

L^∞ -ERROR ESTIMATE FOR A FINITE VOLUME APPROXIMATION OF LINEAR ADVECTION

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Abstract. We study the convergence of the upwind Finite Volume scheme applied to the linear advection equation with a Lipschitz divergence-free speed in \mathbf{R}^d . We prove a $h^{1/2-\varepsilon}$ -error estimate in the $L^\infty(\mathbf{R}^d \times [0, T])$ -norm for Lipschitz initial data. The expected optimal result is a $h^{1/2}$ -error estimate. In a second part, we also prove a $h^{1/2}$ -error estimate in the $L^\infty(0, T; L^2(\mathbf{R}^d))$ -norm for initial data in $H^1(\mathbf{R}^d)$.

Key words. scalar conservation laws, advection equation, Finite Volume method, error estimate

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1. Introduction. We consider here the numerical approximation by the explicit upwind Finite Volume scheme of the following general non linear hyperbolic scalar problem in d -space dimension.

$$\begin{cases} u_t + \operatorname{div}(Vf(u)) & = & 0 & \text{on} & \mathbf{R}_x^d \times \mathbf{R}_{+t} \\ u(x, 0) & = & u_0(x) & \text{on} & \mathbf{R}^d, \end{cases} \quad (1.1)$$

where the vector field V is a Lipschitz function of x and t with values in \mathbf{R}^d and is divergence free.

Finite Volume schemes are widely used for the numerical approximation of such equations and more generally of conservative hyperbolic systems. They have several good behaviours: they can handle general meshes, they have a compact numerical stencil, they conserve the quantity $\int_{\mathbf{R}^d} u$ and under some hypothesis (CFL condition) they have stability properties. Moreover, in general, the solutions to (1.1) are not unique, and, again under some hypothesis concerning the time step and the regularity of the mesh, FV Schemes do converge towards the physical (entropic) solution when the mesh-size h tends to 0.

However, the convergence theory for these schemes is far from being complete. The first result is due to Kuznetsov [7], who proves a $h^{1/2}$ -error estimate in the $L^\infty(0, T; L_x^1)$ norm for BV -data and a family of structured cartesian meshes of mesh-size h tending to 0. The method of Kuznetsov has been extended to the case of non-regular cartesian meshes or almost cartesian meshes (see [2]). This method had also been extended to family of unstructured uniformly regular meshes, but in this last case only a $h^{1/4}$ -convergence rate is obtained (see Cockburn, Coquel, Lefloch [1], Vila [9] and Eymard, Gallouët, Herbin [6]). We emphasize that numerical experiments show that the error behaves like $h^{1/2}$ regardless of whether the family of meshes is structured.

In this paper, we focus on the advection equation (i.e. the linear case $f(u) := u$) with $W^{1,p}$ initial data. Even in the linear case, it is more difficult to establish convergence results than in the setting of Finite Difference methods. Indeed, in general, FV schemes are not consistent in the FD classical sense and Lax's theorem cannot be used. Provided that the Courant-Friedrich-Levy condition is satisfied and that the meshes are uniformly regular (see the condition (1.5) below), we expect a $h^{1/2}$ convergence rate in the $L^\infty(0, T; L_x^p)$ -norm.

In [8] the author and Vovelle prove this result for $p = 1$ (BV -data) under a strict CFL condition in the general case and under a sharp CFL condition when the speed does not depend on the time variable.

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In [4] Després shows the result in the case $p = 2$ under a strict CFL and for a constant speed. His proof is based on the study of the dissipation of the consistency error by the scheme and on a generalization of the Lax theorem. The lack of consistency of the scheme is compensated by the particular structure of the consistency error.

We also refer to Vila and Villedieu [10], who proved (under a strict CFL) a $h^{1/2}$ -error estimate in the L^2_{loc} space-time norm for the FV approximation of Friedrichs hyperbolic systems with H^1 data — the linear advection equation appears to be a particular case of these systems. Their proof is based on energy estimates and on the weak consistency of the scheme in the dual of H^1_{loc} .

The case $p > 2$ has been attacked by Després. In particular, in [5] a $h^{1/4-\varepsilon}$ -estimate is obtained in the L^∞ -norm for Lipschitz data.

Here, we study the case $2 \leq p \leq \infty$. For $p = 2$, we obtain the expected result under a strict CFL condition — Theorem 1.8. The ingredients of the proof are essentially contained in [8]: the conservation of the L^2 -energy of the exact solution and energy estimates for the approximate solution coming from the dissipative properties of the scheme.

In the case $p = +\infty$, we prove a $h^{1/2-\varepsilon}$ -error estimate in the L^∞ -norm for Lipschitz data under a strict CFL condition — Theorem 1.6. In the proof, the fact that the scheme is consistent of order 1 for the dual norm of the BV -seminorm is crucial — see Section 2.3 for a more precise description. This result is again the consequence of the structure of the consistency error: the size of the error is $\mathcal{O}(1)$ but the integral of this error on any finite union of cells ω is controlled by the measure of the boundary of ω times h — and not only by the measure of ω .

The proof of Theorem 1.6 may be sketched as follows. By duality, we are led to prove Theorem 1.7 concerning the behaviour of the FV approximation when the initial data is a probability measure concentrated on one cell. Let K_0 be a cell, let $u_0 := 1/|K_0| \mathbf{1}_{K_0}$ be the initial condition, u be the exact solution to the advection equation and \mathbf{u} be the corresponding FV approximation. Roughly speaking, Theorem 1.7 says that the expectation of the distance $\|x - y\|$ with respect to the probability

$$dP(x, y) := u(x, T)d\lambda(x) \times \mathbf{u}(y, T)d\lambda(y)$$

in $\mathbf{R}_x^d \times \mathbf{R}_y^d$ is bounded by $Ch^{1/2-\varepsilon}$, namely $\int_{\mathbf{R}^d \times \mathbf{R}^d} \|x - y\| u(x, T) \mathbf{u}(y, T) dx dy \leq Ch^{1/2-\varepsilon}$.

For simplicity, assume that the speed is constant: $V(x, s) = \mathbf{V}$; in this case the exact solution is given by $u(\cdot, t) = 1/|K_0| \mathbf{1}_{K_0 + \mathbf{V}t}$. Let $x_0 \in K_0$ and let B_0 be the ball centered at x_0 with radius $k\sigma h^{1/2}$ where k is a positive integer and σ is a parameter depending on V , T and the regularity of the mesh. We consider the solution to the exact advection equation with initial data: $\mathbf{1}_{B_0}$. At time t this solution is still the characteristic function of a ball $B_t = B_0 + \mathbf{V}t$. We have to prove that the integral of \mathbf{u} on B_t^c remains small.

The evolution of the L^2 -energy of \mathbf{u} on B_t^c , is driven by two phenomenons:

- 1) the dissipation of the scheme decreases the energy and provides energy estimates,
- 2) a source term created by the consistency error and located on the boundary of B_t

may increase the energy.

We use the energy estimates and the weak consistency of the scheme to prove (recursively on k) that the L^2 -energy of \mathbf{u} on B_t^c is bounded by $(t/T)^k h^{-d}/k!$ and we conclude by the Cauchy-Schwarz inequality. In fact, in the proof, the characteristic function of the set B_t^c is replaced by a smooth cut-off function in the neighborhood of ∞ .

The sequel is organized as follows. First we supplement this introduction by describing the linear advection problem (§1.1), the explicit upwind Finite Volume scheme (§1.2) and the main results of the paper (§1.3).

Section 2 is devoted to the proof of Theorems 1.6 and 1.7. In §2.1, we prove by duality that Theorem 1.6 ensues from Theorem 1.7. In §2.2, we establish an identity satisfied by the $L^2(\Psi^2 d\lambda(x))$ -energy of the numerical solution — in the applications the weight Ψ^2 will be a cut-off function in the neighborhood of ∞ . In §2.3 we prove that the implicit in time downwind FV scheme is consistent of order 1 in a weak sense. Applying the results of §§2.2-2.3, we prove in §2.4 that if two solutions of the advection equation $\Psi, \bar{\Psi}$ satisfy $0 \leq \mathbf{1}_{\text{supp } \Psi} \leq \bar{\Psi}$, the growth of the $L^2(\Psi^2 d\lambda(x))$ -energy of the numerical solution between times 0 and T is controlled by its $L^2(\bar{\Psi}^2 d\lambda(x)dt)$ -energy. Finally, in §2.5, we build a decreasing sequence of cut-off functions, we apply recursively the result of §2.4 and we end the proof of Theorem 1.7.

Theorem 1.8 is proved in Section 3. In §3.1, we recall that the FV approximation satisfies the weak formulation of the advection equation up to an error term, we prove an energy inequality and we give some technical results concerning some projections of the exact solution on the mesh. In §3.2 we derive successive estimates yielding Theorem 1.8.

1.1. The linear advection equation. We consider the linear advection problem in \mathbf{R}^d :

$$\begin{cases} u_t + \text{div}(Vu) = 0, & x \in \mathbf{R}^d, t \in \mathbf{R}_+, \\ u(x, 0) = u_0(x), & x \in \mathbf{R}^d, \end{cases} \quad (1.2)$$

where we suppose that $V \in W^{1,\infty}(\mathbf{R}^d \times \mathbf{R}_+, \mathbf{R}^d)$ satisfies

$$\text{div } V(\cdot, t) = 0 \quad \forall t \in \mathbf{R}_+. \quad (1.3)$$

The problem (1.2) is well-posed in $L^1_{loc}(\mathbf{R}^d)$. Let us recall the following classical result (see Theorem 1 in [8] for a sketch of proof.)

THEOREM 1.1. *For every $u_0 \in L^1_{loc}(\mathbf{R}^d)$, the problem (1.2) has a unique weak solution u , in the sense that $u \in L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^d)$ such that: for every compactly supported test function $\phi \in W^{1,\infty}(\mathbf{R}^d \times \mathbf{R}_+)$,*

$$\int_{\mathbf{R}_+} \int_{\mathbf{R}^d} u(\phi_t + V \cdot \nabla \phi) dx dt + \int_{\mathbf{R}^d} u_0(x) \phi(x, 0) dx = 0. \quad (1.4)$$

This solution is given by the characteristic formula

$$u(x, t) = u_0(X(x, t)), \quad \forall (x, t) \in \mathbf{R}^d \times \mathbf{R}_+,$$

with X defined by $X(\cdot, t) := Y(\cdot, t)^{-1}$, where then Y solves the Cauchy Problem $\partial_t Y(x, t) = V(Y(x, t), t)$; $Y(x, 0) = x$.

Moreover, for every $t \geq 0$, $X(\cdot, t) : \mathbf{R}^d \rightarrow \mathbf{R}^d$ is one to one and onto, the map X belongs to $W^{1,\infty}(\mathbf{R}^d \times \mathbf{R}_+, \mathbf{R}^d)$. More precisely, for every $T \geq 0$, $X - Id_{\mathbf{R}^d}$ and $Y - Id_{\mathbf{R}^d}$ belong to $W^{1,\infty}(\mathbf{R}^d \times [0, T])$ and there exists $C_0 \geq 1$ only depending on $T, \|V\|_{W^{1,\infty}}$ and d , such that, $\forall x \in \mathbf{R}^d, \forall t \in [0, T]$,

$$\begin{aligned} \|X(x, t) - x\| &\leq C_0 t, & \|\nabla X(x, t)\| &\leq C_0, & \|\partial_t X(x, t)\| &\leq C_0, \\ \|Y(x, t) - x\| &\leq C_0 t, & \|\nabla Y(x, t)\| &\leq C_0, & \|\partial_t Y(x, t)\| &\leq C_0. \end{aligned}$$

Finally, for every $t \geq 0$, $X(\cdot, t)$ preserves the Lebesgue measure λ on \mathbf{R}^d , i.e.:

$$\lambda(X(E, t)) = \lambda(E), \quad \text{for every Borel subset } E \text{ of } \mathbf{R}^d.$$

For $t \geq 0$, the map $u_0 \mapsto u(\cdot, t)$ acts on $W^{1,\infty}(\mathbf{R}^d)$ as a continuous linear operator and if $u_0 \in W^{1,\infty}(\mathbf{R}^d)$, then $t \mapsto u(\cdot, t)$ is locally Lipschitz from \mathbf{R}_+ to $L^\infty(\mathbf{R}^d)$. More precisely, the characteristic formula yields

COROLLARY 1.2. *Let $0 \leq t \leq T$, there exists a constant C_0 depending on $\|V\|_{W^{1,\infty}}$ and T such that if $u_0 \in W^{1,p}(\mathbf{R}^d)$, for some $1 \leq p \leq \infty$, then*

$$\begin{aligned} \|u(\cdot, s) - u(\cdot, t)\|_{L^p} &\leq C_0 \|\nabla u_0\|_{L^p} |s - t|, & 0 \leq s, t \leq T, \\ \|\nabla u(\cdot, t)\|_{L^p} &\leq C_0 \|\nabla u_0\|_{L^p}, & 0 \leq t \leq T. \end{aligned}$$

1.2. Finite Volume scheme. The Finite Volume scheme which approximates (1.2) is defined on a mesh \mathcal{T} which is a family of closed connected polygonal subsets with disjoint interiors covering \mathbf{R}^d ; the time half-line is meshed by regular cells of size $\delta t > 0$. We also assume that the partition \mathcal{T} satisfies the following properties: the common interface of two control volumes is included in an hyperplane of \mathbf{R}^d and

$$\text{there exists } \alpha > 0 \text{ such that } \begin{cases} \alpha h^d \leq |K|, \\ |\partial K| \leq \alpha^{-1} h^{d-1}, \end{cases} \quad \forall K \in \mathcal{T}, \quad (1.5)$$

where h is the size of the mesh: $h := \sup\{\text{diam}(K), K \in \mathcal{T}\}$, $|K|$ is the d -dimensional Lebesgue measure of K and $|\partial K|$ is the $(d-1)$ -dimensional Lebesgue measure of ∂K . If K and L are two control volumes having a common edge, we say that L is a neighbour of K and denote (quite abusively) $L \in \partial K$. We also denote $K \setminus L$ the common edge and \mathbf{n}_{KL} the unit normal to $K \setminus L$ pointing outward K . We set

$$V_{KL}^n := \frac{1}{\delta t} \int_{n\delta t}^{(n+1)\delta t} \int_{K \setminus L} V \cdot \mathbf{n}_{KL}$$

We denote by $K_n := K \times [n\delta t, (n+1)\delta t)$ a generic space-time cell, by $\mathcal{M} := \mathcal{T} \times \mathbf{N}$ the space-time mesh. The sets $\partial K_n^-, \partial K_n^+$ are defined by

$$\partial K_n^- := \{L \in \partial K, V_{KL}^n < 0\}, \quad \partial K_n^+ := \{L \in \partial K, V_{KL}^n \geq 0\}.$$

In the same way we set $K \setminus L_n := K \setminus L \times [n\delta t, (n+1)\delta t)$.

We will assume that a strict Courant-Friedrich-Levy condition is satisfied:

$$\sum_{L \in \partial K_n^-} \delta t |V_{KL}^n| \leq (1 - \xi) |K|, \quad \forall K_n \in \mathcal{M}, \quad \text{for some } 1 \geq \xi > 0. \quad (1.6)$$

Moreover, in order to avoid the occurrence of terms with factor $\delta t/h$ in our estimates, we add to (1.6) the following condition: there exists $c_0 \geq 0$ such that

$$\delta t \leq c_0 h. \quad (1.7)$$

The upwind Finite Volume method with explicit time-discretization is defined by the following set of equations:

$$u_K^0 := \frac{1}{|K|} \int_K u_0(x) dx \quad \forall K \in \mathcal{T}, \quad (1.8)$$

$$u_K^{n+1} := u_K^n + \frac{\delta t}{|K|} \sum_{L \in \partial K_n^-} -V_{KL}^n (u_L^n - u_K^n), \quad \forall K_n \in \mathcal{M}. \quad (1.9)$$

We then denote by \mathbf{u} the approximate solution of (1.2) defined by the FV scheme:

$$\mathbf{u}(\cdot, t) := u_K^n \quad \text{a.e in } K \quad \text{for every } t \in [n\delta t, (n+1)\delta t); \quad K_n \in \mathcal{M}. \quad (1.10)$$

Classical results. The divergence free assumption (1.3) leads to the following identity

LEMMA 1.3. *Under the hypothesis (1.3) on the divergence of V , one has*

$$\sum_{L \in \partial K} V_{KL}^n = \sum_{L \in \partial K_n^-} V_{KL}^n + \sum_{L \in \partial K_n^+} V_{KL}^n = 0 \quad \forall K_n \in \mathcal{M}.$$

Under the CFL condition (1.6) the Finite Volume scheme is order-preserving:

PROPOSITION 1.4. *Under condition (1.6), the linear application $\mathcal{L} : (u_K^n) \mapsto (u_K^{n+1})$ defined by (1.9) is order-preserving and stable for the L^∞ -norm.*

Proof. Eq. (1.9) gives

$$u_K^{n+1} = \left(1 - \sum_{L \in \partial K_n^-} \frac{-V_{KL}^n \delta t}{|K|} \right) u_K^n + \sum_{L \in \partial K_n^+} \frac{-V_{KL}^n \delta t}{|K|} u_L^n$$

i.e., under (1.6), u_K^{n+1} is a convex combination of $u_K^n, (u_L^n)_{L \in \partial K_n^-}$. The stability result follows from $\mathcal{L}(1) = 1$.

From Lemma 1.3 it is not difficult to see that the quantity $\sum_{K \in \mathcal{T}} |K| u_K^n$ is conserved. By the Crandall-Tartar lemma [3], this fact and Proposition 1.4 imply

PROPOSITION 1.5. *Under condition (1.6), the scheme $\mathcal{L} : (u_K^n) \mapsto (u_K^{n+1})$ is stable for the L^1 -norm:*

$$\sum_{K \in \mathcal{T}} |K| |u_K^{n+1}| \leq \sum_{K \in \mathcal{T}} |K| |u_K^n|, \quad \forall n \geq 0.$$

Interpolating between the L^1 - and L^∞ -stability, the scheme is stable for the L^p -norm, $1 < p < \infty$.

1.3. Main results. We now assume that $V \in W^{1,\infty}(\mathbf{R}^d \times \mathbf{R}_+, \mathbf{R}^d)$ satisfies (1.3), that \mathcal{T} is mesh of mesh-size $h > 0$ satisfying conditions (1.5) and that the time step $\delta t > 0$ satisfies the CFL conditions (1.6)-(1.7). We fix a time $T \geq \delta t$ and we assume that there exists an integer $N \geq 0$ such that $T = (N + 1)\delta t$.

In the sequel, given an initial condition $u_0 \in L^1_{loc}(\mathbf{R}^d)$, the function $u \in C(\mathbf{R}_+, L^1_{loc}(\mathbf{R}^d))$ denotes the exact solution to (1.2) and \mathbf{u} its numerical approximation obtained by the upwind FV method (1.8)-(1.9)-(1.10). In the results and in the proofs, $C_0 \geq 1$ is the constant introduced in Theorem 1.1 and Corollary 1.2; this constant only depends on $T, \|V\|_{W^{1,\infty}}$ and d . The letter C denotes various constants which are non decreasing functions of $\alpha^{-1}, c_0, \|V\|_{W^{1,\infty}}$ and d but do not depend on $h, \delta t, \xi, u_0$ or T .

We have the following error estimate for Lipschitz data.

THEOREM 1.6. *Under a strict CFL condition ($\xi > 0$), there exists $h_\star > 0$ depending only on C_0, ξ, c_0 and T such that if $h \leq h_\star$, we have for every $u_0 \in W^{1,\infty}(\mathbf{R}^d)$,*

$$\|\mathbf{u}(\cdot, T) - u(\cdot, T)\|_{L^\infty} \leq CC_0^2 \xi^{-1/2} \|\nabla u_0\|_{L^\infty} T^{1/2} h^{1/2} \Gamma^{-1}(C_0^2 \xi^{-1} T h^{-1}),$$

where $\Gamma^{-1} : [1, +\infty) \rightarrow [2, +\infty)$ is the inverse Euler Γ function — in particular $\Gamma^{-1}(n!) = n + 1$. This result will be the consequence of

THEOREM 1.7. *Let $K_0 \in \mathcal{T}, x_0 \in K_0$ and $u_0 := 1/|K_0| \mathbf{1}_{K_0}$, then under a strict CFL condition, for $h \leq h_\star$, as above, we have*

$$\int_{\mathbf{R}^d} \|x - Y(x_0, T)\| u(x, T) dx \leq CC_0^2 \xi^{-1/2} T^{1/2} h^{1/2} \Gamma^{-1}(C_0^2 \xi^{-1} T h^{-1}).$$

REMARK 1.1. *The dependency in T in Theorem 1.6 and 1.7 is not interesting in the general case since C_0 depends on T . But in the case of a speed independent of x : $V(x, t) := \mathbf{V}(t)$, we have $X(x, t) = x - \int_0^t \mathbf{V}(s) ds$ and we can choose $C_0 = \|\mathbf{V}\|_\infty$ in Theorem 1.1. This is also valid for Theorem 1.8 below.*

REMARK 1.2. *The function Γ^{-1} is very slowly increasing (for exemple $\Gamma^{-1}(R) = o(\ln R)$) but we do not obtain a $h^{1/2}$ -estimate. Proving this result (or exhibiting a counterexample) remains an open problem.*

In this L^2 setting, we need an additional assumption on the mesh: there is no small interface:

$$\alpha h^{d-1} \leq |K \cap L|, \quad \forall K \in \mathcal{T}, L \in \partial K. \quad (1.11)$$

THEOREM 1.8. *Assume that (1.5), (1.11), (1.6) and (1.7) hold. Then for every u_0 in $H^1(\mathbf{R}^d)$, we have*

$$\begin{aligned} \|u(\cdot, T) - \mathbf{u}(\cdot, T)\|_{L^2}^2 + \|\mathbf{u}(\cdot, 0)\|_{L^2}^2 - \|\mathbf{u}(\cdot, T)\|_{L^2}^2 &\leq C(1 + C_0^2)\xi^{-1} \|u_0\|_{H^1}^2 Th + C\|u_0\|_{H^1}^2 h^2, \\ \mathcal{F}(\mathbf{u}, T) &\leq C(1 + C_0^2)\xi^{-2} \|u_0\|_{H^1}^2 Th + C\|u_0\|_{H^1}^2 h^2, \end{aligned}$$

with

$$\mathcal{F}(\mathbf{u}, T) := \sum_{n=0}^N \sum_{K \in \mathcal{T}} |K| |u_K^{n+1} - u_K^n|^2 + \sum_{n=0}^N \sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^-} \delta t |V_{KL}^n| |u_L^n - u_K^n|^2,$$

REMARK 1.3. *If the speed of advection V is constant, then assumption (1.11) is not necessary — see Remark 3.1.*

REMARK 1.4. *Using the technique of [8], we may prove that in the case of a constant speed $V(x, t) = \mathbf{V}$, the result also holds under a sharp CFL condition.*

REMARK 1.5. *The conditions on the mesh (1.5), (1.11) are not stringent. They are satisfied in practice by every efficient mesh generator.*

REMARK 1.6. *We will see in the sequel that the L^2 -norm of the discrete solution is non increasing. Since the L^2 -energy of the exact solution is conserved, we deduce from the first estimate of Theorem 1.8 that $0 \leq \|u(\cdot, T)\|_{L^2}^2 - \|\mathbf{u}(\cdot, T)\|_{L^2}^2 = \mathcal{O}(h)$. Since $\|(u - \mathbf{u})(\cdot, T)\|_{L^2} = \mathcal{O}(h^{1/2})$, we may split the difference $(u - \mathbf{u})(\cdot, T)$ into a $\mathcal{O}(h^{1/2})$ component which is orthogonal to $u(\cdot, T)$ in L^2 and a $\mathcal{O}(h)$ component.*

Conventions and notations. If (X, μ) is a measurable set with finite (positive) measure and $\phi \in L^1(X)$, we denote the mean value of ϕ over X by

$$\langle \phi \rangle_X := \int_X \phi d\mu := \frac{1}{\mu(X)} \int_X \phi d\mu.$$

The characteristic function of a set X is denoted by $\mathbf{1}_X$.

For $R > 0$, $B(R)$ denotes the open ball centered at 0 with radius R in \mathbf{R}^d .

In the sequel, we will use several functions in $L_{loc}^1(\mathbf{R}^d \times \mathbf{R}_+)$ which are almost everywhere constants on each space-time cell K_n . We will use the following convention, when $(\phi_{K_n})_{K_n \in \mathcal{M}}$ is a family of real numbers, then the bold type letter $\boldsymbol{\phi}$ denotes the function $\boldsymbol{\phi} := \sum_{K \in \mathcal{M}} \phi_K^n \mathbf{1}_{K_n}$. When ϕ belongs to $C(\mathbf{R}^d \times \mathbf{R}_+, \mathbf{R})$, then, for $K_n \in \mathcal{M}$, $\phi_K^n := \langle \phi(\cdot, n\delta t) \rangle_K$ denotes the mean value of ϕ on K at time $n\delta t$ and in this case $\boldsymbol{\phi} = \sum_{K \in \mathcal{M}} \langle \phi(\cdot, n\delta t) \rangle_K \mathbf{1}_{K_n}$. This last convention holds for any letter φ but u (u is always the exact solution of (1.2) and \mathbf{u} its FV approximation).

For every $K_n \in \mathcal{M}$, we set $V_K^n := -\sum_{L \in \partial K_n^-} V_{KL}^n = \sum_{L \in \partial K_n^+} V_{KL}^n$ and for every $t \geq 0$,

$$\mathcal{M}_t := \{K_n \in \mathcal{M} : 0 \leq n < t/\delta t\},$$

in particular, for $T = (N + 1)\delta t$, $\mathcal{M}_T = \{K_n : K \in \mathcal{T}, 0 \leq n \leq N\}$.

Let $(\varphi_K^n)_{K_n \in \mathcal{M}}$ be a bounded family of real numbers. We define an energy associated to the weight φ and the FV solution \mathbf{u} :

$$\mathcal{E}(\varphi, \mathbf{u}, t) := \sum_{K_n \in \mathcal{M}_t} \varphi_K^{n+1} e_K^n(\mathbf{u}),$$

$$\text{where } e_K^n(\mathbf{u}) := \sum_{L \in \partial K_n^-} \left(1 - \frac{V_K^n \delta t}{|K|}\right) |V_{KL}^n| \delta t |u_L^n - u_K^n|^2 + \frac{1}{2} \sum_{L, M \in \partial K_n^-} \frac{V_{KL}^n V_{KM}^n \delta t^2}{|K|} |u_M^n - u_L^n|^2.$$

In Section 3, we will write $\mathcal{E}(\mathbf{u}, t)$ for $\mathcal{E}(\mathbf{1}, \mathbf{u}, t)$.

We also define the local consistency error $\mathcal{Q}_K^n(\Psi)$ of the implicit in time downwind FV scheme associated to (1.2) applied to a function $\Psi \in W^{1,\infty}(\mathbf{R}^d \times \mathbf{R}_+)$:

$$\mathcal{Q}_K^n(\Psi) := \frac{\Psi_K^{n+1} - \Psi_K^n}{\delta t} + \sum_{L \in \partial K_n^+} \frac{V_{KL}^n}{|K|} (\Psi_L^{n+1} - \Psi_K^{n+1}), \quad \forall K_n \in \mathcal{M}.$$

2. Proof of Theorems 1.6 and 1.7.

2.1. Proof of Theorem 1.7. In this section, $u_0 \in W^{1,\infty}(\mathbf{R}^d)$. We have to estimate the quantity

$$\|u(\cdot, T) - u(\cdot, T)\|_\infty = \sup_{K_0 \in \mathcal{T}} \sup_{x_0 \in K_0} |u_{K_0}^{N+1} - u(x_0, T)|.$$

Let $x_0 \in \mathbf{R}^d$ and let $K_0 \in \mathcal{T}$, such that $x_0 \in K_0$. Next, set $\mathbf{v}^0 := 1/|K_0| \mathbf{1}_{K_0}$ and define recursively for $K \in \mathcal{T}$ and $0 \leq n \leq N$:

$$v_K^{n+1} = v_K^n + \frac{\delta t}{|K|} \sum_{L \in \partial K_{N-n}^+} V_{KL}^{N-n} (v_L^n - v_K^n). \quad (2.1)$$

Since $u_{K_0}^{N+1} = \sum_{K \in \mathcal{T}} |K| u_K^{N+1} v_K^0$, (1.9) and repeated discrete integrations by part yield

$$u_{K_0}^{N+1} = \sum_{K \in \mathcal{T}} \left(|K| u_K^N v_K^0 - \sum_{L \in \partial K_n^-} \delta t V_{KL}^N (u_L^N - u_K^N) v_K^0 \right) = \sum_{K \in \mathcal{T}} |K| u_K^N v_K^1 = \dots = \sum_{K \in \mathcal{T}} |K| u_K^0 v_K^{N+1}.$$

Notice that the scheme (2.1) is the FV scheme with explicit time-discretization associated to the dual continuous problem $v_t(x, t) + \operatorname{div}\{-V(x, T-t)v(x, t)\} = 0$, for $(x, t) \in \mathbf{R}^d \times \mathbf{R}_+$; $v(x, 0) = \mathbf{v}^0(x)$ for $x \in \mathbf{R}^d$ whose solution at time T is $v(x, T) = \mathbf{v}^0(Y(x, T))$.

From the monotony and the L^1 -stability of the scheme (2.1), we have $v_K^{N+1} \geq 0$, for every $K \in \mathcal{T}$ and $\sum_{K \in \mathcal{T}} |K| v_K^{N+1} = 1$, thus for every $x_0 \in K_0$, we have

$$\begin{aligned} |u_{K_0}^{N+1} - u(x_0, T)| &\leq \sum_{K \in \mathcal{T}} |K| |u_K^0 - u(x_0, T)| v_K^{N+1} = \sum_{K \in \mathcal{T}} |K| |u_K^0 - u_0(X(x_0, T))| v_K^{N+1} \\ &\leq \|\nabla u_0\|_\infty \left(\int_{\mathbf{R}^d} \|x - X(x_0, T)\| v(x, T) dx + h \right), \end{aligned}$$

and Theorem 1.6 is the consequence of Theorem 1.7 applied to the dual problem. \square

REMARK 2.1. In general, the dual problem associated to (1.2) is

$$\begin{cases} v_t + V(\cdot, T - \cdot) \cdot \nabla v = 0, & x \in \mathbf{R}^d, t \in [0, T], \\ v(x, 0) = v_0(x), & x \in \mathbf{R}^d, \end{cases}$$

but since $\operatorname{div} V = 0$, we have $V(\cdot, T - \cdot) \cdot \nabla v = \operatorname{div}(V(\cdot, T - \cdot)v)$ and the dual problem is similar to the original problem.

2.2. Energy Estimates. We now turn to the proof of Theorem 1.7. We begin by proving an energy identity.

LEMMA 2.1. *Let $u_0 \in L^2(\mathbf{R}^d)$, and $(\varphi_K^n)_{K_n \in \mathcal{M}}$ be a family of real numbers. Then, for every $t \geq 0$,*

$$\int_{\mathbf{R}^d} \mathbf{u}^2(x, t) \boldsymbol{\varphi}(x, t) dx - \int_{\mathbf{R}^d} \mathbf{u}^2(x, 0) \boldsymbol{\varphi}(x, 0) dx + \mathcal{E}(\boldsymbol{\varphi}, \mathbf{u}, t) = - \sum_{K_n \in \mathcal{M}_t} |K| \delta t (u_K^n)^2 \mathcal{Q}_K^n(\boldsymbol{\varphi}).$$

Proof. We assume without loss of generality that $t = (n_0 + 1)\delta t$ for some $n_0 \in \mathbf{N}$.

Let $n \geq 0$, we consider the quantity

$$e^n := \sum_{K \in \mathcal{T}} |K| \varphi_K^{n+1} (u_K^{n+1})^2 - \sum_{K \in \mathcal{T}} |K| \varphi_K^n (u_K^n)^2,$$

Using (1.9), the first sum is equal to: $\sum_{K \in \mathcal{T}} |K| \varphi_K^{n+1} \left(\sum_{L \in \mathcal{V}_n^-(K)} a_{KL}^n u_L^n \right)^2$, where, for $K_n \in \mathcal{M}$, we have set $\mathcal{V}_n^-(K) := \{L \in \partial K_n^-\} \cup \{K\}$,

$$a_{KK}^n := 1 - V_K^n \delta t / |K|, \quad \text{and for } L \in \partial K_n^-, \quad a_{KL}^n := -V_{KL}^n \delta t / |K|.$$

Next, using Lemma 1.3, we rewrite the second sum:

$$\sum_{K \in \mathcal{T}} |K| \varphi_K^n (u_K^n)^2 = \sum_{K \in \mathcal{T}} (|K| - V_K^n \delta t) \varphi_K^n (u_K^n)^2 - \sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^+} V_{LK}^n \delta t \varphi_K^n (u_K^n)^2.$$

In the last sum, we permute the indexes K and L , to get:

$$\begin{aligned} \sum_{K \in \mathcal{T}} |K| \varphi_K^n (u_K^n)^2 &= \sum_{K \in \mathcal{T}} (|K| - V_K^n \delta t) \varphi_K^n (u_K^n)^2 + \sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^-} -V_{KL}^n \delta t \varphi_L^n (u_L^n)^2 \\ &= \sum_{K \in \mathcal{T}} |K| \sum_{L \in \mathcal{V}_n^-(K)} a_{KL}^n \varphi_L^n (u_L^n)^2 \\ &= \sum_{K \in \mathcal{T}} |K| \varphi_K^{n+1} \sum_{L \in \mathcal{V}_n^-(K)} a_{KL}^n (u_L^n)^2 + \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{V}_n^-(K)} |K| a_{KL}^n (\varphi_L^n - \varphi_K^{n+1}) (u_L^n)^2. \end{aligned}$$

Thus, we have, $e^n = I + II$, with

$$\begin{aligned} I &:= \sum_{K \in \mathcal{T}} |K| \varphi_K^{n+1} \left\{ \left(\sum_{L \in \mathcal{V}_n^-(K)} a_{KL}^n u_L^n \right)^2 - \sum_{L \in \mathcal{V}_n^-(K)} a_{KL}^n (u_L^n)^2 \right\}, \\ II &:= \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{V}_n^-(K)} |K| a_{KL}^n (\varphi_K^{n+1} - \varphi_L^n) (u_L^n)^2. \end{aligned}$$

For every $K_n \in \mathcal{M}$, using the identity $\sum_{M \in \mathcal{V}_n^-(K)} a_{KM}^n = 1$, an easy computation yields

$$\left(\sum_{L \in \mathcal{V}_n^-(K)} a_{KL}^n u_L^n \right)^2 - \sum_{L \in \mathcal{V}_n^-(K)} a_{KL}^n (u_L^n)^2 = -\frac{1}{2} \sum_{L, M \in \mathcal{V}_n^-(K)} a_{KL}^n a_{KM}^n (u_L^n - u_M^n)^2.$$

Thus, replacing the terms a_{KL}^n by their definitions and summing on $K \in \mathcal{T}$, we get

$$I = - \sum_{K \in \mathcal{T}} \varphi_K^{n+1} e_K^n(\mathbf{u}).$$

Now, rearranging the terms in II , we compute

$$II = \sum_{K \in \mathcal{T}} |K| \delta t (u_K^n)^2 \left\{ \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t} + \sum_{L \in \partial K_n^+} \frac{V_{KL}^n \delta t}{|K|} (\varphi_L^{n+1} - \varphi_K^{n+1}) \right\}.$$

Summing these two equations and then summing on $0 \leq n \leq n_0$, we get the lemma. \square

2.3. Weak consistency. In this section, the integer $0 \leq n \leq N$ is fixed. Let $(v_K)_{K \in \mathcal{T}}$ be a family of real numbers and set $\mathbf{v} := \sum_K v_K \mathbf{1}_K$, then let us define the weak total variation of \mathbf{v} with respect to the family $(V_{KL}^n)_{K,L \in \mathcal{T}}$ by:

$$\|\mathbf{v}\|_{v^n\text{-wTV}} := \sum_{K \in \mathcal{T}} \sum_{L \in \partial K^-} |V_{KL}^n| |v_K - v_L|.$$

Now for $\eta \in \mathbf{R}$, we define $\omega_\eta(\mathbf{v})$ to be the union of cells K such that $v_K < \eta < 0$ or $0 < \eta < v_K$ and we consider the level set decomposition $\mathbf{v}(x) = \int_{\mathbf{R}} \text{sgn}(\eta) \mathbf{1}_{\omega(\mathbf{v}, \eta)} d\eta$. When $\omega \subset \mathbf{R}^d$ is a finite union of elements of \mathcal{T} , we have $\|\mathbf{1}_\omega\|_{v^n\text{-wTV}} = \frac{1}{2} \sum_{K|L \subset \partial \omega} |V_{KL}^n|$, and it is not difficult to see that

$$\|\mathbf{v}\|_{v^n\text{-wTV}} = \int_{\mathbf{R}} \|\mathbf{1}_{\omega(\mathbf{v}, \eta)}\|_{v^n\text{-wTV}} d\eta.$$

This suggests the following definition. Let $(w_K)_{K \in \mathcal{T}} \subset \mathbf{R}$ and $\mathbf{w} := \sum_{K \in \mathcal{T}} w_K \mathbf{1}_K$, we set:

$$\|\mathbf{w}\|'_{v^n\text{-wTV}} := \sup_{\omega} \frac{|\int_{\omega} w(x) dx|}{\|\mathbf{1}_\omega\|_{v^n\text{-wTV}}},$$

where the supremum is taken over all subsets $\omega \subset \mathbf{R}^d$ such that ω is a finite union of elements of \mathcal{T} . This norm is the dual norm associated to the seminorm $\|\cdot\|_{v^n\text{-wTV}}$. We do not prove it, we will only need the following inequality, which is a direct consequence of the two last identities.

LEMMA 2.2. *Let \mathbf{v}, \mathbf{w} as above such that $\|\mathbf{v}\|_{v^n\text{-wTV}} < \infty$ and $\|\mathbf{w}\|'_{v^n\text{-wTV}} < \infty$. Then*

$$\int_{\mathbf{R}^d} \mathbf{v}(x) \mathbf{w}(x) dx = \sum_{K \in \mathcal{T}} |K| v_K w_K \leq \|\mathbf{v}\|_{v^n\text{-wTV}} \|\mathbf{w}\|'_{v^n\text{-wTV}}.$$

We now prove a consistency result for the implicit downwind FV scheme.

LEMMA 2.3. *Let $\Psi_0 \in L^\infty(\mathbf{R}^d)$ such that $\nabla \Psi_0$ is bounded. For $(x, t) \in \mathbf{R}^d \times [0, T]$, set $\Psi(x, t) := \Psi_0(X(x, t))$ and $\mathcal{Q}^n(\Psi) := \sum_{K \in \mathcal{T}} \mathcal{Q}_K^n(\Psi) \mathbf{1}_{K_n}$. There exists $R_K^m(\Psi), K_m \in \mathcal{M}$, such that for $n \leq N$, $\mathbf{R}^n(\Psi) := \sum_{K \in \mathcal{T}} R_K^n(\Psi) \mathbf{1}_{K_n}$, and $\mathbf{S}^n(\Psi) := \mathcal{Q}^n(\Psi) - \mathbf{R}^n(\Psi)$ satisfy*

$$\|\mathbf{R}^n(\Psi)\|_\infty \leq CC_0 (\|\nabla V\|_\infty + \|\partial_t V\|_\infty) \|\nabla \Psi_0\|_\infty h, \quad \|\mathbf{S}^n(\Psi)\|'_{v^n\text{-wTV}} \leq CC_0 \|V\|_\infty \|\nabla \Psi_0\|_\infty h.$$

Proof. For $K_m \in \mathcal{M}$, let us note Ψ_K^m the mean value of Ψ on K at time $m\delta t$: $\Psi_K^m = \langle \Psi(\cdot, m\delta t) \rangle_K$. Let $K \in \mathcal{T}$, since Ψ solves $\partial_t \Psi + \text{div}(V\Psi) = 0$ on $\mathbf{R}^d \times \mathbf{R}_+$, we have

$$\frac{\Psi_K^{m+1} - \Psi_K^m}{\delta t} = \frac{1}{|K|\delta t} \int_{K_n} \partial_t \Psi(x, s) dx ds = -\frac{1}{|K|\delta t} \sum_{L \in \partial K} \int_{K|L_m} \Psi(x, s) V(x, s) \cdot \mathbf{n}_{KL} dx ds.$$

For $K_m \in \mathcal{M}$, we set

$$R_K^m(\Psi) := \frac{1}{|K|\delta t} \sum_{L \in \partial K} \int_{K|L_m} \Psi(x, s) \{ \langle V \cdot \mathbf{n}_{KL} \rangle_{K|L_m} - V(x, s) \cdot \mathbf{n}_{KL} \} dx ds.$$

Since, for every space-time edge $K|L_m$, we have $\int_{K|L_m} \{ \langle V \cdot \mathbf{n}_{KL} \rangle_{K|L_m} - V(x, s) \cdot \mathbf{n}_{KL} \} dx ds = 0$, we may replace $\Psi(x, s)$ by $\Psi(x, s) - \langle \Psi \rangle_{K|L_m}$ in the above equation. Then, for every $K \in \mathcal{T}$,

$$\begin{aligned} |R_K^m(\Psi)| &\leq \frac{1}{|K|\delta t} \sum_{L \in \partial K} \left| \int_{K|L_m} \{ \Psi(x, s) - \langle \Psi \rangle_{K|L_m} \} \{ V(x, s) \cdot \mathbf{n}_{KL} - \langle V \cdot \mathbf{n}_{KL} \rangle_{K|L_m} \} dx ds \right| \\ &\leq \frac{|\partial K|}{|K|} \left(\|\nabla \Psi\|_{L^\infty(\mathbf{R}^d \times [0, T])} h + \|\partial_t \Psi\|_{L^\infty(\mathbf{R}^d \times [0, T])} \delta t \right) (\|\nabla V\|_\infty h + \|\partial_t V\|_\infty \delta t). \end{aligned}$$

The regularity of the mesh (1.5), leads to $|\partial K|/|K| \leq \alpha^{-2}h^{-1}$, and using the bounds on $\|\nabla X\|$ and $\|\partial_t X\|$ of Theorem 1.1 and the CFL condition (1.7), we get the first part of the lemma.

Next, since $V_{KL}^n = |K|L|\langle V \cdot \mathbf{n}_{KL} \rangle_{K|L_n}$, we have

$$\frac{\Psi_K^{n+1} - \Psi_K^n}{\delta t} - R_K^n = - \sum_{L \in \partial K} \frac{V_{KL}^n}{|K|} \langle \Psi \rangle_{K|L_n},$$

and using the identity of Lemma 1.3,

$$\begin{aligned} Q_K^n(\Psi) - R_K^n(\Psi) &= - \sum_{L \in \partial K} \frac{V_{KL}^n}{|K|} \langle \Psi \rangle_{K|L_n} + \sum_{L \in \partial K_n^+} \frac{V_{KL}^n}{|K|} (\Psi_L^{n+1} - \Psi_K^{n+1}) \\ &= \frac{1}{|K|} \left(\sum_{L \in \partial K_n^+} V_{KL}^n (\Psi_L^{n+1} - \langle \Psi \rangle_{K|L_n}) + \sum_{L \in \partial K_n^-} V_{KL}^n (\Psi_K^{n+1} - \langle \Psi \rangle_{K|L_n}) \right). \end{aligned}$$

Let $\omega = \cup_{K \in I_\omega} K$ be a finite union of elements of \mathcal{T} , we have

$$\int_\omega S^n(\Psi) = \sum_{K \in I_\omega} |K| (Q_K^n(\Psi) - R_K^n(\Psi)).$$

If the cells K, L with $L \in \partial K_n^+$ are such that K and L belong to I_ω , then the edge $K|L$ has two contributions in the sum above: $V_{KL}^n (\Psi_L^{n+1} - \langle \Psi \rangle_{K|L_n})$ (due to $K \in I_\omega, L \in \partial K_n^+$) and $V_{LK}^n (\Psi_L^{n+1} - \langle \Psi \rangle_{L|K_n})$ (due to $L \in I_\omega, K \in \partial L_n^-$). These contributions cancel each other. Thus, if $\partial\omega^+$ (resp. $\partial\omega^-$) denotes the set $\{K|L, K \in I_\omega, L \in \partial K_n^+ \setminus I_\omega\}$ (resp. $\{K|L, K \in I_\omega, L \in \partial K_n^- \setminus I_\omega\}$), we have:

$$\int_\omega S^n(\Psi) = \sum_{K|L \in \partial\omega^+} V_{KL}^n (\Psi_L^{n+1} - \langle \Psi \rangle_{K|L_n}) + \sum_{K|L \in \partial\omega^-} V_{KL}^n (\Psi_K^{n+1} - \langle \Psi \rangle_{K|L_n}),$$

and from the definition of $\|\cdot\|_{V^n-WTV}$, we get

$$\begin{aligned} \left| \int_\omega S^n(\Psi) \right| &\leq \left(\frac{1}{2} \sum_{K|L \in \partial\omega} |V_{KL}| \right) \left(\|\nabla \Psi\|_{L^\infty(\mathbf{R}^d \times [0, T])} h + \|\partial_t \Psi\|_{L^\infty(\mathbf{R}^d \times [0, T])} \delta t \right) \\ &\leq \|\mathbf{1}_\omega\|_{V^n-WTV} C_0 (1 + c_0) \|\nabla \Psi_0\|_\infty h, \end{aligned}$$

which is the desired estimate. \square

2.4. Key estimate. We use the results of Sections 2.2 and 2.3 to estimate the growth of $\int_{\mathbf{R}^d} u^2(x, t) \Psi^2(x, t) dx$, when ψ is a cut-off function.

LEMMA 2.4. *Let $\Psi_0, \bar{\Psi}_0 \in W_\infty^1(\mathbf{R}^d, [0, 1])$, such that $\nabla \Psi_0$ is bounded. For $(x, t) \in \mathbf{R}^d \times \mathbf{R}_+$, let us define $\Psi(x, t) := \Psi_0(X(x, t))$ and $\bar{\Psi}(x, t) := \bar{\Psi}_0(X(x, t))$. Assume that $\bar{\Psi}_0 \equiv 1$ on $\text{supp } \Psi_0 + B(0, C_0(2 + c_0)h) = \{x \in \mathbf{R}^d : \Psi_0|_{B(x, C_0(1 + c_0)h)} \neq 1\}$. Then, for every $0 \leq t \leq T$,*

$$\int_{\mathbf{R}^d} u^2(\cdot, t) \Psi^2(\cdot, t) - \int_{\mathbf{R}^d} u_0^2 \Psi^2(\cdot, 0) \leq \{C_{1,\star} \|\nabla \Psi_0\|_\infty^2 h + C_{2,\star} \|\nabla \Psi_0\|_\infty h\} \int_0^t \int_{\mathbf{R}^d} \bar{\Psi}^2 u^2,$$

with $C_{1,\star} = CC_0^2 \xi^{-1}$ and $C_{2,\star} = CC_0$.

Proof. Let $0 \leq t \leq T$, if $t < \delta t$, the left hand side vanishes and there is nothing to prove. So we assume that there exists $n_0 \geq 0$ such that $(n_0 + 1)\delta t \leq t < (n_0 + 2)\delta t$, since the left hand side does not depend on t for $(n_0 + 1)\delta t \leq t < (n_0 + 2)\delta t$ and the right hand side is non-decreasing, we may assume that $t = (n_0 + 1)\delta t$. In this case $\mathcal{M}_t = \{K_n \in \mathcal{M} : 0 \leq n \leq n_0\}$.

Now, let $K \in \mathcal{M}$ and $0 \leq n \leq n_0 + 1$ be such that $\Psi_K^n \neq 0$. There exists $x \in K$, such that $\Psi(x, n\delta t) \neq 0$, thus $\Psi_0(Y(x, n\delta t)) \neq 0$. Let $z \in \mathbf{R}^d$ and $0 \leq m \leq N + 1$ such that $|z - x| \leq 2h$ and $|n - m\delta t| \leq 1$. From the estimates on ∇X and $\partial_t X$ of Theorem 1.1, we have $\|\Psi(x, n\delta t) - \Psi(z, s)\| \leq C_0(2h + \delta t) \leq C_0(2 + c_0)h$, thus $\overline{\Psi}(z, s) = \overline{\Psi}_0(Y(z, s)) = 1$. We conclude that for every $K_n, L_m \in \mathcal{M}$ such that $0 \leq n, m \leq n_0 + 1$, we have

$$(L \in \partial K \cup \{K\}, \quad |m - n| \leq 1, \quad \Psi_K^n > 0) \implies \overline{\Psi}_L^m = 1. \quad (2.2)$$

Let us write the identity of Lemma 2.1, with $\varphi = \Psi^2$.

$$\int_{\mathbf{R}^d} \mathbf{u}^2(x, t) \Psi^2(x, t) dx - \int_{\mathbf{R}^d} u_0^2(x) \Psi^2(x, 0) dx + \mathcal{E}(\Psi^2, \mathbf{u}, t) = - \sum_{K_n \in \mathcal{M}_t} |K| \delta t (u_K^n)^2 \mathcal{Q}_K^n(\Psi^2). \quad (2.3)$$

Let $K_n \in \mathcal{M}$, a direct computation yields

$$\mathcal{Q}_K^n(\Psi^2) = 2\Psi_K^{n+1} \mathcal{Q}_K^n(\Psi) - \frac{(\Psi_K^{n+1} - \Psi_K^n)^2}{\delta t} + \sum_{L \in \partial K_n^+} \frac{V_{KL}^n}{|K|} (\Psi_L^{n+1} - \Psi_K^{n+1})^2.$$

Let I denotes the right hand side of (2.3), we then have

$$I \leq \sum_{K_n \in \mathcal{M}_t} |K| (u_K^n)^2 (\Psi_K^{n+1} - \Psi_K^n)^2 - 2 \sum_{K_n \in \mathcal{M}_t} \delta t |K| \Psi_K^{n+1} (u_K^n)^2 \mathcal{Q}_K^n(\Psi) =: A - 2 \sum_{n=0}^{n_0} \delta t B_n. \quad (2.4)$$

First, from the estimate on $\partial_t X$ of Theorem 1.1, we have for every $K_n \in \mathcal{M}_t$, $|\Psi_K^{n+1} - \Psi_K^n| \leq C_0 \|\nabla \Psi_0\|_\infty \delta t \leq C_0 c_0 \|\nabla \Psi_0\|_\infty h$. Using (2.2), we deduce

$$A \leq CC_0^2 \|\nabla \Psi_0\|_\infty^2 h \sum_{K_n \in \mathcal{M}_t} |K| \delta t (u_K^n)^2 (\overline{\Psi}_K^n)^2 = CC_0^2 \|\nabla \Psi_0\|_\infty^2 h \int_0^t \int_{\mathbf{R}^d} \overline{\Psi}^2 \mathbf{u}^2. \quad (2.5)$$

Next, let us fix $0 \leq n \leq n_0$ and let us define $\Psi^n := \sum_{K \in \mathcal{T}} \Psi_K^n \mathbf{1}_K$. With the notations of Lemma 2.3, we have

$$B_n = \sum_{K \in \mathcal{T}} |K| \Psi_K^{n+1} (u_K^n)^2 R_K^n(\Psi) + \sum_{K \in \mathcal{T}} |K| \Psi_K^{n+1} (u_K^n)^2 S_K^n(\Psi),$$

where $\|R^n(\Psi)\|_\infty \leq CC_0 \|\nabla \Psi_0\|_\infty h$ and $\|S^n(\Psi)\|'_{v^n-wTV} \leq CC_0 \|\nabla \Psi_0\|_\infty h$. Thus,

$$\left| \sum_{K \in \mathcal{T}} |K| \Psi_K^{n+1} (u_K^n)^2 R_K^n(\Psi) \right| \leq CC_0 \|\nabla \Psi_0\|_\infty h \int_{\mathbf{R}^d} \overline{\Psi}^2 \mathbf{u}^2(\cdot, n\delta t).$$

Next, using Lemma 2.2 with $\mathbf{v}_K^n = \Psi_K^{n+1} (u_K^n)^2$, for $K \in \mathcal{T}$ and $\mathbf{w} = S^n(\Psi)$, we get

$$\begin{aligned} & \left| \sum_{K \in \mathcal{T}} |K| \Psi_K^{n+1} (u_K^n)^2 S_K^n(\Psi) \right| \leq CC_0 \|\nabla \Psi_0\|_\infty h \sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^-} |V_{KL}^n| |\Psi_K^{n+1} (u_K^n)^2 - \Psi_L^{n+1} (u_L^n)^2| \\ & \leq CC_0 \|\nabla \Psi_0\|_\infty h \left(\sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^-} |V_{KL}^n| |\Psi_L^{n+1} - \Psi_K^{n+1}| (u_L^n)^2 + \sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^-} |V_{KL}^n| \Psi_K^{n+1} |(u_K^n)^2 - (u_L^n)^2| \right) \end{aligned}$$

From (2.2), the estimate on $\|\nabla X\|$ of Theorem 1.1 and (1.5), the first sum is bounded by

$$\begin{aligned} & \sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^-} (\|V\|_\infty |K|L) (C_0 \|\nabla \Psi_0\|_\infty 2h) (u_L^n)^2 (\bar{\Psi}_L^n)^2 \\ & = 2C_0 \|\nabla \Psi_0\|_\infty \|V\|_\infty \sum_{K \in \mathcal{T}} \left(\sum_{L \in \partial K_n^+} |K|L|h \right) (u_K^n)^2 (\bar{\Psi}_K^n)^2 \\ & \leq 2C_0 \|\nabla \Psi_0\|_\infty \|V\|_\infty \alpha^{-2} \sum_{K \in \mathcal{T}} |K| (u_K^n)^2 (\bar{\Psi}_K^n)^2 \leq CC_0 \|\nabla \Psi_0\|_\infty \int_{\mathbf{R}^d} \mathbf{u}^2 \bar{\Psi}^2(\cdot, n\delta t). \end{aligned}$$

Writing

$$\Psi_K^{n+1} |(u_K^n)^2 - (u_L^n)^2| = |\Psi_K^{n+1} (u_K^n - u_L^n)(u_K^n + u_L^n)| \leq |\Psi_K^{n+1} (u_K^n - u_L^n)| \times (|\bar{\Psi}_K^n u_K^n| + |\bar{\Psi}_L^n u_L^n|)$$

and using the Cauchy Schwarz inequality, we bound the second sum by

$$\begin{aligned} & \left(\sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^-} |V_{KL}^n| |u_K^n - u_L^n|^2 (\Psi_K^{n+1})^2 \right)^{1/2} \left(2 \sum_{K \in \mathcal{T}} \sum_{L \in \partial K} |V_{KL}^n| |u_K^n|^2 (\bar{\Psi}_K^{n+1})^2 \right)^{1/2} \\ & \leq \delta t^{-1/2} \left(\sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^-} |V_{KL}^n| \delta t |u_L^n - u_K^n|^2 (\Psi_K^{n+1})^2 \right)^{1/2} \|V\|_\infty^{1/2} \alpha^{-1} h^{-1/2} \left(2 \sum_{K \in \mathcal{T}} |K| (u_K^n)^2 (\bar{\Psi}_K^n)^2 \right)^{1/2} \\ & \leq Ch^{-1/2} \delta t^{-1/2} \left(\sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^-} |V_{KL}^n| \delta t |u_L^n - u_K^n|^2 (\Psi_K^{n+1})^2 \right)^{1/2} \left(\int_{\mathbf{R}^d} \mathbf{u}^2 \bar{\Psi}^2(\cdot, n\delta t) \right)^{1/2}. \end{aligned}$$

Collecting the three preceding estimates, we have

$$\begin{aligned} |B_n| & \leq CC_0 \|\nabla \Psi_0\|_\infty h^{1/2} (C_0 \|\nabla \Psi_0\|_\infty h^{1/2} + Ch^{1/2}) \int_{\mathbf{R}^d} \mathbf{u}^2 \bar{\Psi}^2(\cdot, n\delta t) \\ & + CC_0 \|\nabla \Psi_0\|_\infty h^{1/2} \delta t^{-1/2} \left(\sum_{K \in \mathcal{T}} \sum_{L \in \partial K_n^-} |V_{KL}^n| \delta t |u_L^n - u_K^n|^2 (\Psi_K^{n+1})^2 \right)^{1/2} \left(\int_{\mathbf{R}^d} \mathbf{u}^2 \bar{\Psi}^2(\cdot, n\delta t) \right)^{1/2}. \end{aligned} \quad (2.6)$$

Plugging (2.6) and (2.5) in (2.4) and using again the Cauchy Schwarz inequality yields

$$\begin{aligned} I & \leq CC_0 \|\nabla \Psi_0\|_\infty h^{1/2} (C_0 \|\nabla \Psi_0\|_\infty h^{1/2} + Ch^{1/2}) \int_0^t \int_{\mathbf{R}^d} \bar{\Psi}^2 \mathbf{u}^2 \\ & + CC_0 \|\nabla \Psi_0\|_\infty h^{1/2} \left(\sum_{K_n \in \mathcal{M}_t} \sum_{L \in \partial K_n^-} |V_{KL}^n| \delta t |u_L^n - u_K^n|^2 (\Psi_K^{n+1})^2 \right)^{1/2} \left(\int_0^t \int_{\mathbf{R}^d} \bar{\Psi}^2 \mathbf{u}^2 \right)^{1/2}. \end{aligned} \quad (2.7)$$

To end the proof, we introduce the following

LEMMA 2.5. *For every $t \geq 0$, we have $\sum_{K_n \in \mathcal{M}_t} \sum_{L \in \partial K_n^-} |V_{KL}^n| \delta t |u_L^n - u_K^n|^2 (\Psi_K^{n+1})^2 \leq \xi^{-1} \mathcal{E}(\Psi^2, \mathbf{u}, t)$.*

Proof. It is sufficient to prove that for every $K_n \in \mathcal{M}$, we have $1 \leq \xi^{-1} (1 - V_K^n \delta t / |K|)$ which is a direct consequence of the CFL condition (1.6). \square

Plugging (2.7) in (2.3) and using Lemma 2.5 and then Young's inequality ($ab \leq \varepsilon a^2/2 + \varepsilon^{-1} b^2/2$) to absorb the term $\mathcal{E}(\Psi^2, \mathbf{u}, t)$ in the left hand side of (2.3), we obtain Lemma 2.4. \square

2.5. Cut-off functions. Let $K_0 \in \mathcal{T}$, $x_0 \in K_0$ and define $u_0 := 1/|K_0| \mathbf{1}_{K_0}$. Without loss of generality, we assume that $x_0 = 0$.

With the notations of Lemma 2.4, we set

$$\sigma := 3C_{1,\star}^{1/2} T^{1/2}. \quad (2.8)$$

Let $\Psi_{0,0}$ be the constant function equal to 1 on \mathbf{R}^d , and for $k \geq 1$, let $\Psi_{k,0} \in W^{1,\infty}(\mathbf{R}^d, [0, 1])$ be the unique cut-off function satisfying:

$$\Psi_{k,0}(B(k+1/2)\sigma h^{1/2}) = \{0\}, \quad \Psi_{k,0}(\mathbf{R}^d \setminus B((k+1)\sigma h^{1/2})) = \{1\}, \quad \|\nabla \Psi_{k,0}\|_\infty \leq \sigma^{-1} h^{-1/2}.$$

Then for $k \geq 0$ and $(x, t) \in \mathbf{R}^d \times \mathbf{R}_+$, set $\Psi_k(x, t) := \Psi_{k,0}(X(x, t))$.

We want to apply Lemma 2.4 with $(\Psi, \bar{\Psi}) = (\Psi_k, \Psi_{k-1})$, $k \geq 1$. The required hypothesis on the support of Ψ_k reads $C_0(2 + c_0)h \leq \sigma h^{1/2}/2$ and is true for $h \leq h_{1,\star}$ small enough. Moreover, for $h \leq h_{2,\star}$, we have $\int_{\mathbf{R}^d} u_0^2 \Psi_k^2(\cdot, 0) = 0$, for $k \geq 1$, and from (2.8), for $h \leq h_{3,\star}$, we have $4C_{1,\star}/\sigma^2 + 2C_{2,\star}h^{1/2}/\sigma \leq T^{-1}$. We now assume $h \leq h_\star := \min(h_{1,\star}, h_{2,\star}, h_{3,\star})$.

So, applying the lemma with $(\Psi, \bar{\Psi}) = (\Psi_k, \Psi_{k-1})$, we get

$$\int_{\mathbf{R}^d} \mathbf{u}^2(\cdot, t) \Psi_k^2(\cdot, t) \leq T^{-1} \int_0^t \int_{\mathbf{R}^d} \mathbf{u}^2 \Psi_{k-1}^2, \quad 0 \leq t \leq T, \quad k \geq 1.$$

Now, for $k = 0$, the L^2 -stability of the scheme and (1.5) yield $\int_{\mathbf{R}^d} \mathbf{u}^2(\cdot, t) \Psi_0^2(\cdot, t) \leq \int_{\mathbf{R}^d} u_0^2 = 1/|K_0| \leq \alpha^{-1} h^{-d}$, for every $t \geq 0$, and we obtain, recursively

$$\int_{\mathbf{R}^d} \mathbf{u}^2(\cdot, t) \Psi_k^2(\cdot, t) \leq \alpha^{-1} \frac{(t/T)^k}{k!} h^{-d}, \quad 0 \leq t \leq T, \quad k \geq 0. \quad (2.9)$$

Let k_0 be a positive integer. Let $x \in \mathbf{R}^d$ such that $\|x\| \geq k_0 \sigma h^{1/2}$, and k denotes the integer satisfying $k\sigma h^{1/2} \leq \|x\| < (k+1)\sigma h^{1/2}$. From the definition of $(\Psi_{j,0})_{j \geq 0}$, we have $(1 - \Psi_{k+1,0}(x)) \Psi_{k-1,0}(x) = 1$ and since $\|x\| \leq (k+1)\sigma h^{1/2}$, we have,

$$\|x\| \leq \sigma h^{1/2} \sum_{k \geq k_0} (k+1) \Psi_{k-1,0}(x) (1 - \Psi_{k+1,0}(x)).$$

So, for every $x \in \mathbf{R}^d$,

$$\|x\| \leq \sigma h^{1/2} \left(k_0 \mathbf{1}_{B(k_0 \sigma h^{1/2})}(x) + \sum_{k \geq k_0} (k+1) (\Psi_{k-1,0} (1 - \Psi_{k+1,0}))(x) \right).$$

From the estimate on $\|\nabla Y\|$ of Theorem 1.1, we have $\|x - Y(0, T)\| \leq C_0 \|X(x, T)\|$. Thus,

$$\|x - Y(0, T)\| \leq C_0 \sigma h^{1/2} \left(k_0 \mathbf{1}_{B(k_0 \sigma h^{1/2})}(X(x, T)) + \sum_{k \geq k_0} (k+1) (\Psi_{k-1,0} (1 - \Psi_{k+1,0}))(X(x, T)) \right).$$

Multiplying by $\mathbf{u}(x, T)$ and integrating on \mathbf{R}^d , we get

$$\begin{aligned} \int_{\mathbf{R}^d} \|x - Y(0, T)\| \mathbf{u}(x, T) dx &\leq C_0 \sigma h^{1/2} k_0 \int_{\mathbf{R}^d} \mathbf{1}_{B(\sigma h^{1/2})}(X(x, T)) \mathbf{u}(x, T) dx \\ &\quad + C_0 \sigma h^{1/2} \sum_{k \geq k_0} (k+1) \int_{\mathbf{R}^d} (\mathbf{u} \Psi_{k-1,0} (1 - \Psi_{k+1,0}))(x, T) dx. \end{aligned}$$

From the monotony and conservation properties of (1.2), the first integral is bounded by 1. For the second term, we use the Cauchy Schwarz inequality and (2.9). For every $k \geq 2$, we have

$$\begin{aligned} \int_{\mathbf{R}^d} (\mathbf{u} \Psi_{k-1} (1 - \Psi_{k+1})) (\cdot, T) &\leq \left(\lambda \left\{ \text{supp} \left(\Psi_{k-1} (1 - \Psi_{k+1}) \right) (\cdot, T) \right\} \right)^{1/2} \left(\int_{\mathbf{R}^d} (\mathbf{u}^2 \Psi_{k-1}^2) (\cdot, T) \right)^{1/2} \\ &\leq C \sigma^{d/2} (k+2)^{d/2} h^{d/4} \left(\int_{\mathbf{R}^d} (\mathbf{u}^2 \Psi_{k-2}^2) (\cdot, T) \right)^{1/2} \leq C \sigma^{d/2} (k+2)^{d/2} h^{-d/4} ((k-2)!)^{-1/2}. \end{aligned}$$

Thus,

$$\int_{\mathbf{R}^d} \|x - Y(x_0, T)\| \mathbf{u}(x, T) dt \leq C C_0 \sigma h^{1/2} \left(k_0 + \sigma^{d/2} h^{-d/4} \sum_{k \geq k_0} (k+2)^{(d-1)/2} ((k-2)!)^{-1/2} \right).$$

Clearly, $\sum_{k \geq k_0} (k+2)^{(d-1)/2} ((k-2)!)^{-1/2} \approx ((k_0 - d - 1)!)^{-1/2} = (\Gamma(k_0 - d))^{-1/2}$. Choosing k_0 to be the first integer, such that $\sigma^d h^{-d/2} (\Gamma(k_0 - d))^{-1} \leq 1$, we have

$$\int_{\mathbf{R}^d} \|x - Y(x_0, T)\| \mathbf{u}(x, T) dx \leq C C_0 \sigma h^{1/2} (k_0 + 1) \leq C C_0^2 \xi^{-1/2} T^{1/2} h^{1/2} (k_0 + 1).$$

And since, $k_0 + 1 \leq \Gamma^{-1}(\sigma^d h^{-d/2}) + d + 2 \leq C \Gamma^{-1}(C_0^2 \xi^{-1} T h^{-1})$, Theorem 1.7 is proved. \square

3. Proof of Theorem 1.8. In this section $u_0 \in L^2(\mathbf{R}^d)$. The function $u \in C(\mathbf{R}_+, L^2(\mathbf{R}^d))$ is the exact solution to (1.2) and \mathbf{u} is the numerical approximation given by the scheme (1.8)-(1.9)-(1.10). We assume that (1.5) and the CFL conditions (1.6)-(1.7) are satisfied with $\xi > 0$.

3.1. Weak formulation and technical results. The next lemma says that \mathbf{u} satisfies the weak formulation (1.4) up to an error term, related to the consistency error of the scheme.

LEMMA 3.1 (Lemma 3.1 in [8]).

For every compactly supported test function $\phi \in W^{1,\infty}(\mathbf{R}^d \times \mathbf{R}_+)$, we have

$$\int_{\mathbf{R}_+} \int_{\mathbf{R}^d} \mathbf{u}(\phi_t + V \cdot \nabla \phi) + \int_{\mathbf{R}^d} \mathbf{u}(\cdot, 0) \phi(\cdot, 0) = \mu(\phi) + \nu(\phi), \quad (3.1)$$

$$\mu(\phi) := \sum_{K_n \in \mathcal{M}} |K| (u_K^{n+1} - u_K^n) (\langle \phi \rangle_{K_n} - \langle \phi \rangle_K ((n+1)\delta t)), \quad (3.2)$$

$$\nu(\phi) := \sum_{K_n \in \mathcal{M}} \sum_{L \in \partial K_n^-} \delta t (u_L^n - u_K^n) (V_{KL}^n \langle \phi \rangle_{K_n} - |K| L \langle V \cdot \mathbf{n} \phi \rangle_{K_n L_n}), \quad (3.3)$$

where $\langle \phi \rangle_{K_n L_n} := \int_{K_n L_n} \phi V \cdot \mathbf{n}_{KL}$. This equality is similar to the entropy inequality used in Kuznetsov method (see e.g. [6]). Here, the initial problem (1.2) is linear and the notion of entropy is useless.

LEMMA 3.2. Assume that the CFL condition (1.6) is satisfied with $\xi \in (0, 1)$, then

$$\mathcal{F}(\mathbf{u}, T) \leq C \xi^{-1} \left(\|\mathbf{u}(\cdot, 0)\|_{L^2(\mathbf{R}^d)}^2 - \|\mathbf{u}(\cdot, T)\|_{L^2(\mathbf{R}^d)}^2 \right).$$

Proof. From (1.9) and the Cauchy-Schwarz inequality, we have for every $K_n \in \mathcal{M}$,

$$|K| |u_K^{n+1} - u_K^n|^2 \leq \left(\sum_{L \in \partial K_n^-} \frac{|V_{KL}^n| \delta t}{|K|} \right) \sum_{L \in \partial K_n^-} |V_{KL}^n| \delta t |u_L^n - u_K^n|^2.$$

Due to the CFL condition (1.6), we have $\sum_{L \in \partial K_n^-} |V_{KL}^n| \delta t \leq |K|$; consequently, summing the preceding inequality on $K_n \in \mathcal{M}_T$, we obtain

$$\mathcal{F}(\mathbf{u}, T) \leq 2 \sum_{K_n \in \mathcal{M}_T} \sum_{L \in \partial K_n^-} |V_{KL}^n| \delta t |u_L^n - u_K^n|^2,$$

and we conclude by Lemma 2.5 applied with $\Psi = 1$ and $t = T$. \square

By continuity in H^1 of the L^2 -projection on the space of functions which are constant with respect to the mesh \mathcal{T} , we have:

LEMMA 3.3. *There exists $c \geq 0$ only depending on d such that for every $u_0 \in H^1(\mathbf{R}^d)$,*

$$\begin{aligned} \sum_{K \in \mathcal{M}, L \in K_n^-} |K| |L| |u_L^0 - u_K^0|^2 &\leq c \alpha^{-4} \|\nabla u_0\|_{L^2}^2 h, \\ \|\mathbf{u}(\cdot, 0) - u_0\|_{L^2} &\leq c \alpha^{-1/2} \|\nabla u_0\|_{L^2} h. \end{aligned}$$

Proof. Let $K \in \mathcal{T}$, $L \in \partial K$. We have

$$|u_K^0 - u_L^0| \leq \frac{1}{|K| |L|} \int_{K \times L} |u_0(x) - u_0(y)| dx dy$$

Now from $|u_0(x) - u_0(y)| \leq \int_{[x,y]} |\nabla u_0(z)| dz$ and the Cauchy-Schwarz inequality:

$$|u_K^0 - u_L^0|^2 \leq \frac{2h}{|K| |L|} \int_{K \times L} \int_{[x,y]} |\nabla u_0(z)|^2 dz dx dy.$$

We write

$$\int_{K \times L} \int_{[x,y]} |\nabla u_0(z)|^2 dz dx dy = \int_{K \times L} |x - y| \int_0^1 |\nabla u_0((1-t)x + ty)|^2 dt dx dy.$$

We bound $|x - y|$ by $2h$ and we perform the change of variables of Jacobian determinant $(-1)^d$, $(x, y, t) \mapsto (z = (1-t)x + ty, w = x - y, t = t)$. We have

$$\int_{K \times L} \int_{[x,y]} |\nabla u_0(z)|^2 dz dx dy \leq 2h \int_{B(x_K, 2h)} |\nabla u_0(z)|^2 \left(\int_0^1 \int_{\mathbf{R}^d} g(z, w, t) dw dt \right) dz,$$

where we have chosen $x_K \in K$ and where $g(z, w, t) = 1$ if $z + tw \in K$ and $z - (1-t)w \in L$, and $g(z, w, t) = 0$ otherwise. For $(z, t) \in B(x_K, 2h) \times [0, 1]$, we have $\int_{\mathbf{R}^d} g(z, w, t) dw \leq 2^d |K|$ if $t \geq 1/2$ and $\int_{\mathbf{R}^d} g(z, w, t) dw \leq 2^d |L|$ if $t < 1/2$. Finally, we obtain

$$\int_{K \times L} \int_{[x,y]} |\nabla u_0(z)|^2 dz dx dy \leq 2^{d+2} h \max(|K|, |L|) \int_{B(x_K, 2h)} |\nabla u_0(z)|^2 dz,$$

and the first inequality of (1.5) yields

$$|u_K^0 - u_L^0|^2 \leq 2^{d+2} h^{2-d} \alpha^{-1} \int_{B(x_K, 2h)} |\nabla u_0(z)|^2 dz.$$

Multiplying by $|K| |L|$, summing on $L \in \partial K$ and using the second part of (1.5), we get

$$\sum_{L \in \partial K} |K| |L| |u_K^0 - u_L^0|^2 \leq M(K) 2^{d+1} h \alpha^{-2} \int_{B(x_K, 2h)} |\nabla u_0(z)|^2 dz,$$

where $M(K)$ is the cardinal of $\{L : L \in \partial K\}$. From (1.5), we have

$$M(K)\alpha h^d \leq |\cup_{L \in \partial K} L| \leq |B(x_K, 3h)|,$$

and $M(K) \leq c\alpha^{-1}$. Summing on $K \in \mathcal{T}$, we obtain

$$\sum_{K, L \in \partial K} |K_1 L| |u_K^0 - u_L^0|^2 \leq c h \alpha^{-3} \int_{\mathbf{R}^d} |\nabla u_0(z)|^2 N(z) dz,$$

where $N(z)$ is the cardinal of $\{K : d(K, z) \leq 3h\}$. As above $N(z)$ is bounded by $c\alpha^{-1}$ and summing on $K \in \mathcal{T}$, we get the first inequality. Now let $K \in \mathcal{T}$. Similarly we have

$$\begin{aligned} \int_K |u(\cdot, 0) - u_0|^2 &\leq \frac{1}{|K|^2} \int_K \left(\int_K \int_{[x,y]} |\nabla u(z)| dz dy \right)^2 dx \\ &\leq c h^2 \int_{B(x_K, h)} |\nabla u(z)|^2 dz. \end{aligned}$$

Summing on $K \in \mathcal{T}$ and using the fact that the cardinal of the set $\{K : d(K, z) \leq h\}$ is bounded by $c\alpha^{-1}$, we get the second estimate. \square

In the sequel, we will need some estimates on the differences between u and its mean values on cells or on edges. They are contained in the next lemma. The three estimates are based on the Poincaré-Wirtinger inequality and on the trace theorem for H^1 -functions. In the proof of (3.6), the conditions (1.5) are not sufficient, we need the additional hypothesis (1.11) to avoid the blow up of the constant in the trace theorem.

LEMMA 3.4. *Assume that (1.5) is satisfied. Then, for every $u_0 \in H^1(\mathbf{R}^d)$ we have*

$$\sum_{K_n \in \mathcal{M}_T} \sum_{L \in \partial K_n^-} \delta t |K_1 L| \left(\langle u \rangle_{K_n} - \fint_{K_1 L_n} u \right)^2 \leq C C_0^2 \|\nabla u_0\|_{L^2}^2 T h, \quad (3.4)$$

$$\sum_{K_n \in \mathcal{M}_T} \sum_{L \in \partial K_n^-} \delta t |K_1 L| \left(\fint_{K_1 L_n} \fint_{n\delta t}^{(n+1)\delta t} |u(x, s) - u(x, t)| ds dt dx \right)^2 \leq C C_0^2 \|\nabla u_0\|_{L^2}^2 T \delta t. \quad (3.5)$$

If moreover assumption (1.11) is satisfied, then

$$\sum_{K_n \in \mathcal{M}_T} \sum_{L \in \partial K_n^-} \delta t |K_1 L| \left(\fint_{K_1 L_n} \fint_{K_1 L} |u(x, t) - u(y, t)| dx dy dt \right)^2 \leq C C_0^2 \|\nabla u_0\|_{L^2}^2 T h \quad (3.6)$$

Proof. We prove (3.4) (3.5) (3.6) successively.

Let $K_n \in \mathcal{M}_T$, and let $L \in \partial K_n^-$. Let us fix $x_K \in K$ and $t \in [n\delta t, (n+1)\delta t)$. As in the proof of Lemma 3.3, (with L replaced by $B(x_K, 2h)$) we have

$$\left| \fint_K u(\cdot, t) - \fint_{B(x_K, 2h)} u(\cdot, t) \right|^2 \leq C h^{2-d} \int_{B(x_K, 2h)} |\nabla u(\cdot, t)|^2.$$

On the other hand, from the trace theorem we have

$$\fint_{KL} \left(u(\cdot, t) - \fint_{B(x_K, 2h)} u(\cdot, t) \right)^2 \leq C \frac{h}{|K_1 L|} \int_{B(x_K, 2h)} |\nabla u(\cdot, t)|^2$$

Thus, using (1.5), we have

$$\delta t |K_1 L| \left(\langle u \rangle_{K_n} - \fint_{K_1 L_n} u \right)^2 \leq C \int_{B(x_K, 2h)} |\nabla u(\cdot, t)|^2 \delta t h.$$

Summing on $L \in \partial K_n^-$ and $K_n \in \mathcal{M}_T$ and using the fact that the cardinals of the sets $\{K : d(K, z) \leq 2h\}$ are uniformly bounded (see the proof of Lemma 3.3), we find that the left hand side of (3.4) is bounded by $C \int_{\mathbf{R}^d \times [0, T]} |\nabla u|^2 h$. We conclude by the second estimate of Corollary 1.2 \square

Before proving (3.5) and (3.6) we recall the denition of a $H^{1/2}$ -seminorm. Let B be a ball in \mathbf{R}^q centered at x_0 , let P be a an affine subspace of dimension $q - 1$ containing x_0 and let $v \in H^1(B, \mathbf{R})$. Then γ , the trace of v on P satisfies

$$|\gamma|_{H^{1/2}(B \cap P)}^2 := \int_{(B \cap P)^2} \frac{|\gamma(x) - \gamma(y)|^2}{|x - y|^q} dx dy \leq C \int_B |\nabla v|^2.$$

Let $K_n \in \mathcal{M}_T$ and $L \in \partial K_n^-$. Let x_{KL} be any element of $K \cap L$ so that $K \cap L \subset B_P(x_{KL}, h)$. We introduce new cartesian coordinates: let P be the affine subspace containing $K \cap L$, we define (x'_1, \dots, x'_d) , such that $x' \in P \Leftrightarrow x'_d = 0$ and we define $K \cap L'$ to be the set such that $x' \in K \cap L \Leftrightarrow (x'_d = 0 \text{ and } \bar{x}' = (x'_1, \dots, x'_{d-1}) \in K \cap L')$.

From the Cauchy-Schwarz inequality and the trace theorem for H^1 functions, we have for every $\bar{x}' \in K \cap L'$

$$\begin{aligned} & \left(\int_{n\delta t}^{(n+1)\delta t} \int_{n\delta t}^{(n+1)\delta t} |u(\bar{x}', 0, s) - u(\bar{x}', 0, t)| ds dt \right)^2 \\ & \leq \delta t^2 \int_{n\delta t}^{(n+1)\delta t} \int_{n\delta t}^{(n+1)\delta t} \frac{|u(\bar{x}', 0, s) - u(\bar{x}', 0, t)|^2}{|s - t|^2} ds dt \\ & \leq C \int_0^{\delta t} \int_{n\delta t}^{(n+1)\delta t} |\partial_{x'_d} u(\bar{x}', x'_d, s)|^2 + |\partial_t u(\bar{x}', x'_d, s)|^2 ds dx'_d. \end{aligned}$$

Integrating on $K \cap L'$, using Jensen inequality and the CFL condition $\delta t \leq c_0 h$, we obtain

$$\left(\int_{n\delta t}^{(n+1)\delta t} \int_{K \cap L_n} |u(x, s) - u(x, t)| ds dt dx \right)^2 \leq C \int_{n\delta t}^{(n+1)\delta t} \int_{B(x_{KL}, (1+c_0)h)} |\nabla u|^2 + |\partial_t u|^2.$$

Multiplying by $\delta t |K \cap L|$ and summing on L, K_n , inequality (3.5) follows from the fact that the cardinal of the neighbours of K is uniformly bounded and from Corollary 1.2. \square

We assume (1.11). Let $K_n \in \mathcal{M}_T$, $L \in \partial K_n^-$ and $t \in [n\delta t, (n+1)\delta t]$. We denote again by P the affine subspace containing $K \cap L$ and $x_{KL} \in K \cap L$. From the Cauchy-Schwarz inequality, we have

$$\left(\int_{(K \cap L)^2} |u(x, t) - u(y, t)| dx dy \right)^2 \leq h^d \int_{(K \cap L)^2} \frac{|u(x, t) - u(y, t)|^2}{|x - y|^d} dx dy,$$

and from the trace theorem,

$$\left(\int_{(K \cap L)^2} |u(x, t) - u(y, t)| dx dy \right)^2 \leq C \frac{h^d}{|K \cap L|^2} \int_{B(x_{KL}, h)} |\nabla u(z, t)|^2 dz.$$

Multiplying by $\delta t |K \cap L|$ and using assumption (1.11), we get

$$\delta t |K \cap L| \left(\int_{K \cap L_n} \int_{K \cap L} |u(x, t) - u(y, t)| dx dy dt \right)^2 \leq C \int_{B(x_{KL}, h)} |\nabla u(z, t)|^2 dz \delta t h.$$

Summing on $L \in \partial K_n^-$ and $K_n \in \mathcal{M}_T$, the second inequality of Corollary 1.2 yields (3.6). \square

3.2. End of the Proof. We now assume that $u_0 \in H^1(\mathbf{R}^d)$ and end the proof of Theorem 1.8. Let us notice that from the L^2 -stability of the scheme and of the evolution problem (1.2), it is sufficient to prove Theorem 1.8 for $u_0 \in C_c^\infty(\mathbf{R}^d)$ and conclude by density of such functions in $H^1(\mathbf{R}^d)$. In this case, the formula $u(x, t) = u_0(X(x, t))$ and the definition of X imply that $u \in W^{1,\infty}(\mathbf{R}^d \times [0, T + \delta t])$ and is compactly supported.

From now on, $u_0 \in C_c^\infty(\mathbf{R}^d)$. We want to apply the weak formulations (1.4) and (3.1) to a test function ϕ defined by $\phi(x, t) := \psi(t)u(x, t)$, for $x \in \mathbf{R}^d$, $s \geq 0$ with $\psi := \mathbf{1}_{[0, T]}$. Since the function $\mathbf{1}_{[0, T]}$ is not Lipschitz, we have to introduce a mollifying sequence to approximate ψ .

Let $(\psi_q)_{q \leq 1}$ be a sequence of $C^\infty(\mathbf{R}_+)$ functions satisfying $\psi_q \equiv 1$ on $[0, T]$, $\psi_q \equiv 0$ on $[T + \frac{\delta t}{q}, +\infty)$ and $0 \geq \psi'_q \geq -\frac{2q}{\delta t}$. We set $\phi_q(x, t) := u(x, t)\psi_q(t)$ for every $(x, t) \in \mathbf{R}^d \times \mathbf{R}_+$ and $q \geq 1$. Since u solves (1.2) and $\operatorname{div} V \equiv 0$, we have $\partial_t u + V \cdot \nabla u = \partial_t u + \operatorname{div}(Vu) = 0$ on $\mathbf{R}^d \times \mathbf{R}_+$. Thus, the weak formulation (3.1) reads

$$\int_T^{T+\delta t} \left(\int_{\mathbf{R}^d} u(x, t)u(x, t)dx \right) \psi'_q(t)ds + \int_{\mathbf{R}^d} u(\cdot, 0)u(\cdot, 0) = \mu(\phi^q) + \nu(\phi^q),$$

The sequence of Radon measures $(\psi'_q(t)dt)_q$ converges to $-\delta_T$, so the first term of the left hand side converges to $-\int_{\mathbf{R}^d} u(\cdot, T)u(\cdot, T)$. Clearly, for every $K_n \in \mathcal{M}$ and $L \in \partial K_n^-$, we have $(\langle \phi_q \rangle_K)_q \rightarrow \langle \phi \rangle_{K_n}$, $(\langle V \cdot \mathbf{n} \phi_q \rangle_{K_n L_n})_q \rightarrow \langle V \cdot \mathbf{n} \phi \rangle_{K_n L_n}$ and $(\langle \phi_q \rangle_K(n\delta t))_q \rightarrow \langle \phi \rangle_K(n\delta t)$. Thus, passing to the limit on q , we get

$$-\int_{\mathbf{R}^d} u(\cdot, T)u(\cdot, T) + \int_{\mathbf{R}^d} u(\cdot, 0)u(\cdot, 0) = \mu(\psi u) + \nu(\psi u).$$

Similarly, we obtain from (1.4) (or more directly from the conservation of λ by the flow), that

$$-\int_{\mathbf{R}^d} u(\cdot, T)^2 + \int_{\mathbf{R}^d} u(\cdot, 0)^2 = 0.$$

Subtracting these equalities, we get

$$\int_{\mathbf{R}^d} ((u - \mathbf{u})u)(\cdot, T) - \int_{\mathbf{R}^d} ((u - \mathbf{u})u)(\cdot, 0) = \mu(\psi u) + \nu(\psi u).$$

Using the identity $(a - b)a = [(a - b)^2]/2 + [a^2 - b^2]/2$, we obtain

$$\begin{aligned} \|(u - \mathbf{u})(\cdot, T)\|_{L^2}^2 + \|\mathbf{u}(\cdot, 0)\|_{L^2}^2 - \|\mathbf{u}(\cdot, T)\|_{L^2}^2 \\ = 2\mu(\psi u) + 2\nu(\psi u) + \|(u - \mathbf{u})(\cdot, 0)\|_{L^2}^2 + (\|\mathbf{u}(\cdot, 0)\|_{L^2}^2 - \|\mathbf{u}(\cdot, T)\|_{L^2}^2). \end{aligned}$$

The last term vanishes and Lemma 3.3 implies $\|(u - \mathbf{u})(\cdot, 0)\|_{L^2}^2 \leq C\|\nabla u_0\|_{L^2}^2 h^2$. Thus we have

$$\|(u - \mathbf{u})(\cdot, T)\|_{L^2}^2 + \|\mathbf{u}(\cdot, 0)\|_{L^2}^2 - \|\mathbf{u}(\cdot, T)\|_{L^2}^2 \leq 2\mu(\psi u) + 2\nu(\psi u) + C\|\nabla u_0\|_{L^2}^2 h^2. \quad (3.7)$$

From the definition (3.2) of μ and the Cauchy-Schwarz inequality, we get

$$|\mu(\psi u)| \leq \mathcal{F}(\mathbf{u}, T)^{1/2} \left(\sum_{K_n \in \mathcal{M}_T} |K| |\langle u \rangle_{K_n} - \langle u \rangle_K((n+1)\delta t)|^2 \right)^{1/2}.$$

By Jensen inequality, we have

$$|\langle u \rangle_{K_n} - \langle u \rangle_K((n+1)\delta t)|^2 \leq \langle |u - u((n+1)\delta t, \cdot)|^2 \rangle_{K_n}$$

and by the first estimate of Corollary 1.2 we get

$$|\mu(\psi u)| \leq C_0 \mathcal{F}(\mathbf{u}, T)^{1/2} \|\nabla u_0\|_{L^2} T^{1/2} \delta t^{1/2}. \quad (3.8)$$

Let us write

$$v(\psi u) = \sum_{K_n \in \mathcal{M}_T} \sum_{L \in \partial K_n^-} \delta t (u_L^n - u_K^n) (V_{KL}^n \langle u \rangle_{K_n} - |K_1 L| \langle V \cdot \mathbf{n} u \rangle_{K_1 L_n}) = \text{I} + \text{II},$$

$$\begin{aligned} \text{with} \quad \text{I} &:= \sum_{K_n \in \mathcal{M}_T} \sum_{L \in \partial K_n^-} \delta t (u_L^n - u_K^n) V_{KL}^n \left(\langle u \rangle_{K_n} - \int_{K_1 L_n} u \right), \\ \text{II} &:= \sum_{K_n \in \mathcal{M}_T} \sum_{L \in \partial K_n^-} \delta t (u_L^n - u_K^n) |K_1 L| \int_{K_1 L_n} \left(-u + \int_{K_1 L_n} u \right) V \cdot \mathbf{n}_{KL}. \end{aligned}$$

From the Cauchy-Schwarz inequality, we deduce

$$|\text{II}| \leq \mathcal{F}(\mathbf{u}, T)^{1/2} \left(\sum_{K_n \in \mathcal{M}_T} \sum_{L \in \partial K_n^-} \delta t |V_{KL}^n| \left(\langle u \rangle_{K_n} - \int_{K_1 L_n} u \right)^2 \right)^{1/2},$$

and since $|V_{KL}^n| \leq \|V\|_\infty |K_1 L|$, (3.4) yields

$$|\text{II}| \leq C C_0 \mathcal{F}(\mathbf{u}, T)^{1/2} \|\nabla u_0\|_{L^2} T^{1/2} h^{1/2}. \quad (3.9)$$

We now estimate II. We split the integral in space and time differences:

$$\begin{aligned} \int_{K_1 L_n} \left(-u + \int_{K_1 L_n} u \right) V \cdot \mathbf{n}_{KL} &= \int_{K_1 L_n} \int_{K_1 L_n} (u(y, t) - u(x, s)) V(x, s) \cdot \mathbf{n}_{KL} dx dr dy ds =: A + B, \\ \text{with} \quad A &:= \int_{K_1 L_n} \int_{n\delta t}^{(n+1)\delta t} (u(y, s) - u(x, s)) V(x, s) \cdot \mathbf{n}_{KL}, \\ B &:= \int_{K_1 L_n} \int_{K_1 L_n} (u(y, t) - u(y, s)) V(x, s) \cdot \mathbf{n}_{KL}. \end{aligned}$$

Since $\int_{n\delta t}^{(n+1)\delta t} \int_{n\delta t}^{(n+1)\delta t} (u(y, t) - u(y, s)) dr ds = 0$, we have

$$\begin{aligned} |B| &= \left| \int_{K_1 L_n} \int_{K_1 L_n} (u(y, s) - u(y, t)) \left(V(x, s) \cdot \mathbf{n}_{KL} - \int_{n\delta t}^{(n+1)\delta t} V(x, \cdot) \cdot \mathbf{n}_{KL} \right) dx dy dr ds \right| \\ &\leq C \delta t \int_{K_1 L_n} \int_{n\delta t}^{(n+1)\delta t} |u(y, s) - u(y, t)| dy dr ds. \end{aligned}$$

Similarly, $|A| \leq Ch \int_{K_1 L_n} \int_{K_1 L} |u(x, s) - u(y, s)|$, and from the Cauchy-Schwarz inequality,

$$\begin{aligned} |\text{III}|^2 &\leq Ch \left(\sum_{K_n \in \mathcal{M}_T, L \in \partial K_n^-} \delta t h^d |u_L^n - u_K^n|^2 \right) \left(\sum_{K_n \in \mathcal{M}_T, L \in \partial K_n^-} \delta t |K_1 L| \left(\int_{K_1 L_n} \int_{K_1 L} |u(x, t) - u(y, t)| \right)^2 \right) \\ &+ C \delta t \left(\sum_{K_n \in \mathcal{M}_T, L \in \partial K_n^-} \delta t^2 |K_1 L| |u_L^n - u_K^n|^2 \right) \left(\sum_{K_n \in \mathcal{M}_T, L \in \partial K_n^-} \delta t |K_1 L| \left(\int_{K_1 L_n} \int_{n\delta t}^{(n+1)\delta t} |u(x, s) - u(x, t)| \right)^2 \right) \\ &=: Ch A_1 B_1 + C \delta t A_2 B_2. \end{aligned}$$

Using the L^2 -stability of the scheme and assumption (1.5), we find that A_1 and A_2 are bounded by $CT\|u_0\|_{L^2}^2$. We use (3.6), (3.5) respectively to estimate B_1 and B_2 and obtain

$$|\text{III}| \leq CC_0 \|\nabla u_0\|_{L^2} \|u_0\|_{L^2} Th. \quad (3.10)$$

REMARK 3.1. Notice that II vanishes if the speed of advection V is constant and that this is the only term for which (3.6) and therefore assumption (1.11) is needed.

Plugging (3.8), (3.9), (3.10) in (3.7) (and using $\delta t \leq c_0 h$), we get

$$\begin{aligned} \|(u - \mathbf{u})(\cdot, T)\|_{L^2}^2 + \|\mathbf{u}(\cdot, 0)\|_{L^2}^2 - \|\mathbf{u}(\cdot, T)\|_{L^2}^2 \\ \leq CC_0 \left(\|u_0\|_{H^1}^2 Th + \|u_0\|_{H^1} \mathcal{F}(\mathbf{u}, T)^{1/2} T^{1/2} h^{1/2} \right) + C \|u_0\|_{H^1}^2 h^2. \end{aligned}$$

And from Lemma 3.2,

$$\begin{aligned} \|(u - \mathbf{u})(\cdot, T)\|_{L^2}^2 + \|\mathbf{u}(\cdot, 0)\|_{L^2}^2 - \|\mathbf{u}(\cdot, T)\|_{L^2}^2 \\ \leq CC_0 \left(\|u_0\|_{H^1}^2 Th + \xi^{-1/2} \|u_0\|_{H^1} \left(\|\mathbf{u}(\cdot, 0)\|_{L^2}^2 - \|\mathbf{u}(\cdot, T)\|_{L^2}^2 \right)^{1/2} T^{1/2} h^{1/2} \right) + C \|u_0\|_{H^1}^2 h^2. \end{aligned}$$

We then use the Young inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ with

$$a = (CC_0)^{-1/2} \left(\|\mathbf{u}(\cdot, 0)\|_{L^2}^2 - \|\mathbf{u}(\cdot, T)\|_{L^2}^2 \right)^{1/2}, \quad b = (CC_0)^{1/2} \xi^{-1/2} \|u_0\|_{H^1} T^{1/2} h^{1/2}$$

to derive

$$\|(u - \mathbf{u})(\cdot, T)\|_{L^2}^2 + \frac{1}{2} \left(\|\mathbf{u}(\cdot, 0)\|_{L^2}^2 - \|\mathbf{u}(\cdot, T)\|_{L^2}^2 \right) \leq C(1 + C_0^2) \xi^{-1} \|u_0\|_{H^1}^2 Th + C \|u_0\|_{H^1}^2 h^2,$$

yielding the first estimate of Theorem 1.8 and by Lemma 3.2, the second estimate. \square

REFERENCES

- [1] Bernardo Cockburn, Frédéric Coquel, and Philippe LeFloch. An error estimate for finite volume methods for multidimensional conservation laws. *Math. Comp.*, 63(207):77–103, 1994.
- [2] Bernardo Cockburn, Pierre-Alain Gremaud, and Jimmy Xiangrong Yang. A priori error estimates for numerical methods for scalar conservation laws. III. Multidimensional flux-splitting monotone schemes on non-Cartesian grids. *SIAM J. Numer. Anal.*, 35(5):1775–1803 (electronic), 1998.
- [3] Michael G. Crandall and Luc Tartar. Some relations between nonexpansive and order preserving mappings. *Proc. Amer. Math. Soc.*, 78(3):385–390, 1980.
- [4] Bruno Després. Convergence of non-linear finite volume schemes for linear transport. In *Notes from the XIth Jacques-Louis Lions Hispano-French School on Numerical Simulation in Physics and Engineering (Spanish)*, pages 219–239. Grupo Anal. Teor. Numer. Modelos Cienc. Exp. Univ. Cádiz, Cádiz, 2004.
- [5] Bruno Despres. Lax theorem and finite volume schemes. *Math. Comp.*, 73(247):1203–1234 (electronic), 2004.
- [6] Robert Eymard, Thierry Gallouët, and Raphaële Herbin. Finite volume methods. In *Handbook of numerical analysis, Vol. VII*, Handb. Numer. Anal., VII, pages 713–1020. North-Holland, Amsterdam, 2000.
- [7] Nikolai Nikolaïevitch Kuznecov. The accuracy of certain approximate methods for the computation of weak solutions of a first order quasilinear equation. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 16(6):1489–1502, 1627, 1976.
- [8] Benoît Merlet and Julien Vovelle. Error estimate for a finite volume scheme applied to the linear advection equation. *preprint*, 2005.
- [9] Jean-Paul Vila. Convergence and error estimates in finite volume schemes for general multidimensional scalar conservation laws. I. Explicit monotone schemes. *RAIRO Modél. Math. Anal. Numér.*, 28(3):267–295, 1994.
- [10] Jean-Paul Vila and Philippe Villedieu. Convergence of an explicit finite volume scheme for first order symmetric systems. *Numer. Math.*, 94(3):573–602, 2003.