Two remarks on liftings of maps with values into S^1

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Abstract

Given a map $u \in L^1_{loc}(\Omega, S^1)$ having some regularity: $|u|_X = R < \infty$, we consider the problem of finding a lifting φ of u (*i.e.* a measurable function satisfying $u = e^{i\varphi}$) with the same regularity and with a control $|\varphi|_X \le g(R)$ which is optimal. Two cases are treated here:

1) $|\cdot|_X$ is a $W^{s,p}(0,1)$-seminorm, with $0 < s < 1 < p$ and $sp > 1$. We find a lifting φ such that $|\varphi|_{W^{s,p}(I)} \le C(R + R^{1/s})$ and we show that the exponent $1/s$ cannot be improved.

2) $|\cdot|_X$ is the $BV(\Omega)$-seminorm where $\Omega \subset \mathbf{R}^d$ is a smooth open set. We give a simplified proof of a preexisting result [J. Dàvila, R. Ignat, C. R. Acad. Sci. Paris, Ser. I 337 (2003)]: there exists $\varphi \in BV(\Omega)$ satisfying $|\varphi|_{BV} \le 2R$.

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Résumé

Deux remarques sur les relèvements d'applications à valeurs dans S^1. Étant donnée une application $u \in L^1_{loc}(\Omega, S^1)$ ayant une certaine régularité : $|u|_X = R < \infty$, nous cherchons un relèvement φ de u (*i.e.* une fonction mesurable telle que $u = e^{i\varphi}$) ayant la même régularité et avec le meilleur contrôle possible de $|\varphi|_X$ en fonction de R. On traite deux cas :

1) $|\cdot|_X$ est une seminorme $W^{s,p}(0,1)$, avec $0 < s < 1 < p$ et $sp > 1$. Nous trouvons un relèvement φ satisfaisant $|\varphi|_{W^{s,p}(I)} \le C(R + R^{1/s})$ et nous montrons que l'exposant $1/s$ ne peut être amélioré.

2) $|\cdot|_X$ est la seminorme $BV(\Omega)$ où $\Omega \subset \mathbf{R}^d$ est un ouvert régulier. Nous donnons une preuve simplifiée d'un résultat préexistant [J. Dàvila, R. Ignat, C. R. Acad. Sci. Paris, Ser. I 337 (2003)] : il existe $\varphi \in BV(\Omega)$ telle que $|\varphi|_{BV} \le 2R$.

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Version française abrégée

Soit $\Omega \subset \mathbf{R}^d$, un ouvert régulier et $u : \Omega \to S^1$ une application mesurable. On appelle relèvement de u une fonction mesurable $\varphi : \Omega \to \mathbf{R}$ telle que $u = e^{i\varphi}$ presque partout dans Ω.

Nous abordons ici le problème de trouver un relèvement φ dont la régularité est contrôlée par la régularité de u. Ce problème a été traité par Bourgain, Brezis et Mironescu dans [1,3] pour des régularités H^s et $W^{s,p}$.

On considère tout d'abord le cas $\Omega = I = (0,1)$ et $u \in W^{s,p}(I, S^1)$ avec $0 < s < 1 < p$ et $sp > 1$, muni de la seminorme :

$$|f|^p_{W^{s,p}(I)} := \int_{I^2} \frac{|f(x) - f(y)|^p}{|x-y|^{1+sp}} dxdy.$$

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Par l'inégalité de Morey, $W^{s,p}(I)$ s'injecte dans un espace de fonctions continues et pour $u \in W^{s,p}(I,S^1)$ on a l'existence d'un relèvement continu φ. On peut vérifier facilement que φ appartient à $W^{s,p}(I)$. Mais, à cause de la forme non locale de la seminorme $W^{s,p}$, on n'a pas un contrôle linéaire de $|\varphi|_{W^{s,p}(I)}$ en fonction de $|u|_{W^{s,p}(I)}$. Nous montrons :

Théorème 0.1 *Soit $u \in W^{s,p}(I,S^1)$, notons φ un relèvement continu de u (unique à une constante additive près), alors $\varphi \in W^{s,p}(I,\mathbf{R})$ et on a l'estimation*

$$|\varphi|_{W^{s,p}(I)} \;\leq\; C(s,p)\left(|u|_{W^{s,p}(I)} + |u|^{1/s}_{W^{s,p}(I)}\right).$$

De plus l'exposant $1/s$ est optimal.

Dans une seconde partie, nous nous interessons au cas d'une application $u\ :\ \Omega \to S^1$, avec Ω domaine régulier de $\mathbf{R}^d$ et u à variation bornée : $|u|_{BV} := \int_\Omega |\nabla u| < \infty$ (il faut comprendre l'intégrale comme la masse totale de la mesure de Radon ∇u). Nous donnons une preuve alternative du résultat suivant dû à Dávila et Ignat [8].

Théorème 0.2 *Soit u dans $L^1_{loc}(\Omega,S^1)$, tel que u soit à variation bornée. Il existe $\varphi \in L^\infty(\Omega,\mathbf{R})$, relèvement de u, à variation bornée, satisfaisant l'estimation :* $\quad |\varphi|_{BV} \leq 2|u|_{BV}$.

Remarque 1 Le coefficient 2 est optimal, il est dû à un obstacle topologique. Par exemple (voir [8]), si Ω est le disque unité de $\mathbf{C}$ et si $u(re^{i\theta}) := e^{i\theta}$, pour $0 \leq r < 1$, $0 \leq \theta < 2\pi$, alors $\varphi(re^{i\theta}) = \theta$ minimise la seminorme BV parmi les relèvements de u et on a $|\varphi|_{BV} = 4\pi = 2|u|_{BV}$.

Remarque 2 Il existe des résultats plus précis en dimension $d = 2$ — voir [5,9].

Notre preuve utilise une définition équivalente de la seminorme BV donnée par Dávila dans [7]. Cette définition est basée sur une caractérisation des espaces de Sobolev $W^{1,p}$ due à Bourgain, Brezis et Mironescu [2,4]. Cette nouvelle définition permet d'éviter l'introduction des outils de la Théorie Géométrique de la Mesure. En particulier, on évite l'identification et le traitement séparé des parties absolument continue, Cantor et saut des mesures ∇u et $\nabla\varphi$.

1. Introduction

Let $\Omega \subset \mathbf{R}^d$ be an open set and $u\ :\ \Omega \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$ a measurable function. A lifting of u is a measurable function $\varphi\ :\ \Omega \to \mathbf{R}$ such that $u = e^{i\varphi}$.

This note concerns the following question: let u be given with some regularity ; can we find a lifting φ with the same regularity and such that the regularity of u controls the regularity of φ ? This problem has been extensively treated by Bourgain, Brezis and Mironescu in [1,3] for $W^{s,p}$ and H^s data.

1.1. $u \in W^{s,p}(I,S^1)$, $0 < s < 1 < p$ and $sp > 1$.

First, we consider the case $\Omega = I = (0,1)$ and $u \in W^{s,p}(I,S^1)$ with $0 < s < 1 < p$ and $sp > 1$. We define the $W^{s,p}(I)$-seminorm of a measurable map $f \in L^1_{loc}(I,\mathbf{R}^q)$ by

$$|f|^p_{W^{s,p}(I)} \;:=\; \int_{I^2} \frac{|f(x)-f(y)|^p}{|x-y|^{1+sp}}dxdy,$$

where $|\cdot|$ is the euclidian norm in $\mathbf{R}^q$ or in $\mathbf{R}$. If $|f|_{W^{s,p}(I)} < \infty$, then, from the Sobolev embedding theorem ($sp > 1$), f is almost everywhere equal to a $C^{0,\alpha}$ function $\tilde{f}$, with $\alpha = s - 1/p$ — in the sequel, we always consider $f = \tilde{f}$. Moreover, there exists $c_0 > 0$ depending only on s and p, such that for every x, y in I, $x \neq y$,

$$\frac{|f(x)-f(y)|}{|x-y|^{s-1/p}} \leq c_0 |f|_{W^{s,p}([x,y])}, \qquad \text{where} \qquad |f|^p_{W^{s,p}([x,y])} := \int_{[x,y]^2} \frac{|f(x')-f(y')|^p}{|x'-y'|^{1+sp}}dx'dy'. \tag{1}$$

Let $u \in W^{s,p}(I,S^1)$. Since u is continuous, there exists a continuous lifting φ of u — which is unique up to an additive constant. But, due to the non local character of the $W^{s,p}$-seminorm, we do not have a linear control of $|\varphi|_{W^{s,p}(I)}$ by $|u|_{W^{s,p}(I)}$. We show:

Theorem 1.1 *Let u in $W^{s,p}(I,S^1)$, let φ be a continuous lifting of u, then*

$$|\varphi|_{W^{s,p}(I)} \ \leq\ C(s,p)\left(|u|_{W^{s,p}(I)} + |u|^{1/s}_{W^{s,p}(I)}\right).$$

Moreover, the power $1/s$ in this estimate is optimal.

1.2. $u \in L^1_{loc}(\Omega, S^1)$, $\int |\nabla u| < \infty$.

Let $\Omega \subset \mathbf{R}^d$ be an open set with a smooth boundary, we say that $f \in L^1_{loc}(\Omega, \mathbf{R}^q)$ has a bounded variation and we write $f \in BV(\Omega)$ if the gradient of f in the sense of distributions is a Radon measure of finite total mass. In this case, we define the BV-seminorm of f by $|f|_{BV} := \int_\Omega |\nabla f|$.

We give an alternative proof of the following result of Dávila and Ignat [8].

Theorem 1.2 *Let $u \in BV(\Omega, S^1)$. There exists a lifting of u, $\varphi \in BV(\Omega, \mathbf{R})$ satisfying $|\varphi|_{BV} \leq 2|u|_{BV}$.*

Remark 1 The coefficient 2 cannot be removed, it is due to a topological obstruction. For example (see [8]), let Ω be the unit disc in $\mathbf{C}$ and let us define the map $u(re^{i\theta}) := e^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta < 2\pi$. The function $\varphi(re^{i\theta}) := \theta$ is a lifting of u of minimal BV-seminorm and we have $|\varphi|_{BV} = 4\pi = 2|u|_{BV}$.

Remark 2 There exist more precise results in dimension $d = 2$ — see [5,9].

Our proof uses an equivalent definition of the BV-seminorm (Dávila [7]). This definition is based on the characterization of the Sobolev spaces $W^{1,p}$ introduced by Bourgain, Brezis et Mironescu [2,4]. Using this new definition, we avoid introducing the tools of Geometric Measure Theory. In particular, we avoid distinct treatments for the absolutely continuous, Cantor and jump parts of the measures ∇u et $\nabla\varphi$.

2. Lifting of $W^{s,p}(I, S^1)$-maps, $sp > 1$ – Proof of Theorem 1.1

In this Section, the reals $0 < s < 1$ and $p > 1$ are such that $sp > 1$, c_0 is the constant introduced in (1). The letter C denotes various positive constants only depending on s ansd p.

Counterexample

We show that the power $1/s$ is optimal by exhibiting an unbounded sequence (u_n) in $W^{s,p}(I, S^1)$ with associated liftings (φ_n) in $W^{s,p}$ such that $|u_n|^{1/s}_{W^{s,p}(I)} \ \leq\ C|\varphi_n|_{W^{s,p}(I)}$, $n \geq 0$.
Namely, for $n \geq 1$, we set $\varphi_n(x) := nx$, and $u_n(x) := e^{inx}$, for $0 \leq x \leq 1$. Clearly, φ_n belongs to $W^{s,p}(I, \mathbf{R})$ and is a lifting of $u_n \in W^{s,p}(I, S^1)$. Then we compute

$$|\varphi_n|^p_{W^{s,p}(I)} \ =\ n^p \int_{I^2} |x-y|^{-1+(1-s)p} dxdy = Cn^p.$$

Next, considering the integral on $(x, y) \in I^2$ in the definition of $|u_n|^p_{W^{s,p}(I)}$, we use the estimate $|u_n(x) - u_n(y)| \leq n|x-y|$ in the case $|x-y| < 1/n$ and $|u_n(x) - u_n(y)| \leq 2$ in the other case. We obtain,

$$|u_n|^p_{W^{s,p}(I)} \ \leq\ Cn^p \int_0^{1/n} r^{-1+(1-s)p} dr + C \int_{1/n}^1 r^{-1-sp} dr \ \leq\ Cn^{sp} + Cn^{sp},$$

and we have the desired estimate. □

Proof of the estimate

Let $u \in W^{s,p}(I, S^1)$. From the Sobolev embedding theorem, u is continuous and there exists a continuous lifting of u denoted φ. Next, define

$$E_0 \ :=\ \{(x,y) \in I^2 \ :\ a(x,y) \leq c_0^{-p}\} \qquad \text{where} \quad a(x,y) := |u|^p_{W^{s,p}([x,y])}|x-y|^{sp-1}, \qquad \text{for } x, y \in I.$$

Let $(x, y) \in E_0$. From (1), for every $x', y' \in [x, y]$, we have $|u(x') - u(y')| \leq 1$ and the set $u([x, y])$ is contained in a half circle of S^1. By continuity, we have $|\varphi(x) - \varphi(y)| \leq (\pi/2)|u(x) - u(y)|$. Thus

$$\int_{E_0} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{1+sp}} dxdy \ \leq\ (\pi/2)^p \int_{E_0} \frac{|u(x) - u(y)|^p}{|x-y|^{1+sp}} dxdy \ \leq\ C|u|^p_{W^{s,p}(I)}. \tag{2}$$

Now let $(x, y) \in I^2$, we set

$$k_{x,y} \ := \ \inf\{2^q \ : \ \exists\, x = x_0, x_1, \cdots, x_{2^q} = y \in I \quad \text{with } (x_{i-1}, x_i) \in E_0 \quad \text{for } 1 \le i \le 2^q\}.$$

Clearly, $|\varphi(x) - \varphi(y)| \le (\pi/2) k_{x,y}$, and

$$\int_{I^2 \setminus E_0} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{1+sp}} dxdy \ \le \ (\pi/2)^p \int_{I^2 \setminus E_0} \frac{k^p_{x,y}}{|x-y|^{1+sp}} dxdy. \tag{3}$$

To bound the quantities $k_{x,y}$, we use the following estimate

$$k^{sp}_{x,y} \ \le \ 2^{sp} c_0^p a(x,y), \qquad \forall (x,y) \in I^2 \setminus E_0. \tag{4}$$

Inequality (4) is obtained by applying recursively the next lemma.

Lemma 2.1 *Let $(x,y) \in I^2$. There exists $z \in (x,y)$ such that* $\quad \max(a(x,z), a(z,y)) \ \le \ 2^{-sp} a(x,y)$.

Indeed, let $(x,y) \in I^2 \setminus E_0$, so that $a(x,y) \ge c_0^{-p}$ and let $q \ge 0$, such that $(2^{q-1})^{sp} \le c_0^p a(x,y) \le (2^q)^{sp}$. Applying recursively the lemma, there exists $x = x_0, x_1, \cdots, x_{2^q} = y \in I$ such that for $1 \le i \le 2^q$, $a(x_{i-1}, x_i) \le (2^{-sp})^q a(x,y) \le c_0^{-p}$. Therefore $(x_{i-1}, x_i) \in E_0$ for $1 \le i \le 2^q$ and $k_{x,y} \le 2^q$. We compute $k_{x,y} = 22^{q-1} \le 2 c_0^{1/s} a(x,y)^{1/(sp)}$ which proves (4).

Proof of Lemma 2.1

We assume that $a(x,y) > 0$ (otherwise there is nothing to prove) and we define $z, f : [0,1] \to [0,1]$ by

$$z(\lambda) := (1-\lambda)x + \lambda y, \qquad f(\lambda) := |u|^p_{W^{s,p}([x, z(\lambda)])} \,/\, |u|^p_{W^{s,p}([x,y)])}.$$

We have $a(x, z(\lambda)) = f(\lambda)\lambda^{sp-1} a(x,y)$ and $a(z(\lambda), y) \le (1 - f(\lambda))(1-\lambda)^{sp-1} a(x,y)$. Setting $g(\lambda) := f(\lambda)\lambda^{sp-1}$ and $h(\lambda) := (1-f(\lambda))(1-\lambda)^{sp-1}$, it is sufficient to prove that there esists $\lambda_\star \in (0,1)$ such that $g(\lambda_\star) \le 2^{-sp}$ and $h(\lambda_\star) \le 2^{-sp}$.

The functions g and h are continuous and satisfy $g(0) = h(1) = 0$, $g(1) = h(0) = 1$, g is non decreasing and h is non increasing. Therefore, there exists $\lambda_\star \in (0,1)$ such that $g(\lambda_\star) = h(\lambda_\star)$. This identity reads $f(\lambda_\star) = (1-\lambda_\star)^{sp-1} / (\lambda_\star^{sp-1} + (1-\lambda_\star)^{sp-1})$, thus $g(\lambda_\star) = \lambda_\star^{sp-1}(1-\lambda_\star)^{sp-1} \,/\, \left(\lambda_\star^{sp-1} + (1-\lambda_\star)^{sp-1}\right)$. And it is not difficult to check that the right hand side of the preceding identity is smaller than 2^{-sp} for every $\lambda_\star \in I$. $\quad \square$

Plugging (4) in (3), we get

$$\int_{I^2 \setminus E_0} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{1+sp}} dxdy \ \le \ C \int_{I^2} \frac{|u|^{p/s}_{W^{s,p}([x,y])}}{|x-y|^m} dxdy,$$

where $m := 1 + 1/s - (1-s)p$. For $(x,y) \in I^2$, we write

$$|u|^p_{W^{s,p}([x,y])} \ = \ |u|^p_{W^{s,p}(I)} \int_{[x,y]^2} \frac{1}{|u|^p_{W^{s,p}(I)}} \frac{|u(x') - u(y')|^p}{|x'-y'|^{1+sp}} dx'dy' \ \le \ |u|^p_{W^{s,p}(I)}\, \mu([x,y]),$$

where μ is the probability measure on I with density $\rho(y') := \dfrac{1}{|u|^p_{W^{s,p}(I)}} \displaystyle\int_I \frac{|u(x') - u(y')|^p}{|x'-y'|^{1+sp}} dx'$.

With this notation, we have:

$$\int_{I^2 \setminus E_0} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{1+sp}} dxdy \le C |u|^{p/s}_{W^{s,p}(I)} \int_{I^2} \frac{\mu([x,y])^{1/s}}{|x-y|^m} dxdy \le C |u|^{p/s}_{W^{s,p}(I)} \int_{I^2} \frac{\mu([x,y])}{|x-y|^m} dxdy,$$

Writing $\mu([x,y]) = \displaystyle\int_I \mathbf{1}_{[x,y]}(x') d\mu(x')$, we get:

$$\int_{I^2 \setminus E_0} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{1+sp}} dxdy \ \le \ C |u|^{p/s}_{W^{s,p}(I)} \int_I \left(\int_{I^2} \frac{\mathbf{1}_{[x,y]}(x')}{|x-y|^m} dxdy \right) d\mu(x').$$

We use Lemma 2.2 below to bound the term in brackets. We only have to check that $m < 2$ which is equivalent to $(sp-1)(1-s) > 0$ and, therefore, is true. Finally,

$$\int_{I^2 \setminus E_0} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{1+sp}} dxdy \ \le \ C |u|^{p/s}_{W^{s,p}(I)}\,, \tag{5}$$

and summing (2) and (5), we obtain the estimate of Theorem 1.1. $\quad \square$

Lemma 2.2 *Let $m < 2$. Then there exists $C > 0$, such that* $\displaystyle\int_{I^2} \frac{\mathbf{1}_{[x,y]}(x')}{|x-y|^m} dxdy \ \leq C, \qquad \forall x' \in I.$

Proof of Lemma 2.2

Using the change of variables $(w,z) = ((x+y)/2 - x', (x-y)/2)$, we have $x' \in [x,y]$ if and only if $|z| \leq |w|$ and we compute

$$\int_{I^2} \frac{\mathbf{1}_{[x,y]}(x')}{|x-y|^m} dxdy \ \leq \ 4 \int_0^1 \frac{1}{w^m} \int_{-w}^{w} dzdw \leq 8 \int_0^1 \frac{1}{w^{m-1}} dw \leq C. \qquad \square$$

3. Lifting of BV-maps

In this Section $\Omega \subset \mathbf{R}^d$ is an open set with a smooth boundary and $|\cdot|$ denotes the classical Euclidian norm in $\mathbf{R}^q$.

Equivalent definition for the BV-seminorm

The key of the simplification of the proof of [8] is the use of the following theorem (Theorem 1 in [7]). In [7], it is stated for functions $u \in BV(\Omega, \mathbf{R})$ but it is also valid, with the same proof, for maps in $BV(\Omega, \mathbf{R}^q)$. The result for maps in $BV(\Omega, \mathbf{R}^q)$ is also a particular case of a theorem of Chiron (Theorem 2 in [6]) concerning maps with values in a metric space.

Theorem 3.1 *There exists a positive constant $K_{1,d}$ such that for every family of non negative radial mollifiers $(\rho_\varepsilon)_{1 \geq \varepsilon > 0} \subset L^1_{loc}((0,\infty), \mathbf{R}_+)$ satisfying:*

$$\int_0^\infty \rho_\varepsilon(r) r^{d-1} dr \ = \ 1, \quad 1 \geq \rho_\varepsilon > 0 \qquad \textit{and} \qquad \lim_{\varepsilon \to 0} \int_\delta^\infty \rho_\varepsilon(r) r^{d-1} dr \ = \ 0, \quad \textit{for every } \delta > 0,$$

we have

$$\lim_{\varepsilon \to 0} \int_{\Omega^2} \frac{|f(x) - f(y)|}{|x-y|} \rho_\varepsilon(|x-y|) dxdy \ = \ K_{1,d} \int_\Omega |\nabla f|, \qquad \textit{for every } f \in L^1_{loc}(\mathbf{R}^d, \mathbf{R}^q).$$

In particular, this equality means that the limit always exists in $\mathbf{R}_+ \cup \{+\infty\}$ and is finite if and only if ∇f is a Radon measure with a finite total mass.

Proof of Theorem 1.2

Let $u \in BV(\Omega, S^1)$ and let (ρ_ε) be a family of radial mollifiers as above.
As in [8], for $0 \leq \alpha < 2\pi$, we define the lifting φ_α, by $u(x) = e^{i\varphi_\alpha(x)}$, $\alpha \leq \varphi_\alpha(x) < 2\pi + \alpha$, for every $x \in \Omega$ and we prove

$$\int_0^{2\pi} \left(\int_\Omega |\nabla \varphi_\alpha| \right) d\alpha \ \leq \ 4\pi \int_\Omega |\nabla u|. \tag{6}$$

Let $1 \geq \varepsilon > 0$, from Fubini's theorem, we have

$$\int_0^{2\pi} \int_{\Omega^2} \frac{|\varphi_\alpha(x) - \varphi_\alpha(y)|}{|x-y|} \rho_\varepsilon(|x-y|) dxdyd\alpha \ = \ \int_{\Omega^2} \frac{\rho_\varepsilon(|x-y|)}{|x-y|} \left(\int_0^{2\pi} |\varphi_\alpha(x) - \varphi_\alpha(y)| d\alpha \right) dxdy.$$

For $x, y \in \Omega$, there exist $\psi_1, \psi_2 \in \mathbf{R}$, such that $u(x) = e^{i\psi_1}$, $u(y) = e^{i\psi_2}$ and $|\psi_1 - \psi_2| \leq \pi$. It is not difficult to see that

$$\int_0^{2\pi} |\varphi_\alpha(x) - \varphi_\alpha(y)| d\alpha \ = \ 2|\psi_1 - \psi_2|(2\pi - |\psi_1 - \psi_2|).$$

Now, by derivating it is easy to prove that the inequality $\sin\beta \geq \beta(1 - \beta/\pi)$ holds for every $0 \leq \beta \leq \pi/2$. Since $|\psi_1 - \psi_2| \leq \pi$, we have $|u(x) - u(y)| = 2|\sin\frac{\psi_1 - \psi_2}{2}| \geq |\psi_1 - \psi_2|(1 - \frac{|\psi_1 - \psi_2|}{2\pi})$. Thus,

$$\int_0^{2\pi} |\varphi_\alpha(x) - \varphi_\alpha(y)| d\alpha \ \leq \ 4\pi |u(x) - u(y)|.$$

Plugging this estimate in the integral above, we get

$$\int_0^{2\pi}\int_{\Omega^2}\frac{|\varphi_\alpha(x)-\varphi_\alpha(y)|}{|x-y|}\rho_\varepsilon(|x-y|)dxdyd\alpha \ \leq \ 4\pi\int_{\Omega^2}\frac{|u(x)-u(y)|}{|x-y|}\rho_\varepsilon(|x-y|)dxdy.$$

Letting ε going to 0, we obtain (6) from Theorem 3.1 and Fatou's lemma. Finally, (6) implies that there exists $\alpha \in [0, 2\pi)$ such that φ_α is a (bounded) lifting of u satisfying $\int_\Omega |\nabla\varphi_\alpha| \leq 2\int_\Omega |\nabla u|$. $\square$

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