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CONVERGENCE TO EQUILIBRIUM FOR DISCRETIZATIONS OF GRADIENT-LIKE FLOWS ON RIEMANNIAN MANIFOLDS.

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Abstract. In this paper, we consider discretizations of systems of differential equations on manifolds that admit a strict Lyapunov function. We study the long time behavior of the discrete solutions. In the continuous case, if a solution admits an accumulation point for which a Łojasiewicz inequality holds then its trajectory converges. Here we continue the work started in [18] by showing that discrete solutions have the same behavior under mild hypotheses. In particular, we consider the θ-scheme for systems with solutions in $\mathbf{R}^d$ and a projected θ-scheme for systems defined on an embedded manifold. As illustrations, we show that our results apply to existing algorithms: 1/ Alouges' algorithm for computing minimizing discrete harmonic maps with values in the sphere; 2/ a discretization of the Landau-Lifshitz equations of micromagnetism.

1. **Introduction.** In this paper, we consider time discretizations of the non-linear differential system,

$$\dot{u} \;=\; G(u), \qquad t \geq 0,\; u(t) \in M, \tag{1}$$

where $M \subset \mathbf{R}^d$ is a C^2-embedded manifold without boundary and G is a continuous tangent vector field on M. More precisely, we are interested in the long-time behavior and stability properties of the global solutions of (1). If the continuous system (1) admits a strict Lyapunov function $F \in C^1(M, \mathbf{R})$ and if the set of accumulation points

$$\omega(u) \;:=\; \{\varphi \in M : \exists (t_n) \uparrow \infty \text{ such that } u(t_n) \to \varphi\}$$

is non-empty, then $t \mapsto F(u(t))$ in a non-increasing function converging to $F(\varphi)$ where $\varphi \in \omega(u)$.

Under additional assumptions, namely if F satisfies a Łojasiewicz inequality in a neighborhood of φ and if $G(u)$ and $-\nabla F(u)$ satisfy an angle condition, then one can prove that $u(t)$ does indeed converge to φ (see the papers by Lageman [16], by Chill *et al.* [10] and the more recent paper by Barta *et al.* [7]). Under an additional comparability condition between $\|G(u)\|$ and $\|\nabla F(u)\|$, we even have convergence rates depending on the Łojasiewicz exponent.

Notice that if φ is an isolated local minimizer of F, then the above convergence property is almost obvious and the Łojasiewicz inequality is not required. On the other hand these results are not trivial (and wrong in general) when a connected component of the critical

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set of F does not reduce to a single point. A typical example is given by the function $F : \mathbf{R}^2 \mapsto \mathbf{R}$, $x \mapsto (\|x\|^2 - 1)^2$ for which the set of minimizers is S^1.

If we are concerned with numerical simulations, it is of interest to know whether the above asymptotic properties also hold for numerical solutions. Consider, for some time-step $\Delta t > 0$, a sequence $(u_n) \subset M$ such that u_n approximates the exact solution u at time $t_n = n\Delta t$. Such a sequence could be built by means of any standard or reasonable numerical scheme. Mimicking the continuous case, our first goal is to establish the following property:

Result 1. *If φ is an accumulation point of the sequence (u_n), then $u_n \to \varphi$.*

When φ is a local minimizer of F, we expect a more precise stability result:

Result 2. *Let $\varphi^* \in M$ be a local minimizer of F. For every $\eta > 0$ there exists $0 < \varepsilon < \eta$ such that*

$$\|u_N - \varphi^*\| < \varepsilon \quad \Longrightarrow \quad \|u_n - \varphi^*\| < \eta, \ \forall n \geq N.$$

Moreover, in this case, the sequence (u_n) converges to some $\varphi \in M$.

It turns out that this last property leads to a uniform convergence result. Indeed, assume that the exact solution u converges to a local minimizer φ^* of F, then for T large enough, we have

$$\|u(t) - \varphi^*\| \ < \varepsilon/2, \qquad \forall t \geq T.$$

If the scheme is uniformly convergent on finite intervals (a reasonable query) then for $\Delta t > 0$ small enough, we have,

$$\|u_n - u(t_n)\| \ < \ \varepsilon/2, \ \text{for } 0 \leq t_n \leq T+1.$$

In particular, $\|u_N - \varphi^*\| < \varepsilon$ where $T \leq N\Delta t < T + 1$. Applying Result 2, we conclude that for $\Delta t > 0$ small enough, we have

$$\|u_n - u(t_n)\| \ \leq \ \eta, \qquad \forall n \geq N.$$

We then infer,

$$\lim_{\Delta t \downarrow 0} \sup_{n \geq 0} \|u_n - u(t_n)\| \ = \ 0.$$

As a consequence, denoting $\varphi(\Delta t) := \lim_{n\to\infty} u_n$, we also have $\varphi(\Delta t) \to \varphi^*$ as $\Delta t \to 0$. Thus, the numerical scheme provides a method to approximate the limit φ^*. This property motivates our interest for Result 2.

Once the convergence of the sequence is known, we will try to precise the convergence rate and establish:

Result 3. *Let φ be the limit of Result 1. We have the estimate $\|u_n - \varphi\| \leq \kappa f(t_n)$.*

The function $t \mapsto f(t)$ should decrease to 0 as t goes to $+\infty$. Typically f is an exponential or a rational function (see (12) below).

As in the continuous case, the background assumptions on (1) to obtain Results of type 1, 2 and 3 are a Łojasiewicz inequality, the angle condition and (for the convergence rates) the comparability condition that we will describe in the next section. For the scheme, on top of usual consistency property, the basic additional required assumption is that F should be a strict Lyapunov function for the scheme with an estimate of the form

$$F(u_{n+1}) + \mu \frac{\|u_{n+1} - u_n\|^2}{\Delta t} \ \leq \ F(u_n), \tag{2}$$

for some $\mu > 0$. In the case of a gradient flow $G = -\nabla F$, and if ∇F satisfies a one-sided Lipschitz condition, this stability property is naturally satisfied by the backward Euler scheme and, under a regularity assumption on G, by the θ-scheme for $0 \leq \theta < 1$. In this paper, we focus on these schemes and we assume that ∇F satisfies a one sided Lipschitz condition.

In a previous paper, Merlet and Pierre [18] (see also [8, Theorem 24]) have studied the long time behavior of some time-discretizations of the gradient system,

$$\dot{u} \;=\; -\nabla F(u), \qquad t \geq 0, u \in \mathbf{R}^d. \tag{3}$$

Their results have been generalized to some second order perturbations in [13]. A closely related question concerns the convergence of the proximal algorithm associated to the minimization of F in finite or infinite dimension (see [1, 6, 8] and references therein). As in the present work, the key assumption for convergence results in these papers is the Łojasiewicz inequality. Here we extend the results of [18] on gradient flows in $\mathbf{R}^d$ by considering gradient-like systems on a manifold: we establish Results of type 1, 2 and 3 for the θ-schemes associated to such systems.

The sequel is organized as follows. In the next section, we set the notation and the main hypotheses. We also prove the convergence result in the continuous case. Elements of this proof are used in Section 5.

Our results concerning the θ-scheme and the projected θ-scheme will be obtained as a consequence of general abstract results of type 1, 2 and 3 that we first establish in Section 3. We will highlight there the essential hypotheses required for these convergence to equilibrium results. We believe that this general setting enables to quickly check whether convergence to equilibrium properties in the continuous case transpose to the solutions of a numerical scheme in specific situations.

In Section 4.1, we apply the abstract situation to the θ-schemes associated to (1) in the case $M = \mathbf{R}^d$.

In Section 4.2, we consider the case of an embedded manifold by paying attention to the constraint $u(t) \in M$. Of course, under usual hypotheses ensuring the unique solvability of (1) for $u(0) \in M$ (e.g. assuming that G is locally Lipschitz), the trajectory of the solution will remain on M. This is no longer true for general time discretizations. For this reason, we introduce and study a linearized θ-scheme supplemented by a projection step that enforces the constraint $u_n \in M$.

We consider the backward Euler scheme in the case $M = \mathbf{R}^d$ in a separate part (Section 5). The fact that each step of this scheme can be rewritten as a minimization problem (even in the case of the gradient-like system (1)) allows us to weaken the regularity hypotheses on G.

Eventually, we apply our methods to some concrete problems. First, we consider in Section 6 a scheme by Alouges [3] designed for the approximation of minimizing harmonic maps with values in the sphere S^{l-1}. We establish that the sequence built by the algorithm does converge to a discrete harmonic map. The original result was convergence up to extraction. Then, in Section 7, we apply our results concerning the projected θ-scheme to a discretization of the Landau-Lifchitz equations of micromagnetism (again proposed by Alouges [4]). These examples illustrate our general results for in both cases the trajectories lie on a non-flat manifold ($(S^{l-1})^N$). Moreover, in the last example, the underlying continuous system is not a gradient flow but merely a system on the form (1) admiting a strict Lyapunov function.

2. **The continuous case.** From now on, M is a C^2-Riemannian manifold without boundary. Without loss of generality, we assume that M is embedded in $\mathbf{R}^d$ and that the inner product on every tangent space T_uM is the restriction of the euclidian inner product on $\mathbf{R}^d$.
We consider a tangent vector field $G \in C(M, TM)$ and a function $F \in C^1(M, \mathbf{R})$. We assume that $-\nabla F$ and G satisfy the *angle condition* defined below.

Definition 2.1. We say that G and ∇F satisfy the *angle condition* if there exists a real number $\alpha > 0$ such that

$$\langle G(u), -\nabla F(u)\rangle \;\geq\; \alpha\|G(u)\|\|\nabla F(u)\|, \qquad \forall u \in M. \tag{4}$$

Remark 2.2. In this definition and below, when we consider a differentiable function $F : M \to \mathbf{R}$, we write $\nabla_M F$ or simply ∇F to denote the gradient of F with respect to the tangent space of M. In particular, $\nabla F(u) \in T_u M$. If the function F is also defined on a neighborhood Ω of M in $\mathbf{R}^d$, the notation ∇F is ambiguous. In this case $\nabla F(u)$ denotes the gradient of F in $\mathbf{R}^d$, that is $\nabla F(u) = (\partial_{x_1} F(u), \cdots, \partial_{x_d} F(u))$ and $\nabla_M F(u) = \Pi_{T_u M}\nabla F(u)$ where $\Pi_{T_u M}$ denotes the orthogonal projection on the tangent space $T_u M \subset \mathbf{R}^d$.

We assume moreover that F is a strict Lyapunov function for (1):

Definition 2.3. We say that F is a Lyapunov function for (1) if for every $u \in M$, we have $\langle G(u), -\nabla F(u)\rangle \geq 0$. If moreover, $\nabla F(u) = 0$ implies $G(u) = 0$ then we say that F is a *strict* Lyapunov function.

As we already noticed, the key tool of the convergence results presented here is a Łojasiewicz inequality.

Definition 2.4. Let $\varphi \in M$.
1/ We say that the function F satisfies a Łojasiewicz inequality at φ if there exists $\beta, \sigma > 0$ and $\nu \in (0, 1/2]$ such that,

$$|F(u) - F(\varphi)|^{1-\nu} \;\leq\; \beta\|\nabla F(u)\|, \qquad \forall u \in B(\varphi, \sigma) \cap M. \tag{5}$$

The coefficient ν is called a Łojasiewicz exponent.
2/ The function F satisfies a Kurdyka-Łojasiewicz inequality at φ if there exists $\sigma > 0$ and a non-decreasing function $\Theta \in C(\mathbf{R}_+, \mathbf{R}_+)$ such that

$$\Theta(0) = 0, \qquad \Theta > 0 \text{ on } (0, +\infty), \qquad 1/\Theta \in L^1_{loc}(\mathbf{R}_+) \tag{6}$$

and,

$$\Theta\left(|F(u) - F(\varphi)|\right) \;\leq\; \|\nabla F(u)\|, \qquad \forall u \in B(\varphi, \sigma) \cap M. \tag{7}$$

Notice that the first definition is a particular case of the second one with $\Theta(f) = (1/\beta) f^{1-\nu}$. The interest of the first definition relies on the following fundamental result:

Theorem 2.5 (Łojasiewicz [17], see also [15])**.** *If $F : \Omega \subset \mathbf{R}^d \to \mathbf{R}$ is real analytic in some neighborhood of a point φ, then F satisfies the Łojasiewicz inequality at φ.*

Remark 2.6. The Łojasiewicz inequality only provides information at critical points of F. Indeed, if φ is not a critical point of F, then by continuity of ∇F, the Łojasiewicz inequality is satisfied in some neighborhood of φ.

It is well known that the Kurdyka-Łojasiewicz inequality implies the convergence of the bounded trajectories of the gradient flow (3) as t goes to infinity. Here we state the convergence result in the more general case of a gradient-like system:

Theorem 2.7. *Assume that F is a strict Lyapunov function for* (1) *and that $G, \nabla F$ satisfy the angle condition* (4). *Let u be a global solution of* (1) *and assume that there exists $\varphi \in \omega(u)$ such that F satisfies the Kurdyka-Łojasiewicz inequality* (7) *at φ. Then $u(t) \to \varphi$ as $t \to +\infty$.*

Remark 2.8. In many applications, the Kurdyka-Łojasiewicz hypothesis holds at every point. Moreover, for finite dimensional systems the fact that $\omega(u)$ is not empty is often the consequence of a coercivity condition on F,

$$F(u) \longrightarrow +\infty, \qquad \text{as } \|u\| \to \infty. \tag{8}$$

Proof. The proof stated here follows [10]. First we write

$$\frac{d}{dt}\left[F(u(t))\right] \overset{(1)}{=} \langle G(u(t)), \nabla F(u(t))\rangle \overset{(4)}{\leq} -\alpha\|G(u(t))\|\|\nabla F(u(t))\| \leq 0,$$

and the function $F(u)$ is non-increasing. By continuity of F and since $\varphi \in \omega(u)$, $F(u(t))$ converges to $F(\varphi)$ as t goes to $+\infty$. Changing F by an additive constant if necessary, we may assume $F(\varphi) = 0$, so that $F(u(t)) \geq 0$ for every $t \geq 0$.

If $F(u(t_0)) = 0$ for some $t_0 \geq 0$ then $F(u(t)) = 0$ for every $t \geq t_0$ and therefore, (since F is a strict Lyapunov function), u is constant for $t \geq t_0$. In this case, there remains nothing to prove.

Hence we may assume $F(u(t)) > 0$ for every $t \geq 0$. Since F satisfies a Kurdyka-Łojasiewicz inequality at φ, there exist $\sigma > 0$ and a function $\Theta \in C(\mathbf{R}_+, \mathbf{R}_+)$ satisfying (6) and (7). Let us define

$$\Phi(f) = \int_0^f \frac{1}{\Theta(s)} ds, \qquad f \geq 0. \tag{9}$$

Let us take $\varepsilon \in (0, \sigma)$. There exists t_0 large enough such that

$$\|u(t_0) - \varphi\| + \alpha^{-1}\Phi(F(u(t_0))) < \varepsilon.$$

Let us set $t_1 := \inf\{\mathrm{t} \geq \mathrm{t}_0 : \|\mathrm{u}(\mathrm{t}) - \varphi\| \geq \varepsilon\}$. By continuity of u we have $t_1 > t_0$. Then for every $t \in [t_1, t_0)$, using the angle condition (4) and the Kurdyka-Łojasiewicz inequality, we have

$$-\frac{d}{dt}\Phi(F(u(t))) = \frac{\langle G(u), -\nabla F(u)\rangle}{\Theta(F(u(t)))} \geq \alpha\|G(u(t))\| = \alpha\|u'(t)\|. \tag{10}$$

Integrating on $[t_0, t)$ for any $t \in [t_0, t_1)$, we get

$$\begin{aligned}\|u(t) - \varphi\| &\leq \|u(t) - u(t_0)\| + \|u(t_0) - \varphi\| \leq \int_{t_0}^{t_1} \|u'(s)\| ds + \|u(t_0) - \varphi\| \\ &\leq \alpha^{-1}\Phi(F(u(t))) + \|u(t_0) - \varphi\| < \varepsilon.\end{aligned}$$

This inequality implies $t_1 = +\infty$. Eventually, the estimate (10) yields $\dot{u} \in L^1(\mathbf{R}_+)$ and we conclude that $u(t)$ converges to φ as t goes to infinity. □

In the case of a gradient flow and if the Łojasiewicz inequality is satisfied, we have an explicit convergence rate that depends on the Łojasiewicz exponent. In order to extend this result to gradient-like systems, the angle condition is not sufficient:

Definition 2.9. We say that G and ∇F satisfy the angle and comparability condition if there exists a real number $\gamma > 0$ such that

$$\langle G(u), -\nabla F(u)\rangle \geq \frac{\gamma}{2}\left(\|G(u)\|^2 + \|\nabla F(u)\|^2\right), \qquad \forall u \in M. \tag{11}$$

Remark 2.10. Notice that this condition implies the angle condition (4). In fact (11) is equivalent to the fact that there exists $\alpha > 0$ such that for every $u \in M$,

$$\langle G(u), -\nabla F(u)\rangle \geq \alpha\|G(u)\|\|\nabla F(u)\| \qquad \text{and} \qquad \alpha^{-1}\|G(u)\| \geq \|\nabla F(u)\| \geq \alpha\|G(u)\|.$$

Theorem 2.11. *Under the hypotheses of Theorem 2.7, assume moreover that ∇F and G satisfy the angle and comparability condition* (11) *and that F satisfies a Łojasiewicz inequality with exponent $0 < \nu \leq 1/2$, then there exist $c, \mu > 0$ such that,*

$$\|u(t) - \varphi\| \leq \begin{cases} c\, e^{-\mu t} & \text{if } \nu = 1/2, \\ c\, t^{-\nu/(1-2\nu)} & \text{if } 0 < \nu < 1/2, \end{cases} \qquad \forall\, t \geq 0. \tag{12}$$

Proof. By Theorem 2.7, we know that $u(t)$ converges to φ as t goes to infinty. As in the preceding proof, we may assume $F(\varphi) = 0$ and $F(u(t)) > 0$ for every $t \geq 0$. Let Φ be defined by (9) with $\Theta(f) := (1/\beta) f^{1-\nu}$ and let us set $H(t) = \Phi(F(u(t)))$. In this case, we have the explicit formula $\Phi(f) = (\beta/\nu) f^{\nu}$.
Next let $t_1 \geq 0$ such that $\|u(t) - \varphi\| \leq \sigma$ for every $t \geq t_1$. By (10), for every $t \geq t_1$, we have

$$\|u(t) - \varphi\| \;\leq\; \int_t^{+\infty} \|u'(s)\|\, ds \;\leq\; \int_t^{+\infty} \gamma^{-1} H'(s) ds \;=\; \gamma^{-1} H(t). \tag{13}$$

Using the angle and comparability condition (11) and the Łojasiewicz inequality (5), we compute

$$\begin{aligned} -H'(t) \;&=\; \beta \left[F(u(t))\right]^{\nu-1} \left(-\frac{d}{dt}\left[F(u(t))\right]\right) \overset{(11)}{\geq} \beta \left[F(u(t))\right]^{\nu-1} \frac{\gamma}{2} \|\nabla F(u(t))\|^2 \\ &\overset{(5)}{\geq} \frac{\gamma\beta}{2} \left[F(u(t))\right]^{\nu-1} \frac{1}{\beta^2} \left[F(u(t))\right]^{2-2\nu} = \frac{\gamma}{2\beta} \left[F(u(t))\right]^{1-\nu} = \lambda \left[H(t)\right]^{\frac{1-\nu}{\nu}}. \end{aligned}$$

where $\lambda = C(\gamma, \beta, \nu) > 0$. Summing up, we have

$$H'(t) + \lambda \left[H(t)\right]^{\frac{1-\nu}{\nu}} \;\leq\; 0, \qquad \forall t \geq t_1.$$

In the case $\nu = 1/2$, we get $H'(t) + \lambda H(t) \leq 0$. Writing $H(t) = e^{-\lambda t} g(t)$ we conclude that g is non-increasing, so $H(t) \leq c\, e^{-\lambda t}$ for every $t \geq t_1$.
In the case $0 < \nu < 1/2$, we set $K(t) := \left[H(t)\right]^{-(1-2\nu)/\nu}$. This function satisfies $K'(t) \geq \lambda\nu/(1-2\nu)$, which implies $K(t) \geq at$ for some $a > 0$ and for t large enough. Hence $H(t) \leq (at)^{-\nu/(1-2\nu)} = ct^{-\nu/(1-2\nu)}$.
Combining this estimate with (13) completes the proof. □

In the sequel, we use some tools related to Riemannian metrics on $\mathbf{R}^d$ that we introduce here.

Definition 2.12. Let g be a Riemannian metric on $\mathbf{R}^d$. We recall that the gradient $\nabla_g F(u)$ of F with respect to the metric g at a point u is defined by

$$\langle \nabla F(u), X \rangle \;=\; \langle \nabla_g F(u), X \rangle_g, \qquad \forall X \in \mathbf{R}^d$$

We write $\langle .,. \rangle_g$ (to be precise, we should write $\langle .,. \rangle_{g(u)}$) for the inner product on the tangent space at the point u. We also write $\|.\|_g$ for the induced norm.

In [7], Barta *et al.* establish the remarkable result that when (1) admits a strict Lyapunov function then, up to a change of metric, (1) is a gradient system. The following theorem is a direct corollary of Theorem 1 and Theorem 2 in [7].

Theorem 2.13. *Assume that F is a strict Lyapunov function. Then there exists a Riemannian metric g on $\tilde{M} := \left\{u \in \mathbf{R}^d : G(u) \neq 0\right\}$ such that $G = -\nabla_g F$.*
Moreover, if ∇F and G satisfy the angle and comparability condition (11) *then g is equivalent to the Euclidean metric. Namely, there exist $c_1, c_2 > 0$ such that*

$$c_1 \|X\| \leq \|X\|_{g(u)} \leq c_2 \|X\|, \qquad \forall X \in \mathbf{R}^d, \forall u \in \tilde{M}. \tag{14}$$

For some numerical schemes, we are able to obtain Results 1 and 2 under the following additional assumption:

Definition 2.14. We say that ∇F satisfies the one-sided Lipschitz condition if there exists $c \geq 0$ such that:

$$\langle \nabla F(u) - \nabla F(v), u - v \rangle \;\geq\; -c\|u - v\|^2, \qquad \forall u, v \in M. \tag{15}$$

Let us define the set of accumulation points of the sequence $(u_n) \subset M$:

$$\omega((u_n)) \;:=\; \left\{ \varphi \in M : \text{ there exists a subsequence } (u_{n_k}) \text{ such that } u_{n_k} \xrightarrow[k\to\infty]{} \varphi \right\}.$$

To end this section, we recall the main hypotheses introduced above:

- $\omega(u_n)$ is not empty. (H0)

(In applications, this hypothesis is mainly a consequence of (2) and (8).)

- The Kurdyka-Łojasiewicz inequality holds at some $\varphi \in \omega(u_n)$. (H1)
- ∇F and G satisfy the angle and comparability condition (11). (H2)
- ∇F satisfies the one-sided Lipschitz condition (15). (H3)

For convergence rate results, (H1) is replaced by

- The Łojasiewicz inequality holds at some $\varphi \in \omega(u_n)$. (H1')

Other assumptions on the regularity of G and F will be made in the statement of the results.

3. **Abstract convergence results.** In this section we prove Results of type 1, 2 and 3 for abstract sequences $(u_n) \subset M$ satisfying the two additional conditions introduced below. In the case $M = \mathbf{R}^d$, Results of type 1 and 2 are known (see Absil *et al.* [1]). Although, the proofs are identical in the case of an embedded manifold, we provide them for completeness and for they are used in the proof of Result 3.

Let us introduce our first condition:

$$\exists C \geq 0,\, \forall n \geq 0, \quad F(u_n) - F(u_{n+1}) \;\geq\; C\|\nabla_M F(u_n)\|\|u_n - u_{n+1}\|. \tag{H4}$$

We need moreover a discrete version of the strict Lyapunov hypothesis:

$$\forall n \geq 0, \quad F(u_{n+1}) = F(u_n) \quad \Longrightarrow \quad u_{n+1} = u_n. \tag{H5}$$

We first state a Result of type 1.

Theorem 3.1 ([1] Theorem 3.2)**.** *Let $(u_n) \subset M$. Assume that hypotheses $(H0)$, $(H1)$, $(H4)$ and $(H5)$ hold. Then the sequence (u_n) converges to φ as n goes to infinity.*

Proof. By (H4) the sequence $(F(u_n))$ is non-increasing, so by continuity of F and hypothesis $(H0)$, we know that $F(u_n)$ converges to $F(\varphi)$ which can be assumed to be 0. By (H5), we may assume that $F(u_n)$ is decreasing, since in the other case, the sequence is constant and thus converge to φ. Since F satisfies the Kurdyka-Łojasiewicz inequality at φ, there exist $\sigma > 0$ and a non-decreasing function Θ satisfying (6) and (7). Let Φ be the function defined by (9). We have, for $n \geq 0$,

$$\Phi(F(u_n)) - \Phi(F(u_{n+1})) \;=\; \int_{F(u_{n+1})}^{F(u_n)} \frac{1}{\Theta(s)} ds \;\geq\; \frac{1}{\Theta(F(u_n))}\left(F(u_n) - F(u_{n+1})\right).$$

Using (H4), we obtain

$$\Phi(F(u_n)) - \Phi(F(u_{n+1})) \;\geq\; C\frac{\|\nabla_M F(u_n)\|}{\Theta(F(u_n))}\|u_{n+1} - u_n\|. \tag{16}$$

By $(H0)$ and the convergence of $(F(u_n))$ to 0, there exists $\bar{n}$ such that

$$\|u_{\bar{n}} - \varphi\| + \frac{1}{C}\Phi(F(u_{\bar{n}})) \;<\; \sigma.$$

Let us define

$$N \;:=\; \sup\left\{n \geq \bar{n} : \|u_k - \varphi\| < \sigma, \quad \forall \bar{n} \leq k \leq n\right\},$$

and assume by contradiction that N is finite. For every $\bar{n} \leq n \leq N$, we have $\|u_n - \varphi\| < \sigma$, so we can apply (7) with $u = u_n$ and deduce from (16)

$$\|u_{n+1} - u_n\| \ \leq \ (1/C)\left\{\Phi(F(u_n)) - \Phi(F(u_{n+1}))\right\}, \qquad \forall \bar{n} \leq n < N+1. \tag{17}$$

Summing these inequalities, we get

$$\sum_{n=\bar{n}}^{N} \|u_{n+1} - u_n\| \ \leq \ \frac{1}{C}\Phi(F(u_{\bar{n}})). \tag{18}$$

In particular,

$$\|u_{N+1} - \varphi\| \ \leq \ \frac{1}{C}\Phi(F(u_{\bar{n}})) + \|u_{\bar{n}} - \varphi\| \ < \ \sigma,$$

which contradicts the definition of N. So $N = +\infty$, and the convergence of the sequence follows from (18). □

We now establish a result of type 2.

Theorem 3.2 ([1] Proposition 3.3)**.** *Let φ be a local minimizer of F such that F satisfies a Kurdyka-Łojasiewicz inequality in a neighborhood of φ. Consider a sequence $(u_n) \subset M$ and assume that* (H4) *holds. Then, for every $\eta > 0$ there exists $\varepsilon \in (0, \eta)$ only depending on F, η and the constant C in* (H4) *such that*

$$\|u_{\bar{n}} - \varphi\| < \varepsilon \quad \Longrightarrow \quad \|u_n - \varphi\| < \eta, \ \forall n \geq \bar{n}.$$

Moreover, in this case, the sequence (u_n) converges.

Proof. We assume without loss of generality that $F(\varphi) = 0$. Since φ is a local minimizer of F, there exists $\rho > 0$ such that

$$\forall u \in \mathbf{R}^d, \quad \|u - \varphi\| < \rho \ \Rightarrow \ F(u) \geq 0. \tag{19}$$

Moreover, since F satisfies the Kurdyka-Łojasiewicz inequality at φ, there exist $\sigma > 0$ and a function Θ satisfying (6) and (7).
Let $\eta > 0$ and let us set

$$\bar{\eta} \ := \ \min(\rho, \sigma, \eta),$$

We fix $\varepsilon \in (0, \eta)$ such that for every $u \in M$

$$\|u - \varphi\| < \varepsilon \quad \Longrightarrow \quad \|u - \varphi\| + (1/C)\Phi(F(u)) < \bar{\eta}.$$

Then we consider a sequence (u_n) satisfying (H4) and we assume that there exists $\bar{n} \geq 0$ such that $\|u_{\bar{n}} - \varphi\| < \varepsilon$. Then, as in the proof of the previous result, we define

$$N \ := \ \sup\left\{n \geq \bar{n} : \|u_k - \varphi\| < \bar{\eta}, \quad \forall \bar{n} \leq k \leq n\right\},$$

and assume by contradiction that N is finite. As in the proof of the preceding result, we establish and sum the Kurdyka-Łojasiewicz inequalities (17) for $\bar{n} \leq n \leq N$ to get:

$$\sum_{n=\bar{n}}^{N} \|u_{n+1} - u_n\| \ \leq \ \frac{1}{C}\left\{\Phi(F(u_{\bar{n}})) - \Phi(F(u_N))\right\}.$$

By definition of N and (19), we have $F(u_N) \geq 0$ so that $\Phi(F(u_N)) \geq 0$ and (18) holds. We deduce

$$\|u_{N+1} - \varphi\| \ \leq \ \frac{1}{C}\Phi(F(u_{\bar{n}})) + \|u_{\bar{n}} - \varphi\| \ < \ \bar{\eta},$$

which contradicts the definition of N. The convergence of (u_n) then follows from (18). □

Remark 3.3. The result does not hold if we only assume that φ is a critical point of F. In this case even if $u_{\bar{n}}$ is very close to φ, the sequence may escape the neighborhood of φ by taking values $F(u_n) < F(\varphi)$. In this case the proof is no more valid. Indeed, we still have the key estimate

$$\sum_{n=\bar{n}}^{N} \|u_{n+1} - u_n\| \ \leq \ c\{\Phi\left(F(u_{\bar{n}})\right) - \Phi(F(u_N))\}\,,$$

but we can not bound the right hand side by $c\Phi(F(u_{\bar{n}}))$.

In order to prove a convergence rate result of type 3, we need to supplement (H4) with the following hypothesis: there exists $C_2 > 0$ such that for every $n \geq 0$,

$$\|u_{n+1} - u_n\| \ \geq \ C_2\|\nabla F(u_n)\|. \tag{H6}$$

In the case of numerical discretizations of the gradient flow (3) (or more generally of a gradient-like system (1)), the quantity $\|u_{n+1} - u_n\|$ behaves like $\Delta t\,\|\nabla F(u_n)\|$, where Δt is the time step. For these applications, the constant C_2 in the above hypothesis should scale as Δt: we expect $C_2 = C_2'\Delta t$. In this context, the factors $C_2 n$ in the convergence rates (20) below, have the form $C_2' t_n$. So, these rates are uniform with respect to the time step Δt. In fact we recover the convergence rates of the continuous case (with possibly different prefactors).

Theorem 3.4. *Let $(u_n) \subset M$. Assume that hypotheses $(H0)$, $(H1')$ and $(H5)$ hold and that there exist $C, C_2 > 0$ such that* (H4) *and* (H6) *hold for $n \geq 0$. Then there exists $\bar{n} \geq 0$ such that for all $n \geq \bar{n}$*

$$\|u_n - \varphi\| \ \leq \ \begin{cases} \lambda_1 e^{-\lambda_2 C_2 n} & \textit{if } \ \nu = 1/2, \\ \lambda_2\left(C_2 n\right)^{-\nu/(1-2\nu)} & \textit{if } \ 0 < \nu < 1/2. \end{cases} \tag{20}$$

where ν is the Lojasiewicz exponent of F at point φ and λ_1, λ_2 are positive constants depending on C, β and ν.

Proof. First let us recall some facts from the proof of Theorem 3.1. We know that (u_n) converges to φ and that the sequence $(F(u_n))$ is non-increasing and converges to $F(\varphi)$ that we assume again to be zero. Let $\tilde{n} \geq 0$ such that $\|u_n - \varphi\| < \sigma$ for $n \geq \tilde{n}$, we can apply (18) with $\bar{n} = n$ and $N = +\infty$ for every $n \geq \tilde{n}$. Here, the function Φ defined by (9) has the explicit form $\Phi(f) = (\beta/\nu)f^\nu$ and estimate (18) yields

$$\|u_n - \varphi\| \ \leq \ \frac{\beta}{C\nu}[F(u_n)]^\nu, \qquad \forall n \geq \bar{n}. \tag{21}$$

Next, let us define the function $K : (0, +\infty) \to (0, +\infty)$ by

$$K(x) \ = \ \begin{cases} -\ln x & \text{if } \ \nu = 1/2 \\ \dfrac{1}{(1-2\nu)x^{1-2\nu}} & \text{if } \ 0 < \nu < 1/2 \end{cases}$$

The sequence $(K(F(u_n)))$ is non-decreasing and tends to infinity. Using (H4) and (H6), we have

$$\begin{aligned} K(F(u_{n+1})) - K(F(u_n)) \ &= \ \int_{F(u_{n+1})}^{F(u_n)} \frac{dx}{x^{2-2\nu}} \ \geq \ [2F(u_{n+1})]^{2\nu-2}\,[F(u_n) - F(u_{n+1})] \\ &\overset{\text{(H4)(H6)}}{\geq} \ CC_2\|\nabla F(u_n)\|^2/\,[F(u_{n+1})]^{2-2\nu}\,. \end{aligned}$$

Applying (7) in the right hand side of the last inequality, we get

$$K(F(u_{n+1})) - K(F(u_n)) \ \geq \ CC_2/\beta^2, \qquad \forall n \geq \bar{n}.$$

Summing from $\bar{n}$ to $n-1$, we get that there exists $c_1 \in \mathbf{R}$ such that

$$K(F(u_n)) \;\geq\; (CC_2/\beta^2)n + c_1, \qquad \forall n \geq \bar{n}. \tag{22}$$

Now let us consider the case $\nu = 1/2$, we have $K(F(u_n)) = -\ln(F(u_n))$, so we get

$$F(u_n) \;\leq \lambda e^{-(CC_2/\beta^2)n}, \qquad \forall n \geq \bar{n},$$

with $\lambda = e^{-c_1}$. Recalling (21), the Theorem is proved in this case.
Eventually, if $0 < \nu < 1/2$, (22) reads

$$F(u_n) \;\leq\; \left[\frac{1-2\nu}{(CC_2/\beta^2)n + c_1}\right]^{1/(1-2\nu)}, \qquad \forall n \geq \bar{n}.$$

Again, (21) completes the proof. □

4. **The θ-scheme and a projected θ-scheme.** In this section, we show that the convergence results of Section 3 apply to some numerical schemes associated to system (1) under the set of hypotheses $(H0)$, $(H1)$, $(H2)$, $(H3)$ ($(H0)$, $(H1')$, $(H2)$, $(H3)$ for the convergence rate). We also need some regularity assumptions on G and F.

4.1. **The θ-scheme in $\mathbf{R}^d$.** We first consider the θ-scheme in the case $M = \mathbf{R}^d$. Recall that for a fixed $\theta \in [0,1]$, the θ-scheme associated to equation (1) reads:

$$\frac{u_{n+1} - u_n}{\Delta t} \;=\; \theta G(u_{n+1}) + (1-\theta)G(u_n). \tag{23}$$

Lemma 4.1. *Let $\theta \in [0,1]$ and let (u_n) be a sequence that complies to the θ-scheme (23). Assume that G is Lipschitz continuous and that hypotheses (H2) and (H3) hold. Then there exist $\mu_1, \mu_2, \Delta t' > 0$ such that for $\Delta t \in (0, \Delta t')$,*

$$F(u_{n+1}) + \mu_1 \frac{\|u_{n+1} - u_n\|^2}{\Delta t} \;\leq\; F(u_n), \qquad \forall n \geq 0, \tag{24}$$

and

$$\frac{\|u_{n+1} - u_n\|}{\Delta t} \;\geq\; \mu_2 \|\nabla F(u_n)\|, \qquad \forall n \geq 0. \tag{25}$$

Proof. First we establish (25). We rewrite the θ-scheme in the form

$$\frac{u_{n+1} - u_n}{\Delta t} \;=\; G(u_n) + \theta\left[G(u_{n+1}) - G(u_n)\right].$$

Denoting $K \geq 0$ the Lipschitz constant of G on $\mathbf{R}^d$ and using the comparability condition (11), we deduce

$$(1 + K\Delta t)\frac{\|u_{n+1} - u_n\|}{\Delta t} \;\geq\; \|G(u_n)\| \;\overset{(11)}{\geq}\; \frac{\gamma}{2}\|\nabla F(u_n)\|.$$

So (25) holds with $\mu_2 = \gamma/4$ as soon as $\Delta t \in (0, 1/K)$.

Next we prove (24). By assumption $(H2)$ we can apply Theorem 2.13 and there exists a metric g on $\tilde{M} := \mathbf{R}^d \setminus \{v : G(v) = 0\}$ satisfying (14) and such that

$$\langle -\nabla F(u), w\rangle = \langle G(u), w\rangle_{g(u)}, \qquad \forall u \in \tilde{M}, w \in \mathbf{R}^d. \tag{26}$$

Let us set $\delta_n := u_n - u_{n+1}$ and write the Taylor expansion,

$$\begin{aligned} F(u_n) \;&=\; F(u_{n+1}) + \left\langle \int_0^1 \nabla F(u_{n+1} + t\delta_n)\,dt, \delta_n \right\rangle \\ &=\; F(u_{n+1}) + \langle \nabla F(u_{n+1}), \delta_n\rangle + \left\langle \int_0^1 \nabla F(u_{n+1} + t\delta_n) - \nabla F(u_{n+1}), \delta_n \right\rangle dt. \end{aligned}$$

Applying assumption (H3) with $u = u_{n+1} + t\delta_n$ and $v = u_{n+1}$, we get

$$F(u_n) - F(u_{n+1}) - \langle \nabla F(u_{n+1}), \delta_n \rangle \geq -c \int_0^1 t\|\delta_n\|^2 \, dt,$$

that is,

$$F(u_{n+1}) - F(u_n) \ \leq \ \langle \nabla F(u_{n+1}), u_{n+1} - u_n \rangle + (c/2)\|u_{n+1} - u_n\|^2.$$

If $u_{n+1} \in \tilde{M}$, then we may apply (26) with $u = u_{n+1}$ and get

$$F(u_{n+1}) - F(u_n) \ \leq \ \langle G(u_{n+1}), u_n - u_{n+1} \rangle_{g(u_{n+1})} + (c/2)\|u_{n+1} - u_n\|^2. \tag{27}$$

If $u_{n+1} \notin \tilde{M}$, then by the comparability condition we have $G(u_{n+1}) = \nabla F(u_{n+1}) = 0$ and this estimate still holds if we set $\langle \cdot, \cdot \rangle_{g(u_{n+1})}$ to be the usual scalar product. Adding the term

$$0 \ = \ \left\langle \frac{u_{n+1} - u_n}{\Delta t} - \theta G(u_{n+1}) - (1-\theta) G(u_n) \, , \, u_n - u_{n+1} \right\rangle_{g(u_{n+1})}$$

to the right hand side of (27), we get

$$\begin{aligned} F(u_{n+1}) - F(u_n) \ \leq \ & -(1/\Delta t)\|u_{n+1} - u_n\|^2_{g(u_{n+1})} \ + \ (c/2)\|u_{n+1} - u_n\|^2 \\ & + \ (1-\theta) \langle G(u_{n+1}) - G(u_n), u_n - u_{n+1} \rangle_{g(u_{n+1})} . \end{aligned}$$

Combining this estimate with (14), we obtain

$$F(u_{n+1}) - F(u_n) \ \leq \ -\mu_{\Delta t} \frac{\|u_{n+1} - u_n\|^2}{\Delta t},$$

with $\mu_{\Delta t} := c_1^2 - \Delta t(c/2 + (1-\theta)K c_2^2)$ where K is the Lipschitz constant of G. The parameter $\mu_{\Delta t}$ being larger than the positive constant $\mu_1 := c_1^2/2$ for Δt small enough (24) is proved. □

Corollary 4.1. *Let $\theta \in [0,1]$ and let (u_n) be the sequence defined by the θ-scheme* (23). *Assume hypotheses* $(H2)$, $(H3)$ *hold, that G is Lipschitz and that $\Delta t \in (0, \Delta t')$. Then:*

- *If* $(H0)$, $(H1)$ *hold, the sequence (u_n) converges to φ.*
- *If F satisfies a Kurdyka-Łojasiewicz inequality in the neighborhood of some local minimizer φ then for every $\eta > 0$ there exists $\varepsilon \in (0, \eta)$ such that*

$$\|u_{\bar{n}} - \varphi\| < \varepsilon \quad \Longrightarrow \quad \|u_n - \varphi\| < \eta, \ \forall n \geq \bar{n}.$$

- *If* $(H0)$, $(H1')$ *hold, the sequence (u_n) converges to φ with convergence rates given by* (20) *with $C_2 = \mu_1 \mu_2^2 \Delta t$.*

Proof. From (24) and (25) of Lemma 4.1, we easily obtain (H4), (H5) and (H6) with $C = \mu_1 \mu_2$ and $C_2 = \mu_1 \mu_2^2 \Delta t$. The result is then a consequence of Theorems 3.1, 3.2 and 3.4. □

4.2. **A projected θ-scheme.** We now consider an embedded C^2-manifold $M \subset \mathbf{R}^d$ without boundary and with a uniformly bounded curvature. We present a simple scheme for the approximation of (1). This scheme has two steps. The first step requires a family of mappings $\{G_u : u + T_u M \to T_u M\}_{u \in M}$. The mapping G_u should approximate G around u. Natural choices are

$$G_u(u+v) \ := \ G(u) \qquad \text{or} \qquad G_u(u+v) \ := \ G(u) + \langle \nabla G(u), v \rangle \quad \forall v \in T_u M.$$

Starting from $u_n \in M$, the first step of the scheme is just the computation of an approximation v_n of $\dot{u}$ thanks to the classical θ-scheme applied to the system $\dot{u} = G_{u_n}(u)$. We obtain an intermediate iterate $\tilde{u}_{n+1} := u_n + \Delta t \, v_n$ which does not belong to M in general. The second step consists in projecting $\tilde{u}_{n+1}$ on the manifold M. Here, to fix the ideas, we only consider the orthogonal projection

$$\Pi_M(u) \ \in \ \text{argmin}\{\|v - u\|^2 : v \in M\}.$$

Other choices are admissible as soon as (33) holds. If M is the boundary of a convex set S, then $\tilde{u}_{n+1} \in u_n + T_{u_n}M$ belongs to $\mathbf{R}^d \setminus S$, so $\Pi_M(\tilde{u}_{n+1}) = \Pi_{\bar{S}}(\tilde{u}_{n+1})$ and the orthogonal projection is uniquely defined. This is not true for general M. Anyway, here M is of class C^2 and we assume that the projection is uniquely defined as soon as $d(\tilde{u}_{n+1}, M) < \delta$ for some $\delta > 0$.

More precisely, the projected θ-scheme described above is defined as follows. Let us choose a fixed parameter $\theta \in [0,1]$ and let $u_0 \in M$. Then for $n = 0, 1, 2, ...$

$$\left[\begin{array}{l} \textbf{step 1. } \text{Find } v_n \in T_{u_n}M \text{ such that} \\ \qquad\qquad v_n = \theta G_{u_n}(u_n + \Delta t\, v_n) + (1-\theta) G_{u_n}(u_n). \\ \\ \textbf{step 2. } \text{Set } u_{n+1} := \Pi_M(u_n + \Delta t\, v_n). \end{array}\right. \tag{28}$$

In this manifold context, we need to strengthen the previous regularity hypotheses. We will assume for simplicity that the family $\{G_u\}$ satisfies

$$G_u(u) \;=\; G(u) \qquad \forall u \in M. \tag{29}$$

We assume that G is bounded and that G, ∇F and the family of mappings $\{G_u\}$ are uniformly Lipschitz continuoud, *i.e.* there exist $Q, K > 0$ such that

$$\|G(u)\| \;\le\; Q, \qquad \forall u \in M, \tag{30}$$

$$\|G_u(u+v) - G_u(u+v')\| \;\le\; K\|v - v'\| \qquad \forall u \in M,\, \forall v, v' \in T_uM, \tag{31}$$

$$\|\nabla F(u) - \nabla F(u')\| \;\le\; K\|u - u'\| \qquad \forall u, u' \in M. \tag{32}$$

We also assume that the projection acts only at second order, that is there exists $\delta, R > 0$ such that

$$\|\Pi_M(u+v) - (u+v)\| \;\le\; R\|v\|^2 \qquad \forall u \in M,\, \forall v \in T_uM \text{ such that } \|v\| < \delta. \tag{33}$$

With these hypotheses, we have the analogue of Lemma 4.1:

Lemma 4.2. *Let $\theta \in [0,1]$ and let (u_n) be the sequence defined by the projected θ-scheme* (28). *Assume that (29,30,31,32) and that (H2) hold. Then there exist $\mu_1, \mu_2, \Delta t' > 0$ such that for $\Delta t \in (0, \Delta t')$,*

$$F(u_{n+1}) + \mu_1 \frac{\|u_{n+1} - u_n\|^2}{\Delta t} \;\le\; F(u_n), \qquad \forall n \ge 0, \tag{34}$$

and

$$\frac{\|u_{n+1} - u_n\|}{\Delta t} \;\ge\; \mu_2 \|\nabla F(u_n)\|, \qquad \forall n \ge 0. \tag{35}$$

Proof. First we establish that the sequence (v_n) is bounded. Indeed, by (29) the first step of the scheme reads,

$$v_n \;=\; G(u_n) + \theta \left[G_{u_n}(u_n + \Delta t v_n) - G_{u_n}(u_n)\right],$$

and we deduce from (30) and (31) the estimate $(1 - K\Delta t)\|v_n\| \le Q$. So, for $\Delta t \in (0, 1/(2K))$, we have

$$\|v_n\| \;\le\; 2Q.$$

Next, for Δt small enough, we have $\Delta t\|v_n\| < \delta$ and the projection step is well defined. Moreover, by (33), we have

$$v_n \;=\; \frac{u_{n+1} - u_n}{\Delta t} \;+ q_n,$$

with $\|q_n\| \le R\Delta t\|v_n\|^2$, so there exists $\alpha \in (0,1)$ such that for Δt small enough, we have

$$\alpha\|c\|^2 \;\le\; \alpha\left\langle a, b\right\rangle \;\le \alpha^{-1}\|c\|^2 \tag{36}$$

for any triplet of vectors a, b, c in the set $\{v_n, (1/\Delta t)(u_{n+1} - u_n)\}$. Similarly, we deduce from the angle and comparability condition $(H2)$, that (36) holds for any choice a, b, c in the set

$$\{G(u_n),\ -\nabla F(u_n),\ v_n,\ (1/\Delta t)(u_{n+1} - u_n)\}.$$

In particular, (35) holds.

Eventually, since F is of class $C^{1,1}$, for Δt small enough, we have

$$F(u_n) - F(u_{n+1}) \ \geq\ \Delta t\, \langle -\nabla F(u_n), v_n\rangle - H(\Delta t)^2 \|v_n\|^2 \ \overset{(36)}{\geq} \ \left(\alpha - \frac{H}{\alpha}\Delta t\right) \frac{\|u_{n+1} - u_n\|^2}{\Delta t},$$

for some $H > 0$. Thus, for Δt small enough, (34) holds with $\mu_1 = \alpha/2$. □

Corollary 4.2. *Let $\theta \in [0, 1]$ and let (u_n) be the sequence defined by the projected θ-scheme* (28). *Assume that (29,30,31,32) and hypothesis $(H2)$ hold and that $\Delta t \in (0, \Delta t')$. Then:*

- *If moreover the hypotheses $(H0)$, $(H1)$ hold, the sequence (u_n) converges to φ.*
- *If F satisfies a Kurdyka-Łojasiewicz inequality in the neighborhood of some local minimizer φ, then for every $\eta > 0$ there exists $\varepsilon \in (0, \eta)$ such that*

$$\|u_{\bar n} - \varphi\| < \varepsilon \quad \Longrightarrow \quad \|u_n - \varphi\| < \eta,\ \forall n \geq \bar n.$$

- *If the hypotheses $(H0)$, $(H1')$ hold, the sequence (u_n) converges to φ with convergence rates given by* (20) *with $C_2 = \mu_1 \mu_2^2 \Delta t$.*

Proof. The proof is the same as the proof of Corollary 4.1, simply replace Lemma 4.1 by Lemma 4.2. □

5. **The Backward Euler Scheme in $\mathbf{R}^d$.** The results of the previous section are easily extended to general Runge-Kutta schemes. The counterpart of this generality is that quite strong regularity assumptions are made on F , G_u or G. Here, we show that these assumptions may be relaxed in the case of the backward Euler scheme in the case $M = \mathbf{R}^d$

We assume $M = \mathbf{R}^d$. Recall that the backward Euler scheme associated to equation (1) reads:

$$\frac{u_{n+1} - u_n}{\Delta t} \ =\ G(u_{n+1}), \qquad n \geq 0, \tag{37}$$

where $u_0 \in \mathbf{R}^d$ is the initial condition and $\Delta t > 0$ is the time step. We establish convergence Results of type 1 and 2 and a convergence rate result for these schemes under a Łojasiewicz inequality $(H1)$ (or $(H1')$), the angle and comparability condition $(H2)$, and (as unique regularity assumption) the one sided Lipschitz condition $(H3)$. These results extend Theorem 2.4 and Proposition 2.5 in [18] to gradient-like systems.

As a first step, we use Theorem 2.13 to show that the solution of the scheme (37) can be interpreted as a minimizer.

Lemma 5.1. *Assume that ∇F and G satisfy the angle and comparability condition (H2) and that F satisfies the one sided Lipschitz condition (H3). There exists $\Delta t^* > 0$ such that for $\Delta t \in (0, \Delta t^*)$, if $(u_n) \subset \mathbf{R}^d$ complies with the backward Euler scheme* (37)*, then for every $n \geq 0$, u_{n+1} is the unique minimizer of the functional*

$$E^n(v) \ :=\ F(v) + \frac{\|v - u_n\|^2_{g(u_{n+1})}}{2\Delta t},$$

where g is the Riemannian metric provided by Theorem 2.13.

Remark 5.2. The above functional E^n depends on the point u_{n+1} through the local metric $g(u_{n+1})$ so (for a non-constant metric) the minimization problem can not be used as a definition of the scheme.

Proof. Since F is a strict Lyapunov function, we may apply Theorem 2.13. Let g be the Riemannian metric on $\tilde{M} = \{u \in \mathbf{R}^d : G(u) \neq 0\}$ provided by this Theorem. For any $u \in \tilde{M}$, the map $(X,Y) \in \mathbf{R}^d \times \mathbf{R}^d \mapsto \langle X, Y\rangle_{g(u)}$ is a coercive symmetric bilinear form, so there exists a $d \times d$ symmetric positive definite matrix $A(u)$ such that

$$\langle X, Y\rangle_{g(u)} \;=\; \langle A(u)X, Y\rangle\,, \qquad \forall X, Y \in \mathbf{R}^d.$$

With this notation, the gradient of E^n reads

$$\nabla E^n(v) \;=\; \nabla F(v) + (1/\Delta t)A(u_{n+1})(v - u_n).$$

Thus, for every $u, v \in \mathbf{R}^d$

$$\begin{aligned}\langle \nabla E^n(u) - \nabla E^n(v), u - v\rangle \;&=\; \langle \nabla F(u) - \nabla F(v) + (1/\Delta t)A(u_{n+1})(u - v)\,,\, u - v\rangle \\ &=\; \langle \nabla F(u) - \nabla F(v)\,,\, u - v\rangle + (1/\Delta t)\|u - v\|^2_{g(u_{n+1})}.\end{aligned}$$

Using the one-sided Lipschitz condition (15) and (14), we get

$$\langle \nabla E^n(u) - \nabla E^n(v), u - v\rangle \;\geq\; (c_1^2 - c\Delta t)\|u - v\|^2/\Delta t.$$

So E^n is strongly convex for all $\Delta t < \Delta t^* := c_1^2/c$. Hence, it admits a unique minimizer v_{n+1} characterized by $\nabla E^n(v_{n+1}) = 0$, that is:

$$\begin{aligned}0 =&\; \nabla F(v_{n+1}) + (1/\Delta t)A(u_{n+1})(v_{n+1} - u_n) \\ =&\; A(u_{n+1})\left[-G(v_{n+1}) + (1/\Delta t)(v_{n+1} - u_n)\right],\end{aligned}$$

which is equivalent to the fact that v_{n+1} solves (37). Consequenlty $u_{n+1} = v_{n+1}$ is uniquely defined as the unique minimizer of E^n as claimed. □

Notice that Lemma 5.1 implies that for $\Delta t < \Delta t^*$, if $G(u_N) = 0$ for some $N \geq 0$, then $u_n = u_N$ for every $n \geq N$. In such a case, there is nothing to prove concerning convergence. In the sequel, we assume $G(u_n) \neq 0$ (that is $u_n \in \tilde{M}$) for every $n \geq 0$.

We now state Result 1 for the backward Euler scheme.

Theorem 5.3. *Assume the set of hypotheses* (H0), (H1), (H2), (H3), *let* $\Delta t \in (0, \Delta t^*)$, *where* Δt^* *is as in Lemma 5.1, and let* $(u_n)_{n\geq 0}$ *be a sequence defined by* (37), *then the sequence* (u_n) *converges to* φ *as* n *goes to infinity.*

Proof. Assume $\Delta t < \Delta t^*$, by Lemma 5.1, we have $E^n(u_{n+1}) \leq E^n(u_n)$, that is

$$\frac{\|u_{n+1} - u_n\|^2_{g(u_{n+1})}}{2\Delta t} + F(u_{n+1}) \;\leq\; F(u_n) \tag{38}$$

Thus, the sequence $(F(u_n))$ is non-increasing. We assume again without loss of generality that $F(\varphi) = 0$, so $F(u_n) \downarrow 0$ as $n \uparrow \infty$.

Next, since F satisfies the Kurdyka-Lojasiewicz inequality at φ, there exist $\sigma > 0$ and a function Θ satisfying (6) and (7). Let us fix $n \geq 0$ and consider the continuous problem

$$v(0) = u_n, \qquad \dot{v} = -\nabla F(v)$$

From the study of the continuous case, we know that if $\|u_n - \varphi\| < \varepsilon < \sigma/2$, with ε small enough then $v(t)$ remains in $B(\varphi, \sigma/2)$ for any time $t > 0$. Moreover, $v(t)$ converges to $v^* \in B(\varphi, \sigma)$ as t tends to infinity. At the limit, we have $\nabla F(v^*) = 0$ and by the Kurdyka-Łojasiewicz inequality, this leads to $F(v^*) = 0$. Now, since $F(u_{n+1}) \geq 0$, there exists $T \in (0, +\infty]$ such that $F(v(T)) = F(u_{n+1})$. Then, from the optimality of u_{n+1}, we have $\|u_{n+1} - u_n\|_{g(u_{n+1})} \leq \|v(T) - u_n\|_{g(u_{n+1})}$. By (14), this leads to $\|u_{n+1} - u_n\| \leq c_2/c_1\|v(T) - u_n\|$.

On the other hand, using the notation and computations of the proof of Theorem 2.7 (see (9), (10)), we have,

$$\begin{aligned}\Phi\left(F(u_n)\right) - \Phi\left(F(u_{n+1})\right) &= \Phi\left(F(v(0))\right) - \Phi\left(F(v(T))\right) = -\int_0^T \frac{d}{ds}\left[\Phi\left(F(v(s))\right)\right] ds \\ &\overset{(10)}{\geq} \alpha \int_0^T \|\dot{v}(s)\| ds = \alpha \|v(T) - u_n\|.\end{aligned}$$

Therefore, if $\|u_n - \varphi\| < \varepsilon < \sigma/2$, with ε small enough then

$$\|u_{n+1} - u_n\| \leq c_1/(c_2\alpha)\left(\Phi\left(F(u_n)\right) - \Phi\left(F(u_{n+1})\right)\right). \tag{39}$$

Finally, using (39) and summation, we conclude as in the proof of Theorem 3.1 that the sequence (u_n) converges to φ. □

As in Section 3, we also have a result of type 2:

Theorem 5.4. *Assume that hypotheses (H2) and (H3) hold and let φ be a local minimizer of F such that F satisfies a Kurdyka-Łojasiewicz inequality in a neighborhood of φ.*
Then, for every $\eta > 0$ there exists $\varepsilon \in (0, \eta)$ such if $\Delta t \in (0, \Delta t^*)$, where Δt^* is as in Lemma 5.1, and if (u_n) is a solution of the scheme (37)*, we have*

$$\|u_{\bar{n}} - \varphi\| < \varepsilon \quad \Longrightarrow \quad \|u_n - \varphi\| < \eta, \ \forall n \geq \bar{n}.$$

Proof. Theorem (5.4) is proved along the lines of Theorem 3.2. We do not repeat the arguments. □

Eventually, if the Łojasiewicz inequality holds then we can estimate the convergence rate in Theorem 5.3.

Theorem 5.5. *Assume that the set of hypotheses* $(H0)$, $(H1')$, $(H2)$, $(H3)$ *holds, let $\Delta t \in (0, \Delta t^*)$ and let $(u_n) \subset \mathbf{R}^d$ be a solution of* (37)*. There exist $\bar{n} \geq 0$ and $\lambda_1, \lambda_2 > 0$ such that for all $n \geq \bar{n}$*

$$\|u_n - \varphi\| \leq \begin{cases} \lambda_1 e^{-\lambda_2 n \Delta t} & \text{if } \nu = 1/2 \\ \lambda_2 \left(n\Delta t\right)^{-\nu/(1-2\nu)} & \text{if } 0 < \nu < 1/2 \end{cases}$$

where ν is the Łojasiewicz exponent of F at the point φ.

Proof. The proof is the same as the proof of Proposition 2.5 in [18]. □

6. **Harmonic maps and harmonic map flow.** In this section, we consider a discretization of the following problem: given $\Omega \subset \mathbf{R}^d$ a bounded domain with a Lipschitz boundary and given $g \in H^{1/2}(\partial\Omega, S^{l-1})$, find a critical point of the Dirichlet energy

$$\mathcal{D}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2,$$

under the constraint

$$u \in H^1_g(\Omega, S^{l-1}) := \left\{v \in H^1(\Omega, \mathbf{R}^l) : v = g \text{ on } \partial\Omega, \, |v(x)| = 1 \text{ a.e. in } \Omega\right\}.$$

Remark 6.1. For $d = 3$ and $l = 2$, the energy $\mathcal{D}$ appears as a simplified model for the Oseen-Franck energy of nematic liquid crystals. In this context the mapping $u : \Omega \to S^2$ represents the orientation of the molecules.

It is well known that such maps exist (for example, we may solve the minimization problem by considering a minimizing sequence and using the relative weak compactness of bounded subsets of H^1_g). Such maps are called harmonic maps with values in S^{l-1}. They are characterized by the following condition: $u \in H^1_g(\Omega, S^{l-1})$ satisfies the non-linear system:

$$-\Delta u \;=\; |\nabla u|^2 u \text{ in } \mathcal{D}'(\Omega). \tag{40}$$

During the preceding decades, many authors have considered existence and regularity problems related to these harmonic maps (see e.g. [9, 11, 14, 19] and a rather complete overview in [12]).

F. Alouges proposed in [2, 3] an efficient algorithm for finding numerical approximations of minimizing harmonic maps. In the continuous case, the algorithm reads as follows: Given an initial guess $u_0 \in H^1_g(\Omega, S^{l-1})$, compute for $n = 0, 1, \cdots$

$$\left[\begin{array}{l} \textbf{step 1.} \text{ Find } v_n \text{ minimizing } v \mapsto \mathcal{D}(u_n + v) \text{ in } K_{u_n} \text{ with} \\[1em] \qquad K_u \;:=\; \left\{v \in H^1_0(\Omega, \mathbf{R}^l),\; u(x)\cdot v(x) = 0 \text{ for a.e. } x \in \Omega\right\}. \\[1em] \textbf{step 2.} \text{ Set } u_{n+1}(x) := \dfrac{u_n(x) + v_n(x)}{\|u_n(x) + v_n(x)\|} \quad \forall x \in \Omega. \end{array}\right. \tag{41}$$

By construction, we have $\mathcal{D}(u_n + v_n) \le \mathcal{D}(u_n)$ so the energy decreases during the first step. Notice that for every $x \in \Omega$, $\|u_n(x) + v_n(x)\|^2 = 1 + \|v_n(x)\|^2 \ge 1$, so, the second step is the projection of $u_n(x)+v_n(x)$ on the closed unit ball of $\mathbf{R}^l$. This ball being convex, this projection is a contraction and we have $\mathcal{D}(u_{n+1}) \le \mathcal{D}(u_n + v_n)$. Consequently $\mathcal{D}(u_{n+1}) \le \mathcal{D}(u_n)$ and the algorithm is energy decreasing. It is also established in [3] that, up to extraction, the sequence (u_n) weakly converges to a harmonic map in $H^1(\Omega)$.

Let us now discretize in space. As in [3], we choose a finite difference approximation. Let us fix $h > 0$ and a finite subset Ω^h of $h\mathbf{Z}^d$ which stands as the discrete domain. The corresponding discrete energy is

$$\mathcal{D}^h(u^h) \;:=\; \frac{h^d}{4} \sum_{x,y\in\Omega^h,\, \|x-y\|=h} \frac{\|u^h(x) - u^h(y)\|^2}{h^2},$$

defined for any mapping $u^h \in H^h := (\mathbf{R}^d)^{\Omega^h}$. The discrete boundary is supposed to be a non-empty subset Γ^h of Ω^h such that every point $x \in \Omega^h$ is connected to Γ^h by a finite path in Ω^h, i.e.

$$\text{there exist } x = x^h_0, x^h_1, \cdots, x^h_p \in \Omega^h \text{ with } \|x^h_i - x^h_{i-1}\| = h \text{ and } x^h_p \in \Gamma^h. \tag{42}$$

The discrete boundary condition is a given function $g^h : \Gamma^h \to S^{l-1}$.

Our discrete problem is then: find minimizers or critical points of $\mathcal{D}^h$ in the manifold

$$M^h_{g^h} \;:=\; \left\{u^h \in H^h : u^h(x) = g^h(x), \forall x \in \Gamma^h\,;\; \|u^h(x)\| = 1,\, \forall x \in \Omega^h\right\}.$$

Let us characterized these critical points. For this, we compute the differential of $\mathcal{D}^h$ at some point $u^h : \Omega^h \to \mathbf{R}^l$. Let $u^h, v^h : \Omega^h \to \mathbf{R}^l$, we have

$$\mathcal{D}^h(u^h + v^h) \;=\; \mathcal{D}^h(u^h) + h^d \sum_{x\in\Omega^h} \left\langle \frac{1}{h^2} \sum_{y\in\Omega^h,\, \|x-y\|=h} (u^h(x) - u^h(y))\,,\, v^h(x) \right\rangle + \mathcal{D}^h(v^h),$$

so if we introduce the scalar product

$$\left\langle v^h, w^h\right\rangle_{H^h} \;:=\; h^d \sum_{x\in\Omega^h} \left\langle v^h(x)\,,\, w^h(x)\right\rangle, \qquad v^h, w^h : \Omega^h \to \mathbf{R}^l,$$

and use a discrete integration by parts, we obtain,

$$\left[\nabla_{H^h}\mathcal{D}^h(u^h)\right](x) \;=\; \frac{1}{h^2}\sum_{y\in\Omega^h,\,\|x-y\|=h}(u^h(x)-u^h(y)) \qquad \forall x\in\Omega^h. \tag{43}$$

Let us consider a point $u^h \in M^h_{g^h}$. The tangent space $T_{u^h}M^h_{g^h}$ at this point is

$$K^h_{u^h} \;:=\; \{v^h \in H^h_0 : v^h(x)\cdot u^h(x) = 0,\, \forall x \in \Omega^h \setminus \Gamma^h\},$$

where we have set

$$H^h_0 \;:=\; \left\{v^h \in H^h : v^h(x) = 0,\, \forall x \in \Gamma^h\right\}.$$

Taking into account the constraint $u^h(x) \in S^{l-1}$ for every $x \in \Omega^h$, we see that $u^h \in M^h_{g^h}$ is a critical point of $\mathcal{D}^h$ in $M^h_{g^h}$ if and only if there exists $\lambda^h : \Omega^h \setminus \Gamma^h \to \mathbf{R}$ such that

$$\left[\nabla_{H^h}\mathcal{D}^h(u^h)\right](x) + \lambda^h(x)u^h(x) \;=\; 0, \qquad \forall x\in\Omega^h\setminus\Gamma^h. \tag{44}$$

Definition 6.2. If $u^h \in M^h_{g^h}$ satisfies (44), we say that it is a discrete harmonic map.

The scalar product $\langle\,,\,\rangle_{H^h}$ is consistent with the L^2-scalar product in the space $L^2(\Omega,\mathbf{R}^l)$. Another interesting bilinear form on H^h, associated to the energy, is

$$\left\langle v^h\,,\,w^h\right\rangle_{\mathcal{D}^h} := \;\frac{h^d}{2}\sum_{x,y\in\Omega^h,\,\|x-y\|=h}\left\langle \frac{v^h(x)-v^h(y)}{h},\, \frac{w^h(x)-w^h(y)}{h}\right\rangle.$$

With this definition, we have $\|u^h\|^2_{\mathcal{D}^h} = 2\mathcal{D}^h(u^h)$ for any $u^h \in H^h$. Under the connectivity assumption (42), this bilinear form defines a scalar product on the subspace H^h_0. Eventually, notice that a discrete integration by parts yields

$$\left\langle\nabla_{H^h}\mathcal{D}^h(v^h)\,,\,w^h\right\rangle_{H^h} \;=\; \left\langle v^h\,,\,w^h\right\rangle_{\mathcal{D}^h}, \qquad \forall v^h, w^h\in H^h_0. \tag{45}$$

The discretized version of (41) reads: Given an ininitial guess $u^h_0 \in M^h_{g^h}$, compute for $n = 0,1,\cdots$

$$\left[\begin{array}{l}\textbf{step 1. } \text{Find } v^h_n \text{ minimizing } v^h \mapsto \mathcal{D}^h(u^h_n + v^h_n) \text{ in } K^h_{u^h_n}.\\[2ex] \textbf{step 2. } \text{Set } u^h_{n+1}(x) := \dfrac{u^h_n(x)+v^h_n(x)}{\|u^h_n(x)+v^h_n(x)\|} \quad \forall x\in\Omega^h.\end{array}\right. \tag{46}$$

Let us state some relevant properties of this algorithm.

Theorem 6.3 ([2, 3])**.**
1/ The algorithm (46) *is well defined. We have for $n \geq 0$,*

$$0 \;\leq\; \mathcal{D}^h(u^h_{n+1}) \;\leq\; \mathcal{D}^h(u^h_n + v^h_n) \;\leq\; \mathcal{D}(u^h_n), \qquad \forall n\geq 0. \tag{47}$$

In particular the sequence $(\mathcal{D}^h(u^h_n))$ is non-increasing and convergent.
Moreover, if equality occurs in one of the above inequalities, then $v^h_p = 0$ for every $p \geq 0$ and u_n is a discrete harmonic map. Conversely, if (u^h_n) is a discrete harmonic map, then $v^h_n = 0$ and the sequence (u^h_n) is stationary.
2/ There exists $\lambda^h_n : \Omega^h \setminus \Gamma^h \to \mathbf{R}$, such that the increment v^h_n satisfies the Euler-Lagrange equation,

$$\left[\nabla\mathcal{D}^h(u^h_n + v^h_n)\right](x) \;=\; \lambda^h_n(x)u^h_n(x), \qquad \forall x\in\Omega^h\setminus\Gamma^h. \tag{48}$$

In particular, we have

$$\mathcal{D}^h(v^h_n) \;=\; \mathcal{D}^h(u^h_n) - \mathcal{D}^h(u^h_n + v^h_h), \tag{49}$$

so that

$$\sum_{n\geq 0} \mathcal{D}^h(v_n^h) \;\leq\; \mathcal{D}^h(u_0^h) - \lim_{n\to\infty} \mathcal{D}^h(u_n^h) \;\leq\; \mathcal{D}^h(u_0^h), \tag{50}$$

and consequently $v_n^h \to 0$ as n goes to infinity.
3/ Up to extraction, the sequence (u_n^h) converges to a discrete harmonic map.

Proof. The proof of all these results can be found in [3]. However, for completeness, we establish (47,48,49,50), that turn out to be useful to our purpose.
The second inequality of (47) is obvious by definition of v_n^h. Now, notice, that if $u_n^h(x) \in S^{l-1}$ and $v_n^h(x)\cdot u_n^h(x) = 0$, then $\|u_n^h(x) + v_n^h(x)\|^2 = 1 + \|v_n^h(x)\|^2 \geq 1$. So, the second step of the algorithm is the projection of $u_n^h(x)+v_n^h(x)$ on the closed unit ball of $\mathbf{R}^d$. By convexity of this ball, this projection is a contraction for the Euclidian distance in $\mathbf{R}^d$, so for every $x, y \in \Omega^h$, we have

$$\|u_{n+1}(x) - u_{n+1}(y)\| \;\leq\; \|(u_n + v_n)(x) - (u_n + v_n)(y)\|,$$

and the first inequality of (47) follows from the very definition of the energy $\mathcal{D}^h$.

Next, for every $n \geq 0$, v_n^h minimizes the quadratic functional $v^h \mapsto \mathcal{D}^h(u_n^h + v^h)$ in the space $K^h_{u_n^h}$, so it satisfies the Euler equation (48) for some $\lambda_n^h : \Omega^h \setminus \Gamma^h \to \mathbf{R}$. We easily compute

$$\mathcal{D}(u_n^h) - \mathcal{D}(u_n^h + v_n^h) \;=\; -\left\langle \nabla\mathcal{D}^h(u_n^h + v_n^h)\,,\, v_n^h\right\rangle_{H^h} + \mathcal{D}^h(v_n^h) \;\overset{(48)}{=}\; \mathcal{D}^h(v_n^h).$$

Summing on $n = 0, \cdots$, and using (47), we obtain (50). The convergence of (v_n^h) to 0 then follows from the fact that $\sqrt{\mathcal{D}}$ is a norm on M_0^h (recall that Γ^h satisfies (42)). □

We can now use the result of Section 3 to improve the convergence result Theorem 6.3 3/ by showing that the whole sequence converges.

Theorem 6.4. *The sequence (u_n^h) built by the algorithm (46) converges to some discrete harmonic map φ^h. Moreover there exists $\nu \in (0, 1/2]$ and constants $\lambda_1, \lambda_2 > 0$, such that for every $n \geq 1$,*

$$\|u_n - \varphi\|_{H^h} \;\leq\; \begin{cases} \lambda_1 e^{-\lambda_2 n} & \text{if } \nu = 1/2, \\ \lambda_1 n^{-\nu/(1-2\nu)} & \text{if } 0 < \nu < 1/2. \end{cases}$$

Proof. We want to apply Theorems 3.1 and 3.4 to the sequence $(u_n) := (u_n^h)$, the manifold $M := M^h_{g^h} \subset H^h$ and the function $F := \mathcal{D}^h_{|M}$. First, recall that we use the scalar product $\langle\cdot,\cdot\rangle_{H^h}$ in the Euclidian space H^h. Since the manifold M is analytic and F is a polynomial function, we deduce (using analytical local charts of M) from Theorem 2.5 that F satisfies the Łojasiewicz inequality in the neighborhood of any point of M. The sequence (u_n^h) is bounded in the Euclidian space H^h so it admits an accumulation point $\varphi^h \in M$. In order to conclude, we only have to check that (H4) and (H6) hold.

By (47) and (48), we have

$$F(u_n^h) - F(u_{n+1}^h) \;\geq\; (1/2)\|v_n^h\|^2_{\mathcal{D}^h}. \tag{51}$$

Next, using the linearity of ∇F, we compute for every $w_n^h \in K^h_{u_n^h}$,

$$\left\langle \nabla F(u_n^h)\,,\, w_n^h\right\rangle_{H^h} \;=\; \left\langle \nabla_{H^h}\mathcal{D}^h(u_n^h + v_n^h)\,,\, w_n^h\right\rangle_{H^h} - \left\langle \nabla_{H^h}\mathcal{D}^h(v_n^h)\,,\, w_n^h\right\rangle_{H^h}.$$

The first term vanishes by (48) and using (45), we get

$$\left\langle \nabla F(u_n^h)\,,\, w_n^h\right\rangle_{H^h} \;=\; \left\langle v_n^h\,,\, w_n^h\right\rangle_{\mathcal{D}^h}.$$

In particular, choosing $w_n^h = \nabla F(u_n^h)$ and using the equivalence of the norms in finite dimension, there exists $\alpha > 0$ such that

$$\|\nabla F(u_n^h)\|_{\mathcal{D}^h} \;\leq\; \alpha\, \|v_n^h\|_{\mathcal{D}^h},$$

So (51) implies

$$F(u_n^h) - F(u_{n+1}^h) \;\geq\; \frac{1}{2\alpha}\|\nabla F(u_n^h)\|_{\mathcal{D}^h}\|v_n^h\|_{\mathcal{D}^h}. \tag{52}$$

Eventually, we know from (50) that v_n^h tends to 0. Since

$$u_{n+1}^h(x) \;=\; u_n^h(x) + v_n^h(x) + O(\|v_n^h(x)\|^2),$$

we have for n large enough

$$2\|v_n\|_{H^h} \;\geq\; \|u_{n+1} - u_n\|_{H^h}.$$

The conditions (H4) and (H6) then follow from (52), (51) and the equivalence of the norms. □

Remark 6.5. The preceding method also applies to the discretization of the harmonic map-flow for functions $u \in L^2((0,+\infty), M_g)$:

$$\partial_t u - \Delta u - \|\nabla u\|^2 u \;=\; 0 \quad t \geq 0, \qquad u(0,t) = u_0 \in H^1(\Omega, S^{l-1}).$$

The corresponding algorithm reads: Given an initial data $u_0^h \in M_{g^h}^h$ and a time step Δt, compute for $n = 0, 1, \cdots$

$$\left[\begin{array}{l} \textbf{step 1.}\ \text{Find } v_n^h \text{ minimizing } v^h \mapsto \dfrac{\Delta t}{2}\|v_n^h\|_{H^h}^2 + \mathcal{D}^h(u_n^h + \Delta t v_n^h) \text{ in } K_{u_n^h}^h. \\ \\ \textbf{step 2.}\ \text{Set } u_{n+1}^h(x) := \dfrac{u_n^h(x) + \Delta t v_n^h(x)}{\|u_n^h(x) + \Delta t v_n^h(x)\|} \quad \forall x \in \Omega^h. \end{array}\right. \tag{53}$$

Remark 6.6. We do not know wether Theorem 6.4 still holds in the continuous case. In fact, in order to reproduce the proof above in the continuous case, we should establish the following Łojasiewicz inequality

$$|\mathcal{D}(u) - \mathcal{D}(\varphi)|^{1-\nu} \;\leq\; \beta\|\Delta u + \|\nabla u\|^2 u\|_{L^2(\Omega)},$$

in the H^1-neighborhood of any harmonic map φ. This is an open issue.

7. **Application to the Landau-Lifschitz equations.** In this section, we show that our results concerning the abstract projected θ-scheme of Section 4 apply to some discretization of the Landau-Lifschitz equations. These equations describe the evolution of the magnetization $m : \Omega \times (0,+\infty) \to S^2$ inside a ferromagnetic body occupying an open region $\Omega \subset \mathbf{R}^3$. This system of equations reads

$$\alpha\partial_t m - m \times \partial_t m = (1+\alpha^2)(\Delta m - |\nabla m|^2 m), \quad \text{in } \Omega, \tag{54}$$

where $\alpha > 0$ is a damping parameter and $'\times'$ denotes the three dimensional cross product. It is supplemented with initial and boundary conditions

$$\left\{\begin{array}{ll} \dfrac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega \\ m(x,0) = m_0(x) \in S^2. & \end{array}\right.$$

Notice that, at least formally, this evolution system preserves the constraint $|m(x,t)| = 1, \forall x \in \Omega$.

We will consider a discretization of the following variational formulation of (54),

$$\alpha\int_\Omega \partial_t m \cdot \psi - \int_\Omega m \times \partial_t m \cdot \psi = -(1+\alpha^2)\int_\Omega \nabla m \cdot \nabla\psi, \tag{55}$$

for every $\psi \in H^1(\Omega, \mathbf{R}^3)$ which furthermore satisfies $\psi(x).m(x) = 0$ a.e. in Ω. It is known that for every initial data $m_0 \in H^1(\Omega, S^2)$, this variational formulation admits a solution for all time (see [5]).

Before coming to discretization, let us show that, formally, the Dirichlet energy $\mathcal{D}(m) = (1/2)\int_\Omega |\nabla m|^2$ is a Lyapunov function for (55). Indeed, considering a smooth solution $m(x,t)$, we compute,

$$\frac{d}{dt}\mathcal{D}(m(\cdot,t)) \;=\; \int_\Omega \nabla m \cdot \nabla \partial_t m(x,t)\,dx.$$

Since, for every $x \in \Omega$, $t \mapsto \|m(x,t)\|^2$ is constant, we have $\partial_t m(x,t)\cdot m(x,t) = 0$. So, we can choose $\psi = \partial_t m(\cdot,t)$ in (55) and deduce,

$$\frac{d}{dt}\mathcal{D}(m(\cdot,t)) \;=\; -\frac{\alpha}{1+\alpha^2}\int_\Omega \|\partial_t m\|^2(x,t)\,dx \;\le\; 0,$$

as claimed.

7.1. **Space discretization.** We discretize the problem in space using P1-Finite Elements. Let us introduce some notation. Let $(\tau_h)_h$ be a regular family of conformal triangulations of the domain Ω parameterized by the space step h. Let $(x_i^h)_i$ be the vertices of τ_h and $(\phi_i^h)_{1\le i\le N_{(h)}}$ the set of associated basis functions of the so-called $P^1(\tau_h)$ discretization. That is to say the functions $(\phi_i^h)_i$ are globally continuous and linear on each triangle (or tetrahedron in 3D) and satisfy $\phi_i^h(x_j^h) = \delta_{ij}$. We define

$$V^h = \left\{ m = \sum_{i=1}^{N_h} m_i\phi_i^h,\ s.t.\ \forall i, m_i \in \mathbf{R}^3 \right\}, \qquad M^h = \left\{ m \in V^h,\ s.t.\ \forall i, m_i \in S^2 \right\}.$$

Notice that M^h is a manifold isomorphic to $(S^2)^{N_h}$. For any $m = \sum_{i=1}^{N} m_i\phi_i^h \in M^h$, we introduce the tangent space

$$T_{m^h}M^h = \left\{ v = \sum_{i=1}^{N} v_i^h\phi_i^h,\ s.t.\ \forall i, m_i^h\cdot v_i^h = 0 \right\}.$$

The space discretization of the variational formulation (55) reads,

$$\begin{cases} m^h(0) = m_0^h \in M^h, \quad \text{ and } \quad \forall \psi^h \in T_{m^h(t)}M^h,\, \forall t>0, \\ \alpha\displaystyle\int_\Omega \partial_t m^h.\psi^h - \sum_{i=1}^{N^h}(m_i^n \times \partial_t m_i^h)\cdot\psi_i^h\int_\Omega \phi_i^h = -(1+\alpha^2)\int_\Omega \nabla m^h.\nabla\psi^h. \end{cases} \tag{56}$$

Remark 7.1. We have replaced the term

$$\int_\Omega (m^n \times p^n)\cdot\psi^h \quad \text{ in the original scheme of [4] by } \quad \sum_{i=1}^{N^h}(m_i^n\times p_i^n)\cdot\psi_i^h\int_\Omega\phi_i^h.$$

This modification is equivalent to using the quadrature formula:

$$\int_\Omega f\,dx \;\simeq\; \sum_{i=1}^{N_h} f(x_i^h)\int_\Omega \phi_i^h,$$

for the computation of this integral. The convergence to equilibrium results below are still true with an exact quadrature formula, but the proof is slightly more complicated, see Remark 7.4.

We now interpret this variational formulation as a gradient-like differential system of the form (1). For this we introduce the Lyapunov functional $F : M^h \subset H^1(\Omega, \mathbf{R}^3) \to \mathbf{R}$ defined by

$$F(m^h) = \frac{1}{2}\int_\Omega |\nabla m^h|^2.$$

As usual, the gradient of this functional is $q^h = \nabla F(m^h) = A^h m^h$, where A^h is the rigidity matrix associated to the P^1-FE discretization:

$$\left\langle q^h, \psi^h \right\rangle_{L^2} \;=\; \int_\Omega \nabla m^h \cdot \nabla \psi^h \;=\; \sum_{i,j} m_i^h \psi_j^h \int \nabla\phi_i^h \cdot \nabla\phi_j^h \;=:\; \left\langle A^h m^h, \psi^h \right\rangle_{L^2}. \tag{57}$$

We also introduce the section $G:\ M^h \to TM^h$ defined by $G(m^h) := p^h$ where $p^h \in T_{m^h}M^h$ solves

$$\alpha\int_\Omega p^h\cdot\psi^h - \sum_{i=1}^{N^h}(m_i^h\times p_i^h)\cdot\psi_i^h\int_\Omega \phi_i^h = -(1+\alpha^2)\int_\Omega \nabla m^h\cdot\nabla\psi^h, \quad \forall \psi^h \in T_{m^h}M^h. \tag{58}$$

The function G is well defined. Indeed, it is sufficient to check that the bilinear form b_{m^h} defined on $T_{m^h}M^h \times T_{m^h}M^h$ by

$$b_{m^h}(p^h,\psi^h) = \alpha\int_\Omega p^h\cdot\psi^h - \sum_{i=1}^{N^h}(m_i^h\times p_i^h)\cdot\psi_i^h\int_\Omega \phi_i^h \tag{59}$$

has a positive symmetric part. Using $p_i^h \times p_i^h = 0$, we see that $b_{m^h}(p^h,p^h) = \alpha\|p^h\|^2_{L^2(\Omega)^2}$ and b_{m^h} is coercive on $T_{m^h}M^h \times T_{m^h}M^h$. So, by definition, $m^h \in C^1(\mathbf{R}_+, M^h)$ solves the variational formulation (56) if and only if

$$\frac{d}{dt}m^h \;=\; G(m^h) \quad \forall t>0, \qquad m^h(0) \;=\; m_0^h.$$

We now check that the hypotheses of Theorem 2.11 hold.

Lemma 7.2. *The functions G and ∇F defined above satisfy the angle and comparability condition* (11)*. Moreover, the Lyapunov function F satisfies a Lojasiewicz inequality* (5) *in the neighborhood of any point m^h of the manifold $M = M^h$.*

Proof. For the first point, let us fix $m^h \in M^h$ and write $p^h = G(m^h)$ and $q^h = \nabla F(m^h)$. Choosing $\psi^h = q^h$ in (58) and using (57), we obtain

$$\alpha\left\langle p^h\,,\,q^h\right\rangle_{L^2} \;=\; -\sum_{i=1}^{N^h}(m_i^h\times p_i^h)\cdot q_i^h\int_\Omega\phi_i^h \,-\,(1+\alpha^2)\|q^h\|^2_{L^2}.$$

Then the Cauchy-Schwarz inequality, the identities $\|m_i^h\| = 1$ and the equivalence of norms in finite dimension yield

$$\|q^h\|_{L^2} \;\le\; C\|p^h\|_{L^2}.$$

On the other hand, choosing $\psi^h = p^h$ in (58), we get

$$\alpha\|p^h\|^2_{L^2} \;=\; -(1+\alpha^2)\int_\Omega \nabla m^h\cdot\nabla p^h \;=\; -(1+\alpha^2)\left\langle q^h\,,\,p^h\right\rangle_{L^2}.$$

So, we have

$$\left\langle -q^h\,,\,p^h\right\rangle_{L^2} \;\ge\; \frac{\gamma}{2}\left(\|p^h\|^2_{L^2} + \|q^h\|^2_{L^2}\right),$$

with $\gamma = \alpha/(C(1+\alpha^2))$: i.e. the pair $(-\nabla F, G)$ satisfies the tangential angle condition and comparability condition (11).

For the second point, $F(m^h)$ is a polynomial function of $(m_i^h)_{1\le i\le N_h} \in (S^2)^{N_h}$, hence it is analytic. The manifold $M^h = (S^2)^{N_h}$ being analytic, we can use an analytic chart φ (for example a product of stereographic projections) defined in a neighborhood of m^h. We apply Theorem 2.5 to the analytic function $F\circ\varphi^{-1}$ and deduce that it satisfies a Łojasiewicz inequality in the neighborhood of $\varphi(m^h)$. □

We deduce from the lemma:

Corollary 7.1. *Assume $m^h(t)$ is a solution of* (56)*. Since $M = M^h$ is compact $\omega(m^h)$ is not empty. Consequently there exists $\varphi \in M^h$ such that $u = m^h$ satisfies all the conclusions of Theorems 2.7, 2.11.*

7.2. **Time-space discretization of the Landau-Lifchitz equations.** We now consider the θ-scheme proposed by F.Alouges in [4]:

$$\left\{\begin{array}{l} m^0 \in M^h \\ \text{For } n=0,1,... \\ \left[\begin{array}{l} \text{Find } p^n \in T_{m^n}M^h \text{ such that} \\ \qquad \alpha\displaystyle\int_\Omega p^n\cdot\psi^h - \sum_{i=1}^{N^h}(m_i^n\times p_i^n)\cdot\psi_i^h\int_\Omega\phi_i^h = -(1+\alpha^2)\int_\Omega\nabla(m^n+\theta\Delta t\,p^n)\cdot\nabla\psi^h \\ \text{Set } m^{n+1} \;:=\; \displaystyle\sum_{i=1}^{N_h}\frac{m_i^n+\Delta t\,p_i^n}{|m_i^n+\Delta t\,p_i^n|}\phi_i^h, \quad \text{and iterate.} \end{array}\right. \end{array}\right. \tag{60}$$

Let us rewrite this scheme as a projected θ-scheme of the form (28). For this we introduce the family of mappings $\left\{G_{m^h} : m^h + T_{m^h} \to T_{m^h}M^h\right\}$ defined by $G_{m^h}(u^h) = p^h$ where $p^h \in T_{m^h}M^h$ solves the variational formulation.

$$\alpha\int_\Omega p^h\cdot\psi^h - \sum_{i=1}^{N^h}(u_i^h\times p_i^h)\cdot\psi_i^h\int_\Omega\phi_i^h = -(1+\alpha^2)\int_\Omega\nabla u^h\cdot\nabla\psi^h, \qquad \forall\psi^h\in T_{m^h}M^h.$$

Notice that G_{m^h} only depends on m^h through the space of test functions $T_{m^h}M^h$. As above, we see that p^h is well defined and uniquely defined by this variational formulation through the coercivity of the bilinear form b_{m^h} (see (59)).

Lemma 7.3. *Let m^n, p^n be defined in the scheme* (60)*. Then,*

$$p^n \;=\; \theta G_{m^n}(m^n+\Delta t p^n) + (1-\theta)G_{m^n}(m^n). \tag{61}$$

Proof. Let us set $q^h = G_{m^n}(m^n+\Delta t p^n)$, $r^h = G_{m^n}(m^n)$. By definition of G_{m^n} and linearity, we see that the function $p^h = \theta q^h + (1-\theta)r^h$ satisfies

$$\begin{aligned}\alpha\int_\Omega p^h\cdot\psi^h - \sum_{i=1}^{N^h}(m_i^h\times p_i^h)\cdot\psi_i^h\int_\Omega\phi_i^h - \theta\Delta t\sum_{i=1}^{N^h}(p_i^n\times r_i^h)\cdot\psi_i^h\int_\Omega\phi_i^h& \\ = \;-(1+\alpha^2)\int_\Omega\nabla(m^h+\theta\Delta t\,p^n)\cdot\nabla\psi^h, \qquad \forall\psi^h\in T_{m^h}M^h.&\end{aligned}$$

We see that in the third term of the left hand side, the triple product $(p_i^n\times r_i^h)\cdot\psi_i^h$ vanishes. Indeed, the three vectors p_i^n, r_i^h, ψ_i^h belong to the two dimensional tangent space $\{v_i^h\in\mathbf{R}^3 : v_i^h\cdot m_i^h = 0\}$. So, it turns out that p^h and p^n solve the same (well-posed) variational formulation. We conclude that $p^h = p^n$ as claimed. □

Remark 7.4. If we had used the original variational formulation, with obvious changes in the definition of G_{m^h}, then the term $\theta\Delta t\int_\Omega(p^n\times r^h)\cdot\psi^h$ would not vanish in general and the identity (61) would be wrong. In this case, we can not link the scheme of [4] to our projected θ-scheme. However, this term is of small magnitude and using the present ideas, it is not difficult to establish that Theorems 3.1, 3.2 and 3.4 apply to this scheme and conclude to the convergence to equilibrium of the sequence (m^n).

This difficulty does not appear if we consider a Finite Difference discretization as in Section 6.

Lemma 7.5. *The functions F, G and $\{G_{m^h}\}$ satisfy the hypotheses (29,30,31,32). Moreover, the projection step $\Pi_{M^h}(z^h) := \sum_{i=1}^{N_h} \frac{z_i^h}{|z_i^h|}\phi_i^h$, satisfies (33).*

Proof. First, the identity (29) is obvious. Next, for $m^h \in M^h$ and $p^h = G(m^h)$, using $\psi_h = p^h$ in (58), we obtain

$$\alpha \|p_h\|_{L^2}^2 \ \leq \ (1+\alpha^2)\|\nabla m^h\|_{L^2}\,\|\nabla p^h\|_{L^2},$$

and we conclude from the equivalence of the norms in finite dimensional spaces, that G is bounded on the compact manifold M^h (that is (29) holds). The Lipschitz estimate (31) is also a consequence of this fact and of the uniform coercivity of the bilinear forms b_{m^h}. The Lipschitz estimate (32) on ∇F is also obvious since F is smooth on the compact manifold M^h.

Eventually, we easily see that (33) holds. Indeed, if $v^h \in T_{m^h}M^h$, then $|m_i^h + v_i^h|^2 = |m_i^h|^2 + |v_i^h|^2 \geq 1$, so $\Pi_{M^h}(m^h + v^h)$ is just the L^2-projection of $(m_i^h + v_i^h)$ on the product of balls $(\overline{B}(0,1))^{N_h} \subset (\mathbf{R}^3)^{N^h}$. □

The previous Lemmas 7.3 and 7.5 show that the sequence $(u_n = m^n)$ satisfies all the hypotheses for Corollary 4.2. Hence, we have:

Corollary 7.2. *There exists $\Delta t' > 0$ such that if $\Delta t \in (0, \Delta t')$ and $(m^n) \subset M^h$ is a sequence that complies to the scheme (60), then there exists $\varphi \in M^h$ such that (m^n) converges to φ. Moreover, there exist $C_3 > 0$ and $\nu \in (0, 1/2]$ depending on φ such that the convergence rate given by (20) holds with $C_2 = C_3\Delta t$.*

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