(z) \; ; \; y' \in D_1'(n(z)) \right\}.$$

*(c) As a consequence,*

$$0 \le \mathscr{H}^{d-1}(\Sigma^s \cap D) - \mathscr{H}^{d-1}(D_1') \le c(\eta),$$

*with $c(\eta) \downarrow 0$ as $\eta \downarrow 0$.*

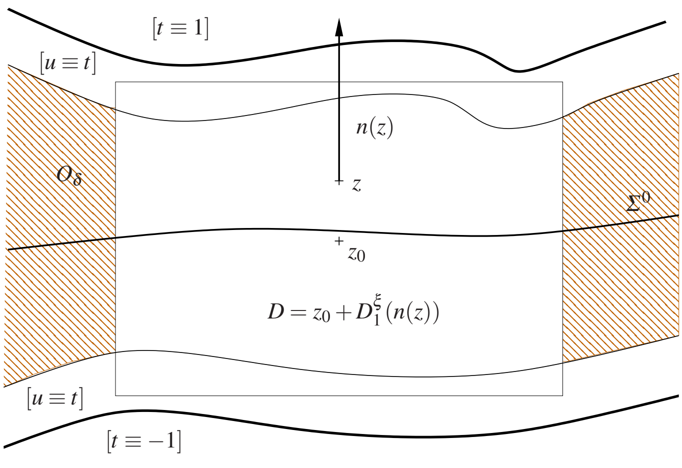


**Fig. 2.2** Example of "good" cylinder in dimension $d = 2$.

*Proof* Without loss of generality, we assume $z = 0$ and $\xi \in (0,1/20)$. Let $\eta > 0$. We establish by contradiction that (a) holds for $\beta > 0$ small enough.

If (a) were not true for any value of $\beta > 0$, there would exist two sequences $(a_k)$, $(a_k')$ with $a_k = (\sigma_k, \nabla t_k, O_k)$, $a_k' = (\sigma_k, \nabla u_k, O_k')$ such that:

(i) $(a_k)$ and $(a_k')$ both satisfy the hypotheses of Theorem I.1.1 with $O = B_3$. In particular, $(u_k)$ and $(t_k)$ are uniformly equicontinuous.

(ii) Either $|\nabla u_k|(0) < 1/2$;
or $|\nabla u_k|(0) \geq 1/2$ and using the notation

$$\bar{n}_k := \nabla u_k / |\nabla u_k|(0), \quad D_k := -u_k(0)\bar{n}_k + D_1^{\xi}(\bar{n}_k), \quad \varphi_k(y) = u_k(0) - z \cdot \bar{n}_k,$$

we have either:
- $D_k \not\subset O_k \cap O'_k$;
- or $D_k \subset O_k \cap O'_k$ and

$$\|t_k - \varphi_k\|_{L^\infty(\overline{D}_k)} + \|u_k - \varphi_k\|_{L^\infty(\overline{D}_k)} + \|\nabla t_k - \nabla \varphi_k\|_{L^2(D_k)} \;>\; \eta; \tag{2.11}$$

(iii) Moreover, $|u_k|(0) \;\leq\; 1/2$ and setting $\widetilde{F}_k := [|t_k| \geq 9/10] + \overline{B_\delta}$, we have (taking into account (2.9)) $u_k \equiv t_k$ on $\widetilde{F}_k$ and $u_k$ is harmonic in $B_3 \setminus \widetilde{F}_k$.

First, up to extraction, we may assume that both $(t_k(0))_k$ and $(u_k(0))_k$ converge towards respectively $s$ and $s'$ with $|s| \leq 9/10$ and $|s'| \leq 1/2$. By Theorem I.1.1, there exists $\bar{n}, \bar{n}' \in S^{d-1}$ such that

$$t_k \overset{k\uparrow\infty}{\longrightarrow} \varphi \text{ on } C_{loc}(O_\star), \qquad u_k \overset{k\uparrow\infty}{\longrightarrow} \varphi' \text{ on } C_{loc}(O'_\star),$$

with $\varphi(y) := s + \bar{n} \cdot y$, $\varphi'(y) := s' + \bar{n}' \cdot y$, $O_\star = B_2 \cap [|\varphi| < 1]$, $O'_\star = B_2 \cap [|\varphi'| < 1]$.

Now, for any $k \geq 1$, $u_k$ is harmonic in $B_{\delta^\star}$. By harmonic regularity, we see that $\nabla u_k(0) \to \nabla \varphi'(0) = \bar{n}'$. In particular, for $k$ large enough, there holds

$$|\nabla u_k|(0) \;\geq\; 1/2.$$

Next, considering the segment $I = \{y + (r - s)\bar{n} : -1 < r < 1\} \subset O'_\star$, and the mapping $p : (-1, 1) \to I$ defined by $p(r) = y + (r - s)\bar{n}$, we have $u_k \circ p \to Id_{(-1,1)}$ as $k \uparrow \infty$.
In particular, for $k$ large enough $|u_k| > 9/10$ on $p(19/20, 1)$ which implies $t_k \equiv u_k$ on $p(19/20, 1)$. Passing to the limit we conclude that $\bar{n} = \bar{n}'$, $s = s'$ and $\varphi = \varphi'$. By uniform equicontinuity of the sequence $(t_k)$, and convergence of $(t_k)$ towards $\varphi'$ on $O'_\star$ we then see that

$$\widetilde{D}_k := -u_k(0)\bar{n}_k + D_{1+\xi}^{\xi/2}(\bar{n}_k) \;\subset\; [|t_k| < 1 - \xi/2], \qquad \text{for } k \text{ large enough.}$$

We already know that the $L^\infty$ norms in (2.11) converge to 0.
Applying the weak rigidity inequality of Theorem I.2.2 to $t_k$ in $\widetilde{D}_k \supset D_k$ we see that the $W^{1,2}$-seminorms also converge to 0 which contradicts (2.11).

Parts (b) and (c) of the lemma follow from part (a) and the harmonicity of $u$ in the domain $[u < 4/5]$. □

When establishing the lower bound of Theorem 1.2 in Section 5, we apply the weak and strong rigidity inequalities in the cylindrical boxes of Lemma 2.4. It is convenient to gather the relevant estimates here.

**Lemma 2.5** *Let $\xi \in (0, 1/4)$. There exists $\beta = \beta_2(\omega, \xi)$ such that if $x \in \Sigma^0$ satisfies $\mathscr{E}_3(x) \leq \beta$, then using the notation $n(x) = \bar{n}$ and*

$$D_{int} \;:=\; x + D_{1-2\xi}^{2\xi}(\bar{n}) \quad \subset \quad D \;:=\; x + D_1^{\xi}(\bar{n}),$$

*we have:*
*(a)* $\overline{D} \;\subset\; \Omega$;

*(b)*
$$|II|^4(x) + |h|^2(x) \;\leq\; C\mathscr{E}(D). \tag{2.12}$$

*(c) There exists a harmonic function* $\psi : D_{int} \to \mathbf{R}$ *satisfying* $\nabla\psi(x) = 0$, $\bar{n}\cdot\nabla\psi \equiv 0$,

$$\int_{D_{int}} |\nabla\psi|^2 \;\le\; C(\xi)\sqrt{\mathscr{E}(D)}, \tag{2.13}$$

*and such that*

$$\int_{D_{int}} |\nabla u - \bar{n} - \nabla\psi|^2 + |\nabla t - \bar{n} - \nabla\psi|^2 + |\sigma - \bar{n} - \nabla\psi|^2 \;\le\; C(\xi)\mathscr{E}(D). \tag{2.14}$$

*Proof* Part (a) is a direct consequence of the preceding lemma. Part (b) follows from Corollary I.3.1 and the harmonicity of $u$ in $B_{\delta^\star}$.

Let us establish part (c). Without loss of generality, we assume $x = 0$ and $\xi \in (0, 1/20)$. We apply Theorem I.2.2 to the function $u$ in the cylinder $D_{\rm int} \subset D$ and in $B_{\delta^\star} \subset D$. There exists $e'_\star, e_\star \in S^{d-1}$ such that

$$\int_{D_{\rm int}} |\nabla u - e'_\star|^2 \;\le\; C(\xi)\sqrt{\mathscr{E}(D)}, \tag{2.15}$$
$$\int_{B_{\delta^\star}} |\nabla u - e_\star|^2 \;\le\; C(\delta^\star)\sqrt{\mathscr{E}(D)}.$$

In particular, we have $|e_\star - e'_\star|^2 \le C(\xi,\delta^\star)\sqrt{\mathscr{E}(D)}$. Since $u$ is harmonic in $B_{\delta^\star}$, we see as in the proof of Corollary I.3.1 that by the mean value property $e_\star = n(0) =: \bar{n}$. Inequality (2.15) then implies

$$\int_{D_{\rm int}} |\nabla u - \bar{n}|^2 \;\le\; C(\xi,\delta^\star)\sqrt{\mathscr{E}(D)}. \tag{2.16}$$

Now, let us apply Theorem I.3.1 to $u$ in $D_{\rm int} \subset D$. There exists a harmonic function $\psi_1 \in L^2(D_{\rm int})$ such that $\bar{n}\cdot\nabla\psi_1 \equiv 0$ and

$$\int_{D_{\rm int}} |\nabla u - \bar{n} - \nabla\psi_1|^2 \;\le\; C(\xi,\delta^\star)\mathscr{E}(D). \tag{2.17}$$

Now, notice that by orthogonality, we have

$$|\nabla\psi_1|^2(0) + |\nabla u(0) - \bar{n}|^2 \;=\; |\nabla u(0) - \nabla\psi_1(0) - \bar{n}|^2 \overset{(2.17)}{\le} C(\xi,\delta^\star)\mathscr{E}(D).$$

In particular,

$$|\nabla\psi_1|^2(0) \;\le\; C(\xi,\delta^\star)\mathscr{E}(D). \tag{2.18}$$

Eventually, setting $\psi(y) := \psi_1(y) - y\cdot\nabla\psi_1(0)$, the function $\psi$ is harmonic in $D_{\rm int}$, it satisfies $\bar{n}\cdot\nabla\psi \equiv 0$ and (2.14) holds thanks to (2.17) and (2.18). Inequality (2.13) then follows from (2.14) and (2.16). □

In the sequel, we perform some changes in the order of integration for which we are led to consider the sets

$$\Gamma^\xi(z) \;:=\; \{x \in \Sigma^0 : z \in x + D_1^\xi(n(x))\}.$$

The purpose of next lemma is to estimate the $(d-1)$-volumes of these sets.

**Lemma 2.6** *Let $\xi \in (0,1/2)$ and $\eta > 0$. There exists $\beta = \beta_3(\xi,\eta) > 0$ such that if $z \in \mathbf{R}^d$ satisfies*

$$|t(z)| \le 1-\xi \qquad \textit{and} \qquad \mathscr{E}_3(z) \le \beta,$$

*then the function $x \in \Sigma^0 \mapsto |x-z|^2$ admits a unique minimizer $x \in \Sigma^0$ and we have $z = x + sn(x)$ with $|s - t(z)| < \eta$. Moreover, using the notation of Lemma 2.4,*

$$\Gamma^\xi(z) = \{x + y + \Psi(y,0)n(x) : y \in X\},$$

*for some open subset $X$ of $n(x)^\perp$ such that*

$$D'_{1-\eta}(n(x)) \subset X \subset D'_{1+\eta}(n(x)).$$

*In particular,*

$$\left| \mathscr{H}^{d-1}(\Gamma^\xi(z)) - \mathscr{H}^{d-1}(D'_1) \right| \le C\eta. \tag{2.19}$$

*Proof* We can prove the lemma by arguing by contradiction as in the proof of Lemma 2.4. The only difference is that we have to take into account all the connected components of $\{y : \lim_{k\to\infty} u_k(y) \in (-1,1)\}$ (see Figure I.1.4). Details are left to the reader. □

We now bound the total volume of points which do not satisfy the assumption $\mathscr{E}_3(x) < \beta$. Let us fix $\beta > 0$ and let us define the sets of good and bad points as

$$\mathscr{G}_\beta := \{x \in \mathbf{R}^d : \mathscr{E}_3(x) \le \beta\}, \qquad \mathscr{B}_\beta := \mathbf{R}^d \setminus \mathscr{G}_\beta.$$

**Lemma 2.7** *i) There exists a finite number of disjoint balls $B_3(y_1), \cdots, B_3(y_N) \subset \mathbf{R}^d$ such that*

$$\mathscr{B}_\beta \subset U_\beta := \bigcup_{i=1}^N B_9(y_i), \qquad \textit{with} \qquad N \le C_{\not{\varepsilon}} \varepsilon^{3-d}/\beta.$$

*ii) Moreover, there exists a finite number of balls $\{B_{9\alpha_i}(z_i)\}_{1\le i\le N'} \subset \mathbf{R}^d$ with*

$$1 \le \alpha_i \le 2^{N-1} \quad \textit{and} \quad N' \le N$$

*such that the balls $\{B_{9(1+\alpha_i)}(z_i)\}$ do not intersect and*

$$\mathscr{B}_\beta \subset U_\beta \subset U'_\beta := \bigcup_{i=1}^{N'} B_{9\alpha_i}(z_i).$$

*Proof* Since the total energy $\mathscr{E}_{+\infty}(0)$ is bounded by $CE_0\varepsilon^{3-d}$, the first part classically follows from Vitali covering theorem. For the second part, it is enough to prove that we can cover $\cup B_{18}(y_i)$ with $N' \le N$ non intersecting balls with radii bounded by $18 \cdot 2^{N-1}$. This ensues from the following claim.

*Claim* Let $N \ge 1$ and $\rho > 0$, if $U$ is a union of $N$ open balls with radius $\rho$, we can cover $U$ with $N$ (or less) disjoint open balls with radii at most $2^{N-1}\rho$.

We proceed recursively on $N$. For $N=1$ the result is obvious. Let us assume $N \geq 2$ and consider the union of $N$ balls with radius $\rho$, $U_N = \cup_{i=1}^N B_\rho(z_i)$. If these balls are disjoint, there is nothing to prove. If at least two balls intersect, say $B_\rho(z_{N-1}) \cap B_\rho(z_N) \neq \emptyset$, then $U_N$ is contained in the union of $N-1$ balls of radius $2\rho$,

$$U_N \;\subset\; U_{N-1} \;:=\; B_{2\rho}(z_1) \cup \cdots \cup B_{2\rho}(z_{N-2}) \cup B_{2\rho}\left(\frac{z_{N-1}+z_N}{2}\right).$$

By the induction hypothesis, we cover $U_{N-1}$ with $M \leq N-1$ balls with radii at most $2^{N-2}(2\rho) = 2^{N-1}\rho$. □

As a consequence of the bound on $N$ we have $\mathscr{H}^d(U_\beta) \;\leq\; C_{\not\xi}(\beta)\varepsilon^{3-d}$ and more generally, for $\beta > 0$, $\lambda \geq 1$,

$$\mathscr{H}^d\left(\cup_{i=1}^N B_{9\lambda}(y_i)\right) \;\leq\; C_{\not\xi}(\beta,\lambda)\varepsilon^{3-d}.$$

In the sequel, we use this inequality without further reference.

### 2.4 Definition and properties of the hypersurface $\Sigma_\varepsilon$. Proof of Proposition 2.1.b

Before beginning, let us sketch our construction of the hypersurface $\Sigma_{(\varepsilon)}$.

- *Step 0.* We introduce a bad set $U^\flat_{(\varepsilon)}$ of points with local energy larger than a fixed value $\beta > 0$. We also define the larger open set $U^\star_{(\varepsilon)} := U^\flat_{(\varepsilon)} + B_9$ and the good set $G^\star_{(\varepsilon)} := \mathbf{R}^d \setminus U^\star_{(\varepsilon)}$.
- *Step 1.* Then, we define $\Sigma_{(\varepsilon)}$ in $G^\star_{(\varepsilon)}$ as $\Sigma^\star_{(\varepsilon)} := G^\star_{(\varepsilon)} \cap \Sigma^0_{(\varepsilon)} = G^\star_{(\varepsilon)} \cap [u_{(\varepsilon)} \equiv 0]$.
- *Step 2.* Next, we use Corollary I.2.1 to find a level set $\Sigma^{s^\flat}_{(\varepsilon)}$ such that $\Sigma^{s^\flat}_{(\varepsilon)} \cap U_{(\varepsilon)}$ satisfies convenient bounds. We then set $\Sigma^\flat_{(\varepsilon)} := \Sigma^s_{(\varepsilon)} \cap U_{(\varepsilon)}$.
- *Step 3.* Eventually, we complete the construction of $\Sigma^\flat_{(\varepsilon)} \cup \Sigma^\star_{(\varepsilon)}$ by adding a smooth hypersuface in the gap $U^\star_{(\varepsilon)} \setminus U^\flat_{(\varepsilon)}$ with boundary $\partial\Sigma^\star_{(\varepsilon)} \cup \partial\Sigma^\flat_{(\varepsilon)} \subset \partial U^\star_{(\varepsilon)} \cup \partial U^\flat_{(\varepsilon)}$.

Let us fix $\xi = \eta = 1/4$. With the notation of Corollary I.3.1 , Lemma 2.4 and Lemma 2.7, we define,

$$\beta \;:=\; \min\Big(\beta_0 \delta^{\star(d+2)}, \beta_1(\omega,\xi,\eta)\Big), \qquad U^\flat \;:=\; U_\beta \;=\; \bigcup_{i=1}^N B_9(y_i). \tag{2.20}$$

Let us recall that $N \leq C_{\not\xi}\varepsilon^{3-d}$. We also note

$$U^\star \;:=\; \bigcup_{i=1}^N B_{18}(y_i), \qquad G^\star \;:=\; \mathbf{R}^d \setminus U^\star.$$

*Step 1.* We set,

$$\Sigma^\star \;:=\; \Sigma^0 \cap G^\star.$$

By Lemma 2.4, $\Sigma^\star$ is an analytic hypersurface. Moreover, $\Sigma^\star \subset B_{R/\varepsilon}$ and its boundary lies on the spheres $\partial B_{18}(y_i)$.

Now, for every $x \in \Sigma^\star$, we have by Corollary 2.1, $B_{\delta^\star}(x) \subset O_\delta$ and applying Corollary I.3.1 with $\varphi = u$ in the ball $B_{\delta^\star}(x)$, we obtain

$$|h|^2(x) \;\leq\; \frac{C}{\delta^{\star(2+d)}} \int_{B_{\delta^\star}(x)} (|\nabla u| - 1)^2,$$

where $h$ denotes the mean curvature on $\Sigma^0$. Integrating on $\Sigma^\star$, and using Fubini, we get

$$\int_{\Sigma^\star} |h|^2 \, d\mathscr{H}^{d-1} \;\le\; \frac{C}{\delta^{\star(2+d)}} \int_{G^\star + B_{\delta^\star}} \mathscr{H}^{d-1}(B_{\delta^\star}(y)\cap\Sigma^0)(|\nabla u|(y)-1)^2\,dy.$$

By Lemma 2.4.b, we have

$$\mathscr{H}^{d-1}(B_{\delta^\star}(y)\cap\Sigma^0) \;\le\; C \;\text{ for every } y \text{ such that } d(y,\Sigma^\star) < \delta^\star.$$

Hence,

$$\int_{\Sigma^\star} |h|^2 \, d\mathscr{H}^{d-1} \;\le\; \frac{C}{\delta^{\star(2+d)}} \int_{\mathbf{R}^d} (|\nabla u|(y)-1)^2\,dy \;\le\; C_{\not{g}}\varepsilon^{3-d}. \tag{2.21}$$

Similarly, using Lemma 2.4.b and Lemma 2.1, we obtain,

$$\mathscr{H}^{d-1}(\Sigma^\star) \;\le\; C\mathscr{H}^d(\Omega_{(\varepsilon)}) \;\le\; CS\varepsilon^{1-d} + C_{\not{g}}\varepsilon^{2-d}. \tag{2.22}$$

*Step 2.* We now define the hypersurface $\Sigma^\star$ inside the balls $B_{18}(y_i)$. Let us first consider the union of interior balls $U^\flat = \cup B_9(y_i) \subset\subset U^\star$. By Corollary 2.1, for any point $y\in\mathbf{R}^d$ such that $|u(y)| < 1/2$, we have $B_{\delta^\star}(y) \subset O_\delta$, that is $u$ is harmonic in $B_{\delta^\star}(y)$. For such a point $y$, we can apply Corollary I.2.1 .b to $u$ in $B_{\delta^\star}(y)$. We get

$$\int_{\mathbf{R}} \int_{\Sigma^s\cap B_{\delta^\star/2}(y)} |II_s|^2 \, d\mathscr{H}^{d-1}\,ds \;\le\; (C/\delta^{\star 2}) \int_{B_{\delta^\star}(y)} ||\nabla u|-1|,$$

where $II_s$ denotes the second fundamental form on $\Sigma^s$. Applying Vitali covering theorem to a cover of $\{y\in U \;:\; |u(y)|<1/2\}$ with balls of the form $B_{\delta^\star/3}(y)$ and summing the estimates, we deduce,

$$\int_{-1/2}^{1/2} \int_{\Sigma^s\cap U} |II_s|^2 \, d\mathscr{H}^{d-1}\,ds \;\le\; (C/\delta^{\star 2}) \int_{U^\star} ||\nabla u|-1|.$$

Next, by Cauchy-Schwarz inequality,

$$\int_{-1/2}^{1/2} \int_{\Sigma^s\cap U} |II_s|^2 \, d\mathscr{H}^{d-1}\,ds \;\le\; (C/\delta^{\star 2}) \sqrt{\mathscr{H}^d(U^\star) \int_{\mathbf{R}^d} (|\nabla u|-1)^2} \;\le\; C_{\not{g}}\varepsilon^{3-d}.$$

On the other hand, by Corollary I.2.1.a, we have

$$\int_{\mathbf{R}} \mathscr{H}^{d-1}(\Sigma^s\cap U)\,ds \;\le\; \mathscr{H}^d(U) + C_{\not{g}}\varepsilon^{3-d} \;\le\; C_{\not{g}}\varepsilon^{3-d}.$$

Let us now introduce a small parameter $s_0 \in (0,1/4)$ to be fixed later. From the above bounds, there exists $s^\flat \in [-s_0, s_0]$ such that $\Sigma^\flat := \Sigma^{s^\flat} \cap U^\flat$ is an analytic hypersurface satisfying

$$\mathscr{H}^{d-1}(\Sigma^\flat) + \int_{\Sigma^\flat} |II^\flat|^2 \, d\mathscr{H}^{d-1} \;\le\; C_{\not{g}}\varepsilon^{3-d}/s_0. \tag{2.23}$$

*Step 3.* We have to build a hypersurface in $U^\star \setminus U^\flat$ connecting $\Sigma^\star$ to $\Sigma^\flat$. For this, let us introduce a nonincreasing cut off function $\chi \in C^\infty(\mathbf{R}_+)$ such that $\chi\equiv 1$ on $[0,1]$, $\chi\equiv 0$ on $[3,+\infty)$ and $0\ge\chi'\ge -1$. We also introduce a nondecreasing truncating function $T\in C^\infty(\mathbf{R}_+)$ such that $T(0)=0$, $T\equiv 1$ on $[1,+\infty)$ and $0\le T'\le 2$. We then set

$$\Sigma \;:=\; \{x\in\mathbf{R}^d \;:\; u(x) - s^\flat\theta(x) \;=\; 0\} \quad\text{with}\quad \theta(y) \;:=\; T\left(\sum_{i=1}^N \chi(d(y,B_9(y_i)))\right).$$

Since $\theta \equiv 1$ in $U^\flat$ and $\theta \equiv 0$ in $G^\star$, we obviously have $\Sigma \cap G^\star = \Sigma^\star$, $\Sigma \cap U^\flat = \Sigma^\flat$ and in fact $\Sigma = \Sigma^{s^\flat}$ in some neighborhood of $\overline{U^\flat}$, $\Sigma = \Sigma^0$ in some neighborhood of $G^\star$. Notice that the balls $B_3(y_i)$ are disjoint. Hence, there exists $K \geq 0$, only depending on $d$ such that the number $P(y)$ of non-zero elements in the sum which defines $\theta(y)$ satisfies $P(y) \leq K$.

Let us note $v(y) = u(y) - s^\flat \theta$ and let us fix $z \in \Sigma \cap [U^\star \setminus U]$. We compute at point $z$,

$$|\nabla v|(z) \;\geq\; |\nabla u|(z) - |s^\flat|\,|\nabla \theta|(z) \;\geq\; 1/2 - 2|s^\flat| P(z) \|\chi'\|_\infty \;\geq\; 1/2 - 2K s_0.$$

Now, we fix

$$s_0 \;:=\; 1/(4K),$$

so that

$$|\nabla v|(z) \;\geq\; 1/4.$$

Consequently, $\Sigma = v^{-1}(\{0\})$ is a smooth hypersurface.
Let us now estimate the $(d-1)$-volume and the $L^2$ norm of the second fundamental form of $\Sigma \cap [U^\star \setminus U]$. We consider again a point $z \in \Sigma \cap [U^\star \setminus U]$. We have $|u(z)| \leq |s^\flat| \leq 1/4$ and by construction, $\mathscr{E}_3(z) \leq \beta_1(\omega, 1/4, 1/4)$ so we can apply Lemma 2.4 at this point with $\xi = \eta = 1/4$. Assuming without loss of generality that $z = 0$, $n(z) = e_d$ and using the notation of Lemma 2.4.b we have for $y' \in D'_1$ and $-1/2 < s < 1/2$,

$$v(y' + s e_d) \;=\; 0 \quad \Longleftrightarrow \quad \Psi(y', s^\flat \theta(y' + s e_d)) \;=\; s.$$

Now, let us set,

$$Z(y', s) := s - \Psi(y', s_\star \theta(y' + s e_d)), \qquad \text{for } y' \in D'_1,\; -1/2 \leq s \leq 1/2.$$

We have $\pm Z(y', \pm 1/2) \geq 1/2 - \eta = 1/4 > 0$ and $\frac{d}{ds} Z(y', s) \geq 1 - \eta |s_0| \|\theta'\|_\infty \geq 15/16 > 0$, so for every $y' \in D'_1$, the equation $Z(y', s) = 0$ admits a unique solution $s = \zeta(y')$. In other words, $[D'_1 \times (-1/2, 1/2) e_d] \cap \Sigma$ is the graph of the mapping $\zeta$. By regularity of $\Psi$ and $\theta$, we also have

$$\|\nabla \zeta\|_\infty \leq C, \qquad \|D^2 \zeta\|_\infty \;\leq\; C.$$

We deduce the inequality

$$\mathscr{H}^{d-1}\big(\big[D'_1 \times (-1/2, 1/2) e_d\big] \cap \Sigma\big) \;+ \int_{[D'_1 \times (-1/2,1/2) e_d] \cap \Sigma} |II|^2 \, d\mathscr{H}^{d-1} \;\leq\; C.$$

Using Vitali covering theorem and the bound $\mathscr{H}^d(U^\star \setminus U') \leq C_{\not{g}} \varepsilon^{3-d}$, we get,

$$\mathscr{H}^{d-1}(\Sigma \cap U^\star) + \int_{\Sigma \cap U^\star} |II|^2 \, d\mathscr{H}^{d-1} \;\leq\; C_{\not{g}} \varepsilon^{3-d}. \tag{2.24}$$

Taking into account (2.21),(2.22),(2.23) and (2.24), we have established

$$\mathscr{H}^{d-1}(\Sigma) \;\leq\; C S \varepsilon^{1-d} + C_{\not{g}}, \quad \text{and} \quad \mathscr{W}(\Sigma) \;\leq\; C_{\not{g}} \varepsilon^{3-d}. \tag{2.25}$$

Eventually, we set $M := \{y \in \mathbf{R}^d \;:\; t(y) = -1\}$ and $O := \{y \in \mathbf{R}^d \;:\; v(y) = u(y) - s_\star \theta(y) < 0\}$. We have $O \subset B_{R/\varepsilon}$ and $\partial O = \Sigma$. Moreover, taking into account Lemma 2.1 and the bound $\mathscr{H}^d(U^\star) \leq C_{\not{g}} \varepsilon^{3-d}$, we have

$$\|\mathbf{1}_O - \mathbf{1}_M\|_{L^1} \;\leq\; 2 S \varepsilon^{1-d} + C_{\not{g}} \varepsilon^{2-d}. \tag{2.26}$$

Putting back the subscripts $(\varepsilon)$, we note $\Sigma_{(\varepsilon)} := \Sigma$, $O_{(\varepsilon)} := O$, $M_{(\varepsilon)} := M$, $U^\star_{(\varepsilon)} := U^\star$ and $G^\star_{(\varepsilon)} := G^\star$. Returning to the original variables, we set $\Sigma_\varepsilon := (1/\varepsilon)\Sigma_{(\varepsilon)}$, $O_\varepsilon := (1/\varepsilon)O_{(\varepsilon)}$ and $M_\varepsilon := (1/\varepsilon)M_{(\varepsilon)}$. Unscaling the inequalities (2.25)(2.26), we obtain the estimates of Proposition 2.1.b.

For later use, we remark here that

$$\mathscr{H}^{d-1}(B_1(y)\cap\Sigma_{(\varepsilon)}) \le C, \qquad \text{for every } y \in G^\star. \tag{2.27}$$

Indeed, by construction, $B_1(y)\cap\Sigma_{(\varepsilon)} = B_1(y)\cap\Sigma^0_{(\varepsilon)}$ for every $y \in G^\star_{(\varepsilon)}$ and we can apply Lemma 2.4.b to any point $z \in B_1(y)\cap\Sigma_{(\varepsilon)}$.
In the sequel, we consider bad sets which write, using the notation of Lemma 2.7.a, as $\mathscr{U}_{(\varepsilon)} = U_{(\varepsilon),\beta'}$. We deduce from Lemma 2.7.a (2.24) and (2.27) that

$$\mathscr{H}^{d-1}\left(\Sigma_{(\varepsilon)}\cap\left(U^\star_{(\varepsilon)}\cup\mathscr{U}_{(\varepsilon)}\right)\right) \le C_{\not{e}}(\beta')\varepsilon^{3-d}.$$

More generally, for $\lambda > 0$, we have

$$\mathscr{H}^{d-1}\left(\Sigma_{(\varepsilon)}\cap\left[(U^\star_{(\varepsilon)}\cup\mathscr{U}_{(\varepsilon)})+B_\lambda\right]\right) \le C_{\not{e}}(\beta',\lambda)\varepsilon^{3-d}. \tag{2.28}$$

## 2.5 Alternative construction of $\Sigma_\varepsilon$. Proof of Proposition 2.1.d

In this subsection (and only here) we assume that $d = 3$ and that Hypothesis 2 holds.
Let us fix $\xi = 1/4$ and let $\eta \in (0,1/4)$ to be fixed later. We then define $\beta$ by (2.20). The definition of the bad set is not the same as in the previous subsection: we use here the union of distant open balls provided by Lemma 2.7.b. Let us set

$$U^\star := U'_\beta + B_1 = \cup_{i=1}^{N'} B_{9(\alpha_i+1)}(z_i), \qquad G^\star := \mathbf{R}^3\setminus U^\star. \tag{2.29}$$

In the good set we will define $\Sigma^\star$ as $\Sigma^0\cap G^\star$. For the present alternative construction of $\Sigma^\star$ inside the bad balls, we need that the trace of some $\Sigma^s$ on the boundary of bad balls meets some regularity properties. For this, we establish a quantitative Sard like result.

*Claim* There exists $\kappa > 0$ only depending on $\omega$, $E_0$ and $\eta$ such that for any $1 \le i \le N'$ there exist $s_i \in [-1/8,1/8]$ and $r_i \in [0,1/2]$ satisfying

$$x \in \Sigma^{s_i}\cap\partial B_{9(\alpha_i+r_i)}(z_i) \quad\Longrightarrow\quad |n(x)\times\nu_i(x)| > \kappa, \tag{2.30}$$

where $\nu_i(x) = (x-z_i)/|x-z_i|$ is the outward unit normal on $\partial B_{6(\alpha_i+r_i)}(z_i)$.

The claim means that the intersections $\Sigma^{s_i}\cap\partial B_{9(\alpha_i+r_i)}(z_i)$ are uniformly transverse.

*Proof (of the claim)* Let us recall the properties:

$$1 \le \alpha_i \le 2^{N-1}, \qquad N' \le N \le C_{\not{e}},$$
$$\|u\|_\infty \le 1, \qquad u \text{ is } \omega^\star\text{-continuous and } u \text{ is harmonic in } [|u| < 1/2]+B_{\delta^\star}.$$
$$\text{Moreover, } \forall y \in B_{9(\alpha_i+1)}(z_i)\setminus\overline{B_{9\alpha_i}}(z_i), \qquad |u|(y) \le 1/2 \Rightarrow |\nabla u|(y) \ge 1/2.$$

Let us assume by contradiction that for every $k \ge 1$, there exist

- $\alpha_k \in [1,2^{N-1}]$;
- $u_k : \mathbf{R}^3 \to [-1,1]$, $\omega^\star$-continuous, harmonic in $[|u_k| < 1/2]+B_{\delta^\star}$, satisfying

$$\left[9\alpha_k < |y| < 9(\alpha_k+1) \quad\text{and}\quad |u_k|(y) \le 1/2\right] \quad\Longrightarrow\quad |\nabla u_k|(y) \ge 1/2;$$

- a mapping $(r,s)\in[0,1/2]\times[-1/8,1/8]\mapsto x_k(r,s)\in\mathbf{R}^3$ such that for every $(r,s)\in[0,1/2]\times[-1/8,1/8]$,

$$u(x_k(r,s))=s,\quad |x_k(r,s)|=9(\alpha_k+r)\quad\text{and}\quad |\nabla u(x_k(r,s))\times x_k(r,s)|\ \le\ 1/k.$$

By diagonal extraction, we may assume that $\alpha_k\to\alpha_\star\in[1,2^{N-1}]$, $x_k\to x_\star$ on

$$P\ :=\ (\mathbf{Q}\times\mathbf{Q})\cap([0,1/2]\times[-1/8,1/8])$$

and by equicontinuity of $(u_k)$ that $u_k\to u_\star$ in $C_{loc}(\mathbf{R}^3)$. Since $u_k$ is harmonic in $[|u_k|<1/2]+B_{\delta^\star}$, we also have $u_k\to u_\star$ in $C^1_{loc}([|u|<1/2]+B_{\delta^\star})$ and $u_\star$ is harmonic in $[|u|<1/2]+B_{\delta^\star}$. Passing to the limit, we conclude that for every $(r,s)\in P$,

$$u_\star(x_\star(r,s))=s,\quad |x_\star(r,s)|=9(\alpha_\star+r),\quad \nabla u_\star(x_\star(r,s))\times x_\star(r,s)\ =\ 0. \tag{2.31}$$

Now, for every $(r,s)\in[0,1/2]\times[-1/8,1/8]\setminus P$, we can extract a sequence of elements of $P$, $(r_j,s_j)\to(r,s)$ such that $x(r_j,s_j)$ converges towards some limit denoted $x_\star(r,s)$. By continuity, (2.31) holds for every $(r,s)\in[0,1/2]\times[-1/8,1/8]$.

To conclude, we consider the mapping $\phi\ :\ y\mapsto(|y|,u(y))$ defined on the open set

$$\left\{y\in\mathbf{R}^3\ :\ 9\alpha_\star<|y|<9(\alpha_\star+1/2),\,|u_\star|(y)<1/2\right\}.$$

This mapping is smooth and by (2.31) every element of

$$(9\alpha_\star,9\alpha_\star+9/2)\times(-1/8,1/8)$$

is a critical value of $\phi$. Indeed, for every $(r,s)\in[0,1/2]\times[-1/8,1/8]$,

$$\phi(x_\star(r,s))\ =\ (9(\alpha_\star+r),s)\quad\text{and}\quad \operatorname{rank} D\phi(x_\star(r,s))\ =\ 1\ <\ 2.$$

This contradicts Sard theorem. □

We now define the set of bad balls as $U^\#:=\cup_i B_{9(\alpha_i+r_i)}(z_i)$ and the good set as $G^\#:=\mathbf{R}^3\setminus U^\#$. Let us notice that since $r_i\le 1/2$, we have

$$d(B_{9(\alpha_i+r_i)}(z_i),B_{9(\alpha_j+r_j)}(z_j))\ge 9\quad\text{for } 1\le i<j\le N'.$$

Using the cut-off function $\chi$ as in the previous subsection, we set

$$\Sigma^\#:=\{x\in G^\#\ :\ u(x)-\theta^\#(x)\ =\ 0\}\quad\text{with}\quad \theta^\#(x)\ :=\ \sum_{i=1}^N s_i\chi(d(x,B_{9(\alpha_i+r_i)}(z_i))).$$

In particular, $\Sigma^\#$ coincides with $\Sigma^0$ in $G^\star$ and $\Sigma^\#=\Sigma^{s_i}$ in the neighborhood of $\overline{B_{9(\alpha_i+r_i)}}(z_i)$. We also define

$$O^\#\ :=\ \{y\in G^\#\ :\ u(y)-\theta^\#(y)\ <\ 0\}.$$

So that $\Sigma^\#\setminus\overline{U^\#}=\partial O^\#\setminus\overline{U^\#}$. Proceeding as in the previous subsection, $\Sigma^\#$ is a smooth surface and we have the estimates

$$\mathscr{H}^{d-1}(\Sigma^\#)\ \le\ CS/\varepsilon^2+C_{\not\varepsilon},\qquad \mathscr{W}(\Sigma^\#)\ \le\ C_{\not\varepsilon} \tag{2.32}$$

$$\text{and}\qquad \|\mathbf{1}_{O^\#}-\mathbf{1}_M\|_{L^1}\ \le\ 2S/\varepsilon^2+C\sqrt{SE_0/\varepsilon^2}+C_{\not\varepsilon}. \tag{2.33}$$

Now, let us consider the mapping

$$\Phi\ :\ (y,t)\in\Sigma^\#\times[-1,1]\mapsto x+(t/4)n(x)\in\mathbf{R}^3.$$

For the moment, the construction depends on the parameter $\eta \in (0,1/4)$. By Lemma 2.4, there exists $\eta_0 \in (0,1/4)$ such that for $\eta \in (0,\eta_0)$,

$$\Phi \text{ is a smooth diffeomorphism, and } \pm\Phi(\Sigma \times \{\pm 1\}) \geq 1/10. \tag{2.34}$$

From now on, we fix $\eta := \eta_0/2$ (notice that $\eta$ is a universal constant, in particular, it does not depend on $\varepsilon$). By Hypothesis 2, $\Sigma^\#$ is homeomorphic to a closed subset of the $g_0$-torus.

We have to complete $\Sigma^\#$ inside $U^\#$ to form a connected compact surface with genus at most $g_0$.

Let us first describe the trace of $\Sigma^\#$ on $\partial U^\# = \cup_i \partial B_{9(\alpha_i+r_i)}(z_i)$. To lighten notation, we note $S_i$ the sphere $\partial B_{9(\alpha_i+r_i)}(z_i)$. Let us fix $1 \leq i \leq N'$ and set

$$\Gamma_i := \Sigma^\# \cap S_i.$$

Let $x \in \Gamma_i$. By Lemma 2.4.b there exists a radius $r > 0$ only depending on $\eta$ such that

$$B^\pm(x) := B_r(x \pm r n(x)) \subset \{y : \pm(u(y) - s_i) > 0\}.$$

Recall that $\alpha_i \geq 1$. By (2.30) the trace of $B^+(x)$ (respectively $B^-(x)$) on $S_i$ contains a geodesic ball of radius $\rho$ which is tangent to $\Gamma_i$ at $x$ and where $\rho > 0$ depends monotonically on $\kappa$ and $r$ (see Figure 2.3). As a consequence, the set $\Gamma_i \subset S_i$ has the exterior and interior ball properties with

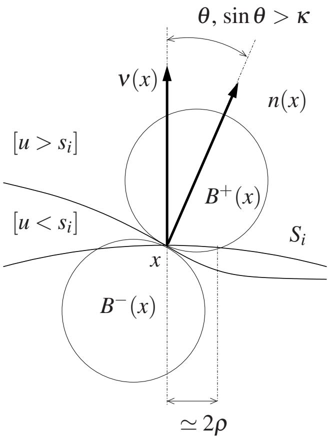


**Fig. 2.3** Cross-sectional view near $x \in \Gamma_i$.

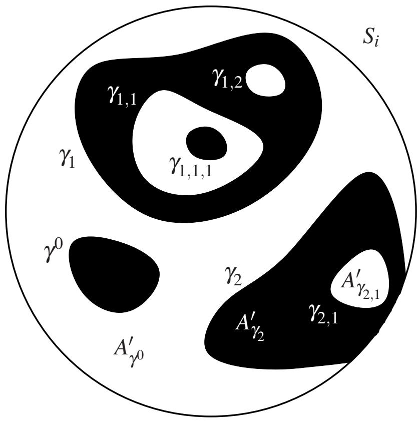


**Fig. 2.4** Labelling the components of $\Gamma_i$.

balls of radius $\rho$ in the sense that for any $x \in \Gamma_i$ there exist two distinct geodesic open balls on $S_i$ with radius $\rho$ which both contain $x$ on their boundary and do not intersect $\Gamma_i$. Hence $\Gamma_i$ is a smooth curve on $S_i$ with curvature bounded by $2/\rho$ (the factor 2 accounts for the difference between the geometries in $\mathbf{R}^3$ and in $S_i$). Moreover, $\Gamma_i$ has a finite number $M_i$ of connected components which are closed Jordan curves on $S_i$ and the geodesic distance between two distinct components is at least $2\rho$. Notice that $M_i \leq C(\kappa) \leq C_{\not{\varepsilon}}$. Let us denote by $\mathscr{C}_i$ the set of connected components of $\Gamma_i$.

Before describing further the construction, let us discuss what has to be proved. By Hypothesis 2, the set $[|t| < 1/10]$ is connected. By Lemma 2.4 and the choice of $\eta$, this implies that the union $\Sigma^\# \cup U^\#$ is connected. Let us denote by

$$\mathscr{C}^\# = \{\Sigma_1^\#, \cdots, \Sigma_{M^\#}^\#\}$$

the set of connected components of $\Sigma^\#$. It may happen that $U^\#$ is empty. In this case $\Sigma^\#$ has only one connected component and is a smooth surface of genus $g_0$, we then set $\Sigma = \Sigma^\#$, $O = O^\#$ and with (2.32)(2.33) Proposition 2.1.b is proved. In the other case, the boundary of any element of $\mathscr{C}^\#$ is formed by a finite union of curves of $\cup_i \mathscr{C}_i$. Let us consider a connected component $\Sigma_j^\# \in \mathscr{C}^\#$ and let us denote by $\gamma_{j,1},\cdots,\gamma_{j,K_j}$ the topological circles forming its boundary. Considering $\Sigma_j^\#$ as an abstract manifold, we can fill its holes by adding $K_j$ topological disks with boundaries $\gamma_{j,1},\cdots,\gamma_{j,K_j}$. We obtain an abstract compact surface $\Sigma_j'$. The genus of $\Sigma_j^\#$ is then defined as the genus of the $\Sigma_j'$. More generally, we can define the genus of any relatively compact surface whose boundary is made of a finite union of topological circles. For example, the genus of the periodic strip $S^1\times(0,1)$ is 0 and the genus of a torus from which we have removed a topological disk is 1. Let us recall the basic genus arithmetic which is used thereafter. Given two disjoint connected surfaces $\Sigma^1$, $\Sigma^2$ with geni $g^1$, $g^2$ and common boundary $\cup_{i=1}^I \gamma_i$ where the $\gamma_i$ are disjoint Jordan curves, there holds:

$$\text{genus }\left[\Sigma_1\cup\Sigma_2\cup\left(\cup_{i=1}^I\gamma_i\right)\right] \;=\; g_1+g_2+(I-1).$$

From the above discussion, a way to connect the surfaces $\Sigma_j^\#$ without increasing the total genus $\sum g_j^\#$ is to add recursively exactly $(M^\#-1)$ necks (periodic strips) connecting $\Sigma_j^\#$ to $\cup_{j'<j}\Sigma_{j'}^\#$ for $j=2,\cdots,M^\#$. Our construction is not that simple because we can not choose the order of the connections. We proceed as follows.
*Step 1.* We close every hole of the connected components of $\Sigma^\#$ with topological disks contained in $U^\#$.
*Step 2.* We define a family of possible necks which connect two components of $\Sigma^\#$, we show that there exists a subfamily of necks connecting $\Sigma^\#$ without redundant connections.
*Step 3.* Eventually, we regularize the resulting surface.

*Step 1. Closing the holes.* Let us fix $1\le i\le N'$ and let us consider the ball $B_{9(\alpha_i+r_i)}(z_i)$. Without loss of generality, we assume $z_i=0$ and to lighten notation, we set $R_i:=9(\alpha_i+r_i)\in[9,9(2^{N-1}+1)]$.
Notice first that every curve $\gamma\in\mathscr{C}_i$ splits $S_i$ into two topological disks $D_{\gamma,1}$, $D_{\gamma,2}$. Let us order these sets such that $\mathscr{H}^2(D_{\gamma,1})\ge\mathscr{H}^2(D_{\gamma,2)}$. Now let $\gamma^0$ be a curve maximizing $\mathscr{H}^2(D_{\gamma^0,1})$ and let us note $A_{\gamma^0}=D_{\gamma^0,1}$. By maximality, $A_{\gamma^0}$ contains all the elements of $\mathscr{C}_i\setminus\{\gamma^0\}$.
Next, every curve of $\gamma\in\mathscr{C}_i\setminus\{\gamma^0\}$ is the boundary of a topological disk in $A_{\gamma^0}$, that we denote by $A_\gamma$. We consider the partial order on $\mathscr{C}_i$ defined by $\gamma\le\gamma'$ if and only if $A_\gamma\subset A_{\gamma'}$. This order defines an oriented graph with the elements of $\mathscr{C}_i$ as vertices. In fact, the corresponding non-oriented graph do not have loops, that is: if $\gamma_1\le\gamma_2\le\gamma_3$ and $\gamma_1\le\gamma_2'\le\gamma_3$ then either $\gamma_2\le\gamma_2'$ or $\gamma_2'\le\gamma_2$.
Indeed, we have $A_{\gamma_2}\cap A_{\gamma_2'}\supset A_{\gamma_1}\ne\emptyset$ which implies $\gamma_2<\gamma_2'$ or $\gamma_2'<\gamma_2$ or $\gamma_2\cap\gamma_2'\ne\emptyset$ which in turn implies $\gamma_2'=\gamma_2$.
As a consequence, the above oriented graph is a finite oriented tree. The only maximal element is $\gamma^0$, so the tree is connected with root $\gamma^0$.
We use the paths from $\gamma^0$ to the elements of $\mathscr{C}_i\setminus\{\gamma^0\}$ in the oriented tree for labelling these elements. More precisely, every element $\gamma$ of $\mathscr{C}_i\setminus\{\gamma^0\}$ is labelled $\gamma=\gamma_{j_1,\cdots,j_k}$ where $k$ is the height of $\gamma$ in the tree and $(j_1,\cdots,j_k)$ describe the path from $\gamma^0$ to $\gamma$. Then the maximal chain from $\gamma$ to $\gamma^0$ is

$$\gamma\;=\;\gamma_{j_1,\cdots,j_k}\;<\;\gamma_{j_1,\cdots,j_{k-1}}\;<\;\cdots\;<\;\gamma_{j_1}\;\le\gamma^0.$$

See an example in Figure 2.4.

We are now able to perform the construction of Step 1. Let $K$ be the height of the tree and for $k=0,\cdots,K$, let $\lambda_k:=(k+1)/(K+1)$. We define a "closing" surface $\Sigma_i'$ as the union over all the elements $\gamma_{i_1,\cdots,i_k} \in \mathscr{C}_i$ of

$$\Sigma_{\gamma_{j_1,\cdots,j_k}} := \lambda_k A_{\gamma_{j_1,\cdots,j_k}} \cup \Big\{ \lambda \partial A_{\gamma_{j_1,\cdots,j_k}} \,:\, \lambda_k \leq \lambda < 1 \Big\}.$$

As required, $\Sigma_{\gamma_{j_1,\cdots,j_k}}$ is a tological open disk with boundary $\partial A_{\gamma_{j_1,\cdots,j_k}}$. By construction, these surfaces do not intersect, see the example Figure 2.5.

Eventually, we define the surface

$$\tilde{\Sigma} := \Sigma^{\#} \cup \left[ \cup_{i=1}^{N'} \Sigma_i' \right]$$

which is the disjoint union of $M^{\#}$ closed surfaces with genii $g_1^{\#}, \cdots, g_M^{\#}$.

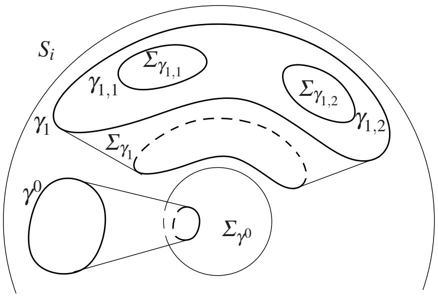


**Fig. 2.5** Step 1. Example, with $K = 2$.

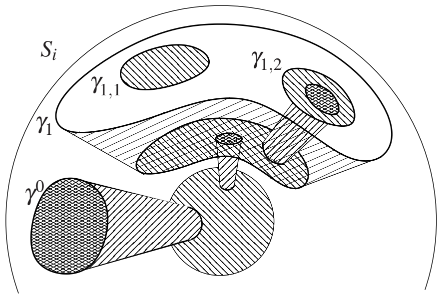


**Fig. 2.6** Step 2. $\gamma_1, \gamma_{1,2} \in \mathscr{N}$, $\gamma_{1,1} \notin \mathscr{N}$.

*Step 2. Connections.* Let $1 \leq i \leq N'$ and let us use the notation of Step 1. For $\gamma \in \mathscr{C}_i$, we call $A'_\gamma$ the connected component of $S_i \setminus \Gamma_i$ such that $A'_\gamma \subset A_\gamma$ and $\gamma \in \partial A'_\gamma$. Equivalently, (See again Figure 2.4)

$$A'_{\gamma_{j_1,,\cdots,j_k}} = A_{\gamma_{j_1,\cdots,j_k}} \setminus \left[ \bigcup_{l>k} A_{\gamma_{j_1,\cdots,j_k,j_{k+1},\cdots,j_l}} \right].$$

For every element $\gamma \in \mathscr{C}_i$, there exists a geodesic open disk $D_\gamma \subset A'_\gamma$ with radius $\rho/2$ and such that $d(D_\gamma, \Gamma_i) \geq \rho/2$.

Given $\gamma \in \mathscr{C}_i$ with height $k$ in the tree and a surface $X$ which contains $\lambda_{k-1} A'_\gamma$ and $\lambda_k A'_\gamma$ we define the operation of adding the neck $N_\gamma$ as

$$Y = X \setminus [\lambda_{k-1} D_\gamma \cup \lambda_k D_\gamma] \cup \{\lambda \,:\, \lambda_{k-1} < \lambda < \lambda_k\} \partial D_\gamma.$$

We will write $Y = X + N_\gamma$. Notice that by construction $\tilde{\Sigma}$ contains $\lambda_{k-1} A'_\gamma$ and $\lambda_k A'_\gamma$ for any $\gamma \in \mathscr{C}_i$ with height $k$. Moreover, the additions of different necks $N_\gamma$ do not create crossings since the $A'_\gamma$ are disjoint sets. Now, we see that

$$\tilde{\Sigma} + \sum_{i=1}^{N} \sum_{\gamma \in \mathscr{C}_i} N_\gamma \quad \text{is a connected surface without boundary.}$$

Hence, there exists a minimal subset $\mathscr{N}$ of $\{\gamma \in \mathscr{C}_i \,:\, 1 \leq i \leq N\}$ such that

$$\Sigma^\flat := \tilde{\Sigma} + \sum_{\gamma \in \mathscr{N}} N_\gamma$$

is connected (see Figure 2.6). If there were redundant connections, then we could remove a neck and still have a connected surface, so the genus of $\Sigma^\flat$ is exactly

$$g \;:=\; \sum_{i=1}^{M^\#} g_i^\# \;\le\; g_0.$$

*Step 3. Smoothing.*

By construction, $\Sigma^\flat$ is piecewise smooth and the smooth pieces meet on a finite number of smooth curves and form angles larger than $\arcsin\kappa$. Moreover, there exists $\rho^\star>0$ only depending on $E_0$ and $\omega$ such that every singular curve has curvature bounded by $1/\rho^\star$ and the distance between two distinct singular curves is at least $\rho^\star$. From the regularity of $\Sigma^\#$ in $G^\#$ and the regularity properties of the singular curves, we can mollify $\Sigma^\flat$ in the neighborhood of the singular curves to obtain a surface $\Sigma$ homeomorphic to $\Sigma^\flat$ such that $\Sigma^\#\setminus U^\# = \Sigma\setminus U^\#$ and such that the total area and maximal curvature of $\Sigma$ inside $U^\#$ are bounded by $C_g$. Notice that the smooth surface $\Sigma\subset\mathbf{R}^3$ is compact and connected, hence it is orientable and splits $\mathbf{R}^3$ into exactly one unbounded and one bounded component. We define $O$ as the bounded connected component of $\mathbf{R}^3\setminus\Sigma$. We have $O^\#\subset O$ and $O\setminus O^\#\subset U^\#$. Taking into account (2.32) and (2.33), we see that $\Sigma$ and $O$ satisfy (2.25) and (2.26) (with $d=3$). We conclude by unscaling as in the previous subsection.
We also see that (2.27) and (2.28) still hold for this construction with $U^\star$ and $G^\star$ given by (2.29).

## 2.6 Proof of Proposition 2.1.c

In Sections 2.4 and 2.5, we have built the open sets $O_\varepsilon$ and $\Sigma_\varepsilon$ satisfying the estimates of Proposition 2.1.b. We now establish Proposition 2.1.c. Let $\varphi\in C(\mathbf{R}^d\times\mathbf{R}^d)$ such that $\sup\left[|\varphi(y,v)|/(1+|v|^2)\right]<\infty$. In scaled variables, we have to prove that

$$\varepsilon^{d-1}\left|\frac{1}{2}\int_{\Omega_{(\varepsilon)}}\varphi(\varepsilon z,\nabla t_{(\varepsilon)}(z))\,dz-\int_{\Sigma_{(\varepsilon)}}\varphi(\varepsilon x,\nu_{(\varepsilon)}(x))\,d\mathscr{H}^{d-1}(x)\right|\;\overset{\varepsilon\downarrow 0}{\longrightarrow}\;0. \tag{2.35}$$

*Proof (of* (2.35)*)*
*Step 1. Mollification*

Let us introduce the quantities,

$$Q_{(\varepsilon)}[\varphi]:=\frac{1}{2}\int_{\Omega_{(\varepsilon)}}\varphi(\varepsilon z,\nabla t_{(\varepsilon)}(z))\,dz,\qquad Q'_{(\varepsilon)}[\varphi]:=\int_{\Sigma_{(\varepsilon)}}\varphi(\varepsilon x,\nu_{(\varepsilon)}(x))\,d\mathscr{H}^{d-1}(x).$$

Let $\chi\in C_c^\infty(\mathbf{R}^d,[0,1])$ be a cut-off function satisfying $\chi\equiv 1$ in $B_2$ and let us set $\varphi'(y,v)=\chi(v)\varphi(y,v)$. We obviously have $Q'_{(\varepsilon)}[\varphi']=Q'_{(\varepsilon)}[\varphi]$. On the other hand, we compute

$$\varepsilon^{d-1}\left|Q_{(\varepsilon)}[\varphi'-\varphi]\right|\;\le\;\frac{\varepsilon^{d-1}}{2}\int_{[|\nabla t_\varepsilon|>2]}|\varphi|(\varepsilon z,\nabla t_{(\varepsilon)}(z))\,dz.$$

Since $|\varphi|(y,v)\;\le\;C(\varphi)(|v|-1)^2$ for $|v|>2$, this leads to,

$$\varepsilon^{d-1}\left|Q_{(\varepsilon)}[\varphi'-\varphi]\right|\;\le\;\frac{C(\varphi)\varepsilon^{d-1}}{2}\int_{\Omega_{(\varepsilon)}}(|\nabla t_{(\varepsilon)}|-1)^2\;\le\;\frac{C(\varphi)E_0\varepsilon^2}{2}\overset{\varepsilon\downarrow 0}{\to}0.$$

As a consequence, we may assume that $\varphi$ is compactly supported in $B_{R+1}\times B_4$.

Let $\widetilde{\varphi} \in C_c(\mathbf{R}^d \times \mathbf{R}^d)$, we deduce from the estimates of Proposition 2.1.b,

$$\varepsilon^{d-1} \left| Q_{(\varepsilon)}[\widetilde{\varphi} - \varphi] \right| \;\le\; \varepsilon^{d-1} \mathscr{H}^d(\Omega_{(\varepsilon)}) \|\widetilde{\varphi} - \varphi\|_\infty / 2 \;\le\; C_{\not{e}} \|\widetilde{\varphi} - \varphi\|_\infty,$$
$$\varepsilon^{d-1} \left| Q'_{(\varepsilon)}[\widetilde{\varphi} - \varphi] \right| \;\le\; \varepsilon^{d-1} \mathscr{H}^{d-1}(\Sigma_{(\varepsilon)}) \|\widetilde{\varphi} - \varphi\|_\infty \le\; C_{\not{e}} \|\widetilde{\varphi} - \varphi\|_\infty.$$

Thus, by density, we may also assume that $\varphi$ is smooth and compactly supported.

*Step 2. Cut-off procedure.*

From now on, we consider a fixed test function $\varphi \in \mathscr{D}(\mathbf{R}^d \times \mathbf{R}^d)$. In order to ease the estimate below, we perform partitions of $\mathbf{R}^d$ into good sets and bad sets. Let us introduce a small parameter $\xi \in (0, 1/2)$ and let us set

$$\Omega_{(\varepsilon)}^{\xi/2} \;:=\; \left\{ z \in \Omega_{(\varepsilon)} \;:\; |t_{(\varepsilon)}|(z) < 1 - \xi/2 \right\}.$$

By Lemma 2.4 and Lemma 2.5, there exists $\beta_a > 0$ only depending on $\xi$ such that if $x \in \Sigma^0_{(\varepsilon)}$ satisfies $\mathscr{E}_{(\varepsilon),3}(x) < \beta_a$ then

$$x + D_1^\xi(n_{(\varepsilon)}(x)) \;\subset\; \Omega_{(\varepsilon)}^{\xi/2},$$

and we have the estimate,

$$\frac{1}{\mathscr{H}^d(D_1^0)} \int_{\left[x + D_1^\xi(n_{(\varepsilon)}(x))\right]} |\nabla t_{(\varepsilon)}(z) - n_{(\varepsilon)}(x)| \, dz \;\le\; \xi. \tag{2.36}$$

Now, for $z \in \Omega_{(\varepsilon)}^{\xi/2}$, we set

$$q^\xi_{(\varepsilon)}(z) \;:=\; \frac{1}{\mathscr{H}^d(D_1^0)} \int_{\Sigma^0_{(\varepsilon)}} \theta^\xi(z - x, n_{(\varepsilon)}(x)) \, d\mathscr{H}^{d-1}(x),$$

where for $\bar{n} \in S^{d-1}$, $y \mapsto \theta^\xi(y, \bar{n})$ denotes the characteristic function of $D_1^\xi(\bar{n})$.
By Lemma 2.6 there exists $\beta_b > 0$ only depending on $\xi$ such that if $z \in \Omega_{(\varepsilon)}^{\xi/2}$ satisfies $\mathscr{E}_{(\varepsilon),3}(x) < \beta_b$, then

$$|q^\xi_{(\varepsilon)}(z) - 1| \;\le\; \xi. \tag{2.37}$$

Let us set $\beta := \min(\beta_a, \beta_b)$ and let us define the bad sets

$$\mathscr{U}_{(\varepsilon)} \;:=\; \left[ U^\star_{(\varepsilon)} \cup U_{(\varepsilon),3,\beta} \right] + B_{\sqrt{2}} \quad \subset \quad \mathscr{U}'_{(\varepsilon)} \;:=\; \mathscr{U}_{(\varepsilon)} + B_2,$$

and the corresponding good sets

$$\mathscr{G}_{(\varepsilon)} \;:=\; \mathbf{R}^d \setminus \mathscr{U}_{(\varepsilon)} \quad \supset \quad \mathscr{G}'_{(\varepsilon)} \;:=\; \mathbf{R}^d \setminus \mathscr{U}'_{(\varepsilon)}.$$

By Lemma 2.7, we have

$$\mathscr{H}^{d-1}(\mathscr{U}'_{(\varepsilon)}) \;\le\; C_{\not{e}}(\xi)\, \varepsilon^{3-d}. \tag{2.38}$$

We now introduce a cut-off function $\chi_{(\varepsilon),\xi} \in C^\infty(\mathbf{R}^d, [0,1])$ such that

$$\chi_{(\varepsilon),\xi} \;\equiv\; 1 \;\text{ on } \mathscr{G}'_{(\varepsilon)}, \qquad \chi_{(\varepsilon),\xi} \;\equiv\; 0 \;\text{ on } \mathscr{U}_{(\varepsilon)}, \qquad \|\nabla \chi_{(\varepsilon),\xi}\|_\infty \;\le\; 1.$$

We set $\varphi_{\varepsilon,\xi}(y,v) := \chi_{(\varepsilon),\xi}((1/\varepsilon)y)\varphi(y,v)$. We have,

$$\varepsilon^{d-1}\left|Q_{(\varepsilon)}[\varphi_{\varepsilon,\xi}-\varphi]\right| \;\le\; \|\varphi\|_\infty \varepsilon^{d-1}\mathscr{H}^d(\mathscr{U}'_{(\varepsilon)})/2 \overset{(2.38)}{\le} C_{\not{g}}(\varphi,\xi)\varepsilon^2,$$
$$\varepsilon^{d-1}\left|Q'_{(\varepsilon)}[\varphi_{\varepsilon,\xi}-\varphi]\right| \;\le\; \|\varphi\|_\infty \varepsilon^{d-1}\mathscr{H}^{d-1}(\Sigma_{(\varepsilon)}\cap\mathscr{U}'_{(\varepsilon)}) \overset{(2.28)}{\le} C_{\not{g}}(\varphi,\xi)\varepsilon^2.$$

Therefore,

$$\varepsilon^{d-1}\left|\left\{Q_{(\varepsilon)}[\varphi_{\varepsilon,\xi}]-Q'_{(\varepsilon)}[\varphi_{\varepsilon,\xi}]\right\}-\left\{Q_{(\varepsilon)}[\varphi]-Q'_{(\varepsilon)}[\varphi]\right\}\right| \;\le\; C_{\not{g}}(\varphi,\xi)\varepsilon^2 \overset{\varepsilon\downarrow 0}{\longrightarrow} 0.$$

As a consequence, we can substitute $\varphi_{\varepsilon,\xi}$ for $\varphi$ in (2.35). The rest of the proof consists in establishing

$$\limsup_{\varepsilon\downarrow 0}\varepsilon^{d-1}\left|Q_{(\varepsilon)}[\varphi_{\varepsilon,\xi}]-Q'_{(\varepsilon)}[\varphi_{\varepsilon,\xi}]\right| \;\le\; C_{\not{g}}(\varphi)\xi. \tag{2.39}$$

Sincem $\xi\in(0,1/2)$ is arbitrary, this proves the desired convergence result (2.35).

*Step 3. Proof of* (2.39).

Let us enumerate the benefits of the cut-off procedure. Since $\varphi_{\varepsilon,\xi}$ is supported in $\mathscr{G}_{(\varepsilon)}\times\mathbf{R}^d$, we may substitute $\Sigma^0_{(\varepsilon)}$ for $\Sigma_{(\varepsilon)}$ in the definition of $Q'_{(\varepsilon)}[\varphi_{\varepsilon,\xi}]$. Moreover (2.36) hold for any $x\in\Sigma^0_{(\varepsilon)}\cap\mathscr{G}_{(\varepsilon)}$ and (2.37) hold for any $z\in\Omega^{\xi/2}_{(\varepsilon)}\cap\mathscr{G}_{(\varepsilon)}$. Eventually, $\varphi_{\varepsilon,\xi}$ satisfies

$$\|\varphi_{\varepsilon,\xi}\|_\infty \;\le\; \|\varphi\|_\infty, \qquad \|\nabla\varphi_{\varepsilon,\xi}\|_{L^\infty(\mathscr{G}'_\varepsilon\times\mathbf{R}^d)} \;\le\; \|\nabla\varphi\|_\infty.$$

To lighten notation we write $Q_{(\varepsilon)}$ for $Q_{(\varepsilon)}[\varphi_{\varepsilon,\xi}]$ and $Q'_{(\varepsilon)}$ for $Q'_{(\varepsilon)}[\varphi_{\varepsilon,\xi}]$. With this notation, we have to show that

$$\limsup_{\varepsilon\downarrow 0}\varepsilon^{d-1}\left|Q_{(\varepsilon)}-Q'_{(\varepsilon)}\right| \;\le\; C_{\not{g}}(\varphi)\xi.$$

For this, we introduce the intermediate quantities

$$Q_{(\varepsilon)}(\xi) := \frac{1}{2}\int_{\Omega^{\xi/2}_{(\varepsilon)}}\varphi_{\varepsilon,\xi}(\varepsilon z,\nabla t_{(\varepsilon)}(z))\,dz,$$

and

$$Q'_{(\varepsilon)}(\xi) := \int_{\Sigma^0_{(\varepsilon)}}\left\{\frac{1}{\mathscr{H}^d(D^0_1)}\int_{\left[x+D^\xi_1(n_{(\varepsilon)}(x))\right]}\varphi_{\varepsilon,\xi}(\varepsilon z,\nabla t_{(\varepsilon)}(z))\,dz\right\}d\mathscr{H}^{d-1}(x).$$

We prove (2.39) by estimating successively $Q_{(\varepsilon)}-Q_{(\varepsilon)}(\xi)$, $Q_{(\varepsilon)}(\xi)-Q'_{(\varepsilon)}(\xi)$ and $Q'_{(\varepsilon)}(\xi)-Q'_{(\varepsilon)}$. In these estimates, apart from the conclusions, we drop the subscripts $(\varepsilon)$.

*Step 3.1. Estimating $Q_{(\varepsilon)}-Q_{(\varepsilon)}(\xi)$.*

Using (2.3) in Lemma 2.1 with $(\alpha_-,\alpha_+)=(-1,-1+\xi/2)$ and $(\alpha_-,\alpha_+)=(1-\xi/2,1)$, we obtain,

$$Q-Q(\xi)| \;\le\; \frac{\|\varphi\|_\infty}{2}\mathscr{H}^d\left(\Omega^{-1,-1+\xi/2}\cup\Omega^{1-\xi/2,1}\right) \;\le\; \frac{\|\varphi\|_\infty}{2}\left(\frac{S\xi}{\varepsilon^{d-1}}+\frac{C_{\not{g}}(\xi)}{\varepsilon^{(d-1)/2}}\right).$$

Hence,

$$\limsup_{\varepsilon\downarrow 0}\varepsilon^{d-1}|Q_{(\varepsilon)}-Q_{(\varepsilon)}(\xi)| \;\le\; \|\varphi\|_\infty S\xi/2. \tag{2.40}$$

*Step 3.2. Estimating $Q_{(\varepsilon)}(\xi)-Q'_{(\varepsilon)}(\xi)$.*

Changing the order of integration in the definition of $Q''(\xi)$, we obtain,

$$Q'(\xi) \;=\; \int_{\Omega^{\xi/2}} \varphi_{\varepsilon,\xi}(\varepsilon z, \nabla t(z)) q^{\xi}(z) dz,$$

Hence, by (2.37), we have,

$$|Q(\xi)-Q'(\xi)| \;\leq\; \|\varphi\|_\infty \mathscr{H}^d(\Omega)\xi \;\overset{(2.3)}{\leq}\; \left(\frac{S}{\varepsilon^{d-1}}+\frac{C_{\not{\xi}}}{\varepsilon^{(d-1)/2}}\right)\xi.$$

Again, this leads to

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{d-1}|Q_{(\varepsilon)}(\xi)-Q'_{(\varepsilon)}(\xi)| \;\leq\; \|\varphi\|_\infty S\xi. \tag{2.41}$$

*Step 3.3. Estimating $Q'(\xi)-Q'$.*

We have

$$\left|Q'(\xi)-Q'\right| \;\leq\; \int_{\Sigma^0} q(x) d\mathscr{H}^{d-1}(x),$$

with

$$q(x) \;:=\; \left|\varphi_{\varepsilon,\xi}(\varepsilon x, n(x)) - \frac{1}{\mathscr{H}^d(D_1^0)} \int_{\left[x+D_1^{\xi}(n(x))\right]} \varphi_{\varepsilon,\xi}(\varepsilon z, \nabla t(z))\, dz\right|.$$

Let us set $\mathscr{U}''=\mathscr{U}'+B_{\sqrt{2}}$ and $\mathscr{G}'':=\mathbf{R}^d\setminus\mathscr{U}''$. For $x\in\Sigma^0\cap\mathscr{U}''$, we use the rough estimate $q(x)\leq 2\|\varphi\|_\infty$. For $x\in\Sigma^0\cap\mathscr{G}''$, we have $\varphi_{\varepsilon,\xi}\equiv\varphi$ in $[x+D_1^{\xi}(n(x))]\times\mathbf{R}^d$ and we estimate $q(x)$ as follows.

$$\begin{aligned}
q(x) \;&\leq\; |\varphi(\varepsilon x, n(x))|\left[1-\mathscr{H}^d(D_1^{\xi})/\mathscr{H}^d(D_1^0)\right] \\
&\qquad\qquad + \frac{1}{\mathscr{H}^d(D_1^0)} \int_{\left[x+D_1^{\xi}(n(x))\right]} |\varphi(\varepsilon z, \nabla t(z)) - \varphi(\varepsilon x, n(x))|\, dz \\
&\leq\; 2\|\varphi\|_\infty \xi + \left(\sqrt{2}\|\nabla_x\varphi\|_\infty \varepsilon + \frac{\|\nabla_\tau\varphi\|_\infty}{\mathscr{H}^d(D_1^0)} \int_{\left[x+D_1^{\xi}(n(x))\right]} |\nabla t(z)-n(x)|\, dz\right) \\
&\overset{(2.36)}{\leq}\; \left(2\|\varphi\|_\infty + \sqrt{2}\|\nabla_x\varphi\|_\infty\varepsilon + \|\nabla_\tau\varphi\|_\infty\right)\xi.
\end{aligned}$$

Thus,

$$\left|Q'(\xi)-Q'\right| \;\leq\; 2\|\varphi\|_\infty \mathscr{H}^{d-1}(\Sigma^0\cap\mathscr{U}'') + 2\|\varphi\|_{W^{1,\infty}}\mathscr{H}^{d-1}(\Sigma)\xi \;\leq\; C_{\not{\xi}}(\varphi)(\varepsilon^{2-d}+\varepsilon^{1-d}\xi).$$

We conclude that

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{d-1}|Q'_{(\varepsilon)}(\xi)-Q'_{(\varepsilon)}| \;\leq\; C_{\not{\xi}}(\varphi)\xi. \tag{2.42}$$

*Step 3.4. Conclusion.*

Gathering (2.40)(2.41)(2.42), we have established (2.39). This proves (2.35). □

## 2.7 Proof of Theorem 1.1

Let $R,S>0$ and let us consider a family $\{a_\varepsilon\}_{0<\varepsilon\leq 1}$ with $a_\varepsilon=(\sigma_\varepsilon,\nabla t_\varepsilon,\Omega_\varepsilon)\in\mathscr{A}_\varepsilon(R,S)$ satisfying the hypotheses of Theorem 1.1. In this case, the conclusions of Proposition 2.1 hold. We use the notation of the proposition.

Let us call $\mathscr{V}'_\varepsilon=\mathscr{V}(\Sigma_\varepsilon,n_\varepsilon)$ the oriented $(d-1)$-varifold associated to $\Sigma_\varepsilon=\partial O_\varepsilon$ and let $\varphi\in C(\mathbf{R}^d\times S^{d-1})$. Proposition 2.1.c applied to the test function $\overline{\varphi}(y,v)=\varphi(y,v/|v|)\chi^\star(|v|)$ reads

$$\langle\mathscr{V}^\star_\varepsilon(a_\varepsilon);\varphi\rangle-\left\langle\mathscr{V}'_\varepsilon;\overline{\varphi}\right\rangle\ \to\ 0\ \text{ as }\varepsilon\downarrow 0.$$

Now, since $\chi^\star(1)=1$, we have $\langle\mathscr{V}'_\varepsilon;\overline{\varphi}\rangle=\langle\mathscr{V}'_\varepsilon;\varphi\rangle$ and we conclude that

$$\mathscr{V}^\star_\varepsilon(a_\varepsilon)-\mathscr{V}'_\varepsilon\ \to\ 0\quad\text{ weakly in }\mathscr{M}(\mathbf{R}^d\times S^{d-1})\text{ as }\varepsilon\downarrow 0.\tag{2.43}$$

As a consequence, the compactness and limit properties of $\{\mathscr{V}'_\varepsilon\}$ transfer to the family $\{\mathscr{V}^\star_\varepsilon(a_\varepsilon)\}$ and it is sufficient to study the former.

The Radon measures $\{\mathscr{V}'_\varepsilon\}$ are non negative and supported in $B_{R+C}\times S^{d-1}$. By the second estimate of Proposition 2.1.b, their total mass $\|\mathscr{V}'_\varepsilon\|(\mathbf{R}^d\times S^{d-1})=\langle\mathscr{V}'_\varepsilon;1\rangle$ is uniformly bounded. Thus, there exists a (not relabelled) sequence $\varepsilon\downarrow 0$ and an oriented $(d-1)$-varifold $\mathscr{V}_0$ such that up to extraction,

$$\mathscr{V}'_\varepsilon\ \to\ \mathscr{V}_0\quad\text{ in }\mathscr{M}(\mathbf{R}^d\times S^{d-1})\text{ as }\varepsilon\downarrow 0.$$

The estimates of Proposition 2.1.b show that the sequence of varifolds $\mathscr{V}'_\varepsilon=\mathscr{V}(\Sigma_\varepsilon,\nu_\varepsilon)$ satisfies the requirements of (1.3). There fore, $\mathscr{V}_0\in\mathscr{A}_0(R,S)$. Moreover, in the case $d=3$ and under Hypothesis 2, the surface $\Sigma_\varepsilon$ has genus smaller than $g_0$¿ By definition, we have $\mathscr{V}_0\in\mathscr{A}_{00}(R,S,g_0)$ in this case. This establishes parts (c) and (d) of the theorem.

We easily see that $\mathscr{V}_0$ is non negative and compactly supported in $\overline{B_R}\times S^{d-1}$ and that its total mass is given by

$$\|\mathscr{V}_0\|(\mathbf{R}^d\times S^{d-1})\ =\ \langle\mathscr{V}_0;1\rangle=\lim_{\varepsilon\downarrow 0}\left\langle\mathscr{V}'_\varepsilon;1\right\rangle\ \overset{(2.43)}{=}\ \lim_{\varepsilon\downarrow 0}\langle\mathscr{V}^\star_\varepsilon(a_\varepsilon);1\rangle\ =\ S.$$

The last identity follows from the second estimate of Proposition 2.1.a. Taking into account $\mathscr{V}^\star(a_\varepsilon)-\mathscr{V}'_\varepsilon\to 0$, we have established part (a) of the theorem.

Next, let us study the compactness properties of the family $T_\varepsilon=(\varepsilon-t_\varepsilon)/2\varepsilon$. These functions are compactly supported in $\overline{B_R}$ and by Proposition 2.1.a the sequence $(T_\varepsilon)$ is uniformly bounded in $BV(\mathbf{R}^d)$. Hence, up to a second extraction, we may assume that $(T_\varepsilon)$ (weakly) converges in $BV(\mathbf{R}^d)$ towards a function $T\in BV(\mathbf{R}^d)$ which is compactly supported in $\overline{B_R}$. Moreover, by Proposition 2.1.a, $\mathscr{H}^d(\Omega_\varepsilon)\sim S\varepsilon$ and since $\{y\in\mathbf{R}^d: T_\varepsilon(y)\notin\{0,1\}\}\subset\Omega_\varepsilon$, we have $T\in\{0,1\}$ almost everywhere in $\mathbf{R}^d$. Therefore, there exists a set of finite perimeter $O_0\subset\overline{B_R}$ such that

$$T_\varepsilon\ \to\ T=\mathbf{1}_{O_0}\text{ weakly in }BV(\mathbf{R}^d)\text{ as }\varepsilon\downarrow 0.$$

(For properties of BV functions and sets of finite perimeter, we refer to the books by Ambrosio, Fusco and Palara [3] or Evans and Gariepy [5]).
Let us now apply Proposition 2.1.c with $\varphi(y,n)=\psi(y)\cdot n$ for any vector field $\psi\in C_c(\mathbf{R}^d,\mathbf{R}^d)$. Passing to the limit $\varepsilon\downarrow 0$, we get

$$\int_{\mathbf{R}^d}\nabla\mathbf{1}_{O_0}\cdot\psi\ =\ -\langle\Lambda\mathscr{V}_0\,;\,\psi\rangle,\quad\text{for every }\psi\in C_c(\mathbf{R}^d,\mathbf{R}^d).$$

Hence, $\Lambda\mathscr{V}_0=-\nabla\mathbf{1}_{O_0}$. This identity uniquely defines the limit $T=\mathbf{1}_{O_0}$, so the second extraction was not necessary.
This establishes part (b) and ends the proof of Theorem 1.1.

## 3 Construction of a recovery family in the smooth case. Proof of Theorem 1.3

We consider an energy $\mathscr{F}(\sigma,\tau)=\int_{\tau\neq 0} f(\sigma,\tau)$ with stored energy function $f\in C(\mathbf{R}^d\times,\mathbf{R}^d,\mathbf{R}_+)$. We assume that $f$ satisfies (1.1) and (1.6), (1.7), namely: $f\equiv 0$ on the sphere $\mathbb{S}^{d-1}=\{(e,e): e\in S^{d-1}\}$, $f$ is of class $C^2$ in some neighborhood $\mathscr{N}$ of $\mathbb{S}^{d-1}$ in $\mathbf{R}^d\times\mathbf{R}^d$ and $f$ is invariant under change of orthonormal coordinates in this neighborhood, i.e. $f(Qv_1,Qv_2)=f(v_1,v_2)$ for every $(v_1,v_2)\in\mathscr{N}$ and every (orientation preserving) $Q\in\mathrm{SO}(d)$.

We begin first describe the structure of the Hessian matrix of $f$ at some point of $\mathbb{S}^{d-1}$.

**Lemma 3.1** *Let $e\in S^{d-1}$ and $b_1,\cdots,b_{d-1}\in S^{d-1}$ such that $(b_1,\cdots,b_{d-1},e)$ is an orthonormal basis of $\mathbf{R}^d$. We define an orthonormal basis of $\mathbf{R}^d\times\mathbf{R}^d$ as*

$$\mathscr{B} \;=\; \big((b_1,0),\cdots,(b_{d-1},0),(0,b_1),\cdots,(0,b_{d-1}),(e,0),(0,e)\big).$$

*Then there exists $a\geq 0$ and a $2\times 2$ non negative symmetric matrix $L$ which do not depend on $e$, such that, in the basis $\mathscr{B}$, the Hessian matrix of $f$ at point $(e,e)$ reads*

$$D^2 f(e,e) \;=\; \begin{pmatrix} a\,\mathbf{I}_{d-1} & -a\,\mathbf{I}_{d-1} & 0\\ -a\,\mathbf{I}_{d-1} & a\,\mathbf{I}_{d-1} & 0\\ 0 & 0 & L\end{pmatrix}. \tag{3.1}$$

*($\mathbf{I}_{d-1}$ denotes the identity matrix of size $(d-1)\times(d-1)$.)*

*Moreover, if (1.2) also holds ($f\geq\kappa f_0$ for some $\kappa>0$), then $a$ and $L$ are positive.*

*Proof* In the basis $\mathscr{B}$, we can write $D^2 f(e,e)$ on the form,

$$D^2 f(e,e) \;=\; \begin{pmatrix} A & B & M_1\\ B^T & C & M_2\\ M_1^T & M_2^T & L\end{pmatrix}, \tag{3.2}$$

where $A,B,C,M_1,M_2,L$ are real matrices, $A,B,C\in\mathscr{M}_{d-1,d-1}(\mathbf{R})$, $M_1,M_2\in\mathscr{M}_{d-1,2}(\mathbf{R})$, $L\in\mathscr{M}_{2,2}(\mathbf{R})$ and $A$, $C$ and $L$ are symmetric.

Let us consider the action of rotations with axis $e$ : for every $Q\in\mathrm{SO}(d-1)$ and every $x,y\in e^\perp=\mathbf{R}^{d-1}\subset\mathbf{R}^d$, we have $f((e+Qx),(e+Qy))=f((e+x),(e+y))$, which yields,

$$D^2 f(e,e) \;=\; \begin{pmatrix} Q^TAQ & Q^TBQ & M_1Q\\ Q^TB^TQ & Q^TCQ & M_2Q\\ (M_1Q)^T & (M_2Q)^T & L\end{pmatrix}\qquad \text{for every } Q\in\mathrm{SO}(d-1).$$

Identifying this expression with (3.2), we see that $M_1=M_2=0$ and that for $K=A,B,C$ we have

$$Q^TKQ \;=\; K \qquad \text{for every } Q\in\mathrm{SO}(d-1), \tag{3.3}$$

Let us split $B$ into its symmetric and skew-symmetric parts: $B=B_{sy}+B_{sk}$, so that (3.3) also holds for $K=B_{sy}$ and $K=B_{sk}$. Since $A$, $B_{sy}$ and $C$ are symmetric, hence diagonal in some orthonormal basis, this implies that these operators are diagonal in any orthonormal basis. Hence, there exists $a,b,c\in\mathbf{R}$ such that

$$A \;=\; a\,\mathbf{I}_{d-1},\qquad B_{sy} \;=\; b\,\mathbf{I}_{d-1},\qquad C \;=\; c\,\mathbf{I}_{d-1}.$$

At this point, in the basis $\mathscr{B}$, $D^2 f(e,e)$ writes

$$D^2 f(e,e) \;=\; \begin{pmatrix} a\,\mathbf{I}_{d-1} & b\,\mathbf{I}_{d-1}+B_{sk} & 0\\ b\,\mathbf{I}_{d-1}-B_{sk} & c\,\mathbf{I}_{d-1} & 0\\ 0 & 0 & L\end{pmatrix}.$$

Now, for a fixed $z \in S^{d-1} \cap e^\perp$, we observe that $d((e,e)+r(z,z),\mathbb{S}^{d-1}) = O(r^2)$. Since $f$ is minimal on $\mathbb{S}^{d-1}$, we have $f((e,e)+r(z,z)) = O(r^4)$ and the second order derivative of $r \mapsto f((e,e)+r(z,z))$ vanishes at $r=0$. From the last expression for $D^2 f(e,e)$, this second order derivative equals $a|z|^2+b|z|^2+z^T B_{sk} z + b|z|^2 - z^T B_{sk} z + c|z|^2 = a+2b+c$. Hence

$$b \;=\; -(a+c)/2.$$

Next, the space $T = \mathrm{span}\,\{(e,0)\,,(0,e)\}^\perp$ is stable under the action of the symmetric operator $D^2 f(e,e)$. Since this operator is non negative, for every $v \in T$ we have $v^T D^2 f(e,e) v \geq 0$ . Choosing $v = (z,\lambda z')$ with $z,z' \in e^\perp$, this inequality reads

$$c|z'|^2\lambda^2 + 2\left(bz\cdot z' + z^T B_{sk} z'\right)\lambda + a|z|^2 \;\geq\; 0 \qquad \text{for every } \lambda \in \mathbf{R}. \tag{3.4}$$

Choosing $z = z' \in S^{d-1}\cap e^\perp$, the term $z^T B_{sk} z'$ vanishes and we end up with the condition $c\lambda^2 + 2b\lambda + a \geq 0$ for every $\lambda \in \mathbf{R}$. This leads to $b^2 \leq ac$ which together with the identity $b = -(a+c)/2$ yields

$$a \;=\; c \;=\; -b.$$

Let us now establish that $B_{sk}=0$. Since $B_{sk} \in \mathscr{M}_{d-1,d-1}(\mathbf{R})$ is skew-symmetric, we have $B_{sk}=0$ if $d=2$. Now, we assume $d \geq 3$ and we apply (3.4) with $\lambda = 1$, $z \in S^{d-1}\cap e^\perp$ and $z' = z+ry$ for some $y \in e^\perp \cap z^\perp$. We deduce $z^T B_{sk} y = O(r)$. Hence, $z^T B_{sk} y = 0$ for every $y \in e^\perp \cap z^\perp$ which yields the conclusion:

$$B_{sk} = 0.$$

We have established that $D^2 f(e,e)$ has the form stated in the lemma. The non-negativity of $a$ and $L$ follow from that of $D^2 f(e,e)$. Eventually, assuming $f \geq \kappa f_0$ for some $\kappa > 0$, we obtain $D^2 f(e,e) \geq \kappa D^2 f_0(e,e)$ which easily yields $a>0$ and $L$ positive definite. □

Now let us establish Theorem 1.3. Let $O_0 \subset \mathbf{R}^d$ be a smooth bounded open subset of $\mathbf{R}^d$, let $\nu_0$ the outward unit normal to $O_0$ and let $\Sigma_0 := \partial O_0$.
We introduce the mapping

$$\psi : \Sigma_0 \times \mathbf{R} \to \mathbf{R}^d, \qquad \psi(x,s) \;:=\; x + s\nu_0(x).$$

There exists $\varepsilon_\star > 0$ such that $\psi$ is a smooth (and bi-Lipschitz) diffeomorphism from $\Sigma_0 \times (-\varepsilon_\star, \varepsilon_\star)$ onto its range $\Omega_\star$. The inverse mapping is given by $\Psi^{-1} = (\pi_0, Z)$ where $\pi_0$ is the orthogonal projection on $\Sigma_0$ and where $Z(y) := d(y,O_0) - d(y,\mathbf{R}^d\setminus O_0)$ is the signed distance function to $\Sigma_0$.
Let us extend $\nu_0$ as a mapping $n_0 : \Omega_\star \to S^{d-1}$ by $n_0(y) = \nu_0(\pi_0(y)) = \nabla Z(y)$.
We are going to build a quasi optimal family $\{a_\varepsilon\} = \{(\sigma_\varepsilon, \nabla t_\varepsilon, \Omega_\varepsilon)\}$ such that $\sigma_\varepsilon$ and $\nabla t_\varepsilon$ are the restrictions to $\Omega_\varepsilon$ of vector fields $\sigma$ and $\nabla t$ defined on $\Omega_\star$.

For symmetry reasons, we look for vector fields $\sigma$ and $\nabla t$ which are collinear to $n_0$ in $\Omega_\star$ and equal to $\nu_0$ on $\Sigma_0$. Under this ansatz the only possibility for the divergence free vector field $\sigma : \Omega_\star \to \mathbf{R}^d$ is to set

$$\sigma(y) \;:=\; \left(\det\left[\mathbf{I}_d + Z(y) D\nu_0(\pi_0(y))\right]\right)^{-1} n_0(y) \qquad \text{for every } y \in \Omega_\star. \tag{3.5}$$

Let us check that $\sigma$ is divergence free in $\Omega_\star$. Equivalently, we have to show that $\int_{\Omega_\star} \nabla\varphi\cdot\sigma$ vanishes for every $\varphi \in \mathscr{D}(\Omega_\star)$. We perform the change of variable $y = \Psi(x,s)$. The Jacobian determinant of $\Psi$ is $J_\Psi(x,s) = \det\left[\mathbf{I}_d + sD\nu_0(x)\right]$ and we indeed obtain,

$$\int_{\Omega_\star} \nabla\varphi(y)\cdot\sigma(y)\,dy \;=\; \int_{\Sigma_0}\left\{\int_{-\varepsilon_\star}^{\varepsilon_\star} [\nu_0(x)\cdot\nabla]\varphi(x+s\nu_0(x))\,ds\right\} d\mathscr{H}^{d-1}(x) \;=\; 0.$$

Then we set

$$t(y) \;:=\; \left(1+\frac{\alpha(\pi_0(y))}{2}Z(y)\right)Z(y),$$

where $\alpha \in C^1(\Sigma_0)$ has to be optimized. Eventually, for $\varepsilon \in (0,\varepsilon_\star)$, we define $\Omega_\varepsilon := \Sigma_0 + B_\varepsilon = \Psi(\Sigma_0 \times (-\varepsilon,\varepsilon))$,

$$\sigma_\varepsilon(y) \;:=\; \begin{cases} \sigma(y) & \text{if } y \in \Omega_\varepsilon, \\ 0 & \text{if } y \in \mathbf{R}^d \setminus \Omega_\varepsilon, \end{cases} \qquad t_\varepsilon(y) \;:=\; \begin{cases} t(y) & \text{if } x \in \Omega_\varepsilon, \\ \pm\varepsilon & \text{if } \pm t(y) \geq \varepsilon, \end{cases}$$

By construction, we have $a_\varepsilon := (\sigma_\varepsilon, \nabla t_\varepsilon, \Omega_\varepsilon) \in \mathscr{A}_\varepsilon(R)$ for some large $R>0$. Writing $y = \Psi(x,s)$, we compute the expansions

$$\begin{aligned} \sigma_\varepsilon(x+s\nu_0(x)) &= \nu_0(x) - s\,h_0(x)\nu_0(x) + O(s^2), \\ \nabla t_\varepsilon(x+s\nu_0(x)) &= \nu_0(x) + s\alpha(x)\nu_0(x) + O(s^2), \end{aligned}$$

where $h_0(x) = \nabla\cdot n_0(x)$ is the scalar mean curvature of $\Sigma_0$. Taking into account Lemma 3.1, we have for every $y = \Psi(x,s) \in \Omega_\varepsilon$,

$$f(\sigma_\varepsilon(y), \nabla t_\varepsilon(y)) \;=\; \frac{s^2}{2}(-h_0(x),\alpha(x))^T L(-h_0(x),\alpha(x)) + o(s^2).$$

Optimizing with respect to $\alpha(x)$, we get $\alpha(x) = (L_{1,2}/L_{2,2})h_0(x)$ which yields

$$f(\sigma_\varepsilon(y), \nabla t_\varepsilon(y)) \;=\; h_0^2(x)\frac{\det L}{2L_{2,2}}\,s^2 + o(s^2).$$

The energy of $a_\varepsilon$ then expands as

$$\mathscr{F}(a_\varepsilon) \;=\; \frac{\det L}{3L_{2,2}}\mathscr{W}(\Sigma_0)\,\varepsilon^3 + o(\varepsilon^3).$$

This is the expected expansion, however, the volume constraint is not exactly satisfied. We only have

$$\mathscr{Q}(\sigma_\varepsilon, \nabla t_\varepsilon) \;=\; \int_{\Omega_\varepsilon} \sigma\cdot\nabla t \;=\; 2\varepsilon\mathscr{H}^{d-1}(\Sigma_0) + O(\varepsilon^2).$$

We recover the exact constraint $\mathscr{Q}(\sigma'_\varepsilon, \nabla t'_\varepsilon) = 2\varepsilon\mathscr{H}^{d-1}(\Sigma_0)$ by substituting $a'_\varepsilon = (\sigma'_\varepsilon, \nabla t'_\varepsilon, \Omega'_\varepsilon)$ for $a_\varepsilon$ with

$$\sigma'_\varepsilon(y) \;:=\; \sigma_\varepsilon((1/\lambda_\varepsilon)y), \qquad t'_\varepsilon(y) \;:=\; \lambda_\varepsilon t_\varepsilon((1/\lambda_\varepsilon)y), \qquad \Omega'_\varepsilon \;:=\; \lambda_\varepsilon\Omega_\varepsilon,$$

where the magnification factor is defined as

$$\lambda_\varepsilon \;:=\; \left[\frac{\mathscr{Q}(\sigma_\varepsilon, \nabla t_\varepsilon)}{2\varepsilon\mathscr{H}^{d-1}(\Sigma)}\right]^{1/d} \;=\; 1 + O(\varepsilon).$$

The convergence properties of $\{\mathscr{V}^\star_\varepsilon(a'_\varepsilon)\}$ towards $\mathscr{V}(\Sigma_0,\nu_0)$ and of the sets $\{[t'_\varepsilon = -\varepsilon]\}$ towards $O_0$ are straightforward by construction. This ends the proof of Theorem 1.3.

## 4 Oriented varifolds with $L^2$-generalized mean curvature

In order to prepare the proof of Theorem 1.2, we gather here well known facts about oriented integer varifolds. In particular, we extend the Willmore functional to the elements of $\mathscr{A}_0$.

Let us first introduce a suitable generalization of Definition 1.1.b extending the notion of hypersurface.

**Definition 4.1 (oriented integer rectifiable $(d-1)$-varifolds)**
a) Given:

i. a $(d-1)$ rectifiable set $\Sigma \subset \mathbf{R}^d$, that is:
- $\mathscr{H}^{d-1}(\Sigma) < \infty$ and $\Sigma$ is $\mathscr{H}^{d-1}$-measurable,
- there exists a countable set of Lipschitz continuous functions $L_k : \mathbf{R}^{d-1} \to \mathbf{R}^d$, such that $\mathscr{H}^{d-1}(\Sigma \setminus \cup_k L_k(\mathbf{R}^{d-1})) = 0$;

ii. two positive integer valued $\mathscr{H}^{d-1}$-integrable functions $\theta^-, \theta^+ : \Sigma \to \mathbf{N} \setminus \{0\}$.

We define the oriented $(d-1)$-varifold $\mathscr{V} = \mathscr{V}(\Sigma, \nu, \theta^+, \theta^-)$ by

$$\langle \mathscr{V} ; \varphi \rangle \ := \ \int_\Sigma \left\{ \theta^+(x)\varphi(x,\nu(x)) + \theta^-(x)\varphi(x,-\nu(x)) \right\} d\mathscr{H}^{d-1}(x).$$

b) The set of all such oriented *integer rectifiable* $(d-1)$-varifolds is denoted by $\mathbf{IV}^o$. In the sequel, we only write *integer varifold* for oriented integer rectifiable $(d-1)$-varifold. We also introduce the set

$$\mathbf{IV}^o_0 \ := \ \left\{ \mathscr{V} \in \mathbf{IV}^o : \Lambda\mathscr{V} = \nabla U \text{ for some } U \in \mathscr{D}'(\mathbf{R}^d) \right\}.$$

Equivalently, this set is formed by integer varifolds whose associated current has vanishing boundary (see Remark 1.1.b).

*Remark 4.1*
*i.* The total mass of $\mathscr{V} = \mathscr{V}(\Sigma, \nu, \theta^+, \theta^-)$ is $\langle \nu \ ; \ \mathbf{1} \rangle = \int_\Sigma (\theta^+ + \theta^-)\, d\mathscr{H}^{d-1}$ and $\Lambda\mathscr{V} = (\theta^+ - \theta^-)\nu\mathscr{H}^{d-1} \llcorner \Sigma$.

*ii.* The functions $\theta^+, \theta^-$ account for multiplicity: the above formula means that the generalized surface pass at point $x \in \Sigma$, $\theta^+$ times with orientation $\nu(x)$ and $\theta^-(x)$ times with opposite orientation.

*iii.* By continuity of the mapping $\Lambda : \mathscr{M}(\mathbf{R}^d \times S^{d-1}) \to \mathscr{D}'(\mathbf{R}^d)^d$, the space $\{\mathscr{V} \in \mathscr{M}(\mathbf{R}^d \times S^{d-1}) : \Lambda\mathscr{V} \equiv 0\}$ is closed in $\mathscr{M}(\mathbf{R}^d \times S^{d-1})$.

Another important tool in the context of area minimization is the *first variation* $\delta\mathscr{V}(X)$ of a $(d-1)$-varifold. If $\mathscr{V} = \mathscr{V}(\Sigma, \nu)$ is the oriented $(d-1)$ varifold associated to a smooth hypersurface $\Sigma = \partial O$, this quantity describes the initial rate of change of the total mass of $\mathscr{V}$ under the flow generated by $X \in \mathscr{D}(\mathbf{R}^d, \mathbf{R}^d)$. It is defined as

$$\delta\mathscr{V}(X) \ := \ \int_\Sigma \operatorname{div}_\Sigma X \, d\mathscr{H}^{d-1}, \tag{4.1}$$

Recall that for every $x \in \Sigma$, if $(e_1, \cdots, e_{d-1})$ denotes an orthonormal basis of $\nu(x)^\perp$, we have

$$\operatorname{div}_\Sigma X(x) \ = \ \sum_{i=1}^{d-1} e_i^T \nabla X(x) e_i.$$

The right hand side of (4.1) naturally extends as a linear continuous functional defined on the space of $(d-1)$-varifolds: for every $\mathscr{V} \in \mathscr{M}(\mathbf{R}^d \times S^{d-1})$, we set

$$\delta\mathscr{V}(X) := \langle \mathscr{V}\,;(y,n) \mapsto \mathrm{div}_{n^\perp} X(y)\rangle, \quad \text{for every } X \in \mathscr{D}(\mathbf{R}^d,\mathbf{R}^d).$$

In general, $\delta\mathscr{V}$ is a distribution of $\mathscr{D}'(\mathbf{R}^d)^d$. However, if $\Sigma \subset \mathbf{R}^d$ is a smooth $(d-1)$-submanifold with tangent hyperplane $\nu^\perp(x)$, then the first variation of $\mathscr{V} = \mathscr{V}(\Sigma,\nu)$ can be related to the vectorial mean curvature $H : \Sigma \to \mathbf{R}^d$ defined by $H = h\nu = (\nabla\cdot\nu)\nu$. The vector field $H$ only depends on the local geometry of $\Sigma$: it does not depend on the local orientation $\pm\nu$. With this notation, the tangential Green formula reads,

$$\int_\Sigma H\cdot X\, d\mathscr{H}^{d-1} = \int_\Sigma \mathrm{div}_\Sigma X\, d\mathscr{H}^{d-1} \text{ for every } X \in \mathscr{D}(\mathbf{R}^d,\mathbf{R}^d). \tag{4.2}$$

In particular, the first variation of $\mathscr{V}$ is a measure given by

$$\delta\mathscr{V}(X) = \int_\Sigma H\cdot X\, d\mathscr{H}^{d-1}. \tag{4.3}$$

Now let $R,S>0$ and let us consider an element $\mathscr{V}_0 \in \mathscr{A}_0(R,S)$ and an associated sequence of varifolds $\mathscr{V}_k = \mathscr{V}(\Sigma_k,\nu_k)$ complying to (1.5). The estimates of (1.5) and the Cauchy-Schwarz inequality easily imply

$$|\delta\mathscr{V}_k(X)| \le K\|X\|_\infty \text{ for every } X \in \mathscr{D}(\mathbf{R}^d,\mathbf{R}^d),$$

for some constant $K \ge 0$. By the Riesz representation theorem, this implies $\delta\mathscr{V}_k \in \mathscr{M}(\mathbf{R}^d)^d$ for any $k \ge 1$ with the uniform bound

$$\|\delta\mathscr{V}_k\|(\mathbf{R}^d) \le K. \tag{4.4}$$

We can now state a compactness result due to Hutchinson [10]. It extends Allard's compactness theorem for (non-oriented) integer varifolds [1] to oriented integer varifolds.

**Theorem 4.1 ([10] Theorem 3.1.[1])** *Let $K>0$ and let $E_K$ be the set of varifolds $\mathscr{V} \in \mathbf{IV}_0^o$ such that $\delta\mathscr{V}$ is a Radon measure of $\mathscr{M}(\mathbf{R}^d)^d$ and*

$$\|\mathscr{V}\|(\mathbf{R}^d \times S^{d-1}) + \|\delta\mathscr{V}\|(\mathbf{R}^d) \le K.$$

*Then $E_K$ is compact in $\mathscr{M}(\mathbf{R}^d \times S^{d-1})$.*

We can apply this result to the sequence $(\mathscr{V}_k)$ and deduce,

$$\mathscr{V}_0 = \mathscr{V}(\Sigma_0, n_0, \theta_0^+, \theta_0^-) \in \mathbf{IV}_0^o. \tag{4.5}$$

We now have to prove that $\mathscr{V}_0$ admits a $L^2$-generalized mean curvature. Let us introduce this notion.

**Definition 4.2 ($L^2$-generalized mean curvature. Willmore energy of a varifold)**
a) Let $\mathscr{V} \in \mathscr{M}(\mathbf{R}^d \times S^{d-1})$ and let $\pi\mathscr{V} \in \mathscr{M}(\mathbf{R}^d)$ be the pushforward of $\mathscr{V}$ by the mapping $(y,n) \in \mathbf{R}^d \times S^{d-1} \mapsto y \in \mathbf{R}^d$, that is $[\pi\mathscr{V}](E) = \mathscr{V}(E \times S^{d-1})$ for every Borel set $E \subset \mathbf{R}^d$.
We say that $\mathscr{V}$ admits a $L^2$-generalized curvature, if there exists $K \ge 0$ such that

$$\delta\mathscr{V}(X) \le K\|X\|_{L^2(\mathbf{R}^d,\pi\mathscr{V})} \quad \text{for every } X \in C_c^1(\mathbf{R}^d)^d.$$

[1] We only use the special case of oriented integer varifolds *without* boundary

Since $\pi\mathscr{V}$ is a Radon measure, it is regular and the space $C^1_c(\mathbf{R}^d)^d$ is dense in $L^2(\mathbf{R}^d,|\mathscr{V}|)^d$. Consequently, if we assume the above bound, we can apply the (Hilbert space) Riesz representation theorem to deduce the existence of a $\pi\mathscr{V}$-measurable mapping $H : \mathrm{supp}(\pi\mathscr{V}) \to \mathbf{R}^d$, such that

$$\delta\mathscr{V}(X) \;=\; \int_{\mathbf{R}^d} H\cdot X\, d[\pi\mathscr{V}] \qquad \text{for every } X \in C^1_c(\mathbf{R}^d)^d.$$

This formula extends (4.3) and we will say that $H$ is the ($L^2$-)*generalized (vectorial) mean curvature* of $\mathscr{V}$.

b) If $\mathscr{V} \in \mathscr{M}(\mathbf{R}^d \times S^{d-1})$ admits a $L^2$-generalized mean curvature $H$, we define its Willmore energy as

$$\mathscr{W}(\mathscr{V}) \;:=\; \int_{\mathbf{R}^d} |H|^2\, d[\pi\mathscr{V}].$$

If $\mathscr{V}$ does not admit a $L^2$-generalized mean curvature, we set $\mathscr{W}(\mathscr{V}) = +\infty$.

*Remark 4.2*

a) Of course if $\mathscr{V} = \mathscr{V}(\Sigma,\nu)$, where $\Sigma$ is a smooth compact surface oriented by $\nu$, then it admits a generalized vectorial mean curvature which matches the usual definition $H = (\mathrm{div}_\Sigma\,\nu)\nu$. Moreover, $[\pi\mathscr{V}] = \mathscr{H}^{d-1}\llcorner\Sigma$ so $\mathscr{W}(\mathscr{V}(\Sigma,\nu)) = \mathscr{W}(\Sigma)$.

b) For $\mathscr{V} = \mathscr{V}(\Sigma,\nu,\theta^+,\theta^-) \in \mathbf{IV}^o$, then $[\pi\mathscr{V}] = (\theta^+ + \theta^-)\mathscr{H}^{d-1}\llcorner\Sigma$. If $\mathscr{V}$ admits a $L^2$-generalized mean curvature $H : \Sigma \to \mathbf{R}^d$, then $H$ is $\mathscr{H}^{d-1}$-measurable and we have,

$$\mathscr{W}(\mathscr{V}) \;=\; \int_\Sigma |H|^2(\theta^+ + \theta^-)\, d\mathscr{H}^{d-1}.$$

c) If a rectifiable varifold $\mathscr{V} = \mathscr{V}(\Sigma,\nu,\theta^+,\theta^-)$ admits a generalized mean curvature $H$, then $H$ is $[\pi\mathscr{V}]$-almost everywhere collinear to $\nu$ (see K. Brakke [4]).

It is easily seen that the extension of the Willmore energy defined above is lower semi-continuous with respect to the convergence of Radon measures. Indeed, let $(\mathscr{V}_k) \subset \mathscr{M}(\mathbf{R}^d \times S^{d-1})$ converging to $\mathscr{V}$ and assume without loss of generality that $\liminf W(\mathscr{V}_k) < \infty$. Using the Cauchy-Schwarz inequality, we have, with obvious notation: for every $X \in C^1_c(\mathbf{R}^d)^d$,

$$\delta\mathscr{V}_k(X) = \langle \pi\mathscr{V}_k\,;\, H_k\cdot X\rangle \;\le\; \sqrt{\mathscr{W}(\mathscr{V}_k)}\|X\|_{L^2(\mathbf{R}^d,\pi\mathscr{V}_k)} \;=\; \sqrt{\mathscr{W}(\mathscr{V}_k)}\sqrt{\langle \pi\mathscr{V}_k; |X|^2\rangle}.$$

By the convergence $\mathscr{V}_k \to \mathscr{V}$ as $k\uparrow\infty$, the left hand side converges towards $\delta\mathscr{V}(X)$ and $\|X\|_{L^2(\mathbf{R}^d,\pi\mathscr{V}_k)} \to \|X\|_{L^2(\mathbf{R}^d,\pi\mathscr{V})}$. So,

$$\delta\mathscr{V}(X) \;\le\; \sqrt{\liminf \mathscr{W}(\mathscr{V}_k)}\;\|X\|_{L^2(\mathbf{R}^d,\pi\mathscr{V})} \tag{4.6}$$

Hence, $\mathscr{V}$ admits a $L^2$-generalized vectorial mean curvature $H \in L^2(\mathbf{R}^d,\pi\mathscr{V})^d$. Now, by density of $C^1_c(\mathbf{R}^d)^d$ in $L^2(\mathbf{R}^d,\pi\mathscr{V})^d$, we have

$$\sqrt{\mathscr{W}(\mathscr{V})} \;=\; \|H\|_{L^2(\mathbf{R}^d,\pi\mathscr{V})} \;=\; \sup\langle\mathscr{V}; X\cdot H\rangle = \;\sup\delta\mathscr{V}(X),$$

where the suprema are taken over all $X \in C^1_c(\mathbf{R}^d)$ such that $\|X\|_{L^2(\mathbf{R}^d,\pi\mathscr{V})} \le 1$. With (4.6), this yields

$$\mathscr{W}(\mathscr{V}) \;\le\; \liminf_{k\uparrow\infty} \mathscr{W}(\mathscr{V}_k), \tag{4.7}$$

as claimed.

In our context, if $\mathscr{V}_0 \in \mathscr{A}_0$, then it is the limit of a sequence $\mathscr{V}_k = \mathscr{V}(\Sigma_k,\nu_k)$ of smooth oriented integer varifolds with uniformly bounded Willmore energy. We conclude that $\mathscr{V}_0 =$

$\mathscr{V}(\Sigma_0, \nu_0, \theta_0^+, \theta_0^-) \in \mathbf{IV}_0^o$ admits a $L^2$-generalized mean curvature $H_0$, the formula of Remark 4.2.b provides a more explicit expression for $\mathscr{W}(\mathscr{V}_0)$:

$$\mathscr{W}(\mathscr{V}_0) \;=\; \int_{\Sigma_0} |H_0|^2 \,(\theta_0^+ + \theta_0^-) d\mathscr{H}^{d-1}. \tag{4.8}$$

## 5 Lower bound. Proof of Theorem 1.2

In this Section, we assume that the stored energy function $f$ satisfies (1.1), (1.2) and (1.6), (1.7) and we consider an oriented $(d-1)$-varifold $\mathscr{V}_0$ and a sequence $(a_{\varepsilon_k})_{k\geq 0}$ with $\varepsilon_k \downarrow 0$ such that $a_{\varepsilon_k} \in \mathscr{A}_{\varepsilon_k}(R,S)$ for some $R,S > 0$ and $\mathscr{V}_{\varepsilon_k}(a_{\varepsilon_k}) \to \mathscr{V}_0$ as $\varepsilon_k \downarrow 0$.
The lower bound result being obvious in the case $\liminf_k \mathscr{F}(a_{\varepsilon_k})/\varepsilon_k^3 = +\infty$, we assume without loss of generality, that there exists $E_0 > 0$ such that

$$\mathscr{F}_0(a_{\varepsilon_k}) \;\leq\; E_0\, \varepsilon_k^3, \qquad \text{for every } k \geq 0.$$

Consequently, the constructions and estimates of Proposition 2.1 and Theorem 1.1 apply to the sequence $\{a_{\varepsilon_k}\}$ and $\mathscr{V}_0$. In particular $\mathscr{V}_0 \in \mathscr{A}_0(R,S)$ and by Section 4 $\mathscr{V}_0 = \mathscr{V}(\Sigma_0, \nu_0, \theta_0^+, \theta_0^-)$ is an oriented integer rectifiable $(d-1)$-varifold which admits a $L^2$-generalized mean curvature. To lighten notation, we drop the subscripts $k$ in the sequence $\varepsilon_k$. We use the notation of Section 2 for the objects constructed along the proof of Proposition 2.1: $u_\varepsilon$, $O_\varepsilon$, $\Sigma_\varepsilon$, $\nu_\varepsilon$ and the corresponding rescaled objects $u_{(\varepsilon)}$, $O_{(\varepsilon)}$, $\Sigma_{(\varepsilon)}$, $\nu_{(\varepsilon)}$, *etc.*

By lower semi-continuity of the Willmore energy (see (4.6)) we have

$$\mathscr{W}(\mathscr{V}_0) \;\leq\; \liminf_{\varepsilon\downarrow 0} \mathscr{W}(\Sigma_\varepsilon).$$

However, there is little hope for deducing the lower bound of Theorem 1.2 from this inequality. Indeed, the mean curvature $h_\varepsilon$ of $\Sigma_\varepsilon$ is loosely and non locally related to the data $a_\varepsilon$ through the harmonic function $u_\varepsilon$. Remark that $u_\varepsilon$ does not even depend on $\sigma_\varepsilon$. In short (and omitting cut-off issues) our strategy is the following.

*i.* In Section 5.1, we define an *approximate mean curvature* $\hat{h}_\varepsilon(x)$ as the dot product of $\sigma_\varepsilon$ with a particular test function supported in the cylinder $x + \varepsilon D^{2\xi}_{1-2\xi}(\nu(x))$, where $\xi \in (0, 1/4)$ is a small parameter. We also establish that $\hat{h}_\varepsilon$ is indeed an approximation of $h_\varepsilon$ in a weak sense. We deduce the inequality

$$\mathscr{W}(\mathscr{V}_0) \;\leq\; \liminf_{\varepsilon\downarrow 0} \int_{\Sigma_\varepsilon} |\hat{h}_\varepsilon|^2 \, d\mathscr{H}^{d-1}. \tag{5.1}$$

*ii.* In Section 5.2, we introduce local minimization problems of the form

$$c(\eta, \xi, f) \;:=\; \inf\left[\mathscr{F}\left(\sigma^\#, \nabla t^\#, D^{2\xi}_{1-2\xi}(e_d)\right) / \left|\hat{h}^\#\right|^2\right],$$

where the infimum ranges over all admissible vector fields $\sigma^\#$, $\nabla t^\#$ with *approximate mean curvature* $\hat{h}^\# \in (-\eta, \eta) \setminus \{0\}$ at $x = 0$. With this notation, we deduce from (5.1)

$$c(\eta, \xi, f)\, \mathscr{W}(\mathscr{V}_0) \;\leq\; \liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^3} \int_{\Sigma_\varepsilon \cap [|\hat{h}_\varepsilon| < \eta/\varepsilon]} \mathscr{F}\left(\sigma_\varepsilon(x), \nabla t_\varepsilon(x), x + \varepsilon D^{2\xi}_{1-2\xi}(\nu_\varepsilon(x))\right) d\mathscr{H}^{d-1}(x).$$

We then show that the above integral is bounded by $[1+s(\eta)]\mathscr{H}^{d-1}(D_1')\mathscr{F}(a_\varepsilon)$ with $s(\eta) \to 0$ as $\eta \downarrow 0$ and get

$$c(\eta, \xi, f)\, \mathscr{W}(\mathscr{V}_0) \;\leq\; [1+s(\eta)]\mathscr{H}^{d-1}(D_1') \liminf_{\varepsilon\downarrow 0} \frac{\mathscr{F}(a_\varepsilon)}{\varepsilon^3}.$$

Eventually we study these local problems and establish

$$\liminf_{\xi\downarrow 0}\liminf_{\eta\downarrow 0} c(\eta,\xi,f) \;\geq\; \mathscr{H}^{d-1}(D_1')c_0(f),$$

which will lead to the desired lower bound.

The above description leaves aside the following technical difficulties: 1/ we are only able to define the approximate mean curvature in a good set, so we need to perform a cut-off and control the volume of the bad set; 2/ we need also to control the $(d-1)$-volumes of the sets $\Sigma_\varepsilon\cap[|\hat{h}_\varepsilon|\geq\eta/\varepsilon]$; 3/ for compactness issues we introduce a small parameter $\alpha>0$ and (with obvious notation), we substitute

$$\mathscr{F}\left(\sigma^\#,\nabla t^\#,D^{2\xi}_{1-2\xi}\right)+\alpha\int_{B_3\cap\Omega^\#}\left[f_0(\sigma^\#,\nabla t^\#)+|\nabla u^\#-\nabla t^\#|^2\right]$$

for $\mathscr{F}(\sigma^\#,\nabla t^\#,D^{2\xi}_{1-2\xi})$ in the local optimization problem.

## 5.1 Definition and properties of the approximate mean curvature.

We already know that $\sigma_\varepsilon$ approximates $\nu_\varepsilon$, so a naive way to build an approximation of the total curvature on $\Sigma_\varepsilon$ by means of the data $(\sigma_\varepsilon,\nabla t_\varepsilon)$ is to set $\hat{h}_\varepsilon(x):=\operatorname{div}_{\Sigma_\varepsilon}\sigma_\varepsilon(x)$. However, the regularity of $\sigma_\varepsilon$ is too weak to provide a robust definition under sole energy bounds. The next idea consists in using a weak notion for the mean curvature where, thanks to integration by parts, the space derivatives on $\sigma_\varepsilon$ are transferred to a test function. The relevant tool for this is the tangential Green formula.

Let us consider a smooth compact hypersurface $\Gamma\subset\mathbf{R}^d$ with normal $\nu$ and mean curvature $h$. Let $X\in C^1(\Gamma,\mathbf{R}^d)$ of the form $X(x)=\varphi(x)\nu(x)$ and let us extend it as a mapping $X\in C^1_c(\mathbf{R}^d,\mathbf{R}^d)$. For such a test vector field, the tangential Green formula (4.2) reads,

$$\int_\Gamma \varphi\, h\,\nu\, d\mathscr{H}^{d-1} \;=\; \int_\Gamma \nabla_\Gamma\varphi\, d\mathscr{H}^{d-1} \qquad \text{for every } \varphi\in\mathscr{D}(\mathbf{R}^d). \tag{5.2}$$

Assume for simplicity that $0\in\Gamma$ with $\nu(0)=e_d$ and let us localize the above identity around 0. For this, we introduce a function of the form

$$\zeta(y',y_d) \;:=\; \chi_{/\!/}(|y'|)\chi_\perp(y_d),$$

with $\chi_{/\!/},\chi_\perp\in\mathscr{D}(\mathbf{R},\mathbf{R}_+)$ even and such that $\chi_\perp(0)\int_{\mathbf{R}^{d-1}}\chi_{/\!/}(|y'|)\,dy'=1$. Using the test function $\varphi(y)=\eta^{1-d}\zeta((1/\eta)y)$ in (5.2), taking the scalar product with $\nu(0)=e_d$ and sending $\eta$ to 0, we obtain after some Taylor expansions,

$$h(0) \;=\; -\lim_{\eta\downarrow 0}\frac{1}{\eta}\int_{\Gamma_{(\eta)}}\chi_\perp(z_d)\,\nu_{(\eta)}(z)\cdot\nabla\chi_{/\!/}(z')\,d\mathscr{H}^{d-1}(z).$$

where $\Gamma_{(\eta)}$ is the hypersurface $(1/\eta)\Gamma$ and $\nu_{(\eta)}(z):=\nu(\eta z)$.

We mimic this formula for defining an approximate mean curvature on $\Sigma_\varepsilon$. For this, we choose $\eta=\varepsilon$ and we substitute the vector field $\sigma_{(\varepsilon)}$ for the outward unit normal $\nu_{(\eta)}=\nu_{(\varepsilon)}$. This substitution closely connects the approximate curvature to the data. We also thicken the domain of integration by substituting the cylinder $D^0_1\subset\mathbf{R}^d$ for the piece of hypersurface $\Sigma_{(\varepsilon)}\cap\operatorname{supp}\zeta$. The precise construction is detailed below.

**Definition 5.1** Let $\chi_{/\!/}, \chi_\perp \in W^{1,\infty}(\mathbf{R},\mathbf{R}_+)$ be two nonnegative, even cut-off functions which are compactly supported in $(-1,1)$ and satisfy

$$\left(\int_{\mathbf{R}^{d-1}} \chi_{/\!/}(|y'|)\, d\mathscr{H}^{d-1}(y')\right)\left(\int_{\mathbf{R}} \chi_\perp(s)\, ds\right) \;=\; 1.$$

We define a Lipschitz test function $\zeta \in W^{1,\infty}(\mathbf{R}^d \times S^{d-1},\mathbf{R}_+)$, by setting

$$\zeta \;:=\; \zeta_{/\!/}\zeta_\perp \qquad \text{with} \qquad \zeta_\perp(y;\bar n) := \chi_\perp(y\cdot \bar n), \quad \zeta_{/\!/}(y;\bar n) := \chi_{/\!/}\left(|\pi_{\bar n} y|\right).$$

Notice that for $\bar n \in S^{d-1}$, the function $y \mapsto \zeta(y;\bar n)$ is compactly supported in $D_1^0(\bar n)$. In the sequel, we fix $\xi = \xi(\zeta) \in (0,1/4)$ such that

$$\operatorname{supp}\zeta(\cdot;\bar n) \;\subset D^{2\xi}_{1-2\xi}(\bar n) \qquad \text{for } \bar n \in S^{d-1}.$$

Notice also that

$$\int_{\mathbf{R}^d} \zeta(y;\bar n)dy \;=\; 1.$$

**Definition 5.2** Let us introduce a small parameter $\eta \in (0,\xi/2)$ and let us set

$$\beta_\eta \;:=\; \min\left(\beta_1(\omega,\xi,\eta)\,,\beta_3(\omega,\xi/2,\eta)\right), \tag{5.3}$$

where the functions $\beta_1$ and $\beta_3$ are given in Lemma 2.4 and Lemma 2.6. Using the notation of Lemma 2.7, we define the two bad sets (union of bad bad balls),

$$\mathscr{U}_{(\varepsilon),\eta} \;:=\; \left[U^\star_{(\varepsilon)} \cup U_{(\varepsilon),3,\beta_\eta}\right] + B_{\sqrt{2}} \quad \subset \quad \mathscr{U}'_{(\varepsilon),\eta} \;:=\; \mathscr{U}_{(\varepsilon),\eta} + B_{\sqrt{2}},$$

and the corresponding good sets

$$\mathscr{G}_{(\varepsilon),\eta} \;:=\; \mathbf{R}^d \setminus \mathscr{U}_{(\varepsilon),\eta}, \quad \supset \quad \mathscr{G}'_{(\varepsilon),\eta} \;:=\; \mathbf{R}^d \setminus \mathscr{U}'_{(\varepsilon),\eta}.$$

Finally, we define a cut-off function $\chi_{(\varepsilon),\eta} \in \mathscr{D}(\mathbf{R}^d,[0,1])$ satisfying

$$\chi_{(\varepsilon),\eta} \equiv 1 \;\text{ on } \mathscr{G}'_{(\varepsilon),\eta}, \qquad \chi_{(\varepsilon),\eta} \equiv 0 \;\text{ on } \mathscr{U}_{(\varepsilon),\eta}, \qquad \|\nabla \chi_{(\varepsilon),\eta}\|_\infty \le 1.$$

The approximate (scalar) mean curvature of $\Sigma_\varepsilon$ at some point $x \in \Sigma_\varepsilon$ is defined as

$$\hat h_\varepsilon(x) \;:=\; -\frac{\chi_{(\varepsilon),\eta}(\varepsilon x)}{\varepsilon}\int_{\mathbf{R}^d} \sigma_\varepsilon(x+\varepsilon z)\cdot \pi_{\nu_\varepsilon(x)}\nabla_y \zeta\left(z;\nu_\varepsilon(x)\right) dz.$$

Eventually, the approximate mean curvature is defined as $\hat H_\varepsilon(x) := \hat h_\varepsilon(x)\nu_\varepsilon(x)$.

*Remark 5.1* To lighten the notation in the above definitions we did not emphasized the dependencies in $\zeta$ and $\eta$. Below, the main computations are carried out for a fixed test function $\zeta$ and a fixed parameter $\eta$. Only at the end, we send $\eta$ to 0 and we optimize the lower bound with respect to $\zeta$.

The next proposition states that $\hat h_\varepsilon$ approximates $h_\varepsilon$, at least in a weak sense in $L^2(\Sigma_\varepsilon,\mathscr{H}^{d-1})$.

**Proposition 5.1** *Let $\hat{h}_\varepsilon$ be given by Definition 5.2:*
*a) We have the following uniform bound in $\varepsilon$,*

$$\int_{\Sigma_\varepsilon} |\hat{h}_\varepsilon|^2 \, d\mathscr{H}^{d-1} \ \le\ C_{\not\xi}(\zeta,\eta).$$

*b) Moreover, for every $\varphi \in C(\mathbf{R}^d)$,*

$$\int_{\Sigma_\varepsilon} \varphi \left[\hat{h}_\varepsilon - h_\varepsilon\right] \nu_\varepsilon \, d\mathscr{H}^{d-1} \ \stackrel{\varepsilon\downarrow 0}{\longrightarrow}\ 0.$$

*Proof* Let us fix $\varepsilon \in (0,1]$ and for $x \in \Sigma_{(\varepsilon)}$, let us set

$$\hat{h}_{(\varepsilon)}(x) := \varepsilon \hat{h}_\varepsilon(\varepsilon x).$$

We now drop the subscripts $(\varepsilon)$: we write $\Sigma = \Sigma_{(\varepsilon)}$, $U^\star = U^\star_{(\varepsilon)}$, $\chi\eta = \chi_{(\varepsilon),\eta}$, *etc.*
In rescaled variables, we have for $x \in \Sigma$,

$$\hat{h}(x) = -\chi_\eta(x) \int_{\mathbf{R}^d} \sigma(x+y) \cdot (\zeta_\perp \nabla_y \zeta_{/\!/})(y;\nu(x))\, dy. \tag{5.4}$$

Recall that in the above integral we can reduce the domain of integration to $D^{2\xi}_{1-2\xi}(\nu(x))$.

We start with a few remarks. Let us consider the piece of hypersurface,

$$\Gamma_\eta \ :=\ \Sigma \cap \operatorname{supp} \chi_\eta.$$

By definition of $\chi_\eta$, we have $\operatorname{supp} \chi_\eta + B_{\sqrt{2}} \subset U^\star$ so that $\Sigma \cap (\operatorname{supp} \chi_\eta + B_{\sqrt{2}}) \subset \Sigma^0$. In particular, $\Gamma_\eta \subset \Sigma_0$ and $\nu(x) = n(x)$ on $\Gamma_\eta$
By definition of $\beta_\eta$, Lemma 2.4 and Lemma 2.5 apply to any point of $\Gamma_\eta$. Following the notation of these lemmas, we define for every $x \in \Gamma_\eta$,

$$D_{\text{int}}(x) \ :=\ x + D^{2\xi}_{1-2\xi}(n(x)) \quad \subset \quad D(x) \ :=\ x + D^{\xi}_{1-\xi}(n(x)) \quad \subset \quad \Omega.$$

The inequalities (2.12) and (2.14) apply at $x$ and moreover, $\Sigma \cap D(x) = \Sigma^0 \cap D(x)$. Eventually, since $\eta < \xi/2$, Lemma 2.4 implies $|t| \le 1 - \xi/2$ in $D(x)$; so Lemma 2.6 apply to any element $y \in D(x)$, $x \in \Gamma_\eta$.

**(a)** We have to estimate the quantity

$$\int_{\Sigma_\varepsilon} \hat{h}^2_\varepsilon \, d\mathscr{H}^{d-1} \ =\ \varepsilon^{d-3} \int_{\Gamma_\eta} \hat{h}^2 \, d\mathscr{H}^{d-1}.$$

Let $x \in \Gamma_\eta$ and let $\psi_x$ be the harmonic function provided by Lemma 2.5. c). Since $n(x) \cdot \nabla \psi_x \equiv 0$ and $\pi_{n(x)} \nabla \zeta_\perp \equiv 0$, we have

$$\int_{\mathbf{R}^d} \nabla \psi_x \cdot \nabla \zeta_{/\!/}(y;n(x))\, \zeta_\perp(y;n(x))\, dy \ =\ -\int_{\mathbf{R}^d} \Delta \psi_x \cdot \nabla \zeta(y;n(x))\, dy \ =\ 0.$$

This identity and $n(x) \cdot \nabla \zeta_{/\!/} \equiv 0$ allow us to rewrite (5.4) as

$$\hat{h}(x) \ =\ -\chi_\eta(x) \int_{\mathbf{R}^d} \left[\sigma(x+y) - n(x) - \nabla \psi_x(x+y)\right] \cdot (\zeta_\perp \nabla_y \zeta_{/\!/})(y;n(x))\, dy.$$

Then (2.14) and the Cauchy-Schwarz inequality yield $|\hat{h}|^2(x) \le C(\zeta)\mathscr{E}(D(x))$. Integrating on $\Gamma_\eta$ and using Fubini, we obtain

$$\int_{\Gamma_\eta} |\hat{h}|^2 \, d\mathscr{H}^{d-1} \;\le\; C(\zeta) \int_{\Omega^{1-\xi/2}} \left[ f_0(\sigma, \nabla t) + |\nabla u - \nabla t|^2 \right](y) q^\xi(y)\, dy,$$

with the notation $\Omega^{1-\xi/2} := [|t| < 1-\xi/2]$ and

$$q^\xi(y) \;:=\; \int_{\Gamma_\eta} \theta^\xi(y-x, n(x))\, d\mathscr{H}^{d-1}(x).$$

where $\theta^\xi(\cdot, n(x))$ is the characteristic function of $D^\xi_{1-\xi}(n(x))$.
By inequality (2.19) of Lemma 2.6, we have $|q^\xi(y)| \le C(\eta)$. Hence,

$$\varepsilon^{d-3} \int_{\Gamma_\eta} |\hat{h}|^2 \, d\mathscr{H}^{d-1} \;\le\; C(\zeta)\varepsilon^{d-3} \int_{\Omega} \left[ f_0(\sigma, \nabla t) + |\nabla u - \nabla t|^2 \right] \overset{\text{(Lemma 2.2)}}{\le} C_{\not\varepsilon}(\zeta).$$

This establishes Proposition 5.1.a.

**(b)** Let $\varphi \in C(\mathbf{R}^d)$. In scaled variables we have to bound the vector $\hat{Q}_{(\varepsilon)}[\varphi] - Q_{(\varepsilon)}[\varphi] \in \mathbf{R}^d$, with

$$\hat{Q}_{(\varepsilon)}[\varphi] \;:=\; \varepsilon^{d-2} \int_{\Sigma_{(\varepsilon)}} \varphi(\varepsilon x) \hat{h}_{(\varepsilon)}(x) n_{(\varepsilon)}(x)\, d\mathscr{H}^{d-1}(x),$$

$$Q_{(\varepsilon)}[\varphi] \;:=\; \varepsilon^{d-2} \int_{\Sigma_{(\varepsilon)}} \varphi(\varepsilon x) h_{(\varepsilon)}(x) n_{(\varepsilon)}(x)\, d\mathscr{H}^{d-1}(x).$$

The scaling factors $\varepsilon^{d-2}$ come from $\mathscr{H}^{d-1}(\Sigma_\varepsilon) = \varepsilon^{d-1}\mathscr{H}^{d-1}(\Sigma_{(\varepsilon)})$ and $h_\varepsilon(x) = \varepsilon^{-1} h_{(\varepsilon)}(\varepsilon^{-1}x)$.

Let us drop again the subscripts $(\varepsilon)$. Recall that $\Sigma \subset B_{R/\varepsilon}$, so we can assume that $\varphi$ is compactly supported. We can in fact assume that $\varphi$ is Lipschitz continuous by density of $W^{1,\infty}(\overline{B_{R/\varepsilon}})$ in $C(\overline{B_{R/\varepsilon}})$. Indeed, $\varphi \mapsto (Q - \widetilde{Q})[\varphi]$ is linear and by part (a) and the bounds of Proposition 2.1.b,

$$|(\hat{Q} - Q)[\varphi]| \;\le\; \|\varphi\|_\infty \sqrt{2\varepsilon^{d-3} \int_\Sigma |\hat{h}|^2 + |h|^2 \, d\mathscr{H}^{d-1}} \sqrt{\varepsilon^{d-1}\mathscr{H}^{d-1}(\Sigma)} \;\le\; C_{\not\varepsilon}(\zeta, \eta)\|\varphi\|_\infty.$$

Now, we claim that we can somehow assume that $\varphi$ is supported in $\mathscr{G}'_\eta$. Let us introduce a cut-off function $\chi'_\eta \in \mathscr{D}(\mathbf{R}^d, [0,1])$ satisfying

$$\chi'_\eta \equiv 1 \;\text{ on } \mathscr{G}''_\eta, \qquad \chi'_\eta \equiv 0 \;\text{ on } \mathscr{U}'_\eta, \qquad \|\nabla \chi'_\eta\|_\infty \le 1.$$

with the larger bad set and smaller good set:

$$\mathscr{U}''_\eta \;:=\; \mathscr{U}'_\eta + B_{\sqrt{2}}, \qquad \mathscr{G}''_\eta \;:=\; \mathbf{R}^d \setminus \mathscr{U}''_\eta.$$

We then define

$$\varphi_\eta(y) \;:=\; \chi_\eta((1/\varepsilon)y)\varphi(y).$$

Since $|\varphi_\eta - \varphi|$ is supported in $\mathscr{U}''_\eta$ and bounded by $\|\varphi\|_\infty$, we have, proceeding as above and using (2.28):

$$|(\hat{Q} - Q)[\varphi_\eta - \varphi]| \;\le\; \sqrt{2\varepsilon^{d-3} \int_\Sigma |\hat{h}|^2 + |h|^2 \, d\mathscr{H}^{d-1}} \sqrt{\varepsilon^{d-3}\mathscr{H}^{d-1}(\Sigma \cap \mathscr{U}''_\eta)}\, \|\varphi\|_\infty \varepsilon$$

$$\le\; C_{\not\varepsilon}(\zeta, \eta)\|\varphi\|_\infty \varepsilon \overset{\varepsilon \downarrow 0}{\longrightarrow} 0.$$

Therefore, we may and do substitute $\varphi_\eta$ for $\varphi$ and assume $\varphi \in W^{1,\infty}(\mathbf{R}^d)$ in the proof of (b). Remark that albeit not explicitly written, $\varphi_\eta$ depends on $\varepsilon$. Let us summarize the properties of $\varphi_\eta$:

$$\|\varphi_\eta\|_{W^{1,\infty}} \le C(\varphi), \qquad \operatorname{supp}\varphi_\eta \subset \mathscr{G}'_\eta.$$

In particular $\chi_\eta \equiv 1$ on the support of $\varphi_\eta$ and we can drop the factor $\chi_\eta$ and replace $\Sigma$ by $\Gamma_\eta \subset \Sigma_0$ in the definitions of $Q_{(\varepsilon)}[\varphi_\eta]$ and $\hat{Q}_{(\varepsilon)}[\varphi_\eta]$.

In the sequel we write $Q$ and $\hat{Q}$ for $Q[\varphi_\eta]$ and $\hat{Q}[\varphi_\eta]$. Using formula (5.4), we rewrite $\hat{Q}$ as an integral over $(x,y) \in \Sigma \times \mathbf{R}^d$. Then, performing the change of variable $z = x + y \in \mathbf{R}^d$, $x = x \in \Sigma$ and using Fubini, we obtain, for $i = 1, \cdots, d-1$,

$$\hat{Q}_i \;=\; \varepsilon^{d-2} \int_{\mathbf{R}^d} \sigma(z) \cdot \hat{q}_i(z)\, dz,$$

with the notation $\hat{Q} = (\hat{Q}_1, \cdots, \hat{Q}_d)$ and

$$\hat{q}_i(z) \;:=\; \int_{\Gamma_\eta} -\varphi_\eta(\varepsilon x) n_i(x) \left[\zeta_\perp \nabla_y \zeta_{/\!/}\right](z - x; n(x))\, d\mathscr{H}^{d-1}(x). \tag{5.5}$$

Here $n_i(x)$ denotes the $i$-est component of $n(x)$. Next, having in mind the identities $\pi_{\bar{n}} \nabla_y \zeta_{/\!/}(y;\bar{n}) = \nabla_y \zeta_{/\!/}(y;\bar{n})$ and $\pi_{\bar{n}} \nabla_y \zeta_\perp(y;\bar{n}) = 0$, we compute the expansion,

$$\pi_{n(x)} \nabla_x \Big[\varphi_\eta(\varepsilon x) \zeta(z - x; n(x)) n_i(x)\Big] \;=\; a_i(x,z) + b_i(x,z) + \varphi_\eta(\varepsilon x) \pi_{n(x)} c_i(x,z), \tag{5.6}$$

where:

*a.* The term $a_i(x,z)$ involves the gradient of the function $z \mapsto \zeta(z;\bar{n})$,

$$a_i(x,z) \;:=\; -\varphi_\eta(\varepsilon x) n_i(x) \left[\zeta_\perp \nabla_y \zeta_{/\!/}\right](z - x; n(x)).$$

Remark that $a_i(x,z)$ is the integrant in (5.5).

*b.* The term $b_i(x,z)$ involves the gradient of $x \mapsto \varphi_\eta(\varepsilon x)$.

$$b_i(x,z) \;:=\; \varepsilon \zeta(z - x; n(x)) n_i(x)\, \pi_{n(x)} \nabla \varphi_\eta(\varepsilon x).$$

*c.* The last term $c_i(x,z)$ involves the gradient of $n(x)$. It can be written on the following form,

$$c_i(x,z) \;=\; \sum_{k=1}^{d} \left.\frac{\partial[\zeta(z - x; \bar{n}) \bar{n}_i]}{\partial \bar{n}_k}\right|_{\bar{n} = n(x)} \nabla n_k(x) = \sum_{j=1}^{d} c_{i,j}(x;z) \nabla n_j(x). \tag{5.7}$$

We have $\operatorname{supp}[c_{i,j}(\cdot;z)] \subset D_{\sqrt{2}}(z)$ and $\|c_{i,j}\|_\infty \le C(\zeta)$.

Integrating (5.6) on $\Gamma_\eta$ and applying the Green formula (5.2) to the left hand side, we get,

$$\begin{aligned}
\hat{q}_i(z) \;=\; & \int_{\Gamma_\eta} a_i(x,z)\, d\mathscr{H}^{d-1}(z) \\
& = \int_{\Gamma_\eta} \varphi_\eta(\varepsilon x) \zeta(z - x; n(x)) n_i(x) h(x) n(x)\, d\mathscr{H}^{d-1}(x) \\
& \qquad - \int_{\Gamma_\eta} b_i(x,z)\, d\mathscr{H}^{d-1}(x) - \int_{\Gamma_\eta} \varphi_\eta(\varepsilon x) \pi_{n(x)} c_i(x,z)\, d\mathscr{H}^{d-1}(x).
\end{aligned}$$

Taking the dot product with $\sigma(z) = n(x) + [\sigma(z) - n(x)]$, integrating in $z \in \mathbf{R}^d$ and changing back the variables of integration to $y = z - x, x = x$, we obtain,

$$\hat{Q}_i \;=\; \hat{Q}_{i,1} + \hat{Q}_{i,2} + \hat{Q}_{i,3} + \hat{Q}_{i,4},$$

with

$$\hat{Q}_{i,1} := \varepsilon^{d-2} \int_{\Gamma_\eta} \left[ \int_{\mathbf{R}^d} \zeta(\cdot; n(x)) \right] h(x) \varphi_\eta(\varepsilon x) n_i(x)\, d\mathscr{H}^{d-1}(x)$$
$$\hat{Q}_{i,2} := \varepsilon^{d-2} \int_{\Gamma_\eta} \left[ \int_{\mathbf{R}^d} \zeta(\cdot; n(x))(\sigma(x+\cdot) - n(x)) \right] \cdot n(x) h(x) \varphi_\eta(\varepsilon x) n_i(x)\, d\mathscr{H}^{d-1}(x)$$
$$\hat{Q}_{i,3} := -\varepsilon^{d-1} \int_{\Gamma_\eta} \left[ \int_{\mathbf{R}^d} \zeta(\cdot; n(x)) \sigma(x+\cdot) \right]^T \pi_{n(x)} \nabla \varphi_\eta(\varepsilon x)\, d\mathscr{H}^{d-1}(x)$$
$$\hat{Q}_{i,4} := -\varepsilon^{d-2} \sum_{j=1}^{d} \int_{\Gamma_\eta} \left[ \int_{\mathbf{R}^d} c_{i,j}(x, x+\cdot) \sigma(x+\cdot) \right]^T \pi_{n(x)} \nabla n_j(x) \varphi_\eta(\varepsilon x)\, d\mathscr{H}^{d-1}(x).$$

Le us consider $\hat{Q}_{i,1}$. By definition of $\zeta$, the term into brackets is equal to 1. Therefore,

$$\hat{Q}_{i,1} \;=\; Q_i. \tag{5.8}$$

We have to establish that the three remaining terms go to 0 as $\varepsilon \downarrow 0$.

Let us fix a point $x \in \Gamma_\eta$. The harmonic function $\psi_x$ given by Lemma 2.5.c applied in $x + D^{\xi}_{1-\xi}(n(x))$ satisfies $\nabla \psi_x(x) = 0$ and $n(x) \cdot \nabla \psi_x \equiv 0$. In particular, we can substitute $[\sigma(x+\cdot) - n(x) - \nabla \psi_x(x+\cdot)]$ for $[\sigma(x+\cdot) - n(x)]$ in the definition of $\hat{Q}_{i,2}$. Using (2.14) and the Cauchy-Schwarz inequality, we compute

$$\begin{aligned} |\hat{Q}_{i,2}| \;&\leq\; C(\xi) \|\varphi\|_\infty \varepsilon^{d-2} \int_{\Gamma_\eta} \sqrt{\mathscr{E}(B_{\sqrt{2}}(x))}\, |h|(x)\, d\mathscr{H}^{d-1}(x) \\ &\leq\; C(\xi) \|\varphi\|_\infty \varepsilon \sqrt{\varepsilon^{d-3} \mathscr{W}(\Sigma)} \sqrt{\varepsilon^{d-3} \int_{\Gamma_\eta} \mathscr{E}(B_{\sqrt{2}}(x))\, d\mathscr{H}^{d-1}(x)}. \end{aligned}$$

Using Fubini and taking into account Lemma 2.2, we obtain

$$|\hat{Q}_{i,2}| \;\leq\; C(\xi, \eta) \|\varphi\|_\infty \varepsilon \sqrt{\varepsilon^{d-3} \mathscr{W}(\Sigma)} \sqrt{\varepsilon^{d-3} \int_{\Omega} f_0(\sigma, \nabla t)}.$$

Unscaling, we get

$$|\hat{Q}_{i,2}| \;\leq\; C(\xi, \eta) \|\varphi\|_\infty \sqrt{\mathscr{W}(\Sigma_\varepsilon)} \sqrt{E_0}\, \varepsilon \;\overset{\varepsilon \downarrow 0}{\longrightarrow}\; 0. \tag{5.9}$$

Next, we consider $\hat{Q}_{i,3}$. We first notice that $n(x) \cdot \pi_{n(x)} \equiv 0$ so that we can substitute $[\sigma(x+\cdot) - n(x)]$ for $\sigma(x+\cdot)$ in the integral into brackets. Now, since $n(x) \cdot \nabla \psi_x \equiv 0$ and $\zeta(\cdot; n(x))$ is radially symmetric with respect to the direction $n(x)$, the mean value property yields

$$\int_{\mathbf{R}^d} \zeta(y; n(x)) \nabla \psi_x(x+y)\, dy \;=\; \nabla \psi_x(x) \;=\; 0. \tag{5.10}$$

Therefore, we can substitute $[\sigma(x+\cdot)-n(x)-\nabla\psi_x(x+\cdot)]$ for $\sigma(x+\cdot)$ in the definition of $\hat{Q}_{i,3}$. Proceeding as above, we obtain

$$|\hat{Q}_{i,3}| \;\le\; C(\xi)\|\nabla\varphi_\eta\|_\infty \sqrt{\varepsilon^{d-3}\int_\Omega f_0(\sigma,\nabla t)}\sqrt{\varepsilon^{d-1}\mathscr{H}^{d-1}(\Sigma)}\,\varepsilon$$
$$\le\; C_{\not{\varepsilon}}(\xi)\|\nabla\varphi_\eta\|_\infty\,\varepsilon \;\overset{\varepsilon\downarrow 0}{\longrightarrow}\; 0. \quad (5.11)$$

For the last term, we can obviously substitute $[\sigma(x+\cdot)-n(x)]$ for $\sigma(x+\cdot)$ but we can still also substitute $[\sigma(x+\cdot)-n(x)-\nabla\psi_x(x+\cdot)]$ as in fact

$$\int_{\mathbf{R}^d}\sum_{j=1}^{d} c_{i,j}(x,x+y)\nabla\psi_x^T(x+y)\pi_{n(x)}\nabla n_j(x)\,dy$$
$$=\;\int_{\mathbf{R}^d}\nabla\psi_x(x+y)\cdot\pi_{n(x)}c_i(x,x+y))\,dy \;=\; 0. \quad (5.12)$$

To see this, we have to go back to the definition (5.7) of $c_i$. Without loss of generality, we assume that $x=0$ and $n(x)=e_d$. By (5.7), we have for $1\le i\le d-1$,

$$c_i(0,z) \;=\; \zeta(z;e_d)\nabla n_i(0) \qquad \overset{(5.10)}{\Longrightarrow} \qquad \int_{\mathbf{R}^d}\nabla\psi_0(z)\cdot\pi_{e_d}c_i(0,z) \;=\; 0.$$

For $i=d$, we have

$$c_d(0,z) \;=\; \sum_{k=1}^{d-1}\left.\frac{\partial[\zeta(z;\bar{n})]}{\partial\bar{n}_k}\right|_{\bar{n}\,=\,e_d}\nabla n_k(0)$$

Writing $z=(z',z_d)$ and taking into account the relations

$$\zeta((z',z_d);e_d) \;=\; \zeta((z',-z_d);e_d), \qquad \zeta(Rz;R\bar{n})=\zeta(z;\bar{n}) \quad \text{for every } R\in \mathrm{SO}(d),$$

we have using rotations in the plane $(e_k,e_d)$,

$$\left.\frac{\partial\zeta}{\partial\bar{n}_k}((z',z_d);\bar{n})\right|_{\bar{n}\,=\,e_d}+\left.\frac{\partial\zeta}{\partial\bar{n}_k}((z',-z_d);\bar{n})\right|_{\bar{n}\,=\,e_d} \;=\; 0, \qquad \text{for } k=1,\cdots,d-1.$$

Hence, since $\nabla\psi_0(z',z_d)$ does not depend on $z_d$, we have

$$\int_{\mathbf{R}^d}\nabla\psi_0(z)\cdot\pi_{e_d}c_d(0,z) \;=\; 0.$$

Therefore, (5.12) holds.
Substituting $[\sigma(x+\cdot)-n(x)-\nabla\psi_x(x+\cdot)]$ for $\sigma(x+\cdot)$ in the definition of $\hat{Q}_{i,4}$ and using (2.12) to bound $|\nabla n|(x)=|I\!I|(x)$ and (2.14) as above, we deduce

$$|\hat{Q}_{i,4}| \;\le\; C(\xi)\|\varphi\|_\infty\varepsilon^{d-2}\int_{\Gamma_\eta}\left[\mathscr{E}(B_{\sqrt{2}}(x))\right]^{3/4}d\mathscr{H}^{d-1}(x)$$
$$\overset{\text{(Hölder)}}{\le}\; C(\xi)\|\varphi\|_\infty\left[\varepsilon^{d-3}\int_{\Gamma_\eta}\mathscr{E}(B_{\sqrt{2}}(x))\,d\mathscr{H}^{d-1}(x)\right]^{3/4}\left[\varepsilon^{d-1}\mathscr{H}^{d-1}(\Sigma)\right]^{1/4}\sqrt{\varepsilon}$$
$$=\; C_{\not{\varepsilon}}(\xi)\|\nabla\varphi_\eta\|_\infty\sqrt{\varepsilon}\;\overset{\varepsilon\downarrow 0}{\longrightarrow}\; 0. \quad (5.13)$$

Proposition 5.1.b then follows from (5.8),(5.9),(5.11) and (5.13). □

### 5.2 Passage to the limit $\varepsilon \downarrow 0$, computation of the lower bound

Proceeding in the same way as to obtain the lower semicontinuity of $\mathscr{W}$ (see (4.7)), we deduce from Proposition 5.1 that

$$\mathscr{W}(\mathscr{V}_0) \;=\; \int_{\Sigma_0} (\theta_0^+ + \theta_0^-)|H_0|^2\, d\mathscr{H}^{d-1} \;\leq\; \liminf_{\varepsilon\downarrow 0} \int_{\Sigma_\varepsilon} |\hat{h}_\varepsilon|^2\, d\mathscr{H}^{d-1}. \tag{5.14}$$

By definition of $\hat{h}_\varepsilon$, we can replace the domain of integration by $\Gamma_{\varepsilon,\eta} := \varepsilon\Gamma_{(\varepsilon),\eta}$ in the last integral. Let us recall the definition of the local energy at some point $x \in \Gamma_{(\varepsilon),\eta}$,

$$\mathscr{E}_{(\varepsilon)}(x) \;:=\; \int_{B_3(x)\cap\Omega_\varepsilon} \left[f_0(\sigma_{(\varepsilon)}, \nabla t_{(\varepsilon)}) + |\nabla u_{(\varepsilon)} - \nabla t_{(\varepsilon)}|^2\right].$$

We introduce another local energy. Let $\alpha > 0$ be a small parameter, we set,

$$\mathscr{F}_{(\varepsilon),\alpha}(x) \;:=\; \int_{x+D^{2\xi}_{1-2\xi}(n_{(\varepsilon)}(x))} f(\sigma_{(\varepsilon)}, \nabla t_{(\varepsilon)}) + \alpha\mathscr{E}_{(\varepsilon)}(x).$$

*Remark 5.2* We add the term $\alpha\mathscr{E}_{(\varepsilon)}$ for compactness reasons. This terms controls oscillations of $\nabla u$ and thus of $\Sigma^0$.

We then consider local optimization problems associated to this local energy. By frame invariance, we only have to consider the case $x = 0$ and $n(x) = e_d$.

**Definition 5.3** Let $\zeta$, $\xi = \xi(\zeta)$, $\alpha > 0$ and $\eta \in (0, \xi/2)$ as above. We set

$$c_{\zeta,\eta,\alpha}(f) \;:=\; \frac{1}{\mathscr{H}^{d-1}(D_1')} \inf\left\{ \frac{\mathscr{F}^\#_\alpha(a^\#)}{|\hat{h}^\#|^2(a^\#)} \;:\; a^\# \in \mathscr{S}^\#_\eta,\, \hat{h}^\#(a^\#) \neq 0 \right\},$$

where:

(i) $\mathscr{S}^\#_\eta$ is the set of quadruplets $a^\# = (\sigma^\#, t^\#, u^\#, \Sigma^\#)$ with:

$$\sigma^\# \in L^2(B_3, \mathbf{R}^d), \quad t^\#, u^\# \in W^{1,2}(B_3, [-1,1]), \quad \Gamma^\# \subset D_1^{1/2}(e_d),$$

such that:

- $t^\#$ is $\omega$-continuous, $u^\#$ is $\omega^\star$-continuous, $u^\# = t^\#$ in $\{|t^\#| \geq 9/10\}$,
- the vector field $\sigma^\#$ is divergence free in $\mathscr{D}'([|t^\#| < 1])$,
- $u^\#$ is harmonic and $|\nabla u^\#| \geq 1/2$ in $D_1^{1/2}(e_d)$,
- $\Gamma^\#$ is the hypersurface:

$$\Gamma^\# \;=\; \left[u^\# = 0\right] \cap D_1^{1/2}(e_d),$$

  and moreover, $0 \in \Gamma^\#$ and $\nabla u^\#/|\nabla u^\#|(0) = e_d$,
- we have the energy bound (see (5.3))

$$\int_{B_\lambda\cap\Omega^\#} \left(f_0(\sigma^\#, \nabla t_\varepsilon) + |\nabla u^\# - \nabla t^\#|^2\right) \;\leq\; \beta_\eta.$$

(ii) The local energy $\mathscr{F}^\#_\alpha(a^\#)$ is modeled on $\mathscr{F}_{(\varepsilon),\alpha}$:

$$\mathscr{F}^\#_\alpha(a^\#) := \int_{D^{2\xi}_{1-2\xi}} f(\sigma^\#,\nabla t^\#)+\alpha\mathscr{E}^\#(a^\#),$$

with

$$\mathscr{E}^\#(a^\#) := \int_{B_3\cap[|t^\#|<1]} \left[f_0(\sigma^\#,\nabla t^\#)+|\nabla u^\# - \nabla t^\#|^2\right].$$

(iii) The approximate curvature $\hat{h}^\#(a^\#)$ is defined as (compare to (5.4)):

$$\hat{h}^\#(a^\#) := -\int_{\mathbf{R}^d} \sigma^\# \cdot (\zeta_\perp \nabla_y \zeta_{/\!/})(y;e_d)\, dy.$$

The parameter $\eta \in (0,\xi/2)$ being fixed, according to the above definition we have for every $x\in\Gamma_{(\varepsilon),\eta}$, $\varepsilon\in(0,1]$:

$$|\hat{h}_{(\varepsilon)}|^2(x) \le \frac{1}{c_{\zeta,\eta,\alpha}(f)}\frac{1}{\mathscr{H}^{d-1}(D'_1)}\mathscr{F}_{(\varepsilon),\alpha}(x). \tag{5.15}$$

Using this inequality in (5.14), we obtain

$$c_{\zeta,\eta,\alpha}(f)\mathscr{W}(\mathscr{V}_0) \le \frac{1}{\mathscr{H}^{d-1}(D'_1)}\liminf_{\varepsilon\downarrow 0}\ \varepsilon^{d-3}\int_{\Gamma_{(\varepsilon),\eta}} \mathscr{F}_{(\varepsilon),\alpha}(x)\, d\mathscr{H}^{d-1}(x). \tag{5.16}$$

Let us consider the integral in the right hand side. Replacing $\mathscr{F}_{(\varepsilon),\alpha}(x)$ by its definition, applying Fubini and taking into account Lemma 2.4, we get

$$\int_{\Gamma_{(\varepsilon),\eta}} \mathscr{F}_{(\varepsilon)}(x)\, d\mathscr{H}^{d-1}(x) \le \int_{\Omega^{\xi/2}_{(\varepsilon)}} f(\sigma_{(\varepsilon)},\nabla t_{(\varepsilon)})q_{(\varepsilon)} + \alpha\int_{\Omega_{(\varepsilon)}} \left[f_0(\sigma_{(\varepsilon)},\nabla t_{(\varepsilon)})+|\nabla u_{(\varepsilon)}-\nabla t_{(\varepsilon)}|^2\right] q'_{(\varepsilon)}.$$

with $\Omega^{\xi/2}_{(\varepsilon)} = [|t_{(\varepsilon)}|(y)<1-\xi/2]$,

$$q_{(\varepsilon)}(y) := \mathscr{H}^{d-1}\left(\left\{x\in\Gamma_{(\varepsilon),\eta} : y\in x+D^{2\xi}_{1-2\xi}(n(x))\right\}\right).$$

and $q'_{(\varepsilon)}(y)=\mathscr{H}^{d-1}(\Sigma^0_{(\varepsilon)}\cap B_3(y))$.
By inequality (2.19) of Lemma 2.6, we have $q_{(\varepsilon)}(y)\le\mathscr{H}^{d-1}(D'_1)(1+s(\eta))$ with $s(\eta)\to 0$ as $\eta\downarrow 0$. On the other hand, thanks to (2.27), we have $q'_{(\varepsilon)}\le C_{\not\xi}$. Hence,

$$\frac{1}{\mathscr{H}^{d-1}(D'_1)}\int_{\Gamma_{(\varepsilon),\eta}} \mathscr{F}_{(\varepsilon),\alpha}(x)\, d\mathscr{H}^{d-1}(x) \le (1+s(\eta))\mathscr{F}(\sigma_{(\varepsilon)},\nabla t_{(\varepsilon)},\Omega_{(\varepsilon)})+C_{\not\xi}\alpha.$$

Putting this estimate in (5.16), unscaling and sending $\eta$ to 0 and then $\alpha$ to 0, we end with

$$\left[\liminf_{\alpha\downarrow 0}\ \liminf_{\eta\downarrow 0}\ c_{\zeta,\eta,\alpha}(f)\right]\mathscr{W}(\mathscr{V}_0) \le \liminf_{\varepsilon\downarrow 0}\frac{\mathscr{F}(\sigma_\varepsilon,\nabla t_\varepsilon,\Omega_\varepsilon)}{\varepsilon^3}.$$

Theorem 1.2 then follows from the next lemma which states the required lower bound for the local optimization problems.

**Lemma 5.1 (Local optimization)**
*(a) For every $\alpha > 0$, there holds:*

$$\liminf_{\eta\downarrow 0} c_{\zeta,\eta,\alpha}(f) \;\geq\; c_{0,\zeta}(f) \;:=\; \frac{\det L}{2L_{2,2}}\left(\mathscr{H}^{d-1}(D_1') \int_{D_1^1} |\partial_{y_d}\zeta|^2(y;e_d)\,dy\right)^{-1},$$

*where L is the $2\times 2$ matrix defined in Lemma 3.1.*

*(b) There exists a sequence of convenient functions $(\zeta_k) \subset W^{1,\infty}(\mathbf{R}^d \times S^{d-1}, \mathbf{R}_+)$ (i.e. $\zeta_k$ complies to the constraints of Definition 5.1) such that*

$$c_{0,\zeta_k}(f) \;\overset{k\uparrow\infty}{\longrightarrow}\; c_0(f) \;=\; \frac{\det L}{3L_{2,2}}.$$

*Proof* Let us fix $\alpha > 0$ and let $\zeta$ be as in Definition 5.1. We first establish $(a)$. For this, we consider a minimizing sequence $a_k^\# = (\sigma_k^\#, \nabla t_k^\#, \nabla u_k^\#, \Sigma_k^\#) \subset \mathscr{S}_{\eta_k}^\#$ such that $\eta_k \downarrow 0$, $\hat{h}_\zeta^\#(a_k^\#) \neq 0$ and

$$\lim_{k\uparrow\infty} \frac{\mathscr{F}_\alpha^\#(a_k^\#)}{|\hat{h}^\#|^2(a_k^\#)} \;=\; \mathscr{H}^{d-1}(D_1') \liminf_{\eta\downarrow 0} c_{\zeta,\eta}.$$

Using the reflection symmetry with respect to the hyperplane $e_d^\perp$, we assume without loss of generality that $\hat{h}_\zeta^\#(a_k^\#) > 0$.

We start by proving that $c_{\zeta,\eta,\alpha}(f)$ is uniformly bounded with respect to $\eta$. Lemmas 2.4 and 2.5 have been stated for the quadruplets

$$\left(\sigma_{(\varepsilon)}, t_{(\varepsilon)}, u_{(\varepsilon)}, \Sigma_{(\varepsilon)}^0\right)$$

but by definition of the sets $\mathscr{S}_\eta^\#$ they obviously also apply to the elements of the sequence $(a_k^\#)$ at point $x = 0$. In particular, $\sigma_k^\# \to e_d$ in $L^2(D_{1-\xi}^\xi)$, which implies $\hat{h}_\zeta^\#(a_k^\#) \to 0$.
Now let $\hbar \in (0,1]$ and let us define $a^\# := (\sigma^\#, t^\#, u^\#, \Sigma^\#)$, as

$$\sigma^\#(y) = e_d + \hbar\left(\frac{1}{d-1}y' - y_d e_d\right), \quad t^\#(y) = u^\#(y) = y_d, \quad \Sigma^\# = D_1'.$$

We easily check that $\hat{h}_\zeta^\#(a^\#) = \hbar$, that $a^\# \in \mathscr{S}_\eta^\#$ for $\eta \in (0,\eta_0)$ for some $\eta_0 > 0$ depending on $\hbar$ and that

$$\mathscr{F}_\alpha^\#(a^\#) \;\leq\; C(f)\,\hbar^2 \;=\; C(f)\,|\hat{h}^\#(a^\#)|^2.$$

Therefore, $c_{\zeta,\eta,\alpha}(f) \leq C(f)/\mathscr{H}^{d-1}(S^{d-1})$ and since $(a_k^\#)$ is a minimizing sequence, we may assume $\mathscr{F}_\alpha^\#(a_k^\#) \leq C(f)\,|\hat{h}^\#(a_k^\#)|^2$.

Using the notation $\hbar_k \;:=\; \hat{h}^\#(a_k^\#)$, we now assume

$$\mathscr{F}_\alpha^\#(a_k^\#) \;\leq\; C(f)\,|\hbar_k|^2 \;\overset{k\uparrow\infty}{\longrightarrow}\; 0. \tag{5.17}$$

Let us call $\psi_k^\#$ the harmonic function given by Lemma 2.5.c applied to $a_k^\#$ and let us set $\psi_k^\flat := (1/\sqrt{\hbar_k})\psi_k^\#$. We define a vector field $\sigma_k^\flat$ and a function $t_k^\flat$ on $D_{1-2\xi}^{2\xi}$ by

$$\sigma_k^\# \;=\; e_d + \sqrt{\hbar_k}\,\nabla\psi_k^\flat + \hbar_k\sigma_k^\flat, \qquad \nabla t_k^\# \;=\; e_d + \sqrt{\hbar_k}\,\nabla\psi_k^\flat + \hbar_k\nabla t_k^\flat. \tag{5.18}$$

(To fix the the constant, we assume $\int_{D^{2\xi}_{1-2\xi}} t^\flat_k = 0$)
Using (5.17) in the estimates of Lemma 2.5.c, we have

$$\|\nabla\psi^\flat_k\|^2_{L^2(D^0_{1-3\xi/2})} + \|\sigma^\flat_k\|^2_{L^2(D^{2\xi}_{1-2\xi})} + \|\nabla t^\flat_k\|^2_{L^2(D^{2\xi}_{1-2\xi})} \leq C(f,\alpha,\xi). \tag{5.19}$$

Hence, up to extraction, there exist two functions $\psi^\flat_\infty, t^\flat_\infty \in W^{1,2}(D^{2\xi}_{1-2\xi})$ and a vector field $\sigma^\flat_\infty \in L^2(D^{2\xi}_{1-2\xi}, \mathbf{R}^d)$ such that $\psi^\flat_\infty$ is harmonic, $\partial_d \psi^\flat_\infty \equiv 0$, $\sigma^\flat_\infty$ is divergence free and

$$\nabla\psi^\flat_k \overset{k\uparrow\infty}{\longrightarrow} \nabla\psi^\flat_\infty \qquad \text{in } C^\infty\left(\overline{D^0_{1-2\xi}}\right), \tag{5.20}$$

$$\sigma^\flat_k \overset{k\uparrow\infty}{\longrightarrow} \sigma^\flat_\infty, \quad \nabla t^\flat_k \overset{k\uparrow\infty}{\longrightarrow} \nabla t^\flat_\infty \qquad \text{weakly in } L^2(D^{2\xi}_{1-2\xi}). \tag{5.21}$$

To study the asymptotic behavior of $\mathscr{F}^\#_\zeta(a^\#_k)$, we are going to use (5.18) and a Taylor expansion of $f$. The latter requires point-wise bounds. First, by (5.20), there exists a constant $K > 0$ such that

$$\sup_{D^0_{1-2\xi}} |\nabla\psi^\flat_k| \leq K.$$

We do not have pointwise bounds on the sequence $(\sigma^\flat_k, \nabla t^\flat_k)$ and we are led to reduce the domains of definition of these functions.
For this, let us introduce a small parameter $\rho > 0$. By the preceding estimate, for $k$ large enough, say $k \geq k_0(\rho)$, there holds

$$\sqrt{\hbar_k}|\nabla\psi^\flat_k| < \rho \quad \text{ on } D^0_{1-2\xi}.$$

Now, for $k \geq 0$, we set

$$S_{\rho,k} := \left\{ y \in D^{2\xi}_{1-2\xi} : |(\sigma^\flat_k(y), \nabla t^\flat_k(y))| < \rho/|\hbar_k| \right\}.$$

Notice, for later use, that since $(\sigma^\flat_k)$ and $(\nabla t^\flat_k)$ are bounded in $L^2(D^{2\xi}_{1-2\xi})$ and $\hbar_k \to 0$, we have

$$\mathscr{H}^{d-1}(D^{2\xi}_{1-2\xi} \setminus S_{\rho,k}) \overset{k\uparrow\infty}{\longrightarrow} 0. \tag{5.22}$$

Since $f$ is of class $C^2$ in some neighborhood of $\mathbb{S}^{d-1}$, there exists $\rho_0 > 0$ and a sublinear modulus of continuity $\omega_f(r) \overset{r\downarrow 0}{\to} 0$ such that for $\sigma', \tau', \sigma'', \tau'' \in B_{\rho_0}(e_d)$, we have

$$|D^2 f(\sigma'', \tau'') - D^2 f(\sigma', \tau')| \leq \omega_f(|(\sigma'' - \sigma', \tau'' - \tau')|). \tag{5.23}$$

Let us now assume $\rho \in (0, \rho_0/2)$ and let us fix for a moment $k \geq k_0(\rho)$ and $y \in S_k$ so that by construction, we have $\sigma^\#_k(y), \nabla t^\#_k(y) \in B_{\rho_0}(e_d)$). We set

$$m := e_d + \sqrt{\hbar_k}\nabla\psi^\flat_k(y), \qquad n := m/|m|,$$

(we also have $n \in B_{\rho_0}(e_d)$). Let us also introduce the pairs of vectors,

$$M := (m,m), \qquad N := (n,n), \quad \text{ and } \quad Q := \hbar_k \left( \sigma^\flat_k(y)\,, \nabla t^\flat_k(y) \right),$$

We estimate $f(\sigma_k^\#(y),\nabla t_k^\#(y))$ from below by using (5.18) and a Taylor expansion of $f$ at $N$. Taking into account $f(N)=0$, $Df(N)=0$ and (5.23), we obtain

$$f(\sigma_k^\#(y),\nabla t_k^\#(y)) \;=\; f(N+(M-N+Q)) \\ \geq\; \frac{1}{2}(M-N+Q)^T D^2 f(N)\,(M-N+Q)-\frac{1}{2}\omega_f(2\rho)|M-N+Q|^2.$$

We rewrite the quadratic term as

$$D^2 f(N) \;=\; D^2(f(e_d,e_d))+[D^2 f(N)-D^2(f(e_d,e_d)].$$

Since, $|n-e_d|<\rho$, this leads to

$$f(\sigma_k^\#(y),\nabla t_k^\#(y)) \\ \geq\; \frac{1}{2}(M-N+Q)^T D^2 f(e_d,e_d)\,(M-N+Q)-\omega_f(2\rho)|M-N+Q|^2. \quad (5.24)$$

Now performing a Taylor expansion of $m-n$, we get

$$m-n \;=\; \frac{\hbar_k}{2}|\nabla\psi_k^\flat|^2(y)e_d+O(|\hbar_k|^{3/2}K^3).$$

Hence, setting

$$\nu_k^\flat \;:=\; \sigma_k^\flat+\frac{|\nabla\psi_k^\flat|^2}{2}e_d, \qquad \tau_k^\flat :=\; \nabla t_k^\flat+\frac{|\nabla\psi_k^\flat|^2}{2}e_d,$$

we obtain,

$$M-N+Q \;=\; \hbar_k\left(\nu_k^\flat,\tau_k^\flat\right)+O(|\hbar_k|^{3/2}K^3).$$

Substituting this expansion in (5.24), dividing by $(\hbar_k)^2$ and integrating on $S_{\rho,k}$, we get

$$\frac{1}{|\hbar_k|^2}\mathscr{F}_\zeta^\#(a_k^\#) \;\geq\; \frac{1}{2}\int_{S_{\rho,k}}\left(\nu_k^\flat(y),\tau_k^\flat(y)\right)^T D^2 f(e_d,e_d)\left(\nu_k^\flat(y),\tau_k^\flat(y)\right) \\ -2\omega_f(2\rho)\left(2C(K)\mathscr{H}^{d-1}(D_1')+C(\alpha,\xi)\right), \quad (5.25)$$

where we have used (5.19) to bound the remainder.
Let $\chi_{\rho,k}$ be the characteristic function of $S_{\rho,k}$. We deduce from (5.20) (5.21) and (5.22), that

$$\left(\chi_{\rho,k}\nu_k^\flat\,,\,\chi_{\rho,k}\tau_k^\flat\right) \;\overset{k\uparrow\infty}{\longrightarrow}\; (\nu_\infty^\flat,\tau_\infty^\flat) \;:=\; \left(\sigma_\infty^\flat+\frac{|\nabla\psi_\infty^\flat|^2}{2}e_d\,,\,\nabla t_\infty^\flat+\frac{|\nabla\psi_\infty^\flat|^2}{2}e_d\right) \\ \text{weakly in } L^2(D_{1-2\xi}^{2\xi}).$$

By positivity of $D^2 f(e_d,e_d)$, the functional

$$W\in L^2(D_{1-2\xi}^{2\xi}) \;\mapsto\; \int_O W^T D^2 f(e_d,e_d)W$$

is lower semicontinuous with respect to the topology of weak convergence in $L^2$. Therefore, sending $k$ to $+\infty$ in (5.25), then sending $\eta$ to 0 and taking into account the fact that $\rho>0$ is arbitrary, we end with

$$\mathscr{H}^{d-1}(D_1')\left[\liminf_{\eta\downarrow 0}\, c_{\zeta,\eta,\alpha}\right] \;\geq\; \frac{1}{2}\int_{D_{1-2\xi}^{2\xi}}(\nu_\infty^\flat,\tau_\infty^\flat)^T D^2 f(e_d,e_d)(\nu_\infty^\flat,\tau_\infty^\flat).$$

We simplify further the right hand side by using the structure of $D^2 f(e_d, e_d)$ described in Lemma 3.1. Calling $p = (\nu^\flat_{\infty,d}, \tau^\flat_{\infty,d})$ the last components of the vector fields $\nu^\flat_\infty, \tau^\flat_\infty$, we obtain,

$$\mathscr{H}^{d-1}(D_1') \left[\liminf_{\eta\downarrow 0} c_{\zeta,\eta,\alpha}\right] \geq \frac{1}{2}\int_{D^{2\xi}_{1-2\xi}} p^T L p. \tag{5.26}$$

where the positive symmetric $2\times 2$ matrix $L$ is defined by (3.1).
Now we pass to the limit in the affine constraint. Putting the expansion (5.18) of $\sigma^\#_k$ in the definition of $\hbar_k = h^\#_\zeta(a^\#_k)$ and using the facts that $\psi^\#_k$ is harmonic and does not depend on $y_d$, we see that

$$-\int_{\mathbf{R}^d} \sigma^\flat_k \cdot (\zeta_\perp \nabla_y \zeta_{/\!/})(y; e_d)\, dy = 1.$$

Passing to the limit $k \uparrow \infty$, $\sigma^\flat_\infty$ satisfies the constraint

$$-\int_{\mathbf{R}^d} \sigma^\flat_\infty \cdot (\zeta_\perp \nabla_y \zeta_{/\!/})(y; e_d)\, dy = 1. \tag{5.27}$$

Since $\nabla\cdot\sigma^\flat_\infty \equiv 0$ and $\zeta$ is compactly supported, this amounts to

$$\int_{\mathbf{R}^d} \sigma^\flat_{\infty,d}(y)\partial_{y_d}\zeta\,(y; e_d)\, dy = 1.$$

Now, since $p_1 = \sigma^\flat_{\infty,d} + |\nabla\psi^\flat_\infty|^2/2$ and $\nabla\psi^\flat_\infty$ does not depend on $y_d$, we also have

$$\int_{\mathbf{R}^d} p_1(y)\partial_{y_d}\zeta\,(y; e_d)\, dy = 1. \tag{5.28}$$

Eventually, we have to optimize the right hand side of (5.26) with respect to the elements $p \in L^2(D^\xi_{1-2\xi}, \mathbf{R}^2)$ satisfying the constraint (5.28). Since $L$ is positive definite, the existence of a unique minimizer relies on the Lax-Milgram theorem. The Euler-Lagrange equations lead to $p_{opt} = \lambda\partial_{y_d}\zeta(\cdot; e_d)L^{-1}\binom{1}{0}$, where the Lagrange multiplier

$$\lambda = \frac{\det L}{L_{2,2}\|\partial_{y_d}\zeta(\cdot; e_d)\|^2_{L^2(D^0_1)}}$$

is fixed by the constraint. The minimal energy is $\lambda/2$. Together with (5.26), this leads to

$$\mathscr{H}^{d-1}(D_1') \left[\liminf_{\eta\downarrow 0} c_{\zeta,\eta,\alpha}\right] \geq \frac{\det L}{2L_{2,2}\|\partial_{y_d}\zeta(\cdot; e_d)\|^2_{L^2(D^0_1)}},$$

and $(a)$ is proved.

**(b)** We now optimize in $\zeta$. We are looking for the largest possible lower bound so the smallest possible $\|\partial_{y_d}\zeta\|^2_{L^2(D^0_1(e_d))}$. Minimizing this quantity under the constraints $\int_{D^0_1(e_d)} \zeta = 1$, $\partial_{y_d}\zeta \in L^2(D^0_1(e_d))$ and $\zeta = 0$ on $\{y \in \partial D^0_1(e_d) : y_d = \pm 1\}$, we obtain as unique minimizer

$$\zeta_0(y) = \frac{3}{4\mathscr{H}^{d-1}(D_1')}\left(1 - y_d^2\right) \qquad \text{with} \quad \mathscr{H}^{d-1}(D_1')\int_{D^0_1(e_d)} |\partial_{y_d}\zeta_0|^2 = \frac{3}{2}.$$

As required, we have $c_{0,\zeta_0}(f) = (\det\, L)/(3L_{2,2})$ and $\zeta_{opt}$ is a product of smooth functions depending on tangential and thickness coordinates respectively. However, we can not use $\zeta_0$ in

Definition 5.1 because it is not compactly supported in $D_1^0(e_d)$. Anyway, we can approximate $\zeta_0$ by a sequence of compactly supported function $(\zeta_j) \subset W^{1,\infty}(D_1^0(e_d) \times S^{d-1})$ that comply to the requirements of Definition 5.1. For example:

$$\zeta_j(R(y', y_d); Re_d)) \;:=\; \zeta_j((y', y_d); e_d) \;:=\; \frac{j+1}{j} \frac{\theta_j(y')}{\|\theta_j\|_{L^1(D_1')}} \zeta_0\left(y', \frac{j+1}{j} y_d\right),$$

with $\theta_j \in \mathscr{D}(D_1', [0,1])$ satisfying $\theta_j(y') = 1$ for $|y'| \le 1 - 2/j$. For such a sequence, we have $c_{0,\zeta_j}(f) \to (\det L)/(3L_{2,2})$ as $j \uparrow \infty$ which establishes part (b) of the lemma. □

## 6 Concluding remarks

### 6.1 The uniform continuity and confining assumptions

We would like to get rid of the uniform continuity hypothesis on $x \mapsto \varepsilon^{-1} t_\varepsilon(\varepsilon x)$ (Hypothesis 1) and establish the compactness result under sole energy bounds.
We could try to relax the hypothesis by approximating the functions

$$t_\varepsilon \in W^{1,2}_{loc}(\mathbf{R}^d, [-\varepsilon, \varepsilon])$$

by Lipschitz continuous functions. In fact, proceeding as in Evans and Gariepy [5], (Section 6.6.3, Theorem 3, see also the appendix of [6]), it is not difficult to establish that for such function, there exists $\hat{t}_\varepsilon \in W^{1,\infty}(\mathbf{R}^d, [-\varepsilon, \varepsilon])$ and an open set $U_\varepsilon$ such that $\hat{t}_\varepsilon \equiv t_\varepsilon$ in $\mathbf{R}^d \setminus U_\varepsilon$, $\|\nabla \hat{t}_\varepsilon\|_\infty \le 2$ and

$$\mathscr{H}^d(U_\varepsilon) + \int_{\mathbf{R}^d} |\nabla \hat{t}_\varepsilon - \nabla t_\varepsilon|^2 \;\le\; \int_{\mathbf{R}^d} [|\nabla t_\varepsilon| - 1]_+^2 \,.$$

The last term is controlled by the energy $\mathscr{F}_0(\sigma_\varepsilon, \nabla t_\varepsilon, \Omega_\varepsilon)$ and we can think of substituting $\hat{t}_\varepsilon$ for $t_\varepsilon$ to derive our estimates. This implies that we should also substitute $\hat{\Omega}_\varepsilon = [|\hat{t}_\varepsilon| < \varepsilon]$ for the domain $\Omega_\varepsilon$ and find a vector field $\hat{\sigma}_\varepsilon$ which approximates $\sigma_\varepsilon$ and is divergence free on $\hat{\Omega}_\varepsilon$. We did not find a solution to this last problem. In fact, given a vector field $\sigma \in W^{1,2}(\Omega, \mathbf{R}^d)$ and an open set $U \subset \mathbf{R}^d$, it is not true in general that we can extend $\sigma$ as a divergence free vector field in $\Omega \cup U$ or even that we can find a vector field which is reasonably close to $\sigma$ in $\Omega$ and which is divergence free in $\Omega \cup U$. The difficulty arises from the topology of $\Omega \cup U$ which may differ from that of $\Omega$. For instance, if $\sigma = x/|x|^d$ in $B_1 \setminus \overline{B_\rho}$ then we can not extend $\sigma$ as a divergence free vector field in $B_1$.

We would also like to weaken the confining hypothesis: $[t_\varepsilon \equiv \varepsilon]$ in the complement of $B_R$. This is feasible, at least for $d = 2$ and $d = 3$.

In dimension $d = 2$, the bad sets introduced in the compactness step are empty for $\varepsilon$ small enough: indeed, the number of bad balls in Lemma 2.7 is subjected to $N \le C_{\not{e}}(\beta)\varepsilon$. Therefore, the one dimensional set $\Sigma_\varepsilon$ is, for $\varepsilon$ small enough equal to $[u_\varepsilon = 0]$ which is a finite union of non-intersecting analytic Jordan curves. The total length of these curves is uniformly bounded, as well as their elastic energy $\mathscr{W}(\Sigma_\varepsilon)$. Substituting the condition "$t_\varepsilon \equiv \varepsilon$ in the neighborhood of infinity" for the confining assumption, we can track the different connected components by using finitely many translations and obtain at the limit a finite number of $W^{1,2}$ Jordan curves with total mass $S$. Alternatively, we can assume that $[|t_\varepsilon| < 1/2]$ is connected, the lemmas of Section 2.3 then imply that $\Sigma_\varepsilon$ reduces to a single Jordan curve. Assuming $|t_\varepsilon(0)| < 1/2$ to anchor the membranes or using a translation, we obtain a bounded sequence of uniformly bounded Jordan curves with uniformly bounded lengths and uniformly bounded elastic energy.
In these cases, the conclusion of the compactness result is the same as in [13].

In dimension $d=3$, the number of bad balls is only uniformly bounded: $N \leq C_{\not{g}}(\beta)$. Substituting the condition "$t_\varepsilon \equiv \varepsilon$ in the neighborhood of infinity" to the confining assumption and assuming that $[|t_\varepsilon| < 1/2]$ is connected, we see that

$$\operatorname{diam}(\Sigma_\varepsilon) \leq \sum_i \operatorname{diam}(\Sigma_\varepsilon^i) + C_{\not{g}},$$

where the $\Sigma_\varepsilon^i$ are the connected components of $\Sigma_\varepsilon$. We invoke a result of Simon [16], Lemma 1.1 which states that for such connected components, there holds

$$\operatorname{diam}(\Sigma_\varepsilon^i) \leq C\sqrt{\mathscr{H}^{d-1}(\Sigma_\varepsilon^i)\mathscr{W}(\Sigma_\varepsilon^i)}.$$

Therefore,

$$\operatorname{diam}(\Sigma_\varepsilon) \leq C\sqrt{\mathscr{H}^{d-1}(\Sigma_\varepsilon)\mathscr{W}(\Sigma_\varepsilon)} + C_{\not{g}} \leq C_{\not{g}},$$

and, up to translations, we recover the confining hypothesis.

## 6.2 The case of a material with non vanishing spontaneous curvature

The model introduced in [12] is more general than the one we have discussed so far. In Section I.1.3, the divergence free condition is relaxed to

$$\nabla \cdot \sigma = \mu \quad \text{in } \mathscr{D}'(\Omega), \tag{6.1}$$

where $\mu \in \mathbf{R}$ is a parameter which characterizes the spontaneous curvature of the material. As expected in [12], the results of the present paper extend to this case. More precisely, we claim that if we substitute for the Willmore functional, the following energy introduced by Helfrich [9],

$$\mathscr{W}_\mu(\Sigma) := \int_\Sigma |h-\mu|^2 \, d\mathscr{H}^{d-1},$$

then Theorems 1.1, 1.2 and 1.3 generalize to this setting without further changes. Let us indicate the main adjustments that must be made to the proof.

### *6.2.1 Changes in the compactness step*

The proof of Proposition 2.1.a relies on Lemma 2.1. In rescaled variables, (6.1) reads $\nabla \cdot \sigma_{(\varepsilon)} \equiv \mu\varepsilon$ in $\Omega_{(\varepsilon)}$. Proceeding as in the original proof of Lemma 2.1 and taking into account this modification, the identity (2.7) is replaced by the inequality

$$\left| \int_{\partial O} \sigma_{(\varepsilon)} \cdot \nu - \frac{S}{\varepsilon^{d-1}} \right| \leq \varepsilon|\mu|\mathscr{H}^d(\Omega_{(\varepsilon)} \cap O).$$

Continuing the proof of the lemma and performing straightforward modifications to treat the above right hand side, we conclude to a weaker version of Proposition 2.1.a : we have to substitute $C_{\not{g}}(\varepsilon + \varepsilon^{(d-1)/2})$ for $C_{\not{g}}\varepsilon$ in the estimate of Proposition 2.1.a. This only concerns the case $d=2$ and does not affect the rest of the paper — in fact we only need this term to go to 0 as $\varepsilon \downarrow 0$.

For the construction of the hypersurface $\Sigma_\varepsilon$, the definition of the harmonic extension $u_{(\varepsilon)}$ of Section 2.2 does not change. However, the proof of the inequality

$$\int_{\mathbf{R}^d} |\nabla u_{(\varepsilon)} - \nabla t_{(\varepsilon)}|^2 \leq C_{\not{g}}\varepsilon^{3-d} \tag{6.2}$$

is more involved. Indeed, Lemma 2.2 was based on the fact that $u_{(\varepsilon)}$ was a minimizer of $J(\varphi) = \int_{\mathbf{R}^d} |\nabla\varphi - \sigma|^2$ in the set of $W^{1,2}$-functions equal to $t_{(\varepsilon)}$ on $F_{(\varepsilon),\delta}$. Now, we have to consider the functional

$$\widetilde{J}(\varphi) \ := \ \int_{\mathbf{R}^d} |\nabla\varphi - \sigma_{(\varepsilon)}|^2 - 2\mu\varepsilon \int_{O_{(\varepsilon),\delta}} (\varphi - t_{(\varepsilon)}).$$

The inequality $\widetilde{J}(u_{(\varepsilon)}) \leq \widetilde{J}(t_{(\varepsilon)})$ yields

$$\int_{\mathbf{R}^d} |\nabla u_{(\varepsilon)} - \sigma_{(\varepsilon)}|^2 \ \leq \ C_{\not{\varepsilon}}\varepsilon^{3-d} + 2|\mu|\varepsilon \int_{O_{(\varepsilon),\delta}} |u_{(\varepsilon)} - t_{(\varepsilon)}|. \tag{6.3}$$

To bound the last term, we first use the maximum principle which implies $|u_{(\varepsilon)} - t_{(\varepsilon)}| \leq 4$ and the bound $\mathscr{H}^d(\Omega_{(\varepsilon)}) \leq C_{\not{\varepsilon}}\varepsilon^{1-d}$. This leads to the weaker estimate

$$\int_{\mathbf{R}^d} |\nabla u_{(\varepsilon)} - \nabla t_{(\varepsilon)}|^2 \ \leq \ C_{\not{\varepsilon}}\varepsilon^{2-d}. \tag{6.4}$$

To improve this estimate, we notice that $\Omega_{(\varepsilon)}$ is ideally a set of width 2 for which the following Poincaré inquality should hold:

$$\int_{\Omega_{(\varepsilon)}} |\psi|^2 \ \leq \ C_{\not{\varepsilon}} \int_{\Omega_{(\varepsilon)}} |\nabla\psi|^2 \quad \text{for every } \psi \in W_0^{1,2}(\Omega_{(\varepsilon)}).$$

Applying such Poincaré inequality to $\psi = u_{(\varepsilon)} - t_{(\varepsilon)}$ to estimate the right hand side of (6.3) would lead to

$$\int_{\mathbf{R}^d} |\nabla u_{(\varepsilon)} - \sigma_{(\varepsilon)}|^2 \ \leq \ C_{\not{\varepsilon}}\varepsilon^{3-d} + C_{\not{\varepsilon}}|\mu|\varepsilon\sqrt{\mathscr{H}^d(\Omega_{(\varepsilon)})}\left(\int_{\mathbf{R}^d} |\nabla u_{(\varepsilon)} - \sigma_{(\varepsilon)}|^2\right)^{1/2}.$$

We could conclude to the desired estimate:

$$\int_{\mathbf{R}^d} |\nabla u_{(\varepsilon)} - \sigma_{(\varepsilon)}|^2 \ \leq \ C_{\not{\varepsilon}}\varepsilon^{3-d}. \tag{6.5}$$

Unfortunately, the domain $\Omega_{(\varepsilon)}$ may contain large balls and the above Poincaré inequality does not hold in general. Our strategy is to apply the Poincaré inequality in good cylinders.
As in the original proof, we introduce the local energy

$$\mathscr{E}(O) \ := \ \int_{O\cap\Omega_{(\varepsilon)}} \left[f_0(\sigma_{(\varepsilon)}, \nabla t_{(\varepsilon)}) + |\nabla u_{(\varepsilon)} - \nabla t_\varepsilon|^2\right].$$

We also still set $\Sigma^0_{(\varepsilon)} := [u_{(\varepsilon)} = 0]$. For $x \in \Sigma^0_{(\varepsilon)}$ and $y \in \mathbf{R}^d$, we introduce the vector field:

$$\widetilde{\sigma}^x_{(\varepsilon)}(x+y) \ = \ \sigma_{(\varepsilon)}(x+y) - \mu\varepsilon y \cdot \nu_{(\varepsilon)}(x)\nu_{(\varepsilon)}(x).$$

This vector field is divergence free in $\Omega_{(\varepsilon)}$ and will be used in place of $\sigma_{(\varepsilon)}$ to derive slightly modified versions of the "cylinder" lemmas 2.4, 2.5 and 2.6. For this, we define a new local energy:

$$\mathscr{E}^x_{(\varepsilon)}(O) \ := \ \int_{O\cap\Omega\varepsilon} \left[f_0(\widetilde{\sigma}^x_{(\varepsilon)}, \nabla t_{(\varepsilon)}) + |\nabla u_{(\varepsilon)} - \nabla t_\varepsilon|^2\right].$$

We have $\widetilde{\mathscr{E}}^x_\varepsilon(B_3(x)) \leq \mathscr{E}_\varepsilon(B_3(x)) + C\varepsilon^2|\mu|^2$, so that for $\varepsilon > 0$ small enough and $\beta > 0$ small enough, we deduce from Lemma 2.4 applied to $(\widetilde{\sigma}^x_{(\varepsilon)}, \nabla t_{(\varepsilon)}, \Omega_{(\varepsilon)})$ that if $x \in \Sigma^0_{(\varepsilon)}$ is such that $\mathscr{E}(B_3(x)) < \beta$ then

$$D \ := \ x + D_1^{1-1/20}(n_{(\varepsilon)}(x)) \subset \ \Omega_{(\varepsilon)},$$

and $t_{(\varepsilon)}(z)=u_{(\varepsilon)}(z)$ for every $z\in D$ such that $|(z-x)\cdot n_{(\varepsilon)}(x)|>9/10$.
Applying the Poincaré inequality to $u_{(\varepsilon)}-t_{(\varepsilon)}$ in such cylinder $D$, we get

$$\int_D |u_{(\varepsilon)}-t_{(\varepsilon)}|^2 \le C\int_D |\nabla u_{(\varepsilon)}-\nabla t_{(\varepsilon)}|^2. \tag{6.6}$$

Now, from (6.4), the total energy $\mathscr{E}_{(\varepsilon)}(\mathbf{R}^d)$ is bounded by $C_{\not\varepsilon}\varepsilon^{2-d}$. Hence, the volume of bad points such that $\mathscr{E}(B_3(x))\ge\beta$ is also bounded by $C_{\not\varepsilon}\varepsilon^{2-d}$. Using a covering argument and applying (6.6) in good cylinders and the rough bound $|u_{(\varepsilon)}-t_{(\varepsilon)}|\le 4$ in the bad set, we deduce (6.5) from (6.3).

No major difficulties appears in the remaining parts of the proof of Theorem 1.1. The other changes concern the use of $\widetilde{\sigma}^x_{(\varepsilon)}$ to derive adapted versions of the "cylinder" lemmas.

### *6.2.2 Changes in the upper and lower bound steps*

In the proof of the upper bound, we have to modify the construction of the vector field $\sigma$ defined in a neighborhood of $\Sigma_0$. As in [12] (heuristic section I.1.4.), we substitute for (3.5), the formula

$$\sigma(y) \;:=\; \left(\frac{1+\mu\displaystyle\int_0^s \det(\mathbf{I}_d+rD\nu_0(\pi_0(y))\,dr}{\det(\mathbf{I}_d+Z(y)D\nu_0(\pi_0(y)))}\right)\nu_0(y).$$

This yields the expansion $\sigma(x+s\pi_0(x))-\mu\nu_0(x) \;=\; \nu_0(x)-s\,h_0(x)\nu_0(x)+O(s^2)$. The rest of the proof is not modified.

For the proof of the upper bound, no modifications are required until the definition of the local problem: we still define the approximate mean curvature as in Definition 5.2 and the proof of Proposition 5.1 is still valid.
The local optimization problem introduced in Definition 5.3 is modified as follows. We now set,

$$c_{\zeta,\eta,\alpha}(f) \;:=\; \frac{1}{\mathscr{H}^{d-1}(D_1')}\inf\left\{\frac{\mathscr{F}^{\#}_{\alpha}(a^{\#})}{|\hat{h}^{\#}-\mu^{\#}|^2(a^{\#})}\;:\; a^{\#}\in\mathscr{S}^{\#}_{\eta},\,\hat{h}^{\#}(a^{\#})\neq 0\right\},$$

where the infimum now ranges over quintuplets $a^{\#}=(\sigma^{\#},\mu^{\#},t^{\#},u^{\#},\Sigma^{\#})$ with $\mu^{\#}\in\mathbf{R}$ and where the divergence free condition on $\sigma^{\#}$ is replaced by

$$\nabla\cdot\sigma^{\#}\;\equiv\mu^{\#}\quad\text{in } [|t^{\#}|<1].$$

With this definition (5.15) becomes

$$|\hat{h}_{(\varepsilon)}-\mu\varepsilon|^2(x)\;\le\;\frac{1}{c_{\zeta,\eta,\alpha}(f)}\,\frac{1}{\mathscr{H}^{d-1}(D_1')}\mathscr{F}_{(\varepsilon),\alpha}(x).$$

In place of (5.16), we now deduce,

$$c_{\zeta,\eta,\alpha}(f)\,\mathscr{W}_{\mu}(\mathscr{V}_0)\;\le\;\frac{1}{\mathscr{H}^{d-1}(D_1')}\liminf_{\varepsilon\downarrow 0}\,\varepsilon^{d-3}\int_{\Gamma_{(\varepsilon),\eta}}\mathscr{F}_{(\varepsilon),\alpha}(x)\,d\mathscr{H}^{d-1}(x).$$

To conclude we have to solve the new optimization problem.
We claim that Lemma 5.1 is still valid. Indeed proceeding as above, we consider a minimizing sequence

$$(a^{\#}_k)=(\sigma^{\#}_k,\mu^{\#}_k,t^{\#}_k,u^{\#}_k,\Sigma^{\#}_k).$$

The normalizing factor is now define as

$$\hbar_k \ := \ \hat{h}^{\#}_{\zeta}(a^{\#}_k) - \mu^{\#}_k \ \to \ 0.$$

We expand $\sigma^{\#}_k$ and $t^{\#}_k$ around $e_d$ and after extraction of a subsequence, we obtain the same limit objects $\sigma^{\flat}_{\infty}$, $\nabla t^{\flat}_{\infty}$, $\nabla \psi^{\flat}_{\infty}$ with the same properties except that $\sigma^{\flat}_{\infty}$ is now subjected to the constraint,

$$\nabla \cdot \sigma^{\flat}_{\infty} \ = \ \mu^{\flat}_{\infty} \quad \text{in } D^{2\xi}_{1-2\xi}, \tag{6.7}$$

where $\mu^{\flat}_{\infty} \in \mathbf{R}$ is the limit normalized spontaneous curvature. Passing to the limit in the linear constraint, the identity (5.27) now becomes

$$-\int_{\mathbf{R}^d} \sigma^{\flat}_{\infty} \cdot (\zeta_{\perp} \nabla_y \zeta_{/\!/})(y; e_d)\, dy - \mu^{\flat}_{\infty} \ = \ 1.$$

Integrating by parts in the horizontal hyperplanes and using (6.7) we obtain (5.28) as before. The proof then continues without further changes.

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