ON THE CONTRACTIBILITY OF SPACES OF FORMAL BARYCENTERS AND SYMMETRIC SMASH PRODUCTS

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Abstract. We show that barycenter spaces associated to a simplicial complex X can never be contractible unless X is acyclic. In so doing we verify that the reduced n-th symmetric products of X are contractible if and only if X has the homology of a wedge of k-circles with $n > k$.

1. Introduction and Statement of Main Result

Spaces of formal barycenters play an important role in the study of limiting Sobolev exponent problems in non-linear analysis such as the Yamabe and the scalar-curvature equations. They were introduced by Bahri and Coron in seeking solutions to the non-linear elliptic equation $-\Delta u = u^{(N+2)/(N-2)}$ on bounded regular open domains Ω, with u strictly positive on Ω and zero on its boundary.

More recently, they have appeared in the variational study of some mean field equations and systems. Those nonlinear PDE problems can be solved by studying the topology of the low energy sublevels of the associated energy functional. And this study is sometimes accomplished via a suitable space of barycenters. Examples of those applications are the study of the prescribed gaussian curvature with conical points, the prescribed Q-curvature problem in supercritical regimes, and the Toda system on compact surfaces; see [2, 3] and references therein.

The barycenters spaces are defined as the family of "unit measures supported in at most k points of X"

$$\mathcal{B}_n(X) = \left\{ \sum_{j=1}^{k} t_j \sigma_{x_j} \mid \sum_{j=1}^{k} t_j = 1 \ , t_i \geq 0, \ x_i \in X \right\}$$

A more topological definition is given in §2. A property that is desired out of barycenter spaces is that they be non-contractible when X of course is non-contractible. This necessary condition is however not enough in general to ensure the non-contractibility of $\mathcal{B}_n X$. Our main result clarifies the situation entirely. We recall that a space is acyclic if it has trivial reduced homology; i.e. $H_*(X;\mathbb{Z}) = 0$ for $* \geq 1$.

Theorem 1.1. *Assume X is a connected space of the homotopy type of a CW complex and let $n \geq 2$. Then $\mathcal{B}_n(X)$ is contractible if and only if X is acyclic.*

We don't claim the theorem to be true for all Euclidean domains X but only for those of the homotopy type of a CW complex. Note that in this case, $\mathcal{B}_n(X)$ is contractible if and only if it is acyclic. Indeed this is a consequence of the fact that $\mathcal{B}_n(X)$ is always simply connected for $n \geq 2$ [9] and of the homotopy type of a CW complex [2].

Example 1.2. The punctured Poincaré sphere is an example of a non contractible space whose barycenter spaces are contractible for $n \geq 2$. This space is acyclic with non-trivial fundamental group.

Corollary 1.3. [4, 9] *If X is a closed manifold, or of the homotopy type of one, then its barycenter spaces are never contractible.*

The homology of barycenter spaces $\mathcal{B}_n(X)$ is related to the homology of $\overline{\mathrm{SP}}^n(\Sigma X)$, where $\overline{\mathrm{SP}}^n Y$ is the symmetric smash product $Y^{\wedge n}/\mathfrak{S}_n$ with $\mathfrak{S}_n$ the symmetric group on n-letters acting on $Y^{\wedge n}$ by permutations. It is also the quotient of the n-th symmetric product $\mathrm{SP}^n Y$ by its subspace of tuples containing the basepoint (also called *reduced* symmetric product). To prove Theorem 1.1 we therefore need to prove a similar result about reduced symmetric products. The following key result is obtained in §3.

Theorem 1.4. *Let $n \geq 2$ and let X be connected of the homotopy type of a CW complex. Then $\overline{SP}^n(X)$ is contractible if and only if $H_*(X) \cong H_*(\bigvee^k S^1)$ and $1 \leq k < n$.*

To prove Theorem 1.4 we use an approach by R.J. Milgram to the computation of the homology of symmetric products of based CW complexes. This approach [13] is a refinement of [12] and is not readily accessible in the literature. We provide some details in §4. In particular, our discussion explains how to compute the homology of barycenter spaces of any space of the homotopy type of a CW complex.

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2. Proof of the Main Theorem

Given X a space, then its n-th barycenter space can be defined as the identification space [9]

$$\mathcal{B}_n(X) := \coprod_{k=1}^{n} \Delta_{k-1} \times_{\mathfrak{S}_k} X^k/_{\sim}$$

where $\Delta_{k-1} \times_{\mathfrak{S}_k} X^k$ is the quotient of $\Delta_{k-1} \times X^k$ by the symmetric group $\mathfrak{S}_k$ acting diagonally, and where $\sim$ is the equivalence relation generated by:

$$\begin{aligned} (i) \qquad & [t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n; x_1, \dots, x_i, \dots x_n] \\ & \sim [t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n; x_1, \dots, \hat{x}_i, \dots x_n] \end{aligned}$$

where $\hat{x}_i$ means the i-th entry has been suppressed, and by

$$\begin{aligned} (ii) \qquad & [t_1, \dots, t_i, \dots, t_j, \dots, t_n; x_1, \dots, x_i, \dots, x_j, \dots, x_n] \\ & \sim [t_1, \dots, t_{i-1}, t_i + t_j, t_{i+1}, \dots, \hat{t}_j, \dots, t_n; x_1, \dots, x_i, \dots, \hat{x}_j, \dots, x_n] \qquad \text{if} \quad x_i = x_j \end{aligned}$$

Using this presentation, it was shown in ([9], Theorem 1.3) that up to homotopy

$$\Sigma \mathcal{B}_n(X) \simeq \overline{\mathrm{SP}}^n(\Sigma X) \tag{1}$$

where ΣX means the suspension of X, $\mathrm{SP}^n(X)$ is the n-th symmetric product obtained as the quotient of X^n by the permutation action of the n-th symmetric group $\mathfrak{S}_n$, and $\overline{\mathrm{SP}}^n(X)$ is the symmetric smash or equivalently the cofiber of the embedding $\mathrm{SP}^{n-1}(X) \hookrightarrow \mathrm{SP}^n(X)$ sending $[x_1, \dots, x_{n-1}] \mapsto [x_1, \dots, x_{n-1}, x_0]$, where $x_0 \in X$ is a choice of basepoint. The major advantage in this description is that after one suspension the barycenter space transforms into a reduced symmetric product which is a space very well studied. Here's a quick consequence of that.

Proposition 2.1. *Suppose $f : X \to Y$ induces an embedding $H_*(X) \hookrightarrow H_*(Y)$ into a direct summand. Then there is an embedding in homology $H_*(\mathcal{B}_n X) \hookrightarrow H_*(\mathcal{B}_n Y)$.*

Proof. This is a consequence of the fact that if $H_*(X) \hookrightarrow H_*(Y)$ has a retract, then the induced homology map at the level of reduced symmetric products $H_*(\overline{\mathrm{SP}}^n X) \hookrightarrow H_*(\overline{\mathrm{SP}}^n Y)$ is an embedding (Proposition 3.1). We can then invoke (1). □

We turn to the proof of Theorem 1.1. The sufficiency condition is easy to establish. Indeed suppose X is acyclic. An acyclic space is not necessarily simply connected, but its suspension is, and being also acyclic it must be contractible. This means that all symmetric products of ΣX are contractible and $\overline{\mathrm{SP}}^n(\Sigma X) \simeq *$. This implies in turn that $\Sigma \mathcal{B}_n(X)$ is contractible which implies that $\mathcal{B}_n(X)$ is acyclic. By Theorem 3.6 and Theorem 1.1. of [9], $\mathcal{B}_n(X)$ is simply connected for $n \geq 2$. It must follow that $\mathcal{B}_n(X)$ is contractible for $n \geq 2$.

The proof of the necessity is much more tedious. The case $n = 2$ has however an interesting short proof.

Proposition 2.2. *Suppose X is connected of the homotopy type of a finite complex. Then $\mathcal{B}_2(X)$ is contractible if and only if X is acyclic.*

Proof. Suppose $\mathcal{B}_2(X)$ is contractible. By (1), the reduced square $\overline{\mathrm{SP}}^2(\Sigma X)$ is also contractible, and thus we obtain the equivalence $Y \simeq \mathrm{SP}^2(Y)$ for $Y = \Sigma X$. This means we have a map

$$\mathrm{SP}^2 Y \longrightarrow Y$$

such that the composite $Y \hookrightarrow \mathrm{SP}^2 Y \longrightarrow Y$ is an equivalence, where the first inclusion sends $x \mapsto [x, *]$ and $*$ is the basepoint. In other words, this is saying that Y is a homotopy abelian H-space with identify $*$.

Since it is a finite complex, by a classical Theorem of Hubbuck [8], Y must be of the homotopy type of a point or a product of circles (a torus). It cannot be a torus since Y being the suspension ΣX of a connected space is simply connected. To summarize, if $\mathcal{B}_2(X)$ is contractible, then $\Sigma X = Y$ is contractible and X is acyclic. □

The proof of the necessity in Theorem 1.1 for general n takes more space and is a direct consequence of Proposition 3.2 proved in the next section. Assuming Proposition 3.2 holds, suppose that $\mathcal{B}_n(X)$ is contractible. Then $\overline{\mathrm{SP}}^n(\Sigma X)$ is also contractible and hence in particular is acyclic. This means according to Proposition 3.2 that ΣX has the homology of the bouquet of k-circles with $n > k$. But for connected X, ΣX is simply-connected so $k = 0$ and ΣX is acyclic. This concludes the proof of our main result.

3. Acyclicity of Reduced Symmetric Products

We wish to describe $H_*(\overline{\mathrm{SP}}^n X)$ with various coefficients and for X of the homotopy type of a finite CW complex. We will be based mainly on [7, 12, 13] and make light additions. One identifies $\mathrm{SP}^n(X)$ inside $\mathrm{SP}^{n+1}(X)$ as the subset of unordered tuples containing the basepoint. The union of all subspaces is $\mathrm{SP}^\infty(X)$. The starting point is a result of Steenrod ([15],§22) which asserts that $\tilde{H}_*(\mathrm{SP}^n X)$ embeds in

$$H_*(\mathrm{SP}^\infty X;\Gamma) \cong \bigoplus_{n\geq 0} H_*(\mathrm{SP}^n X, \mathrm{SP}^{n-1};\Gamma) \cong \Gamma \oplus \bigoplus_{n\geq 1} \tilde{H}_*(\overline{\mathrm{SP}}^n X;\Gamma) \tag{2}$$

for $\Gamma = \mathbb{Z}$ or Γ any field. Here $\mathrm{SP}^{-1}X = \emptyset$ and $\mathrm{SP}^0 X$ is the basepoint. The symmetric product pairing $\mathrm{SP}^r X \times \mathrm{SP}^s X \longrightarrow \mathrm{SP}^{r+s} X$ induces a pairing

$$H_i(\mathrm{SP}^r X, \mathrm{SP}^{r-1}X) \times H_j(\mathrm{SP}^s X, \mathrm{SP}^{s-1}X) \longrightarrow H_{i+j}(\mathrm{SP}^{r+s}X, \mathrm{SP}^{r+s-1}X)$$

which makes (2) into a *bigraded* ring by both homological degree $*$ and filtration degree n. The subgroup $H_*(\overline{\mathrm{SP}}^n X)$ consists of elements of bidegree $(n,*)$.

When X is a connected based CW complex, then by a result of Dold and Thom, there is a homotopy equivalence

$$\mathrm{SP}^\infty(X) \simeq \prod K(\tilde{H}_k(X;\mathbb{Z}), k) \tag{3}$$

Let $\mathbb{Z}_k$ be a cyclic group (this is $\mathbb{Z}/k\mathbb{Z}$ if $k \geq 1$ and is $\mathbb{Z}$ if $k = 0$). Let $M(k,n)$ be the Moore space defined uniquely up to homotopy for $n \geq 2$ by the fact that it is simply connected with homology $H_n(M(k,n)) \cong \mathbb{Z}_k$ and $H_*(M(k,n)) = 0$ for $* \neq 0, n$. Then $\mathrm{SP}^\infty(M(k,n))$ is a model for the Eilenberg-MacLane space $K(\mathbb{Z}_k, n)$, or we can set directly

$$\mathrm{SP}^\infty(M(k,n)) = K(\mathbb{Z}_k, n)$$

so that according to our previous discussion, $H_*(K(\mathbb{Z}_k,n)) = H_*(\mathrm{SP}^\infty(M(k,n))$ is a bigraded ring.

For a general complex X of finite type, write $H_k(X;\mathbb{Z})$ as a sum of elementary abelian groups $\bigoplus_i \mathbb{Z}_{k_i}$ so that

$$K(H_k(X;\mathbb{Z}), k) \simeq \prod_i K(\mathbb{Z}_{k_i}, k)$$

By combining with (3), we have that over any field coefficients

$$H_*(\mathrm{SP}^\infty(X)) \cong \bigotimes_{k,i} H_*(K(\mathbb{Z}_{k_i}, k))$$

Theorem 5.1 of [12] says this is an isomorphism of *bigraded rings* where the graded ring structure on the right is the standard tensor product structure. Elements of exact filtration degree n in $H_*(\mathrm{SP}^\infty(X))$ correspond to elements of filtration degree n in the tensor product and these describe $H_*(\overline{\mathrm{SP}}^n X)$.

We can extract the following useful consequence

Proposition 3.1. *For X of the homotopy type of a CW complex, $H_*(SP^\infty X)$ is a bigraded ring and $H_*(\overline{SP}^n X)$ consists of elements of filtration degree exactly n. Moreover if $H_*(X) \cong H_*(Y) \oplus H_*(Z)$, then $H_*(\overline{SP}^n Y)$ embeds in $H_*(\overline{SP}^n X)$.*

We wish to describe the structure of $H_*(\overline{\mathrm{SP}}^n Y)$ as a direct summand of $H_*(\mathrm{SP}^\infty Y) = H_*(K(\pi, n))$. This comes down to undestanding the bigrading in the calculations of Cartan [5] and this was carried out in [12]. Milgram's method is explained in the appendix. It is used here to prove the following main result.

Proposition 3.2. *Fix $n \geq 2$. Then $\overline{SP}^n(X)$ is contractible $\Longleftrightarrow H_*(X) \cong H_*(\bigvee^k S^1)$ with $n > k \geq 1$.*

Proof. First we establish the sufficiency:
$\Leftarrow$ If $H_*(X) \cong H_*(\bigvee^k S^1)$ then by [7, 12], $H_*(\mathrm{SP}^n(X)) \cong H_*(\mathrm{SP}^n(\bigvee^k S^1))$ and $H_*(\overline{\mathrm{SP}}^n(X)) \cong H_*(\overline{\mathrm{SP}}^n(\bigvee^k S^1))$. Set $Z = \bigvee^k S^1$, then $\mathrm{SP}^n(Z) \simeq (S^1)^k$ if $n \geq k$ according to [10]. The inclusion $\mathrm{SP}^{n-1}(Z) \hookrightarrow \mathrm{SP}^n(Z)$ is a homotopy equivalence when $n > k$ and $\overline{\mathrm{SP}}^n(Z)$ is contractible in this case. This gives that $H_*(\overline{\mathrm{SP}}^n X)$ is trivial and $\overline{\mathrm{SP}}^n X$ is acyclic. Now $\overline{\mathrm{SP}}^n X$ is the cofiber of the inclusion $\mathrm{SP}^{n-1}X \longrightarrow \mathrm{SP}^n X$. This inclusion induces an isomorphism on fundamental groups when $n \geq 3$ and an epimorphism when $n = 2$ and this is because $\pi_1(\mathrm{SP}^n(X)) \cong H_1(X;\mathbb{Z})$ if $n \geq 2$. It follows that $\overline{\mathrm{SP}}^n(X)$ is simply connected for connected X and $n \geq 2$. Since it is acyclic and of the homotopy type of a CW complex, it must be contractible.

We now turn to the other implication of the Lemma. We assume that $\overline{\mathrm{SP}}^n(X)$ is contractible hence acyclic. We study depending on the structure of $H_*(X)$ when this can be the case. We distinguish some cases.

<u>Case 1</u>: Suppose $H_*(X;\mathbb{Z})$ has a free summand $\oplus^k\mathbb{Z}$ in degree $* = 1$. This term is the homology of the Moore space $\bigvee^k S^1$ and $H_*(\overline{\mathrm{SP}}^n(\bigvee^k S^1))$ embeds in $H_*(\overline{\mathrm{SP}}^n X)$ by Proposition 3.1. As indicated earlier [10], $\mathrm{SP}^n(\bigvee^k S^1)$ is up to homotopy the n-th skeleton of the torus $(S^1)^k$ so that

$$H_*(\overline{\mathrm{SP}}^n \bigvee^k S^1) \cong E(e_1, \ldots, e_k)_n$$

where on the right we have the submodule of an exterior algebra generated by all n-fold products $e_{i_1} \otimes \cdots \otimes e_{i_n}$. When $n \leq k$, this tensor product is not trivial and $H_*(\overline{\mathrm{SP}}^n \bigvee^k S^1)$ cannot be acyclic. However when $n > k$, there must be at least two repeating generators in any n-fold product and the homology must be trivial as desired, so only the case $n > k$ is possible.

<u>Case 2</u>: Suppose $H_*(X;\mathbb{Z})$ has a free summand $\mathbb{Z}$ in degree $k > 1$. Then $H_*(\overline{\mathrm{SP}}^n(S^k))$ embeds in $H_*(\overline{\mathrm{SP}}^n X)$ (Proposition 3.1). By Proposition 4.3, $\overline{\mathrm{SP}}^n S^k$ can never be acyclic if $k > 1$. This implies that $\overline{\mathrm{SP}}^n X$ is not acyclic as well. Note that $\overline{\mathrm{SP}}^n(S^1)$ is contractible (this is $k = 1$).

<u>Case 3</u>: Suppose $H_*(X;\mathbb{Z})$ has a cyclic summand $\mathbb{Z}_{p^r}$ in degree $k \geq 1$, p a prime. This cyclic summand is the homology of the space $M(p^r, k) = S^k \cup_f D^{k+1}$ obtained by attaching the disk D^{k+1} via a map of degree p^r. The reduced homology of $M(p^r, k)$ is concentrated in degree k and is equal to $\mathbb{Z}_{p^r}$ there. If we use $\mathbb{F}_{p^r}$ coefficients, then $H_*(M(p^r, k); \mathbb{F}_{p^r}) \cong H_*(S^k \vee S^{k+1}; \mathbb{F}_{p^r})$. By the main theorem of Dold [7], the infinite symmetric products of these spaces have the same homology with $\mathbb{F}_{p^r}$-coefficients and we have the decomposition

$$H_*(\mathrm{SP}^\infty(M(p^r, k)); \mathbb{F}_{p^r}) \cong H_*(\mathrm{SP}^\infty(S^k \vee S^{k+1}); \mathbb{F}_{p^r}) \cong H_*(\mathrm{SP}^\infty(S^k); \mathbb{F}_{p^r}) \otimes H_*(\mathrm{SP}^\infty(S^{k+1}); \mathbb{F}_{p^r})$$

This decomposition is as bigraded algebras by [12]. In particular we have embeddings $H_*(\overline{\mathrm{SP}}^n(S^k)) \hookrightarrow H_*(\overline{\mathrm{SP}}^n(M(p, k)))$ and $H_*(\overline{\mathrm{SP}}^n(S^{k+1})) \hookrightarrow H_*(\overline{\mathrm{SP}}^n(M(p, k)))$ with $\mathbb{F}_{p^r}$-coefficients. According to Lemma 4.4, one of $\overline{\mathrm{SP}}^n(S^k)$ or $\overline{\mathrm{SP}}^n(S^{k+1})$ is not acyclic and so $\overline{\mathrm{SP}}^n(M(p^r, k))$ is not acyclic for any $n, r, k \geq 1$. Since $H_*(\overline{\mathrm{SP}}^n(M(p^r, k)); \mathbb{F}_{p^r})$ embeds in $H_*(\overline{\mathrm{SP}}^n X; \mathbb{F}_{p^r})$ by Proposition 3.1, the latter group is also acyclic.

In conclusion in order for $\overline{\mathrm{SP}}^n X$ to be acyclic, X must be acyclic or has the homology of a wedge of k circles with $k < n$ (<u>case 1</u>). This concludes the proof of Proposition 3.2. □

As a consequence we have the following result which should be of independent interest.

Proposition 3.3. *Let X be a connected space of the homotopy type of a CW complex, and suppose for some $r, s > 0$, the embedding $SP^r(X) \hookrightarrow SP^{r+s}X$ is a homotopy equivalence. Then X has the homology of a wedge $\bigvee^k S^1$ with $k \leq r$.*

Proof. The fact that this embedding is an equivalence implies that all of $\overline{\mathrm{SP}}^n(X)$ for $r + 1 \leq n \leq r + s$ are acyclic hence contractible. □

4. Appendix: Reduced Symmetric Products

The object of this section is to discuss the acyclicity of $\overline{\mathrm{SP}}^n X$ for spheres X. In the particular case when $\mathbb{F} = \mathbb{F}_p$ with $p > n$ or $\mathbb{F} = \mathbb{Q}$, the cohomology $H^*(\mathrm{SP}^n(S^k); \mathbb{F})$ is a truncated polynomial algebra $P_T(u, m + 1) := \mathbb{F}[u]/u^{m+1}$ with $m = 1$ or $m = n$ according as k is odd or even. The reason behind this is

that the map $(S^k)^n \longrightarrow \mathrm{SP}^n(S^k)$ is a finite branched covering, and as long as the characteristic of $\mathbb{F}$ doesn't divide n, $H_*(\mathrm{SP}^n(S^k);\mathbb{F})$ is identified with the submodule of invariants in $H_*(S^k;\mathbb{F})^{\otimes n}$ under the symmetric group action $\mathfrak{S}_n$. This happens if the characteristic is zero or is p with $p > n$. When k is odd, there are no invariant classes other than the primitive class $\sum 1^{\otimes n-i} \otimes [S^k] \otimes 1^{\otimes i-1}$. When k is even, the class $[S^k]^{\otimes n}$ is invariant and maps to a non-trivial element in $H_{nk}(\overline{\mathrm{SP}}^n(S^k))$.

For general coefficients $\mathbb{F}_p$, the (co)homology of $\mathrm{SP}^n(S^k)$ has been computed a long time ago by Nakaoka. This has been recovered by calculations of Serre and Cartan of the homology of Eilenberg-MacLane spaces. The following approach due to R.J. Milgram has the major advantage of keeping track of the bigrading and thus describing the homology of the reduced products as well.

To describe $H_*(\mathrm{SP}^\infty(S^k);\mathbb{Z})$ one uses induction and the identification

$$\mathrm{SP}^\infty(\Sigma X) = B\mathrm{SP}^\infty X \tag{4}$$

where $B(-)$ is the topological bar construction introduced by Milgram. The main point is that one computes $H_*(\mathrm{SP}^\infty(\Sigma X))$ over field coefficients $\mathbb{F}_p$ by computing the homology of $EC_*(\mathrm{SP}^\infty X) \otimes_{\mathbb{F}_p} \mathbb{F}_p$ where $E(-)$ is the acyclic algebraic bar construction on the commutative algebra $C_*(\mathrm{SP}^\infty X)$. The E_2-term is $Tor^{H_*(SP^\infty(X);\mathbb{F})}(\mathbb{F},\mathbb{F}) := H_*(EH_*(\mathrm{SP}^\infty X) \otimes_{\mathbb{F}} \mathbb{F})$ and the spectral sequence collapses at this E_2-term when $\mathbb{F} = \mathbb{F}_p$ by main observations of [13] so that we have isomorphisms

$$H_*(\mathrm{SP}^\infty \Sigma X, \mathbb{F}_p) \cong H_*(B\mathrm{SP}^\infty(X), \mathbb{F}_p) \cong Tor^{H_*(SP^\infty(X))}(\mathbb{F}_p, \mathbb{F}_p) \tag{5}$$

All terms above are isomorphic as bigraded algebras. The bigraded algebra structure on the E^2-term

$$Tor^{H_*(SP^\infty(X))}(\mathbb{F}_p,\mathbb{F}_p) \otimes Tor^{H_*(SP^\infty(X))}(\mathbb{F}_p,\mathbb{F}_p) \longrightarrow Tor^{H_*(SP^\infty(X))}(\mathbb{F}_p,\mathbb{F}_p)$$

is derived from the abelian and associative shuffle product $*$ in the bar construction. If $a \in H_*(\mathrm{SP}^\infty(X))$, then its suspension class corresponds to $|a|$ in the total complex for Tor (filtration is preserved under suspension). The class $|a_1|\cdots|a_k|$ has filtration degree $\sum fil(a_i)$ and homological degree $\sum \deg a_i + k$.

Proposition 4.1. *If $H_*(SP^\infty X;\mathbb{F}) \cong A \otimes B$ as bigraded algebras, then*

$$H_*(SP^\infty(\Sigma X);\mathbb{F}) \cong Tor^{A\otimes B}(\mathbb{F},\mathbb{F}) \cong Tor^A(\mathbb{F},\mathbb{F}) \otimes Tor^B(\mathbb{F},\mathbb{F})$$

as bigraded algebras as well.

One can retrieve classes in exact filtration n in $H_*(\mathrm{SP}^\infty(\Sigma X))$ from the resolution of $H_*(\mathrm{SP}^\infty(X))$. This method is illustrated in the case of spheres below. We need first general facts from homological algebra which can also be found in [13]. Let $\Lambda(e)$ denote an exterior algebra on a generator e and Γ a divided power algebra on generators $\gamma_1, \gamma_2, \cdots, \gamma_i, \cdots$. We assume e of odd degree and a of even degree.

Theorem 4.2. [13]
(i) $Tor^{\Lambda(e)}(\mathbb{Z},\mathbb{Z})$ is a divided power algebra on $\gamma_i = \underbrace{|a|\cdots|a|}_{i}$, and modulo p this splits into a product of truncated polynomial algebras [5]

$$Tor^{\Lambda(e)}(\mathbb{F}_p,\mathbb{F}_p) = P_T(|a|,p) \otimes P_T(\gamma_p,p) \otimes \cdots P_T(\gamma_{p^i},p) \otimes \cdots$$

with γ_{p^i} of bidegree $(p^i, (\deg e + 1)p^i)$.
(ii) $Tor^{P_T(a,p)}(\mathbb{Z},\mathbb{Z}) \cong \Lambda(|a|) \otimes \Gamma$, with γ_i represented in the bar construction by

$$\gamma_i = \underbrace{|a|a^{p-1}|\cdots|a|a^{p-1}|}_{i}$$

If a has bidegree $(\ell, 2n)$, then $|a|$ has bidegree $(\ell, 2n+1)$ and γ_i has bidegree $(ip\ell, 2nip + 2i)$. Modulo p, the divided power splits again and

$$Tor^{P_T(a,p)}(\mathbb{F}_p,\mathbb{F}_p) = \Lambda|a| \otimes P_T(\gamma_1,p) \otimes P_T(\gamma_p,p) \otimes \cdots \otimes P_T(\gamma_{p^i},p) \otimes \cdots$$

where γ_{p^i} is of bidegree $(\ell p^{i+1}, 2np^{i+1} + 2p^i)$.

As a consequence we have the Lemma

Lemma 4.3. *Fix $n \geq 1, k \geq 2$. Then $H_*(\overline{SP}^n(S^k);\mathbb{F}_2) \neq 0$.*

This is proved next. Of course we can say a lot more and describe the structure of the homology groups entirely (as the proof shows) but all we are concerned with is the non-acyclicity of the spaces $\overline{\mathrm{SP}}^n(S^k)$. Note that the homological degree $*$ in the Lemma above must be at least $2n+k-2$. Indeed if X is an r-connected[1] complex with $r \geq 1$, then it is known that $\overline{\mathrm{SP}}^n(X)$ is $2n+r-2$ connected [9]. This bound is often sharp for spheres as can be seen when $k=2$, $\overline{\mathrm{SP}}^n(S^2) = S^{2n}$, and when $n=2$, $\overline{\mathrm{SP}}^2(S^k) \simeq \Sigma^{k+1}\mathbb{R}P^{k-1}$. With mod 2 coefficients, $H_{2n+k-2}(\overline{\mathrm{SP}}^n(S^k);\mathbb{F}_2)$ might be trivial in some cases. For example $H_7(\overline{\mathrm{SP}}^3(S^3);\mathbb{F}_2) = 0$. This group is however non-trivial with $\mathbb{F}_3$-coefficients.

Proof. The computation is by induction with starting point $H_*(\mathrm{SP}^\infty S^1) = \Lambda(e)$ an exterior algebra on a one dimensional generator e (since $\deg e$ is odd, this algebra is free graded). The suspension class $|e| \in Tor^{\Lambda(e)}(\mathbb{Z},\mathbb{Z})$ generates elements $\gamma_{(i,2i)} = \underbrace{|e|\cdots|e|}_{i}$ of filtration degree i and homological degree $2i$. They form a divided power algebra under the shuffle product $*$;

$$\underbrace{|e|\cdots|e|}_{i} * \underbrace{|e|\cdots|e|}_{j} = \binom{i+j}{i}\underbrace{|e|\cdots|e|}_{i+j}$$

This describes the homology of $H_*(\mathrm{SP}^\infty(S^2);\mathbb{Z})$ which is the homology of the infinite complex projective space $\mathbb{P}^\infty$. The class $\gamma_{(i,2i)}$ lives in $H_{2i}(\overline{\mathrm{SP}}^i S^2)$. According to Theorem 4.2

$$H_*(\mathrm{SP}^\infty(S^2);\mathbb{F}_p) \cong Tor^{\Lambda(e)}(\mathbb{F}_p,\mathbb{F}_p) \cong \bigotimes \mathbb{F}_p[\gamma_{(p^i,2p^i)}]/(\gamma^p_{(p^i,2p^i)}) \ := \bigotimes P_T(\gamma_{(p^i,2p^i)},p) \tag{6}$$

This splitting is as bigraded algebras over $\mathbb{F}_p$. Over $\mathbb{F}_2$, the products $\gamma_{(2^{i-1},2^i)} * \gamma_{(2^{i-1},2^i)}$ are trivial mod-2 and the homology splits into a product of exterior algebras

$$H_*(\mathrm{SP}^\infty(S^2);\mathbb{F}_2) \cong \bigotimes_{i\geq 1} \Lambda(\gamma_{(2^{i-1},2^i)}) \tag{7}$$

One should beware that this decomposition is as bigraded algebras and not as Hopf algebras. More precisely the following holds (as a useful reminder)

- $\mathbb{F}_2[x]^{dual} \cong \Gamma_{\mathbb{F}_2}[x]$ as Hopf algebras
- $\Gamma_{\mathbb{F}_2}[x] \cong \otimes\Lambda_{\mathbb{F}_2}(\gamma_{2^i}(x))$ as graded $\mathbb{F}_2$-algebras
- $\Gamma_{\mathbb{F}_2}[x] \cong \mathbb{F}_2[x]$ as graded $\mathbb{F}_2$-vector spaces ($\gamma_{2^i}(x) \mapsto x^{2^i}$).

Back to equation (7), we see that there is a class in filtration degree 2^i for all i. Since any integer n can be written in base two as $2^{i_1}+\cdots+2^{i_k}$, $i_1 > \cdots > i_k$, it follows that there is a class (a unique one in this case) in filtration degree n with homological degree $2^{i_1+1}+\cdots+2^{i_k+1} = 2n$. This is of course representing the torsion free class in $H_{2n}(\overline{\mathrm{SP}}^n(S^2)) \cong H_{2n}(S^{2n})$.

The construction above can now be iterated. Applying the Tor functor to (7) we obtain

$$H_*(\mathrm{SP}^\infty(S^3),\mathbb{F}_2) \cong \bigotimes Tor^{\Lambda(\gamma_{(2^{i-1},2^i)})}(\mathbb{F}_2,\mathbb{F}_2) \tag{8}$$

Tor of exterior algebras on an element a carries an algebra structure depending on the parity of a. More precisely

- $Tor^{\Lambda(a)}(\mathbb{Z},\mathbb{Z}) \cong \Lambda(|a|)\otimes\Gamma(|a|a|)$ if $\deg a$ is even
- $Tor^{\Lambda(a)}(\mathbb{Z},\mathbb{Z}) \cong \Gamma(|a|)$ if $\deg a$ is odd.

With coefficients in $\mathbb{F}_2$, regardless of the degree of a we have as already indicated $Tor^{\Lambda(a)}(\mathbb{F}_2,\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(|a|) \cong \bigotimes\Lambda(\underbrace{|a|\cdots|a|}_{2^i})$. Replacing into (8) we get

$$H_*(\mathrm{SP}^\infty(S^3),\mathbb{F}_2) \cong \bigotimes_{i,j\geq 1}\left(\Lambda|\gamma_{(2^{j-1},2^j)}|\otimes\cdots\otimes\Lambda(\underbrace{|\gamma_{(2^{j-1},2^j)}|\cdots|\gamma_{(2^{j-1},2^j)}|}_{2^i})\otimes\cdots\right) \tag{9}$$

The class $b_{(2^{j-1},2^j+1)} := |\gamma_{(2^{j-1},2^j)}|$ is primitive and has filtration degree 2^{j-1}. By the same reason as before any integer n can be written uniquely in the form $2^{i_1}+\cdots+2^{i_k}$ so that the class $b_{(2^{i_1},2^{i_1}+1)}\cdots b_{(2^{i_k},2^{i_k}+1)}$ is

[1] All homotopy groups are trivial up to and including degree r.

non-zero in $H_*(\overline{\mathrm{SP}}^n(S^3);\mathbb{F}_2)$. In turn the class $|b_{(2^{i_1},2^{i_1}+1)}|\cdots|b_{(2^{i_k},2^{i_k}+1)}|$ is non-zero in $H_*(\overline{\mathrm{SP}}^n(S^4);\mathbb{F}_2)$, and so on. The claim follows by iterating this process. □

As already indicated in the first paragraph of this section, $H_*(\overline{\mathrm{SP}}^n(S^k);\mathbb{F})$ might be trivial for some values of n when k is odd and $\mathrm{char}(\mathbb{F})\neq 2$. This situation doesn't occur for even spheres and the homology of their reduced symmetric products is always non-trivial, a fact that serves in the proof of Proposition 3.2.

Lemma 4.4. *Suppose $n\geq 1$, $k\geq 2$ with $k=2m$ even. Then for field coefficients $\mathbb{F}$, $H_*(\overline{SP}^n(S^k);\mathbb{F})\neq 0$.*

Proof. The case $\mathbb{F}=\mathbb{F}_2$ (previous lemma) and $\mathbb{F}=\mathbb{Q}$ or $\mathbb{F}_p$ with $p>n$ have been established earlier. In general the main point to observe is that there is an embedding of rings

$$\Lambda(e)\hookrightarrow H_*(\mathrm{SP}^\infty(S^{2m-1});\mathbb{F})$$

where $e=[S^{2m-1}]$ is of bidegree $(1,2m-1)$. There is therefore an embedding

$$Tor^{\Lambda(e)}(\mathbb{F},\mathbb{F})\hookrightarrow H_*(\mathrm{SP}^\infty(S^{2m});\mathbb{F})=Tor^{H_*(SP^\infty(S^{2m-1}))}(\mathbb{F},\mathbb{F}) \tag{10}$$

But $Tor^{\Lambda(e)}(\mathbb{F},\mathbb{F})=Tor^{\Lambda(e)}(\mathbb{Z},\mathbb{Z})\otimes\mathbb{F}$ and $Tor^{\Lambda(e)}(\mathbb{Z},\mathbb{Z})$ is a divided power algebra on generators $\gamma_i=\underbrace{|e|\cdots|e|}_{i}$ of bidegree $(i,2mi)$. For any n, the class γ_n gives a non-zero class in $\overline{\mathrm{SP}}^n(S^{2m})$ with any coefficients. This concludes the proof. □

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