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# Convergence to equilibrium for the backward Euler scheme and applications

Benoît Merlet, Morgan Pierre

1. **Introduction.** In this paper, we are concerned with the asymptotic behaviour, as time goes to infinity, of the solution of the backward Euler scheme applied to gradient flows. As a model example, consider the following gradient flow:

$$U'(t) = -\nabla F(U(t)) \quad t \geq 0, \tag{1}$$

where $U = (u_1, \ldots, u_d)^t$, $F \in C^{1,1}_{loc}(\mathbf{R}^d, \mathbf{R})$ and

$$\nabla F = \left( \frac{\partial F}{\partial u_1}, \frac{\partial F}{\partial u_2}, \ldots, \frac{\partial F}{\partial u_d} \right).$$

A result of Łojasiewicz [22, 23] asserts that if $F$ is real analytic, then any bounded solution $U(t)$ of (1) converges to a critical point of $F$ as $t$ tends to infinity. The proof relies on the so-called Łojasiewicz inequality for real analytic functions. If $F$ is only assumed $C^\infty$ and if $d \geq 2$, the result may fail (see [1] and references therein). The Łojasiewicz inequality also allows to obtain optimal convergence rates to equilibrium. This result has many generalizations, in particular to infinite dimension [19, 27] and to gradient-like flows [15, 24]), and there is a growing literature extending it to other situations (see for instance [6, 14]).

The aim of this paper is to show that these convergence results also apply to some stable discretizations of gradient flows, and first of all, to the backward Euler scheme. We recall that for the model example (1), the backward Euler scheme reads:

$$\frac{U^{n+1} - U^n}{\Delta t} = -\nabla F(U^{n+1}), \quad n \geq 0, \tag{2}$$

where $\Delta t > 0$ is the time step. The implicit Euler scheme is widely used in the numerical or theoretical treatment of flows governed by parabolic semi-linear PDE's, because of its stability properties (see for instance [3, 17]). The question of convergence to equilibrium for the backward Euler scheme is therefore a natural question, and to our knowledge, it seems to be still open (see however Remark 1.1 below). Notice that a related question has been answered for some explicit descent methods by Absil, Mahony and Andrews in a context of optimization [1]. These authors, however, assume a so-called "primary descent condition" which, in the case of the backward Euler scheme, is impossible to check *a priori*. In [11], Gajewski and Griepentrog also prove convergence to equilibrium for a descent method (distinct from the backward Euler scheme) in a Hilbert setting, using the Łojasiewicz inequality.

The starting point of our paper, Theorem 2.4 in Section 2, shows that if $F$ is real analytic and coercive, then any solution $(U^n)_{n\geq 0}$ of (2) converges to a critical point of $F$ as $n$ goes to infinity. We also obtain, in Proposition 2.5, optimal convergence rates, similarly to the continuous case. As a corollary of Theorem 2.4 and under reasonable assumptions, the backward Euler approximation of a gradient flow is shown to converge uniformly in time towards the exact solution, as the time step tends to 0. The rest of the paper is organized as follows: in Section 3, we show how the preceding results apply to a time and space discretization of the Cahn-Hilliard equation. In Section 4, we extend Theorem 2.4 to the $\theta$-scheme, and in the last section, we extend Theorem 2.4 in infinite dimension to the case of the semilinear heat equation.

**Remark 1.1.** After this paper was complete, we learned that Bolte et al. recently proved convergence to equilibrium for the backward Euler scheme with variable step size in a Hilbert setting: see [4, Theorem 24]. Their proof is also based on a Łojasiewicz inequality.

2. **Convergence to equilibrium for the backward Euler scheme.** In this section, we will consider the backward Euler scheme as a descent method. In this regard, we assume the following coercivity condition on $F$:

$$F(V) \to +\infty \text{ as } \|V\| \to +\infty. \tag{3}$$

This assumption, together with the continuity of $F$, implies that $F$ is bounded from below. For convenience, we define the implicit Euler scheme as follows: let $U^0 \in \mathbf{R}^d$ and for all $n \geq 0$,

$$U^{n+1} \text{ is a minimizer of the function } V \mapsto \frac{\|V - U^n\|^2}{2\Delta t} + F(V). \tag{4}$$

Here and in the following, $\|V\|$ denotes the euclidean norm of a vector $V \in \mathbf{R}^d$. Assumption (3) guarantees that problem (4) has at least one solution for all $n$. If $F \in C^1(\mathbf{R}^d)$, then clearly, any solution $U^{n+1}$ of (4) satisfies (2) for all $n$. A solution of (4) may not be unique, but in many applications, uniqueness of solutions for (2) is guaranteed for $\Delta t > 0$ small enough, assuming some additional assumptions on $F$ (see Sections 3 and 4, for instance); in such cases, (4) and (2) are equivalent.

Convergence to equilibrium for (4) will be a consequence of the Łojasiewicz inequality, which is defined as follows.

**Definition 2.1.** We say that $F \in C^1(\mathbf{R}^d, \mathbf{R})$ satisfies the Łojasiewicz inequality at a point $\overline{U} \in \mathbf{R}^d$ if there exist $\nu \in (0, 1/2]$, $\gamma > 0$ and $\sigma > 0$ such that

$$\forall V \in \mathbf{R}^d, \quad \|V - \overline{U}\| < \sigma \Rightarrow |F(V) - F(\overline{U})|^{1-\nu} \leq \gamma \|\nabla F(V)\|. \tag{5}$$

In some (few) cases, the exponent $\nu$ is known explicitly, and it is called the Łojasiewicz exponent at $\overline{U}$. It allows the computation of convergence rates. For instance, $F : \mathbf{R}^d \to \mathbf{R}$ defined by $F(U) = \|U\|^p$, $p \geq 2$, satisfies (5) at $\bar{U} = 0$ with $\nu = 1/p$. Another example is the following result, which is a consequence of the inverse mapping Theorem and of a Taylor's expansion (see for instance [16]):

**Proposition 2.2.** *Let $F : \mathbf{R}^d \to \mathbf{R}$ be $C^2$ in the neighborhood of a point $\overline{U}$. If the hessian $\nabla^2 F_{\overline{U}} = \left(\partial^2 F(\overline{U})/\partial u_i \partial u_j\right)_{1 \leq i,j \leq d}$ is non singular, then $F$ satisfies the Łojasiewicz inequality at $\overline{U}$ with exponent $\nu = 1/2$.*

However, most of the time, the Łojasiewicz inequality is a consequence of analyticity, and the exponent $\nu$ is not known. This is the fundamental result:

**Lemma 2.3** (Łojasiewicz' Theorem [23])**.** *If $F : \mathbf{R}^d \to \mathbf{R}$ is real analytic in the neighborhood of a point $\overline{U}$, then $F$ satisfies the Łojasiewicz inequality at $\overline{U}$.*

We denote

$$\mathcal{S} = \{V \in \mathbf{R}^d \; : \; \nabla F(V) = 0\}$$

the set of critical points of $F$. Our main result is:

**Theorem 2.4.** *Assume that $F \in C^1(\mathbf{R}^d, \mathbf{R})$ satisfies (3) and that the Łojasiewicz inequality (5) holds for every $\overline{U} \in \mathcal{S}$. Let $(U^n)_{n\geq 0}$ be a sequence defined by (4). Then there exists $U^\infty \in \mathcal{S}$ such that $U^n \to U^\infty$ as $n \to +\infty$.*

*Proof.* By (4),

$$\frac{\|U^{n+1} - U^n\|^2}{2\Delta t} + F(U^{n+1}) \leq F(U^n), \quad \forall n \geq 0. \tag{6}$$

Thus, the sequence $(F(U^n))$ is nonincreasing, and since $F$ is bounded from below, $F(U^n)$ tends to some $F^\star \in \mathbf{R}$. Without loss of generality, we assume $F^\star = 0$. By (3), $(U^n)$ is bounded, so there exist $U^\infty \in \mathbf{R}^d$ and a subsequence $(U^{n_k})$ such that $U^{n_k} \to U^\infty$ as $k \to +\infty$. By (6), $\|U^{n+1} - U^n\| \to 0$, and letting $n = n_k$ tend to $\infty$ in (2), we obtain $\nabla F(U^\infty) = 0$, so $U^\infty \in \mathcal{S}$. Since $F$ satisfies the Łojasiewicz inequality at $U^\infty$, there exist $\nu \in (0, 1/2]$ and $\sigma, \gamma > 0$ such that

$$\forall V \in \mathbf{R}^d,\ \|V - U^\infty\| < \sigma \Rightarrow |F(V)|^{1-\nu} \leq \gamma \|\nabla F(V)\|. \tag{7}$$

Let $n$ be such that $\|U^{n+1} - U^\infty\| < \sigma$. We consider two cases:

• Case 1: $F(U^{n+1}) > F(U^n)/2$. On computing,

$$\begin{aligned}
[F(U^n)]^\nu - [F(U^{n+1})]^\nu &= \int_{F(U^{n+1})}^{F(U^n)} \nu x^{\nu-1} dx \ \geq \int_{F(U^{n+1})}^{F(U^n)} \nu \left(F(U^n)\right)^{\nu-1} dx \\
&\overset{\text{case 1}}{\geq} 2^{\nu-1}\nu \left(F(U^{n+1})\right)^{\nu-1} [F(U^n) - F(U^{n+1})].
\end{aligned}$$

Thus,

$$\begin{aligned}
[F(U^n)]^\nu - [F(U^{n+1})]^\nu &\overset{(6)}{\geq} 2^{\nu-2}\nu \frac{\|U^{n+1} - U^n\|^2}{\Delta t [F(U^{n+1})]^{1-\nu}} \\
&\overset{(2)}{\geq} 2^{\nu-2}\nu \|U^{n+1} - U^n\| \frac{\|\nabla F(U^{n+1})\|}{[F(U^{n+1})]^{1-\nu}} \overset{(7)}{\geq} \frac{2^{\nu-2}\nu}{\gamma} \|U^{n+1} - U^n\|.
\end{aligned}$$

• Case 2: $F(U^{n+1}) \leq F(U^n)/2$. Then

$$\begin{aligned}
\|U^{n+1} - U^n\| &\overset{(6)}{\leq} \sqrt{2\Delta t}[F(U^n) - F(U^{n+1})]^{1/2} \leq \sqrt{2\Delta t}[F(U^n)]^{1/2} \\
&\overset{\text{case 2}}{\leq} \left(1 - \frac{1}{\sqrt{2}}\right)^{-1} \sqrt{2\Delta t} \left([F(U^n)]^{1/2} - [F(U^{n+1})]^{1/2}\right).
\end{aligned}$$

In both cases, for all $n$ such that $\|U^{n+1} - U^\infty\| < \sigma$, we have

$$\begin{aligned}
\|U^{n+1} - U^n\| \leq\ & \frac{2^{2-\nu}\gamma}{\nu} \left([F(U^n)]^\nu - [F(U^{n+1})]^\nu\right) \\
& + 5\sqrt{\Delta t} \left([F(U^n)]^{1/2} - [F(U^{n+1})]^{1/2}\right).
\end{aligned} \tag{8}$$

Now, let $\tilde{F} > 0$ be small enough so that

$$\frac{2^{2-\nu}\gamma}{\nu} \tilde{F}^\nu + 5\sqrt{\Delta t} \tilde{F}^{1/2} \leq \sigma/3. \tag{9}$$

We choose $\bar{n}$ large enough such that $\|U^{\bar{n}} - U^\infty\| < \sigma/3$, and $F(U^{\bar{n}}) \leq \tilde{F}$. Let $N \geq \bar{n}$ be the largest integer (including $+\infty$) such that $\|U^n - U^\infty\| < 2\sigma/3$ for all

$\bar{n} \le n \le N$. Assume by contradiction that $N$ is finite. We deduce from (6) that

$$\begin{aligned} \|U^{N+1} - U^\infty\| &\le \|U^{N+1} - U^N\| + \|U^N - U^\infty\|, \\ &\overset{(6)}{\le} \sqrt{2\Delta t F(U^N)} + \|U^N - U^\infty\| \overset{(9)}{<} \sigma. \end{aligned}$$

So we may apply (8) to every $\bar{n} \le n \le N$ and since $(F(U^n))$ is nonincreasing, we obtain

$$\sum_{n=\bar{n}}^{N} \|U^{n+1} - U^n\| \overset{(8)}{\le} \frac{2^{2-\nu}\gamma}{\nu} \left(F(U^{\bar{n}})\right)^\nu + 5\sqrt{\Delta t}\left(F(U^{\bar{n}})\right)^{1/2} \overset{(9)}{\le} \sigma/3.$$

Thus, $\|U^{N+1} - U^\infty\| \le \|U^{\bar{n}} - U^\infty\| + \sigma/3 < 2\sigma/3$, and this contradicts the definition of $N$. So $N = +\infty$ and the whole sequence converges. □

We are also able to estimate the convergence rates to equilibrium: they are analogous to the continuous case (see for instance [16]).

**Proposition 2.5.** *Assume that the assumptions of Theorem 2.4 are satisfied, let $U^\infty$ be the limit of a sequence $(U^n)_{n\ge 0}$ defined by (4), and let $\nu \in (0, 1/2]$ be the Łojasiewicz exponent of $U^\infty$. If $\nu = 1/2$, the convergence is geometric, i.e. there exist $\tilde{n} \ge 0$ and $\lambda, \alpha > 0$, such that*

$$\|U^n - U^\infty\| \le \lambda \exp(-\alpha n \Delta t), \qquad \text{for } n > \tilde{n};$$

*if $0 < \nu < 1/2$, the convergence is polynomial, i.e. there exist $\tilde{n} \ge 0$ and $\lambda > 0$ such that*

$$\|U^n - U^\infty\| \le \frac{\lambda}{(n\Delta t)^{\nu/(1-2\nu)}}, \qquad \text{for } n > \tilde{n}.$$

The convergence rates in Proposition 2.5 are optimal. Indeed, let $F(U) = \|U\|^p$ with $p \ge 2$. The function $F$ satisfies the Łojasiewicz inequality (5) with exponent $\nu = 1/p$ at its unique critical point 0. Since $F$ is rotationally symmetric, in spherical coordinates, the gradient flow $U'(t) = -\nabla F(U(t))$ reduces to $r'(t) = -f'(r(t))$, where $f(r) = r^p$ and $r = \|U\| \ge 0$, i.e.

$$r'(t) = -pr(t)^{p-1}, \quad t \ge 0. \tag{10}$$

Similarly, the Euler scheme, in spherical coordinates, reduces to

$$(r_{n+1} - r_n)/\Delta t = -pr_{n+1}^{p-1}, \quad n \ge 0. \tag{11}$$

For $p = 2$, we have $(1 + 2\Delta t)r_{n+1} = r_n$, so that $r_n = r_0(1 + 2\Delta t)^{-n}$, and we see that the geometric convergence proved in Proposition 2.5 is optimal. If $p > 2$, the solution of (10) with initial condition $r(0) = r_0 > 0$ is

$$r(t) = [r_0^{2-p} + (p-2)pt]^{-1/(p-2)}.$$

On the other hand, it is easily seen that if $r_n \ge r(n\Delta t)$, then the solution of (11) satisfies $r_{n+1} \ge r((n+1)\Delta t)$. By induction, the sequence uniquely defined by (11) and the initial condition $r_0 = r(0)$ satisfies $r_n \ge r(n\Delta t)$, so that

$$r_n \ge Cn^{-1/(p-2)}, \quad n \ge 1,$$

for some constant $C = C(r_0, p, \Delta t) > 0$. Thus the exponent found in Theorem 2.5 is optimal (notice that $1/(p-2) = \nu/(1-2\nu)$).

*Proof of Proposition 2.5.* We argue as in the proof of Theorem 2.4. By (6), the sequence $(F(U^n))$ is nonincreasing and converges to $F(U^\infty) = 0$. There exists $\tilde{n} \geq 0$ such that, for $n \geq \tilde{n}$, $\|U^n - U^\infty\| < \sigma$, where $\sigma$ is defined by the Lojasiewicz inequality (7). If $F(U^{\tilde{n}}) = 0$ for some $\tilde{n}$, then by (6), $U^n = U^{\tilde{n}} = U^\infty$ for all $n \geq \tilde{n}$, and the result is obvious. From now on, we assume that $F(U^n) > 0$ for all $n$. Then, applying estimate (8) with $n, n+1, \cdots$ and summing, we get

$$\|U^n - U^\infty\| \leq \frac{2^{2-\nu}\gamma}{\nu}[F(U^n)]^\nu + 5\sqrt{\Delta t}[F(U^n)]^{1/2} \qquad \forall n \geq \tilde{n}. \tag{12}$$

Next, we estimate the rate of decay of the sequence $(F(U^n))$. Let us define the function $G\ : (0, +\infty) \ \rightarrow\ (0, +\infty)$ by

$$G(f) := \begin{cases} -\ln f & \text{if} \quad \nu = 1/2, \\ \dfrac{1}{(1-2\nu)f^{1-2\nu}} & \text{if} \quad 0 < \nu < 1/2. \end{cases}$$

The sequence $(G(F(U^n)))$ is nondecreasing and tends to $+\infty$. Moreover, for $n \geq \tilde{n}$, we have

- Case 1: $F(U^{n+1}) > F(U^n)/2$. Then

$$\begin{aligned} G(F(U^{n+1})) - G(F(U^n)) \quad &= \quad \int_{F(U^{n+1})}^{F(U^n)} \frac{df}{f^{2-2\nu}}, \\ &\overset{\text{case 1}}{\geq} \quad 2^{2\nu-2}\left[F(U^{n+1})\right]^{2\nu-2}[F(U^n) - F(U^{n+1})], \\ &\overset{(6)}{\geq} \quad 2^{2\nu-3}\Delta t[F(U^{n+1})]^{2\nu-2}\|\nabla F(U^{n+1})\|^2, \\ &\overset{(7)}{\geq} \quad 2^{2\nu-3}\gamma^{-2}\Delta t. \end{aligned}$$

- Case 2: $F(U^{n+1}) \leq F(U^n)/2$. Then

$$(\ln 2)^{-1}2^{2\nu-3}\gamma^{-2}\Delta t(\ln(F(U^n)) - \ln(F(U^{n+1}))) \ \geq\ 2^{2\nu-3}\gamma^{-2}\Delta t.$$

In both cases, we have for $n \geq \tilde{n}$,

$$\begin{aligned} 2^{2\nu-3}\gamma^{-2}\Delta t \quad \leq \quad & G(F(U^{n+1})) - G(F(U^n)) \\ & + (\ln 2)^{-1}2^{2\nu-3}\gamma^{-2}\Delta t\left(\ln(F(U^n)) - \ln(F(U^{n+1}))\right). \end{aligned}$$

Writing this relation for $\tilde{n}, \tilde{n}+1, \cdots, n-1$ and summing, we obtain

$$\begin{aligned} (n-\tilde{n})2^{2\nu-3}\gamma^{-2}\Delta t \ \leq\ & G(F(U^n)) - G(F(U^{\tilde{n}})) \\ & + (\ln 2)^{-1}2^{2\nu-3}\gamma^{-2}\Delta t\left(\ln(F(U^{\tilde{n}})) - \ln(F(U^n))\right). \end{aligned} \tag{13}$$

If $\nu = 1/2$, inequality (13) yields

$$G(F(U^n)) \ \geq\ C_1 n\Delta t + C_2, \qquad \forall n > \tilde{n},$$

with

$$C_1 = \frac{2^{2\nu-3}\gamma^{-2}}{1 + (\ln 2)^{-1}2^{2\nu-3}\gamma^{-2}\Delta t} > 0 \text{ and } C_2 = C_2(\gamma, \nu, \tilde{n}, F(U^{\tilde{n}}), \Delta t).$$

Thus,

$$F(U^n) \ \leq\ \exp(-C_2)\exp(-C_1 n\Delta t), \qquad \forall n > \tilde{n},$$

and recalling (12), the proof is complete in the case $\nu = 1/2$.

If $\nu \in (0, 1/2)$, we notice that for $\tilde{f} > 0$ small enough, we have

$$(\ln 2)^{-1} 2^{2\nu-3} \gamma^{-2} \Delta t \left(\frac{1}{f}\right) \le \frac{1}{f^{2-2\nu}}, \quad \forall f \in (0, \tilde{f}].$$

On integrating,

$$(\ln 2)^{-1} 2^{2\nu-3} \gamma^{-2} \Delta t \left(\ln(\tilde{f}) - \ln(f)\right) \le G(f) - G(\tilde{f}), \qquad \forall f \in (0, \tilde{f}].$$

Thus, replacing $\tilde{n}$ by a larger value if necessary, so that $F(U^{\tilde{n}}) \le \tilde{f}$, we deduce from (13):

$$G(F(U^n)) \;\ge\; C_1 n \Delta t + C_2, \qquad \forall n > \tilde{n},$$

with

$$C_1 = 2^{2\nu-4}\gamma^{-2} \text{ and } C_2 = C_2(\gamma, \nu, \tilde{n}, F(U^{\tilde{n}}), \Delta t).$$

Thus,

$$F(U^n) \;\le\; \left(\frac{1-2\nu}{C_1 + C_2 n \Delta t}\right)^{\frac{1}{1-2\nu}}, \qquad \forall n > \tilde{n}.$$

Recalling (12), the proof is complete. □

In [24, Theorem 2.2], Miranville and Rougirel noticed that the Łojasiewicz inequality can sometimes be used to show the continuity of the limiting state with respect to some parameters. Here, we adapt their remark to show a result concerning convergence of the discrete solution to the continuous solution as $\Delta t \to 0$. For this purpose, for $\Delta t > 0$, we denote $(U^n_{\Delta t})$ a sequence generated by (4), and we define the continuous and piecewise affine function $U_{\Delta t} : [0, +\infty) \to \mathbf{R}^d$ associated to $(U^n_{\Delta t})$ by

$$U_{\Delta t}(t) = \frac{(n+1)\Delta t - t}{\Delta t} U^n + \frac{t - n\Delta t}{\Delta t} U^{n+1}, \quad t \in [n\Delta t, (n+1)\Delta t], \; n \ge 0.$$

We have:

**Proposition 2.6.** *Assume that $F \in C^{1,1}_{loc}(\mathbf{R}^d)$ satisfies* (3) *and let $U$ be a solution of* (1) *in $C^1([0, +\infty), \mathbf{R}^d)$. Assume that $F$ satisfies the Łojasiewicz inequality at $U^\infty$, with $U^\infty = \lim_{t\to+\infty} U(t)$. If $U^\infty$ is a local minimizer of $F$, then $U_{\Delta t} \to U$ uniformly on $[0, +\infty)$, as $\Delta t \to 0$ and $U^0_{\Delta t} \to U(0)$. In particular, $U^\infty_{\Delta t} \to U^\infty$, as $\Delta t \to 0$ and $U^0_{\Delta t} \to U(0)$, where $U^\infty_{\Delta t} := \lim_{n\to+\infty} U^n_{\Delta t}$.*

Recall that "$U^\infty$ is a local minimizer of $F$" means that there exists $\rho > 0$ such that

$$\forall V \in \mathbf{R}^d, \quad \|V - U^\infty\| < \rho \Rightarrow F(V) \ge F(U^\infty). \tag{14}$$

This assumption cannot be removed in general: if $U^\infty \in \mathcal{S}$ has a non trivial unstable set, the conclusion of Proposition 2.6 is hardly true.

*Proof.* By standard results [8], $U_{\Delta t}$ converges uniformly to $U$ on every compact interval $[0, T]$, as $\Delta t \to 0$ and $U^0_{\Delta t} \to U(0)$. Therefore, it is sufficient to prove that $U_{\Delta t}$ converges to $U^\infty$ uniformly on $[0, \infty)$ (for the variable $t$) and on $(0, 1]$ (for the parameter $\Delta t$) as $U^0_{\Delta t} \to U^\infty$: for the general case, divide $[0, +\infty)$ into $[0, T]$ and $[T, +\infty)$ with $T$ large enough.

Since $F$ satisfies the Łojasiewicz inequality at $U^\infty$, assuming as previously that $F(U^\infty) = 0$, there exist $\nu \in (0, 1/2]$ and $\sigma, \gamma > 0$ such that (7) holds. We may assume $\sigma \le \rho$, where $\rho$ satisfies (14). Arguing as in the proof of Theorem 2.4, we find that for all $n$ such that $\|U^{n+1}_{\Delta t} - U^\infty\| < \sigma$, inequality (8) holds.

Now, let us fix $\Delta t \in (0,1]$, $\varepsilon \in (0, \sigma/3]$, and $\tilde{F} > 0$ small enough so that

$$\frac{2^{2-\nu}\gamma}{\nu}\tilde{F}^{\nu} + 5\tilde{F}^{1/2} \le \varepsilon. \tag{15}$$

We assume that $U^0_{\Delta t}$ is close enough to $U^\infty$ so that

$$\|U^0_{\Delta t} - U^\infty\| < \varepsilon \qquad \text{and} \qquad F(U^0_{\Delta t}) \le \tilde{F}.$$

Define $N \ge 0$ as the largest integer (including $+\infty$) such that $\|U^n_{\Delta t} - U^\infty\| < 2\sigma/3$ for all $0 \le n \le N$. Assume by contradiction that $N$ is finite. Then, by (6),

$$\|U^{N+1}_{\Delta t} - U^\infty\| \le \|U^{N+1}_{\Delta t} - U^N_{\Delta t}\| + \|U^N_{\Delta t} - U^\infty\| \le \sqrt{2F(U^N_{\Delta t})} + 2\sigma/3 < \sigma,$$

where we have used that $0 \le F(U^{N+1}_{\Delta t}) \le F(U^N_{\Delta t}) \le \tilde{F}$, and (15). So, we may apply (8) to every $0 \le n \le N$, and obtain

$$\sum_{n=0}^{N} \|U^{n+1}_{\Delta t} - U^n_{\Delta t}\| \le \frac{2^{2-\nu}\gamma}{\nu}\left(F(U^0_{\Delta t})\right)^{\nu} + 5\sqrt{\Delta t}\left(F(U^0_{\Delta t})\right)^{1/2} \overset{(15)}{\le} \varepsilon. \tag{16}$$

Thus, $\|U^{N+1}_{\Delta t} - U^\infty\| \le \|U^0_{\Delta t} - U^\infty\| + \varepsilon < 2\varepsilon \le 2\sigma/3$, and this contradicts the definition of $N$. So, $N = +\infty$, and using (16) again, we find that for every $n \ge 0$,

$$\|U^n_{\Delta t} - U^\infty\| \le 2\varepsilon,$$

and this completes the proof. □

In the next section, we give an application of Theorem 2.4 to a PDE. The following remark will be useful.

**Remark 2.7.** In Theorem 2.4 and Proposition 2.6, we can replace the usual euclidean norm (which arises through (4)) by any euclidean norm. More precisely, let $A$ be a symmetric positive definite matrix and let $\langle V_1, V_2\rangle_A = (AV_1, V_2)$ (where $(\cdot,\cdot)$ is the usual scalar product on $\mathbf{R}^d$) be the scalar product associated to $A$, with norm $\|\cdot\|_A$. In Theorem 2.4 and Proposition 2.6, we can replace assumption (4) by the following definition:

$$U^{n+1} \text{ is a minimizer of the function } V \mapsto \frac{\|V - U^n\|_A^2}{2\Delta t} + F(V).$$

The proofs are similar, replacing $\|\cdot\|$ by $\|\cdot\|_A$. Notice that in this case, $U^{n+1}$ satisfies

$$A\frac{(U^{n+1} - U^n)}{\Delta t} = -\nabla F(U^{n+1}).$$

This is the implicit Euler scheme for the flow

$$AU'(t) = -\nabla F(U(t)), \quad t \ge 0,$$

which is of course the gradient flow of $F$ for the scalar product $\langle\cdot,\cdot\rangle_A$.

## 3. Application to a time and space discretization of the Cahn-Hilliard equation.

Let us consider the Cahn-Hilliard equation

$$u_t = \Delta f(u) - \gamma\Delta^2 u, \quad \text{in } \Omega \times (0, +\infty), \tag{17}$$

where $\Omega$ is a bounded domain of $\mathbf{R}^d$ $(1 \le d \le 3)$ with Lipschitz continuous boundary, $\gamma > 0$, and $f$ is a polynomial of degree $2p+1$ with positive leading coefficient, $p \ge 1$ if $d = 1$ or $d = 2$, and $1 \le p \le 2$ if $d = 3$. Equation (17) is suplemented with an initial condition $u_0$ and Neumann or periodic boundary conditions (if periodic boundary conditions are considered, then $\Omega$ is a parallelepiped). The Cahn-Hilliard equation is mass conservative, and it can be seen as a gradient flow for the $H^{-1}$ scalar product, for the energy

$$E(u) = \int_\Omega \frac{\gamma}{2}|\nabla u|^2 + F(u)dx,$$

where $F(s)$ is an antiderivative of $f(s)$. Convergence to equilibrium for solutions of the Cahn-Hilliard equation, as $t \to +\infty$, has been shown in [26]. This is all the more interesting since stationary solutions of (17) are neither unique nor isolated in general (see for instance [18, 28]). We show below that this convergence still holds for a time and space discretization of (17), where the time discretization is the backward Euler scheme, and the space discretization is a finite element method.

The variational formulation of (17), as introduced by Elliott et al [10], reads as follows. Let $V = H^1(\Omega)$ (for Neumann boundary conditions) or $V = H^1_{per}(\Omega)$ (for periodic boundary conditions), and for a given $u_0 \in V$, find $u, w : [0, +\infty) \to V \times V$ such that $u(0) = u_0$ and

$$\begin{aligned} (u_t, \varphi) &= -(\nabla w, \nabla \varphi), \quad \forall \varphi \in V, \\ -(w, \chi) &= -(f(u), \chi) - \gamma(\nabla u, \nabla \chi), \quad \forall \chi \in V, \end{aligned}$$

where $(\cdot, \cdot)$ denotes the $L^2(\Omega)$-scalar product. The $L^2(\Omega)$-norm is denoted $\|\cdot\|_0$. For the space semi-discretization, we replace $V$ by $V^h$, a $P^k$ finite element approximation approximation of $V$ (see for instance [7, 10]). The space $V^h$ has finite dimension $N$ and it is built on a triangulation $\Omega^h$ of $\Omega$. Notice that if $\Omega^h$ covers $\Omega$ exactly, then $V^h \subset V$, but in general $V^h \not\subset V$.

The total discretization by the backward Euler scheme was considered by Elliott [9]. It reads: let $u^{h,0} \in V^h$, and for $n \geq 1$, find $(u^{h,n}, w^{h,n}) \in V^h \times V^h$ such that

$$\begin{aligned} (\frac{u^{h,n} - u^{h,n-1}}{\Delta t}, \varphi) &= -(\nabla w^{h,n}, \nabla \varphi), \quad \forall \varphi \in V^h, &(18)\\ -(w^{h,n}, \chi) &= -(f(u^{h,n}), \chi) - \gamma(\nabla u^{h,n}, \nabla \chi), \quad \forall \chi \in V^h. &(19) \end{aligned}$$

Since any constant function on $\Omega^h$ belongs to $V^h$, the mass is preserved, i.e.

$$(u^{h,n}, 1) = (u^{h,0}, 1), \quad \forall n \geq 0.$$

A steady state for this problem is a solution $(\bar{u}^h, \bar{w}^h) \in V^h \times \mathbf{R}$ such that

$$\left\{ \begin{aligned} (\bar{u}^h, 1) &= (u^{h,0}, 1) \\ \gamma(\nabla \bar{u}^h, \nabla \chi) + (f(\bar{u}^h), \chi) &= (\bar{w}^h, \chi). \end{aligned} \right.$$

The assumption on the nonlinearity $f$ implies that there exists $c_f \geq 0$ such that

$$f'(s) \geq -c_f, \quad \forall s \in \mathbf{R}. \tag{20}$$

Elliott proved existence and uniqueness of a solution $(u^{h,n}, w^{h,n})_{n\geq 0}$ for this scheme, for the nonlinearity $f(s) = s^3 - \beta^2 s$ [9]. He also proved dynamical stability, which implies convergence to equilibrium under the assumption that the steady states are isolated. However, such an assumption is impossible to check in general. Here, we prove convergence to equilibrium without any knowledge on the structure of steady states.

**Theorem 3.1.** *Assume that $\Delta t \in (0, 4\gamma/c_f^2)$, where $c_f$ is defined by (20). Then, for every $u^{h,0} \in V^h$, the scheme (18)-(19) defines a unique solution $((u^{h,n}, w^{h,n})) \in (V^h \times V^h)^{\mathbb{N}}$. Moreover, there exists a steady state $(u^{h,\infty}, w^{h,\infty})$ such that*

$$(u^{h,n}, w^{h,n}) \to (u^{h,\infty}, w^{h,\infty}) \text{ as } n \to +\infty.$$

*Proof.* The proof of uniqueness is the same as in [9] (it requires the smallness assumption made on $\Delta t$). For the proof of existence and convergence to equilibrium, we apply Theorem 2.4 and Remark 2.7: we show below that the scheme (18)-(19) can be seen as the backward Euler scheme applied to a gradient flow which is real analytic.

Let $(e_j)_{1\le j\le N}$ be an orthonormal basis of $V^h$ for the $L^2(\Omega^h)$-scalar product, with $e_1 \equiv cste$. In matrix representation, (18)-(19) reads:

$$\begin{pmatrix} A & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} W^n \\ (U^n - U^{n-1})/\Delta t \end{pmatrix} = \begin{pmatrix} 0 \\ -F^h(U^n) - \gamma A U^n \end{pmatrix},$$

where $U^n$, (resp. $W^n$) is the vector of the coordinates of $u^{h,n}$ (resp. $w^{h,n}$), $A$ is the matrix of the Laplacian, $I$ is the identity matrix, and

$$(F^h(U^n))_i = \int_{\Omega^h} f(u^{h,n}) e_i dx, \quad 1 \le i \le N.$$

The matrix $A$ is not invertible because of the constants. However, since the mass is preserved, we can eliminate the constants by writing, for any vector $X \in \mathbf{R}^N$, $X = (x_1, \dot{X})$, with $\dot{X} \in \mathbf{R}^{N-1}$. Now, $\dot{A}$ is invertible, because it is the matrix of the scalar product $(\nabla\cdot, \nabla\cdot)$ on $V^h/\mathbf{R}$. The scheme in matrix form becomes

$$\begin{pmatrix} \dot{A} & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \dot{W}^n \\ (\dot{U}^n - \dot{U}^{n-1})/\Delta t \end{pmatrix} = \begin{pmatrix} 0 \\ -\dot{F}^h(\dot{U^n}) - \gamma \dot{A} \dot{U}^n \end{pmatrix}.$$

This is equivalent to

$$\begin{aligned} \dot{W} &= -\dot{A}^{-1}(\dot{U}^n - \dot{U}^{n-1})/\Delta t, \\ \dot{A}^{-1}(\dot{U}^n - \dot{U}^{n-1})/\Delta t &= -\dot{F}^h(\dot{U}^n) - \gamma \dot{A}\dot{U}^n. \end{aligned}$$

We recognize the backward Euler scheme for the flow

$$\dot{A}^{-1}\dot{U}'(t) = -\dot{F}^h(\dot{U}(t)) - \gamma\dot{A}\dot{U}(t), \quad t \ge 0.$$

This is a gradient flow for the function

$$E^h(\dot{U}) = \int_{\Omega^h} \frac{\gamma}{2}|\nabla u^h|^2 + F(u^h) dx,$$

where, for $\dot{U} \in \mathbf{R}^{N-1}$, $u^h$ is defined by $u^h = u_1^0 e_1 + \sum_{i=2}^N u_i e_i$ ($u_1^0$ is the first component of $u^{h,0}$). The assumptions on $f$ imply that $F$ is bounded from below, so $E^h(\dot{U}) \to +\infty$ as $\|\nabla u^h\|_0 \to +\infty$. Since $\dot{U} \mapsto \|\nabla u^h\|_0$ is a norm on $\mathbf{R}^{N-1}$, this means that $E^h$ satisfies assumption (3). The assumption on $f$ implies that $E^h$ is a polynomial of the variables $(u_2, \dots, u_N)$. The proof is complete. □

A similar result holds for the discrete version of the viscous Cahn-Hilliard equation considered in [2]. Notice that in both cases, the convergence rates from Section 2 are valid (for some $\nu \in (0, 1/2]$ which is not known explicitly here).

4. **Convergence to equilibrium for the $\theta$-scheme.** In this section, we extend the preceding results on the backward Euler scheme to a more general family of schemes, namely the $\theta$-scheme. Recall that for a fixed $\theta \in [0,1]$, the $\theta$-scheme associated to the gradient flow (1) reads:

$$\frac{U^{n+1} - U^n}{\Delta t} = -\theta\nabla F(U^{n+1}) - (1-\theta)\nabla F(U^n). \tag{21}$$

For $\theta = 1$ we recover the backward Euler scheme (and for $\theta = 0$, the forward Euler scheme). The $\theta$-scheme has order one, except when $\theta = 1/2$, in which case it has order two. The $1/2$-scheme is also known as the Crank-Nicholson scheme.

In [17], Humphries and Stuart proved that when $\theta \in [1/2, 1]$ (i.e. when the implicit part is dominant) and if $F$ satisfies an additional natural assumption, then the scheme (21) is dynamically stable. For this purpose, they assume the following one-sided Lipschitz condition on $\nabla F$: there exists $c \geq 0$ such that

$$(\nabla F(U) - \nabla F(V)\,,\, U - V) \geq -c\|U - V\|^2, \qquad \forall U, V \in \mathbf{R}^d. \tag{22}$$

Assumption (22) is equivalent to saying that $\nabla F + cI$ is a monotone operator [5], or equivalently, that the function $V \mapsto F(V) + c\|V\|^2/2$ is convex. Notice that if $F \in C^1(\mathbf{R}^d, \mathbf{R})$ satisfies (6) and (22), then for every $U^n$, equation (21) has at least one solution $U^{n+1}$, obtained as a minimizer of the function

$$V \mapsto \frac{\|V - U^n\|^2}{2\Delta t} + \theta F(V) + (1 - \theta)(\nabla F(U^n), V),$$

which is bounded from below and coercive. Moreover, if $\Delta t < 1/(\theta c)$, then this solution is unique, by strict convexity.

Following [17], we now define, for $\Delta t > 0$,

$$F_{\Delta t}(U) := F(U) + \frac{\Delta t}{2}(1 - \theta)\|\nabla F(U)\|^2.$$

The stability result of Humphries and Stuart reads:

**Lemma 4.1** ([17])**.** *Let $\theta \in [1/2, 1]$. Assume that $F \in C^1(\mathbf{R}^d)$ satisfies* (3) *and* (22)*. Then, for every $0 < \Delta t \leq 1/c$, and for every sequence $(U^n)$ defined by* (21)*, the sequence $(F_{\Delta t}(U^n))$ satisfies*

$$F_{\Delta t}(U^{n+1}) + (1 - c\Delta t)\frac{\|U^{n+1} - U^n\|^2}{\Delta t} \leq F_{\Delta t}(U^n), \quad \forall n \geq 0. \tag{23}$$

In particular, the sequence $(U^n)$ is bounded, and this can be used to prove that the $\omega$-limit set of any initial condition $U^0$ is a subset of $\mathcal{S}$. By showing a modified Łojasiewicz inequality for $F_{\Delta t}$, we are able to prove convergence to equilibrium. In the remainder of this section, the parameter $\theta$ is fixed in $[1/2, 1]$.

**Theorem 4.2.** *Assume that $F \in C^{1,1}_{loc}(\mathbf{R}^d, \mathbf{R})$ satisfies* (3)*,* (22) *and that the Łojasiewicz inequality* (5) *holds for every $\bar{U} \in \mathcal{S}$. Let $U^0 \in \mathbf{R}^d$. Then there exists $\Delta t^\star = \Delta t^\star(U^0) \in (0, 1/c)$ such that for every $\Delta t \in (0, \Delta t^\star]$, if $(U^n)_{n\geq 0}$ is the unique sequence defined by* (21)*, then $U^n \to U^\infty$ as $n \to +\infty$, for some $U^\infty \in \mathcal{S}$.*

*Proof.* The proof is similar to that of Theorem 2.4. Let $\Delta t^\star \in (0, 1/c)$, which will be chosen later on, and let $\Delta t \in (0, \Delta t^\star]$. By (23) and (3), $(F_{\Delta t}(U^n))$ is nonincreasing, so $F_{\Delta t}(U^n) \to F^\star$, and we may assume that $F^\star = 0$. By (3), $(U^n)$ is bounded, so there exist $U^\infty \in \mathbf{R}^d$ and $(U^{n_k})$ such that $U^{n_k} \to U^\infty$. By (23), $\|U^{n+1} - U^n\| \to 0$, and letting $n = n_k$ tend to $+\infty$ in (21), we find $\nabla F(U^\infty) = 0$. By Lemma 4.3 below, there exist $\tilde{\sigma}, \tilde{\gamma} > 0$ and $\nu \in (0, 1/2]$ such that $F_{\Delta t}$ satisfies

$$\forall V \in \mathbf{R}^d, \quad \|V - U^\infty\| < \tilde{\sigma} \Rightarrow |F_{\Delta t}(V)|^{1-\nu} \leq \tilde{\gamma}\|\nabla F(V)\|. \tag{24}$$

Let $n$ be such that $\|U^{n+1} - U^\infty\| < \tilde{\sigma}$. We consider two cases:

• Case 1: $F_{\Delta t}(U^{n+1}) > F_{\Delta t}(U^n)/2$. We still have

$$\begin{aligned} F_{\Delta t}(U^n)^\nu - F_{\Delta t}(U^{n+1})^\nu &\overset{\text{case 1}}{\geq} 2^{\nu-1}\nu \left(F_{\Delta t}(U^{n+1})\right)^{\nu-1} \left[F_{\Delta t}(U^n) - F_{\Delta t}(U^{n+1})\right] \\ &\overset{(23)}{\geq} 2^{\nu-1}\nu(1-c\Delta t^\star)\frac{\|U^{n+1}-U^n\|^2}{\Delta t[F_{\Delta t}(U^{n+1})]^{1-\nu}}. \end{aligned} \tag{25}$$

On the other hand,

$$\frac{U^{n+1}-U^n}{\Delta t} \overset{(21)}{=} -\nabla F(U^{n+1}) + (1-\theta)\left[\nabla F(U^{n+1}) - \nabla F(U^n)\right].$$

Thus,

$$\frac{\|U^{n+1}-U^n\|}{\Delta t} \geq \|\nabla F(U^{n+1})\| - M\|U^{n+1}-U^n\|,$$

where $M$ is the Lipschitz constant of $\nabla F$ on the set

$$K_0 := \left\{U \;:\; F(U) \leq F(U_0) + \frac{(1-\theta)}{2c}\|\nabla F(U^0)\|^2\right\}.$$

The set $K_0$ is bounded by assumption (3), and $U^n \in K_0$ for all $n \geq 0$, by (23), so $M < +\infty$. Now, we choose $\Delta t^\star = 1/M$, and this yields, since $1/\Delta t \geq (M+1/\Delta t)/2$,

$$\frac{\|U^{n+1}-U^n\|^2}{\Delta t} \geq \frac{1}{2}\|\nabla F(U^{n+1})\| \cdot \|U^{n+1}-U^n\|.$$

Plugging this inequality into (25) and using (24), we get

$$F_{\Delta t}(U^n)^\nu - F_{\Delta t}(U^{n+1})^\nu \geq \frac{2^{\nu-2}\nu}{\tilde{\gamma}}(1-c\Delta t^\star)\|U^{n+1}-U^n\|.$$

• Case 2: $F_{\Delta t}(U^{n+1}) \leq F_{\Delta t}(U^n)/2$. Using (23), we find

$$\|U^{n+1}-U^n\| \leq \frac{1}{\sqrt{1-c\Delta t^\star}}\left(1-\frac{1}{\sqrt{2}}\right)^{-1}\sqrt{\Delta t}\left(F_{\Delta t}(U^n)^{1/2} - F_{\Delta t}(U^{n+1})^{1/2}\right).$$

We conclude as in the proof of Theorem 2.4. □

We have used the following

**Lemma 4.3.** *Assume that $F \in C^1(\mathbf{R}^d)$ satisfies the Łojasiewicz inequality* (5) *with exponent $\nu$ at a point $\bar{U} \in \mathcal{S}$, and let $\Delta t_1 > 0$. Then there exist constants $\tilde{\sigma}, \tilde{\gamma} > 0$ such that for all $0 < \Delta t \leq \Delta t_1$,*

$$\forall V \in \mathbf{R}^d, \quad \|V-\bar{U}\| < \tilde{\sigma} \Rightarrow |F_{\Delta t}(V) - F_{\Delta t}(\bar{U})|^{1-\nu} \leq \tilde{\gamma}\|\nabla F(V)\|.$$

*Proof.* Let $\Delta t \in (0, \Delta t_1]$ and let $V \in \mathbf{R}^d$ such that $\|V-\bar{U}\| < \sigma$, where $\sigma, \gamma > 0$ are such that the Łojasiewicz inequality (5) holds. We consider two cases:

• Case 1: $\frac{(1-\theta)\Delta t}{2}\|\nabla F(V)\|^2 \leq |F(V)-F(\bar{U})|$. Then,

$$\begin{aligned} \|\nabla F(V)\| &\overset{(5)}{\geq} \gamma^{-1}|F(V)-F(\bar{U})|^{1-\nu}, \\ &\overset{\text{case 1}}{\geq} \frac{\gamma^{-1}}{2^{1-\nu}}\left(|F(V)-F(\bar{U})| + \frac{(1-\theta)\Delta t}{2}\|\nabla F(V)\|^2\right)^{1-\nu}. \end{aligned}$$

• Case 2: $\dfrac{(1-\theta)\Delta t}{2}\|\nabla F(V)\|^2 > |F(V)-F(\bar U)|$. Then,

$$\begin{aligned}
\|\nabla F(V)\|^2 &= \frac{2}{3}\|\nabla F(V)\|^2 + \frac{1}{3}\|\nabla F(V)\|^2,\\
&\overset{\text{case 2}}{\geq} \frac{2}{3}\,\|\nabla F(V)\|^2 + \frac{2}{3(1-\theta)\Delta t}|F(V)-F(\bar U)|,\\
&\geq \frac{2}{3(1-\theta)\Delta t_1}\left(|F(V)-F(\bar U)|?? + \frac{(1-\theta)\Delta t}{2}\|\nabla F(V)\|^2\right).
\end{aligned}$$

Now, we choose $\tilde\sigma \in (0,\sigma]$ small enough so that

$$\forall V \in \mathbf{R}^d, \quad \|V-\bar U\| < \tilde\sigma \Rightarrow |F(V)-F(\bar U)|?? + \frac{(1-\theta)\Delta t_1}{2}\|\nabla F(V)\|^2 < 1.$$

Then, since $\nu \in (0,1/2]$, we see that there exists $\tilde\gamma = \tilde\gamma(\gamma,\nu,\Delta t_1) > 0$ so that in both cases, for all $V \in \mathbf{R}^d$ such that $\|V-\bar U\| < \tilde\sigma$,

$$\begin{aligned}
\|\nabla F(V)\| &\geq \tilde\gamma^{-1}\left(|F(V)-F(\bar U)| + \frac{(1-\theta)\Delta t}{2}\|\nabla F(V)\|^2\right)^{1-\nu},\\
&\geq \tilde\gamma^{-1}\left|F_{\Delta t}(V)-F_{\Delta t}(\bar U)\right|^{1-\nu},
\end{aligned}$$

and the proof is complete. □

Theorem 4.2 can be applied, for instance, to the space and time discretization of the viscous diffusion equation $u_t - \nu\Delta u_t = \Delta f(u)$ ($\nu > 0$, $f(u) = u^3 - u$) used in [25]. It could also be applied, as in Section 3, to the Cahn-Hilliard equation or to the viscous Cahn-Hilliard equation, by replacing the backward Euler scheme by the $\theta$-scheme.

**Remark 4.4.** 1. It is possible to obtain convergence rates for the $\theta$-scheme, by the same proof as in Section 2.
2. It is also possible to adapt Proposition 2.6 to the $\theta$-scheme.

5. **A model problem in infinite dimension.** Our aim, in this section, is to see that Theorem 2.4 can also be generalized, to some extent, in infinite dimension. For this purpose, we consider the following model problem:

$$u_t = \Delta u - f(u), \quad \text{in } \Omega \times [0,+\infty), \tag{26}$$

where $\Omega$ is a bounded domain of $\mathbf{R}^d$ with smooth boundary. Equation (26) is completed with Neumann, Dirichlet, or periodic boundary conditions, and with an initial condition.

In [19, 27], it was proved that a solution of (26) converges to equilibrium, provided that $f$ is real analytic and that the orbit of $u$ is precompact in an adequate function space. This problem has also been considered by many authors (see the references in [19]). Here, we prove that under some assumptions which remind the ones made in Theorem 4.2, any solution of the backward Euler scheme applied to (26) converges to equilibrium. In order to be slightly more general, we first consider an abstract version of (26).

5.1. **A general result.** Let $V$ denote a closed subspace of $H^1(\Omega)$ which is dense in $L^2(\Omega)$; in particular, $V$ is a Hilbert space for the $H^1(\Omega)$-scalar product. Identifying $L^2(\Omega)$ with its dual, we have the continuous inclusions $V \hookrightarrow L^2(\Omega) \hookrightarrow V'$. Let $a : V \times V \to \mathbf{R}$ be a continuous bilinear form on $V$ which is symmetric and coercive, i.e.

$$\exists \alpha > 0, \quad a(u,u) \geq \alpha \|u\|_V^2, \quad \forall u \in V. \tag{27}$$

We denote $A : V \to V'$ the linear isomorphism associated to $a$ by

$$\langle Au, v\rangle_{V',V} = a(u,v), \quad \forall u,\ v \in V. \tag{28}$$

We consider a Caratheodory function $f : \Omega \times \mathbf{R} \to \mathbf{R}$ assumed to be of class $C^1$ in the variable $s \in \mathbf{R}$ and such that

$$f(\cdot, 0) \in L^\infty(\Omega), \tag{29}$$

and we denote $F(x,s) = \int_0^s f(x,\sigma)d\sigma$. We assume the following growth condition:

$$|\partial_s f(x,s)| \leq a(x) + b|s|^{p_1}, \quad \forall s \in \mathbf{R}, \quad \text{a.e. } x \in \Omega, \tag{30}$$

for some $a \in L^{p_0}(\Omega)$ with $p_0 := d/2$ and $p_1 := 4/(d-2)$ if $d \geq 3$, and $p_0 > 1$ and $p_1 < \infty$ if $d = 2$; when $d = 1$, no growth condition is needed. Under these assumptions, the function

$$v \mapsto \int_\Omega F(x, v(x))dx$$

is of class $C^2$ on $H^1(\Omega)$, and its derivative with respect to $v$, $v \mapsto f(\cdot, v)$, is Lipschitz continuous on bounded sets of $H^1(\Omega)$ with values into $[H^1(\Omega)]'$ (see [20, Proposition 17.8 p. 68]). We assume in addition that (compare with (22))

$$\exists c_f \geq 0, \quad \partial_s f(x,s) \geq -c_f, \quad \forall s \in \mathbf{R}, \quad \text{a.e. } x \in \Omega. \tag{31}$$

Under the previous assumptions, by monotonicity, for every initial data $u_0 \in L^2(\Omega)$, the abstract evolution equation

$$u_t + Au + f(\cdot, u) = 0 \tag{32}$$

has a unique weak solution $u \in C^0([0,+\infty); L^2(\Omega))$ such that $u(0) = u_0$ (see [21] or [5, Example 2.3.7 and Theorem 3.17]). Equation (32) can be seen as a gradient flow, for the $L^2(\Omega)$ scalar product, of the energy functional

$$E(u) = \frac{1}{2}a(u,u) + \int_\Omega F(x, u(x))dx. \tag{33}$$

By the assumptions made above, $E$ is of class $C^2$ on $V$, with

$$dE_u(v) = a(u,v) + \int_\Omega f(x,u(x))v(x)dx, \quad \forall u, v \in V,$$

or equivalently, $dE_u = Au + f(\cdot, u)$ in $V'$. Similarly,

$$d^2E_u(v,v) = a(v,v) + \int_\Omega \partial_s f(x, u(x))v(x)^2dx, \quad \forall u, v \in V.$$

Now, consider the backward Euler scheme applied to (32): let $u^0 \in L^2(\Omega)$, and for $n \geq 0$, let $u^{n+1}$ be such that

$$\frac{u^{n+1} - u^n}{\delta t} + Au^{n+1} + f(\cdot, u^{n+1}) = 0 \qquad \text{in } V', \tag{34}$$

where $\delta t > 0$ is the time step. It is easy to see that for $\delta t > 0$ small enough, problem (34) has a unique solution $u^{n+1}$ for all $n$. Indeed, by (27) and (31),

$$d^2E_u(v,v) \geq \alpha\|v\|_1^2 - c_f\|v\|_0^2, \qquad \forall u, v \in V,$$

where $\|\cdot\|_0$ denotes the $L^2(\Omega)$-norm. Thus, if $1/\delta t \geq c_f$, then for every $u^n \in L^2(\Omega)$, the functional

$$v \mapsto \frac{\|v-u^n\|_0^2}{2\delta t} + E(v), \tag{35}$$

is $\alpha$-convex on $V$: it has a global minimizer $u^{n+1}$ in $V$, which is the unique solution of (34).

We will use the following definition, which generalizes Definition 2.1 in this context.

**Definition 5.1.** We say that $E : V \to \mathbf{R}$ defined by (33) satisfies the Łojasiewicz inequality in $V'$ at a point $\bar{u} \in V$ if there exist $\nu \in (0, 1/2]$ and $\sigma > 0$, $C > 0$ such that

$$\forall v \in V, \quad \|v - \bar{u}\|_V < \sigma \Rightarrow |E(v) - E(\bar{u})|^{1-\nu} \leq C\|dE_v\|_{V'}.$$

We also define the set of critical points of $E$

$$\mathcal{S}_E := \{v \in V,\ dE_v = 0 \text{ in } V'\}.$$

We have:

**Theorem 5.2.** *Let $\delta t > 0$ such that $1/\delta t \geq c_f$. Assume that (27)–(31) hold and that $E$ satisfies the Łojasiewicz inequality at every $\bar{u} \in \mathcal{S}_E$. Let $(u^n)$ be a sequence defined by (34). If $\cup_{n\geq 1}\{u^n\}$ is precompact in $V$, then there exists $u^\infty \in \mathcal{S}_E$ such that $u^n \to u^\infty$ in $V$.*

*Proof.* The proof is similar to the finite dimensional case, except that we have to deal with different norms. By uniqueness, $u^{n+1}$ minimizes the functional (35), so

$$E(u^{n+1}) + \frac{\|u^{n+1}-u^n\|_0^2}{2\delta t} \leq E(u^n), \quad \forall n \geq 0. \tag{36}$$

The sequence $(E(u^n))$ is therefore nonincreasing: it converges to some $E^\star$. We assume, without loss of generality, that $E^\star = 0$. By compactness, there exist $u^\infty \in V$ and a subsequence $(u^{n_k})$ such that $u^{n_k} \to u^\infty$ in $V$. By (36), $\|u^{n+1} - u^n\|_0 \to 0$; letting $n = n_k - 1$ tend to $+\infty$ in (34), we find that $dE_{u^\infty} = 0$ in $V'$, i.e. $u^\infty \in \mathcal{S}_E$.

Since $E$ satisfies the Łojasiewicz inequality at $u^\infty$, there exist $\nu \in (0, 1/2]$, $\sigma > 0$ and $\gamma > 0$ such that

$$\forall v \in V, \quad \|v - u^\infty\|_V < \sigma \Rightarrow |E(v)|^{1-\nu} \leq \gamma\|dE_v\|_{V'}. \tag{37}$$

Let $n$ be such that $\|u^{n+1} - u^\infty\|_V < \sigma$. We consider two cases:

- Case 1: $E(u^{n+1}) > E(u^n)/2$. Then, by (36),

$$E(u^n)^\nu - E(u^{n+1})^\nu \geq 2^{\nu-2}\nu\frac{\|u^{n+1}-u^n\|_0^2}{\delta t E(u^{n+1})^{1-\nu}}.$$

The scheme (34) reads $(u^{n+1} - u^n)/\delta t = -dE_{u^{n+1}}$; using (37) and

$$\|dE_v\|_0 \geq \|dE_v\|_{V'}, \quad \forall v \in V$$

(recall that $V$ is equipped with the $H^1(\Omega)$-norm), we find

$$E(u^n)^\nu - E(u^{n+1})^\nu \geq \frac{2^{\nu-2}\nu}{\gamma}\|u^{n-1} - u^n\|_0.$$

- Case 2: $E(u^{n+1}) \leq E(u^n)/2$. Then, by (36),

$$\|u^{n+1} - u^n\|_0 \leq 5\sqrt{\delta t}\left(E(u^n)^{1/2} - E(u^{n+1})^{1/2}\right).$$

In both cases, for all $n$ such that $\|u^{n+1} - u^\infty\|_V < \sigma$, we have

$$\|u^{n+1} - u^n\|_0 \leq \frac{2^{2-\nu}\gamma}{\nu}\left(E(u^n)^\nu - E(u^{n+1})^\nu\right) \\ + 5\sqrt{\delta t}\left(E(u^n)^{1/2} - E(u^{n+1})^{1/2}\right). \quad (38)$$

Now, we notice that $v^n := u^{n+1} - u^n$ tends to 0 in $L^2(\Omega)$, and by compactness, $v^n \to 0$ in $V$. Let $\varepsilon > 0$ ($\varepsilon << \sigma$) and let $\tilde{E} > 0$ be small enough so that

$$\frac{2^{2-\nu}\gamma}{\nu}\tilde{E}^\nu + 5\sqrt{\delta t}\tilde{E}^{1/2} < \varepsilon/2.$$

We choose $\bar{n} = n_k$ large enough so that $E(u^{\bar{n}}) \leq \tilde{E}$,

$$\|u^{\bar{n}} - u^\infty\|_0 < \varepsilon/2, \quad \|u^{\bar{n}} - u^\infty\|_V \leq \sigma/2, \quad \text{and} \quad \|v^n\|_V < \sigma/2, \quad \forall n \geq \bar{n}.$$

Let $N = N_\varepsilon \geq \bar{n}$ be the largest integer (including $+\infty$) such that $\|u^n - u^\infty\|_V < \sigma/2$ for all $\bar{n} \leq n \leq N$. If $N < +\infty$, then

$$\|u^{N+1} - u^\infty\|_V \leq \|u^N - u^\infty\|_V + \|v^N\|_V < \sigma.$$

Thus, by (38),

$$\sum_{n=\bar{n}}^{N} \|u^{n+1} - u^n\|_0 \leq \frac{2^{2-\nu}\gamma}{\nu}E(u^{\bar{n}})^\nu + 5\sqrt{\delta t}E(u^{\bar{n}})^{1/2} < \varepsilon/2. \quad (39)$$

In particular,

$$\|u^{N+1} - u^\infty\|_0 \leq \sum_{n=\bar{n}}^{N} \|u^{n+1} - u^n\|_0 + \|u^{\bar{n}} - u^\infty\|_V < \varepsilon.$$

This proves that $N_\varepsilon = +\infty$, if $\varepsilon > 0$ is small enough. Otherwise, we can find a sequence $\varepsilon_k \to 0$ and $N_{\varepsilon_k} \to +\infty$ (with $N_{\varepsilon_k} < +\infty$ for all $k$), such that $u^{N_{\varepsilon_k}+1} - u^\infty \to 0$ in $L^2(\Omega)$, and therefore in $V$, by compactness. This contradicts the definition of $N_\varepsilon$. Thus, $N_\varepsilon = +\infty$, and by the estimate (39), $(u^n)$ converges in $L^2(\Omega)$; by compactness, $(u^n)$ converges also in $V$ (to $u^\infty$). The proof is complete. □

**Remark 5.3.** Proposition 2.6 could also be generalized in a similar way. It is also possible to obtain convergence rates in $L^2(\Omega)$-norm, by the method used in section 2 and methods used in the continuous case (see for instance [16]).

5.2. **The semilinear heat equation with Dirichlet boundary conditions.** As an illustration, we consider our initial problem (26), with homogeneous Dirichlet boundary conditions. In this case, we have $V = H_0^1(\Omega)$,

$$a(u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx, \quad \forall u, v \in V,$$

and $A = -\Delta$. Assumption (29) is satisfied and the growth assumption (30) is replaced by

$$|f'(s)| \leq C(1 + |s|)^{p_1}, \quad \forall s \in \mathbf{R}, \quad (40)$$

where $C > 0$ and $p_1 < 4/(d-2)$ if $d \geq 3$, $p_1 < \infty$ if $d = 2$, and no growth assumption is needed if $d = 1$. Assumption (31) becomes:

$$\exists c_f \geq 0, \quad f'(s) \geq -c_f, \quad \forall s \in \mathbf{R}. \quad (41)$$

We also assume that

$$\liminf_{|s|\to+\infty} \frac{f(s)}{s} > -\lambda_1, \tag{42}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$:

$$\lambda_1 = \inf_{\|v\|_0=1} a(v,v).$$

By (42), $F(s) := \int_0^s f(\sigma)d\sigma$ satisfies

$$F(s) \geq -\frac{\kappa_1}{2}s^2 - \kappa_2, \quad \forall s \in \mathbf{R},$$

for some $\kappa_1 < \lambda_1$ and some $\kappa_2 \geq 0$. Thus,

$$E(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 + F(v)dx \geq \left(\frac{1}{2} - \frac{\kappa_1}{2\lambda_1}\right) a(v,v) - \kappa_2|\Omega|, \quad \forall v \in V. \tag{43}$$

The backward Euler scheme reads: let $u^0 \in L^2(\Omega)$ and for $n \geq 0$, let $u^{n+1}$ be defined by

$$\frac{u^{n+1} - u^n}{\delta t} - \Delta u^{n+1} + f(u^{n+1}) = 0. \tag{44}$$

We have

**Theorem 5.4.** *Assume that $f : \mathbf{R} \to \mathbf{R}$ is real analytic and satisfies* (40)-(42). *Let $\delta t \leq 1/c_f$, and let $(u^n)$ be a sequence defined by the backward Euler scheme* (44). *Then there exists $u^\infty \in V$ such that $-\Delta u^\infty + f(u^\infty) = 0$ and $u^n \to u^\infty$ as $n \to +\infty$.*

An important example of a function $f$ which satisfies the required assumptions is $f(s) = s^3 - s$, when $1 \leq d \leq 3$. In this case, equation (26) is known as the Allen-Cahn (or Ginzburg-Landau) equation.

*Proof.* We apply Theorem 5.2. We only need to prove that for every $\bar{u} \in \mathcal{S}_E$, the functional $E$ satisfies the Łojasiewicz inequality, and that $\cup_{n\geq 1}\{u^n\}$ is precompact in $H_0^1(\Omega)$.

Let $\bar{u} \in \mathcal{S}_E$. By elliptic regularity [12], $\bar{u} \in C^\infty(\overline{\Omega})$. Corollary 4.5 in [6] shows that $E$ satisfies the Łojasiewicz inequality at $\bar{u}$. This proves the first point.

For the second point, we first notice that $E(u^1) < +\infty$, because $u^1$ is a minimizer in $V$ of the functional

$$v \mapsto \frac{\|v - u^0\|_0^2}{2} + E(v).$$

By (36), $E(u^n) \leq E(u^1)$, for all $n \geq 1$. By (43), $(u^n)$ is bounded in $V$. In particular, there exists a bound $M_1$ such that for every $n \geq 0$,

$$\|u^n\|_0 + \|u^{n+1}\|_V \leq M_1.$$

Let us first assume $d \geq 3$. For every $n \geq 0$, we deduce from the Sobolev imbedding theorem that $u^{n+1} \in L^{2^\star}(\Omega)$ where $2^\star = 2d/(d-2)$ with $\|u^{n+1}\|_{L^{2^\star}(\Omega)} \leq M_2$. So the growth condition (40) and the bound on $p_1$ imply that there exists $2 \geq q > 2d/(d+2)$ such that $\|f(u^{n+1})\|_{L^q(\Omega)} \leq M_3$. By elliptic regularity [12], we deduce from (44) that the sequence $(u^{n+1})$ is bounded in $W^{2,q}(\Omega)$. Finally, from the Sobolev imbedding theorem, $W^{2,q}(\Omega)$ is compactly imbedded in $H^1(\Omega)$: we obtain that the sequence $(u^n)$ is precompact in $H^1(\Omega)$.

In the case $d = 2$, we directly obtain from the Sobolev imbedding theorem that $(f(u^{n+1}))$ is bounded in any $L^q(\Omega)$, $q < \infty$ and we conclude similarly.

In the case $d = 1$, $H^1(\Omega)$ is compactly embedded in $C(\bar{\Omega})$ so $(f(u^{n+1}))$ is uniformly bounded in $L^\infty(\Omega)$. We conclude as above, and this completes the proof. □

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