On inverse scattering at fixed energy for the multidimensional Newton equation in a non-compactly supported field

Alexandre Jollivet

March 19, 2014

Abstract

We consider the inverse scattering problem at fixed and sufficiently large energy for the nonrelativistic and relativistic Newton equation in $\mathbb{R}^n$, $n \geq 2$, with a smooth and short-range electromagnetic field (V, B). Proceeding from different known results we obtain, in particular, that the scattering map at a fixed and sufficiently large energy uniquely determines (V, B) when B is assumed to be zero in a neighborhood of infinity and V is assumed to be spherically symmetric in a neighborhood of infinity.

1 Introduction

Consider the following second order differential equation that is the multidimensional nonrelativistic Newton equation with electromagnetic field

$$\ddot{x}(t) = F(x(t), \dot{x}(t)) := -\nabla V(x(t)) + B(x(t))\dot{x}(t), \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, $\dot{x}(t) = \frac{\mathrm{d}x}{\mathrm{d}t}(t)$ and $n \geq 2$. In this equation we assume that $V \in C^2(\mathbb{R}^n, \mathbb{R})$ and for any $x \in \mathbb{R}^n$, $B(x)$ is a $n \times n$ antisymmetric matrix with elements $B_{i,k}(x)$, $B_{i,k} \in C^1(\mathbb{R}^n, \mathbb{R})$, which satisfy

$$\frac{\partial B_{i,k}}{\partial x_l}(x) + \frac{\partial B_{l,i}}{\partial x_k}(x) + \frac{\partial B_{k,l}}{\partial x_i}(x) = 0, \tag{1.2}$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and for $l, i, k = 1 \ldots n$.

For $n = 3$, the equation (1.1) is the equation of motion in $\mathbb{R}^n$ of a nonrelativistic particle of mass $m = 1$ and charge $e = 1$ in an external and static electromagnetic field described by (V, B) (see, for example, [13, Section 17]). For the electromagnetic field the function V is an electric potential and B is the magnetic field. Then x denotes the position of the particle, $\dot{x}$ denotes its velocity, $\ddot{x}$ denotes its acceleration and t denotes the time.

For the equation (1.1) the energy

$$E = \frac{1}{2}|\dot{x}(t)|^2 + V(x(t)) \tag{1.3}$$

is an integral of motion. In connection with the relativistic Newton equation, see Section 4.1.

We assume that the electromagnetic field (V, B) is short-range. More precisely we assume that (V, B) satisfies the following conditions

$$\begin{aligned} |\partial_x^{j_1} V(x)| &\leq \beta_{|j_1|}(1+|x|)^{-\alpha-|j_1|},\ x \in \mathbb{R}^n, & (1.4)\\ |\partial_x^{j_2} B_{i,k}(x)| &\leq \beta_{|j_2|+1}(1+|x|)^{-\alpha-1-|j_2|},\ x \in \mathbb{R}^n, & (1.5) \end{aligned}$$

for $|j_1| \leq 2, |j_2| \leq 1,\ i,k = 1 \ldots n$ and some $\alpha > 1$ (here j_l is the multiindex $j_l = (j_{l,1}, \ldots, j_{l,n}) \in (\mathbb{N} \cup \{0\})^n$, $|j_l| = \sum_{k=1}^n j_{l,k}$ and $\beta_{|j_l|}$ are positive real constants). We denote by $\|.\|$ the norm on the short-range electromagnetic fields defined by

$$\begin{aligned} \|(V,B)\| = & \sup_{\substack{x\in\mathbb{R}^n,\ j_1\in\mathbb{N}^n\\ |j_1|\leq 2}} \Big((1+|x|)^{\alpha+|j_1|}|\partial_x^{j_1}V(x)|\Big) \\ & + \sup_{\substack{x\in\mathbb{R}^n,\ j_2\in\mathbb{N}^n\\ |j_2|\leq 1,\ i,k=1\ldots n}} \Big((1+|x|)^{\alpha+1+|j_2|}|\partial_x^{j_2}B_{i,k}(x)|\Big). \end{aligned} \tag{1.6}$$

Under conditions (1.4)–(1.5), we have the following properties (see, for example, [16] and [14] where classical scattering of particles in a short-range electric field and in a long-range magnetic field are studied respectively): for any $(v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n,\ v_- \neq 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = tv_- + x_- + y_-(t), \tag{1.7}$$

where $|\dot{y}_-(t)| + |y_-(t)| \to 0,$ as $t \to -\infty$; in addition for almost any $(v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n,\ v_- \neq 0$, the unique solution $x(t)$ of equation (1.1) that satisfies (1.7) also satisfies the following asymptotics

$$x(t) = tv_+ + x_+ + y_+(t), \tag{1.8}$$

where $v_+ \neq 0,\ |\dot{y}_+(t)| + |y_+(t)| \to 0,$ as $t \to +\infty$. At fixed energy $E > 0$, we denote by $\mathbb{S}_E^{n-1}$ the set $\{v_- \in \mathbb{R}^n \mid |v_-|^2 = 2E\}$ and we denote by $\mathcal{D}_E$ the set of $(v_-, x_-) \in \mathbb{S}_E^{n-1} \times \mathbb{R}^n$ for which the unique solution $x(t)$ of equation (1.1) that satisfies (1.7) has also an asymptotics (1.8). For $R > 0$ we also denote by $\mathcal{B}(0, R)$ the Euclidean open ball of center 0 and radius R. We have that $\mathcal{D}_E$ is an open set of $\mathbb{S}_E^{n-1} \times \mathbb{R}^n$ and $\mathrm{Mes}((\mathbb{S}_E^{n-1} \times \mathbb{R}^n) \backslash \mathcal{D}_E) = 0$ for the Lebesgue measure on $\mathbb{S}_E^{n-1} \times \mathbb{R}^n$. The map $S_E : \mathcal{D}_E \to \mathbb{S}_E^{n-1} \times \mathbb{R}^n$ given by the formula

$$S_E(v_-, x_-) = (v_+, x_+), \tag{1.9}$$

is called the scattering map at fixed energy $E > 0$ for the equation (1.1). Note that if $V(x) \equiv 0$ and $B(x) \equiv 0$, then $v_+ = v_-,\ x_+ = x_-,\ (v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n,\ v_- \neq 0$. In connection with the relativistic Newton equation, see Section 4.1.

In this paper we consider the following inverse scattering problem at fixed energy:

Problem 1. Given S_E at fixed energy $E > 0$, find (V, B).

Note that using the conservation of energy we obtain that if $E < \sup_{\mathbb{R}^n} V$ then S_E does not determine uniquely V.

We mention results on Problem 1. When $B \equiv 0$ and V is assumed to be spherically symmetric and monotonuous decreasing in $|x|$ (V is not assumed to be short-range), uniqueness results for Problem 1 were obtained in [5, 12]. The scattering map S_E also uniquely determines (V, B) at fixed and sufficiently large energy when (V, B) is assumed to be compactly supported inside a fixed domain of $\mathbb{R}^n$ (see [15] for $B \equiv 0$ and see [9] for the general case). This latter result relies on a uniqueness result for an inverse boundary kinematic problem for equation (1.1) (see [7, 15] when $B \equiv 0$, and see [3, 9] for the general case) and connection between this boundary value problem and the inverse scattering problem on $\mathbb{R}^n$ (see [15] for $B \equiv 0$, and see [9] for the general case).

To our knowledge it is still unknown whether the scattering map at fixed and sufficiently large energy uniquely determines the electromagnetic field under the regularity and short-range conditions (1.4) and (1.5) (see [15, Conjecture B] for $B \equiv 0$).

In this paper we propose, in particular, a generalization of results in [5, 12] for the short-range case where no decreasing monotonicity is assumed. Combining this with results of [7, 15, 3, 9] we obtain the following uniqueness result:

Theorem 1.1. *Let $(\lambda, R) \in (0, +\infty)^2$ and let (V, B) be an electromagnetic field that satisfies the assumptions* (1.2), (1.4) *and* (1.5) *and* $\|(V, B)\| \le \lambda$. *Assume that $B \equiv 0$ outside $\mathcal{B}(0, R)$ and that V is spherical symmetric outside*

$\mathcal{B}(0,R)$. *Then there exists a positive constant* $E(\lambda,R)$ *(which does not depend on* (V,B)*) so that the scattering map at fixed energy* $E > E(\lambda,R)$ *uniquely determines* (V,B) *on* $\mathbb{R}^n$.

The proof of Theorem 1.1 is obtained by recovering first the electric potential in a neighborhood of infinity developing Firsov or Keller-Kay-Shmoys' result [5, 12] and then by recovering the electromagnetic field on $\mathbb{R}^n$ using the following Theorem which generalizes [9, Theorem 7.2].

Theorem 1.2. *Let* $(\lambda,R)\in(0,+\infty)^2$ *and let* (V,B) *be an electromagnetic field that satisfies the assumptions* (1.2)*,* (1.4) *and* (1.5) *and* $\|(V,B)\|\le\lambda$. *Assume that* (V,B) *is known outside* $\mathcal{B}(0,R)$. *Then there exists a positive constant* $E(\lambda,R)$ *so that the scattering map at fixed energy* $E > E(\lambda,R)$ *uniquely determines* (V,B) *on* $\mathbb{R}^n$.

Concerning the inverse scattering problem for the classical multidimensional nonrelativistic Newton equation at high energies and the inverse scattering problem for a particle in electromagnetic field (with $B\not\equiv 0$ or $B\equiv 0$) in quantum mechanics, we refer the reader to [7, 15, 9, 10] and references therein.

Concerning the inverse problem for (1.1) in the one-dimensional case, we can mention the works [1, 11, 2].

The structure of the paper is as follows. In section 2 we prove Theorem 1.2. In section 3 we prove Theorem 1.1. In section 4 we provide similar results (Theorems 4.1 and 4.2) for the relativistic multidimensional Newton equation with electromagnetic field.

2 Proof of Theorem 1.2

2.1 Nontrapped solutions of equation (1.1)

We will use the standard Lemma 2.1 on nontrapped solutions of equation (1.1). For sake of consistency its proof is given in Appendix.

Lemma 2.1. *Let* $E>0$ *and let* R_E *and* C_E *be defined by*

$$C_E := \frac{2E}{(n\beta_1+2\beta_0)(1+\sqrt{2(E+\beta_0)})}, \tag{2.1}$$

$$\sup_{|x|\ge R_E}(1+|x|)^{-\alpha}\le\frac{C_E}{2}. \tag{2.2}$$

If $x(t)$ is a solution of equation (1.1) of energy E such that $|x(0)| < R_E$ and if there exists a time $T > 0$ such that $|x(T)| = R_E$, then

$$|x(t)|^2 \geq R_E^2 + E|t-T|^2 \text{ for } t \in (T, +\infty), \tag{2.3}$$

and there exists a unique $(x_+, v_+) \in \mathbb{R}^n \times \mathbb{S}_E^{n-1}$ so that

$$x(t) = x_+ + tv_+ + y_+(t),\ t \in \mathbb{R},$$

where $|y_+(t)| + |\dot{y}_+(t)| \to 0$ as $t \to +\infty$.

Note that $C_E \to +\infty$ as $E \to +\infty$ while $\sup_{|x|\geq R}(1+|x|)^{-\alpha}$ is a decreasing function of R that goes to 0 as $R \to +\infty$. Note that Lemma 2.1 is stated for positive times t but a similar result holds for negative times t.

2.2 The inverse kinematic problem for equation (1.1)

Let $R > 0$ and let $\overline{\mathcal{B}(0,R)}$ denote the closed Euclidean ball of center 0 and radius R. We first formulate the inverse kinematic problem for equation (1.1) inside $\overline{\mathcal{B}(0,R)}$. For $(m,l) \in (\mathbb{N}\backslash\{0\})^2$ and for a function f from $\overline{\mathcal{B}(0,R)}$ to $\mathbb{R}^m$ of class C^l we define the C^l norm of f by

$$\|f\|_{C^l,R} = \sup_{\substack{x\in\mathcal{B}(0,R),\ \alpha\in\mathbb{N}^n \\ |\alpha|\leq l}} |\partial_x^\alpha f(x)|.$$

We denote by $\partial\mathcal{B}(0,R)$ the boundary of the ball $\mathcal{B}(0,R)$.

Then we recall that there exists a constant $E(R, \|V\|_{C^2,R}, \|B\|_{C^1,R})$ so that at fixed energy $E > E(R, \|V\|_{C^2,R}, \|B\|_{C^1,R})$ the solutions x of equation (1.1) in $\mathcal{B}(0,R)$ at energy E have the following properties (see for example [9]):

$$\begin{aligned} &\text{for each solution } x(t) \text{ there are } t_1, t_2 \in \mathbb{R}, t_1 < t_2, \text{ such that} \\ &x \in C^3([t_1,t_2],\mathbb{R}^n),\ (x(t_1),x(t_2)) \in \partial\mathcal{B}(0,R)^2,\ x(t) \in \mathcal{B}(0,R) \text{ for } t \in]t_1,t_2[, \\ &x(s_1) \neq x(s_2) \text{ for } s_1, s_2 \in [t_1,t_2], s_1 \neq s_2; \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\text{for any two distinct points } q_0, q \in \partial\mathcal{B}(0,R), \text{ there is one and only one solution} \\ &x(t) = x(t,E,q_0,q) \text{ such that } x(0) = q_0, x(s) = q \text{ for some } s > 0. \end{aligned} \tag{2.5}$$

This is closely related to the property that at fixed and sufficiently large energy E, the compact set $\overline{\mathcal{B}(0,R)}$ endowed with the riemannian metric $\sqrt{E-V(x)}|dx|$ and the magnetic field defined by B is simple (see [3]).

For q_0 and q two distinct points of $\partial\mathcal{B}(0,R)$ we denote by $s(E,q_0,q)$ the time at which $x(t,E,q_0,q)$ reaches q from q_0 and we denote by $k_0(E,q_0,q)$ the velocity vector $\dot{x}(0,E,q_0,q)$ and by $k(E,q_0,q)$ the velocity vector $\dot{x}(s(E,q_0,q), E,q_0,q)$. The inverse kinematic problem is then:

Problem 2. Given $k(E,q_0,q),\ k_0(E,q_0,q)$ for all $q_0,q\in\partial\mathcal{B}(0,R)$, $q_0\neq q,$ at fixed sufficiently large energy $E,$ find (V,B) in $\mathcal{B}(0,R)$.

The data $k_0(E,q_0,q),\ k(E,q_0,q),\ q_0,q\in\partial\mathcal{B}(0,R), q_0\neq q$, are the boundary value data of the inverse kinematic problem, and we recall the following result.

Proposition 2.2 (see, for example, Theorem 7.1 in [9])**.** *At fixed* $E> E(\|V\|_{C^2,R},\ \|B\|_{C^1,R},R)$, *the boundary data* $k_0(E,q_0,q)$, $(q_0,q)\in\partial\mathcal{B}(0,R)\times\partial\mathcal{B}(0,R)$, $q_0\neq q$, *uniquely determine* (V,B) *in* $\mathcal{B}(0,R)$.

2.3 Relation between boundary data of the inverse kinematic problem and the scattering map S_E

We will prove that at fixed and sufficiently large energy E the scattering map S_E determines the boundary data $k_0(E,q_0,q),\ k(E,q_0,q),\ q_0,q\in\partial\mathcal{B}(0,R), q_0\neq q$, when (V,B) is known outside $\mathcal{B}(0,R)$. This will prove that S_E uniquely determines (V,B) in $\mathcal{B}(0,R)$, which will prove Theorem 1.2.

Let $R>0$ and $\lambda>0$ be such that (V,B) is known outside $\mathcal{B}(0,R)$ and $\|(V,B)\|<\lambda$. Note that $\max(\|V\|_{C^2,R},\|B\|_{C^1,R})\leq\|(V,B)\|<\lambda$. Thus there exists a constant $E_0(\lambda,R)$ such that at fixed energy $E>E_0(\lambda,R)$ solutions $x(t)$ of equation (1.1) in $\mathcal{B}(0,R)$ at energy E have properties (2.4) and (2.5) and such that at fixed $E>E_0(\lambda,R)$ the boundary data $k_0(E,q_0,q)$, $(q_0,q)\in\partial\mathcal{B}(0,R)\times\partial\mathcal{B}(0,R)$, $q_0\neq q$, uniquely determine (V,B) in $\mathcal{B}(0,R)$. Then using that the constant $C_E\to+\infty$ as $E\to+\infty$ in Lemma 2.1 we obtain that there exists $E_1(\lambda,R)$ such that for $E>E_1(\lambda,R)$ we have $\sup_{|x|\geq R}(1+|x|)^{-\alpha}\leq\frac{C_E}{2}$ so that R_E can be replaced by R in Lemma 2.1.

Set $E(\lambda,R)=\max(E_0(\lambda,R),E_1(\lambda,R))$ and fix $E>E(\lambda,R)$. Let $(v_-,x_-)\in\mathcal{D}_E$ and $(v_+,x_+)=S_E(v_-,x_-)$. We denote by $x(.,x_-,v_-)$ the solution of equation (1.1) that satisfies (1.7) (and (1.8)). Set

$$\begin{aligned} t_-(x_-,v_-) &= \sup\{t\in\mathbb{R}\mid |x(s,x_-,v_-)|\geq R,\ s\in(-\infty,t)\},\\ t_+(x_-,v_-) &= \inf\{t\in\mathbb{R}\mid |x(s,x_-,v_-)|\geq R,\ s\in(t,+\infty)\}. \end{aligned}$$

Since (V,B) is known outside $\mathcal{B}(0,R)$ we can solve equation (1.1) with initial conditions (1.7) and (1.8) and we obtain that $x(.,x_-,v_-)$ is known on $(-\infty,t_-(x_-,v_-)]\cup[t_+(x_-,v_-),\infty)$.

If $x(s,x_-,v_-) \notin \mathcal{B}(0,R)$ for any $s \in \mathbb{R}$, then $t_\pm(x_-,v_-) = \mp\infty$. If there exists $s \in \mathbb{R}$ such that $x(s,x_-,v_-) \in \mathcal{B}(0,R)$ then set

$$q_0 = x(t_-(x_-,v_-),x_-,v_-),\ q = x(t_+(x_-,v_-),x_-,v_-). \tag{2.6}$$

Using Lemma 2.1 and $E > E_1(\lambda,R)$ we obtain that $|x(s,x_-,v_-)| < R$ for $s \in (t_-(x_-,v_-),t_+(x_-,v_-))$ and $q_0 \neq q$. (Note that if $x(t)$ satisfies equation (1.1) then $x(t+t_0)$ also satisfies (1.1) for any $t_0 \in \mathbb{R}$.) Therefore we have $x(s,x_-,v_-) = x(s-t_-(x_-,v_-),E,q_0,q)$ for $s \in (t_-(x_-,v_-),t_+(x_-,v_-))$ where $x(t,E,q_0,q)$ is the solution of (1.1) given by (2.5), and we have

$$k_0(E,q_0,q) = \dot{x}(t_-(x_-,v_-),x_-,v_-),\ k(E,q_0,q) = \dot{x}(t_+(x_-,v_-),x_-,v_-). \tag{2.7}$$

We proved that the scattering map S_E uniquely determines the data $k_0(E,q_0,q)$, $k(E,q_0,q)$, $(q_0,q) \in \partial\mathcal{B}(0,R)^2, q_0 \neq q$, when $(q_0,q) = (x(t_-(x_-,v_-),x_-,v_-),$ $x(t_+(x_-,v_-),x_-,v_-))$ for $(x_-,v_-) \in \mathcal{D}_E$. And using again Lemma 2.1 we know that for any $(q_0,q) \in \partial\mathcal{B}(0,R)^2$, $q_0 \neq q$, the solution $x(t,E,q_0,q)$ given by (2.5) satisfies (1.7) and (1.8) for some $(x_\pm,v_\pm) \in \mathbb{R}^n \times \mathbb{S}^{n-1}_E$ and that $|x(t,E,q_0,q)| > R$ for $t < 0$ and $t > s(E,q_0,q)$. Thus S_E uniquely determines the data $k_0(E,q_0,q)$, $k(E,q_0,q)$, $(q_0,q) \in \partial\mathcal{B}(0,R)^2, q_0 \neq q$. $\square$

3 Proof of Theorem 1.1

In this section we assume that the electromagnetic field (V,B) in equation (1.1) satisfies (1.4) and (1.5) and is so that $B \equiv 0$ and V is spherically symmetric outside $\mathcal{B}(0,R)$ for some $R > 0$. Let $W \in C^2([R,+\infty),\mathbb{R})$ be defined by $V(x) = W(|x|)$ for $x \notin \mathcal{B}(0,R)$. From (1.4) it follows that

$$\sup_{r>R}(1+r)^\alpha|W(r)| \le \beta_0 \text{ and } \sup_{r>R}(1+r)^{\alpha+1}|W'(r)| \le \beta_1, \tag{3.1}$$

where W' denotes the derivative of W.

3.1 The function $r_{\min,.}$

Set

$$\beta := (2E+2\beta_0)^{\frac{1}{2}} \max\big(R, (\frac{\beta_1+2\beta_0}{2E})^{\frac{1}{\alpha}}\big). \tag{3.2}$$

Then for $q \ge \beta$ consider the real number $r_{\min,q}$ defined by

$$r_{\min,q} = \sup\{r \in (R,+\infty) \mid W(r) + \frac{q^2}{2r^2} = E\}. \tag{3.3}$$

The function $r_{\min,.}$ has the following properties.

Lemma 3.1. *The function $r_{\min,.}$ is a C^2 strictly increasing function from $[\beta,+\infty)$ to $(R,+\infty)$ and we have $r^3_{\min,q}W'(r_{\min,q}) < q^2$ for $q \geq \beta$ and*

$$W(r_{\min,q}) + \frac{q^2}{2r^2_{\min,q}} = E,\ \frac{dr_{\min,q}}{dq} = \frac{qr_{\min,q}}{q^2 - r^3_{\min,q}W'(r_{\min,q})} > 0. \tag{3.4}$$

In addition the following estimates and asymptotics at $+\infty$ hold

$$\frac{q}{\sqrt{2E+2\beta_0}} \leq r_{\min,q} \leq \frac{q}{\left(2E - 2\beta_0 q^{-\alpha}(2\beta_0+2E)^{\frac{\alpha}{2}}\right)^{\frac{1}{2}}},\quad \textit{for}\ \ q \in [\beta,+\infty), \tag{3.5}$$

$$r_{\min,q} = \frac{q}{\sqrt{2E}} + O(q^{1-\alpha}),\ \textit{as}\ q \to +\infty. \tag{3.6}$$

Lemma 3.1 and works [5, 12] allow the reconstruction of the force F in a neighborhood of infinity. In sections 3.2 and 3.3 we develop the reconstruction procedure given in [5, 12].

3.2 The scattering angle

Let $\mathcal{P}$ be a plane of $\mathbb{R}^n$ containing 0 and let (e_1, e_2) be an orthonormal basis of $\mathcal{P}$. For $(v_1, v_2) \in \mathbb{R}^2$ and for $v = v_1e_1 + v_2e_2$ we define $v^\perp \in \mathcal{P}$ by $v^\perp = -v_2e_1 + v_1e_2$.

Let $q \geq \beta$. Then set $x_- := (2E)^{-\frac{1}{2}}qe_1$ and $v_- = \sqrt{2E}e_2$. We have $x_-\cdot v_- = 0$ and $x_- \cdot v_-^\perp = -q$, and for such couple (x_-, v_-) we consider $x_q(t)$ the solution of (1.1) with energy E and with initial conditions (1.7) at $t \to -\infty$. Let $t_- := \sup\{t \in \mathbb{R} \mid |x_q(s)| \geq R \text{ for } s \in (-\infty, t)\}$. We will prove that $t_- = +\infty$. Since the force F in (1.1) is radial outside $\mathcal{B}(0,R)$ we obtain that $x_q(t) \in \mathcal{P}$ for $t \in (-\infty, t_-)$.

We introduce polar coordinates in $\mathcal{P}$. We write $x_q(t) = r_q(t)(\cos(\theta_q(t))e_1 + \sin(\theta_q(t))e_2)$ for $t \in (-\infty, t_-)$ where the functions r_q, θ_q, satisfy the following ordinary differential equations

$$\ddot{r}_q(t) \ = \ -W'(r_q(t)) + \frac{q'^2}{r_q(t)^3}, \tag{3.7}$$

$$r_q(t)^2\dot{\theta}_q(t) \ = \ q',\ \text{for some } q' \in \mathbb{R}. \tag{3.8}$$

Asymptotic analysis of $x_q(t)$ at $t = -\infty$ using the initial conditions (1.7) and $\dot{y}_-(t) = o(t^{-1})$ as $t \to -\infty$ (see for example [15, Theorem 3.1] for this latter property) shows that $q' = q$. We refer the reader to the Appendix for details.

The energy E defined by (1.3) is then written as follows

$$2E = \dot{r}_q(t)^2 + \frac{q^2}{r_q(t)^2} + 2W(r_q(t)). \tag{3.9}$$

Let $t_q = \inf\{t \in (-\infty, t_-) \mid r_q(t) = r_{\min,q}\}$. Then using (3.3) and (3.9) we have $\dot{r}_q(t_q) = 0$. Since r_q satisfies the second order differential equation (3.7) we obtain that $t_- = +\infty$, $r_q(t_q + t) = r_q(t_q - t)$ for $t \in \mathbb{R}$, and $\pm \dot{r}_q(t) > 0$ for $\pm t > t_q$. We thus define

$$g(q) = \int_{-\infty}^{+\infty} \frac{dt}{r_q(t)^2} = 2\int_{t_q}^{+\infty} \frac{dt}{r_q(t)^2}. \tag{3.10}$$

The integral (3.10) is absolutely convergent and from (3.8) it follows that $qg(q) = \int_{\mathbb{R}} \dot{\theta}_q(s)ds$ is the scattering angle of $x_q(t)$, $t \in \mathbb{R}$.

Note also that from (3.8) we have

$$S_{1,E}(\sqrt{2E}e_2, (2E)^{-\frac{1}{2}}qe_1) = \sqrt{2E}\Big(\cos\big(qg(q) - \frac{\pi}{2}\big)e_1 + \sin\big(qg(q) - \frac{\pi}{2}\big)e_2\Big), \tag{3.11}$$

for $q \in [\beta, +\infty)$ and where $S_{1,E}$ is the first component of the scattering map S_E. Note that $qg(q) \to \pi$ as $q \to +\infty$ and that g is continuous on $[\beta, +\infty)$ (these properties can be proven by using [15, Theorem 3.1] and continuity of the flow of (1.1)). Hence using (3.11) we obtain that $S_{1,E}$ uniquely determines the function g.

3.3 Spherically symmetric reconstruction formulas

Let χ be the strictly increasing function from $[0, \beta^{-2})$ to $[0, r_{\min,\beta}^{-1})$, continuous on $[0, \beta^{-2})$ and C^2 on $(0, \beta^{-2})$, defined by

$$\chi(0) = 0, \quad \text{and} \quad \chi(\sigma) = r^{-1}_{\min,\sigma^{-\frac{1}{2}}}, \quad \text{for} \quad \sigma \in (0, \beta^{-2}). \tag{3.12}$$

Let $\phi : (0, \chi(\beta^{-2})) \to (0, \beta^{-2})$ denote the inverse function of χ. From (3.4) and (3.12) it follows that

$$2(E - W(\chi(u)^{-1})) = \frac{\chi(u)^2}{u}, \text{ for } u \in (0, \beta^{-2}), \tag{3.13}$$

$$2(E - W(s^{-1}))\phi(s) = s^2, \text{ for } s \in (0, \chi(\beta^{-2})). \tag{3.14}$$

Define the function H from $(0, \beta^{-2})$ to $\mathbb{R}$ by

$$H(\sigma) := \int_0^{\sigma} \frac{g(u^{-\frac{1}{2}})du}{2\sqrt{u}\sqrt{\sigma - u}} \quad \text{for} \quad \sigma \in (0, \beta^{-2}), \tag{3.15}$$

Hence H is known from the first component of the scattering map S_E.

The following formulas are valid (see Appendix for more details)

$$H(\sigma) = \pi \int_0^{\chi(\sigma)} \frac{ds}{\sqrt{2(E - W(s^{-1}))}}, \quad \frac{1}{\pi\sqrt{\sigma}} \frac{dH}{d\sigma}(\sigma) = \frac{d}{d\sigma}\ln(\chi(\sigma)), \tag{3.16}$$

for $\sigma \in (0, \beta^{-2})$.

Then note that from (3.6) it follows that $\chi(\sigma) = \left((2E)^{\frac{1}{2}}\sigma^{-\frac{1}{2}} + O(\sigma^{\frac{-1+\alpha}{2}})\right)^{-1} = (2E)^{-\frac{1}{2}}\sigma^{\frac{1}{2}} + O(\sigma^{\frac{\alpha+1}{2}})$ as $\sigma \to 0^+$, and $\ln\left((2E)^{\frac{1}{2}}\chi(\sigma)\sigma^{-\frac{1}{2}}\right) = \ln(1+O(\sigma^{\frac{\alpha}{2}})) \to 0$ as $\sigma \to 0^+$ (note that we just need the assumption $\alpha > 0$). Therefore we obtain the following reconstruction formulas

$$\begin{aligned} \chi(\sigma) &= (2E)^{-\frac{1}{2}}\sigma^{\frac{1}{2}} e^{\int_0^\sigma \left(\frac{1}{\pi\sqrt{s}}\frac{dH}{ds}(s) - \frac{1}{2s}\right)ds} \text{ for } \sigma \in (0, \beta^{-2}), && (3.17)\\ W(s) &= E - \frac{1}{2s^2\phi(s^{-1})} \text{ for } s \in (r_{\min,\beta}, +\infty). && (3.18) \end{aligned}$$

Set

$$\beta' = \frac{\beta}{\left(2E - 2\beta_0\beta^{-\alpha}(2\beta_0 + 2E)^{\frac{\alpha}{2}}\right)^{\frac{1}{2}}}. \quad (3.19)$$

Then note that from (3.5) and (3.2), it follows that

$$r_{\min,\beta} \leq \beta'. \quad (3.20)$$

Therefore from (3.17) and (3.18) we obtain that W is determined by the first component of the scattering map S_E on $(\beta', +\infty)$.

The proof of Theorem 1.1 then relies on this latter statement and on Theorem 1.2. □

4 The relativistic multidimensional Newton equation

4.1 Uniqueness results

Let $c > 0$. Consider the relativistic multidimensional Newton equation in an electromagnetic field

$$\begin{aligned} \dot p &= F(x, \dot x) := -\nabla V(x) + \frac{1}{c}B(x)\dot x, && (4.1)\\ p &= \frac{\dot x}{\sqrt{1 - \frac{|\dot x|^2}{c^2}}},\ \dot p = \frac{dp}{dt},\ \dot x = \frac{dx}{dt},\ x \in C^2(\mathbb{R}, \mathbb{R}^n), \end{aligned}$$

where (V, B) satisfies (1.2), (1.4) and (1.5). The equation (4.1) is an equation for $x = x(t)$ and is the equation of motion in $\mathbb{R}^n$ of a relativistic particle of mass $m = 1$ and charge $e = 1$ in an external static electromagnetic field described by the electric potential V and the magnetic field B (see [4] and, for example, [13, Section 17]). In this equation x is the position of the particle, p is its impulse, F is the force acting on the particle, t is the time and c is the speed of light.

For the equation (4.1) the energy

$$E = c^2\sqrt{1+\frac{|p(t)|^2}{c^2}} + V(x(t)) = \frac{c^2}{\sqrt{1-\frac{|\dot{x}(t)|^2}{c^2}}} + V(x(t)), \tag{4.2}$$

is an integral of motion.

Under the conditions (1.4)–(1.5), we have the following properties (see [17]): for any $(v_-, x_-) \in \mathcal{B}(0,c)\times\mathbb{R}^n,\ v_-\neq 0$, the equation (4.1) has a unique solution $x\in C^2(\mathbb{R},\mathbb{R}^n)$ that satisfies (1.7) where y_- in (1.7) satisfies $|\dot{y}_-(t)|+|y_-(t)|\to 0$, as $t\to-\infty$; in addition for almost any $(v_-,x_-)\in\mathcal{B}(0,c)\times\mathbb{R}^n,\ v_-\neq 0$, the unique solution $x(t)$ of equation (1.1) that satisfies (1.7) also satisfies the asymptotics

$$x(t) = tv_+ + x_+ + y_+(t), \tag{4.3}$$

where $v_+\neq 0$, $|\dot{y}_+(t)|+|y_+(t)|\to 0$, as $t\to+\infty$. At fixed energy $E>c^2$, we denote by $\mathbb{S}^{n-1}_{E,c}$ the set $\{v_-\in\mathbb{R}^n \mid |v_-| = c\sqrt{1-\frac{c^4}{E^2}}\}$ and we denote by $\mathcal{D}^{\mathrm{rel}}_E$ the set of $(v_-,x_-)\in\mathbb{S}^{n-1}_{E,c}\times\mathbb{R}^n$ for which the unique solution $x(t)$ of equation (4.1) that satisfies (1.7) has also an asymptotics (4.3). We have that $\mathcal{D}^{\mathrm{rel}}_E$ is an open set of $\mathbb{S}^{n-1}_{E,c}\times\mathbb{R}^n$ and $\mathrm{Mes}((\mathbb{S}^{n-1}_{E,c}\times\mathbb{R}^n)\backslash\mathcal{D}^{\mathrm{rel}}_E)=0$ for the Lebesgue measure on $\mathbb{S}^{n-1}_{E,c}\times\mathbb{R}^n$. The map $S^{\mathrm{rel}}_E:\mathcal{D}^{\mathrm{rel}}_E\to\mathbb{S}^{n-1}_{E,c}\times\mathbb{R}^n$ given by $S^{\mathrm{rel}}_E(v_-,x_-)=(v_+,x_+)$, is called the scattering map at fixed energy $E>c^2$ for the equation (4.1). Note that if $V(x)\equiv 0$ and $B(x)\equiv 0$, then $v_+=v_-,\ x_+=x_-,\ (v_-,x_-)\in\mathcal{B}(0,c)\times\mathbb{R}^n,\ v_-\neq 0$.

We consider the inverse scattering problem at fixed energy for equation (4.1) that is similar to Problem 1:

Problem 3. Given S^{rel}_E at fixed energy $E>c^2$, find (V,B). (4.4)

Note that using the conservation of energy we obtain that if $E<c^2+\sup_{\mathbb{R}^n}V$ then S^{rel}_E does not determine uniquely V.

For Problem 3, we obtain the following uniqueness result that is the analog of Theorem 1.1:

Theorem 4.1. *Let* $(\lambda,R)\in(0,+\infty)^2$ *and let* (V,B) *be an electromagnetic field that satisfies the assumptions* (1.2), (1.4) *and* (1.5) *and* $\|(V,B)\|\leq\lambda$. *Assume that* $B\equiv 0$ *outside* $\mathcal{B}(0,R)$ *and that* V *is spherical symmetric* *outside $\mathcal{B}(0,R)$. Then there exists a positive constant $E^{\mathrm{rel}}(\lambda,R)>c^2$ (which does not depend on (V,B)) so that the scattering map S_E^{rel} at fixed energy $E>E^{\mathrm{rel}}(\lambda,R)$ uniquely determines (V,B) on $\mathbb{R}^n$.*

In Section 4.3 we sketch the proof of Theorem 4.1. It is based on the reconstruction of the electric potential in a neighborhood of infinity developing a relativistic version of Firsov or Keller-Kay-Shmoys' result [5, 12] and on the following analog of Theorem 1.2.

Theorem 4.2. *Let $(\lambda,R)\in(0,+\infty)^2$ and let (V,B) be an electromagnetic field that satisfies the assumptions (1.2), (1.4) and (1.5) and $\|(V,B)\|\le\lambda$. Assume that (V,B) is known outside $\mathcal{B}(0,R)$. Then there exists a positive constant $E^{\mathrm{rel}}(\lambda,R)>c^2$ so that the scattering map S_E^{rel} at fixed energy $E>E^{\mathrm{rel}}(\lambda,R)$ uniquely determines (V,B) on $\mathbb{R}^n$.*

In Section 4.2 we sketch the proof of Theorem 4.2.

For inverse scattering at high energies for the relativistic multidimensional Newton equation and inverse scattering in relativistic quantum mechanics see [8] and references therein.

Concerning the inverse problem for (4.1) in the one-dimensional case, we can mention the work [6].

4.2 Proof of Theorem 4.2

We first consider the analog of Lemma 2.1.

Lemma 4.3. *Let $E>c^2$ and let R_E and C_E^{rel} be defined by*

$$C_E^{\mathrm{rel}}:=\min\Big(\frac{E-c^2}{2\beta_0},\frac{c^2\big(\big(\frac{E-c^2}{4c^2}+1\big)^2-1\big)}{4\beta_1 n\big(\frac{3(E-c^2)}{2c^2}+1\big)^2}\Big),\ \sup_{|x|\ge R_E}(1+|x|)^{-\alpha}\le\frac{C_E^{\mathrm{rel}}}{2}.\quad(4.5)$$

If $x(t)$ is a solution of equation (1.1) of energy E such that $|x(0)|<R_E$ and if there exists a time $T>0$ such that $|x(T)|=R_E$ then

$$|x(t)|^2\ge R_E^2+\frac{1}{2}\frac{c^2\big(\big(\frac{E-c^2}{4c^2}+1\big)^2-1\big)}{\big(\frac{3(E-c^2)}{2c^2}+1\big)^2}|t-T|^2 \text{ for } t\in(T,+\infty),\qquad(4.6)$$

and there exists a unique $(x_+,v_+)\in\mathbb{R}^n\times\mathbb{S}^{n-1}_{E,c}$ so that $x(t)=x_++tv_++y_+(t)$, $t\in\mathbb{R}$, where $|y_+(t)|+|\dot y_+(t)|\to0$ as $t\to+\infty$.

The proof of Lemma 4.3 is similar to the proof of Lemma 2.1.

The solutions $x(t)$ of equation (4.1) in $\mathcal{B}(0,R)$ for some $R>0$ also have properties (2.4) and (2.5) at fixed and sufficiently large energy. Therefore at fixed and sufficiently large energy we consider the inverse kinematic problem in a ball $\mathcal{B}(0,R)$ for equation (4.1) similar to the inverse kinematic problem given in Section 2.2. Then Proposition 2.2 still holds (see [9, Theorem 1.2]) and the connection between boundary data of the inverse kinematic problem and the scattering map $S_E^{\rm rel}$ is similar to the one given for the nonrelativistic case in Section 2.3 (note that the radius R has also to be chosen so that $2\sup_{|x|\geq R}(1+|x|)^{-\alpha} < c^2/(144\beta_1 n) = \lim_{E\to+\infty} C_E^{\rm rel}$). This proves Theorem 4.2. $\square$

4.3 Proof of Theorem 4.1

We assume that the electromagnetic field (V,B) in equation (4.1) satisfies (1.4) and (1.5) and is so that $B\equiv 0$ and V is spherically symmetric outside $\mathcal{B}(0,R)$ for some $R>0$. Let $W\in C^2([R,+\infty),\mathbb{R})$ be defined by $V(x)=W(|x|)$ for $x\notin\mathcal{B}(0,R)$. We give the analog of Lemma 3.1. Let $\tilde\beta = \Big(E(2\beta_0+\beta_1)+\beta_1\beta_0-\Big((E(2\beta_0+\beta_1)+\beta_1\beta_0)^2-4\beta_0^2(E^2-c^4)\Big)^{\frac12}\Big)^{-\frac1\alpha}\frac{c(2\beta_0^2)^{\frac1\alpha}}{E}\big((E+\beta_0)^2-c^4\big)^{\frac12}$ and set

$$\beta := \max\left(\tilde\beta, \left(\frac{\beta_0}{E-c^2}\right)^{\frac1\alpha}E^{-1}c\sqrt{(E+\beta_0)^2-c^4}, \frac{cR\sqrt{(E+\beta_0)^2-c^4}}{E}\right). \tag{4.7}$$

Then for $q\geq\beta$ consider the real number $r_{\min,q}$ defined by

$$r_{\min,q} = \sup\{r\in(R,+\infty)\mid (E-W(r))^2-c^4-\frac{q^2E^2}{c^2r^2}=0\}. \tag{4.8}$$

The function $r_{\min,.}$ has the following properties.

Lemma 4.4. *The function $r_{\min,.}$ is a C^2 strictly increasing function from $[\beta,+\infty)$ to $(R,+\infty)$ and we have $r_{\min,q}^3W'(r_{\min,q})\frac{c^2(E-W(r_{\min,q}))}{E^2} < q^2$ for $q\geq\beta$ and*

$$(E-W(r_{\min,q}))^2-c^4-\frac{q^2E^2}{c^2r_{\min,q}^2}=0, \tag{4.9}$$

$$\frac{dr_{\min,q}}{dq} = \frac{E^2qr_{\min,q}}{-c^2(E-W(r_{\min,q}))r_{\min,q}^3W'(r_{\min,q})+q^2E^2}>0. \tag{4.10}$$

In addition the following estimates and asymptotics at $+\infty$ hold

$$\frac{qE}{c\sqrt{(E+\beta_0)^2-c^4}} \leq r_{\min,q} \leq \frac{Eq}{c\sqrt{\Big(E-\beta_0\big(\frac{qE}{c\sqrt{(E+\beta_0)^2-c^4}}\big)^{-\alpha}\Big)^2-c^4}}, \tag{4.11}$$

for $q\geq\beta$, and

$$r_{\min,q}=\frac{qE}{c\sqrt{E^2-c^4}}+O(q^{1-\alpha}),\ \textit{as } q\to+\infty. \tag{4.12}$$

Then take any plane $\mathcal{P}$ containing 0 and keep notations of Section 3.2. Let $q\geq\beta$. Then set $x_-:=(c\sqrt{1-c^4/E^2})^{-1}qe_1$ and $v_-=c\sqrt{1-c^4/E^2}e_2$, and consider $x_q(t)$ the solution of (4.1) with energy E and with initial conditions (1.7) at $t\to-\infty$. We write x_q in polar coordinates: $x_q(t)=r_q(t)(\cos(\theta_q(t))e_1+\sin(\theta_q(t))e_2)$ for $t\in(-\infty,t_-)$ where $t_-:=\sup\{t\in\mathbb{R}\mid |x_q(s)|\geq R \text{ for } s\in(-\infty,t)\}$ and the functions r_q, θ_q, satisfy

$$\ddot r_q(t)=\frac{-W'(r_q(t))}{\left(\frac{E-W(r_q(t))}{c^2}\right)^3}+q^2E^2\frac{E-W(r_q(t))-r_q(t)W'(r_q(t))}{r_q(t)^3(E-W(r_q(t)))^3}, \tag{4.13}$$

$$\frac{r_q(t)^2\dot\theta_q(t)}{\sqrt{1-\frac{\dot r_q(t)^2+r_q(t)^2\dot\theta_q(t)^2}{c^2}}}=\frac{qE}{c^2}. \tag{4.14}$$

The energy E defined by (4.2) is then written as follows

$$1-\frac{\dot r_q(t)^2}{c^2}-\frac{c^4}{(E-W(r_q(t)))^2}-\frac{q^2E^2}{c^2r_q(t)^2(E-W(r_q(t)))^2}=0. \tag{4.15}$$

We also have

$$\dot\theta_q(t)=\frac{qE}{r_q(t)^2(E-W(r_q(t)))}. \tag{4.16}$$

Similarly to Section 3.2 we have $t_-=+\infty$, $r_q(t_q+t)=r_q(t_q-t)$ for $t\in\mathbb{R}$, and $\pm\dot r_q(t)>0$ for $\pm t>t_q$ where $t_q=\inf\{t\in(-\infty,t_-)\mid r_q(t)=r_{\min,q}\}$.

We thus define

$$g(q)=\int_{-\infty}^{+\infty}\frac{dt}{r_q(t)^2(E-W(r_q(t)))}=2\int_{t_q}^{+\infty}\frac{dt}{r_q(t)^2(E-W(r_q(t)))}. \tag{4.17}$$

From (4.16) $Eqg(q)=\int_{\mathbb{R}}\dot\theta_q(s)ds$ is the scattering angle of $x_q(t)$, $t\in\mathbb{R}$, and

$$S^{\rm rel}_{1,E}(v_-,x_-)=c\sqrt{1-c^4/E^2}\Big(\cos\big(Eqg(q)-\frac{\pi}{2}\big)e_1+\sin\big(Eqg(q)-\frac{\pi}{2}\big)e_2\Big), \tag{4.18}$$

for $q\in[\beta,+\infty)$ and where $S^{\rm rel}_{1,E}$ is the first component of the scattering map $S^{\rm rel}_E$. Since $Eqg(q)\to\pi$ as $q\to+\infty$ and that g is continuous on $[\beta,+\infty)$ $S^{\rm rel}_{1,E}$ uniquely determines the function g.

We now provide the reconstruction formulas for W from g in a neighborhood of infinity . Let χ be the strictly increasing function from $[0,\beta^{-2})$

to $[0, r_{\min,\beta}^{-1})$, continuous on $[0,\beta^{-2})$ and C^2 on $(0,\beta^{-2})$, defined by $\chi(0)=0$ and $\chi(\sigma)=r^{-1}_{\min,\sigma^{-\frac{1}{2}}}$ for $\sigma\in(0,\beta^{-2})$. Let $\phi:(0,\chi(\beta^{-2}))\to(0,\beta^{-2})$ denote the inverse function of χ. From (4.9) it follows that $(E-W(\chi(u)^{-1}))^2-c^4-\frac{E^2\chi(u)^2}{c^2u}=0$ for $u\in(0,\beta^{-2})$, and $\big((E-W(s^{-1}))^2-c^4\big)\phi(s)-\frac{E^2s^2}{c^2}=0$ for $s\in(0,\chi(\beta^{-2}))$.

Define the function H from $(0,\beta^{-2})$ to $\mathbb{R}$ by (3.15). The following formulas are valid

$$H(\sigma)=\frac{\pi}{c}\int_0^{\chi(\sigma)}\frac{ds}{\sqrt{(E-W(s^{-1}))^2-c^4}},\ \ \frac{E}{\pi\sqrt{\sigma}}\frac{dH}{d\sigma}(\sigma)=\frac{d}{d\sigma}\ln(\chi(\sigma)),\tag{4.19}$$

for $\sigma\in(0,\beta^{-2})$. The proof of formulas (4.19) is similar to the proof of formulas (3.16).

Then note that from (4.12) it follows that $\chi(\sigma)=\Big(\frac{E}{c\sqrt{E^2-c^4}}\sigma^{-\frac{1}{2}}+O(\sigma^{\frac{-1+\alpha}{2}})\Big)^{-1}$ $=\frac{c\sqrt{E^2-c^4}}{E}\sigma^{\frac{1}{2}}+O(\sigma^{\frac{\alpha+1}{2}})$, $\sigma\to0^+$, and

$$\ln\Big(\frac{E\chi(\sigma)}{c\sqrt{E^2-c^4}\sigma^{\frac{1}{2}}}\Big)=\ln(1+O(\sigma^{\frac{\alpha}{2}}))\to0,\ \text{ as }\sigma\to0^+.\tag{4.20}$$

Therefore we obtain the following reconstruction formulas

$$\chi(\sigma)\ =\ \frac{c\sqrt{E^2-c^4}}{E}\sigma^{\frac{1}{2}}e^{\int_0^\sigma\left(\frac{E}{\pi\sqrt{s}}\frac{dH}{ds}(s)-\frac{1}{2s}\right)ds},\ \text{ for }\sigma\in(0,\beta^{-2}),\tag{4.21}$$

$$W(s)\ =\ E-\left(c^4+\frac{E^2}{c^2s^2\phi(s^{-1})}\right)^{\frac{1}{2}},\ \text{ for }s\in(r_{\min,\beta},+\infty).\tag{4.22}$$

Set $\beta'=\frac{E\beta}{c\Big(\big(E-\beta_0\beta^{-\alpha}E^{-\alpha}c^{\alpha}((E+\beta_0)^2-c^4)^{\frac{\alpha}{2}}\big)^2-c^4\Big)^{\frac{1}{2}}}$. Then note that from (4.7) and (4.11) it follows that $r_{\min,\beta}\le\beta'$. Therefore using (4.21) and (4.22) we obtain that W is determined by the first component of the scattering map $S_E^{\rm rel}$ on $(\beta',+\infty)$.

The proof of Theorem 4.1 then relies on this latter statement and on Theorem 4.2. $\square$

A Proof of Lemmas 2.1 and 3.1

In this Section we give a proof of Lemmas 2.1 and 3.1, and we give details on the derivation of formulas (3.16) and the equality "$q=q'$" in Section 3.2.

Proof of Lemma 2.1. We will use the following estimate. Under conditions (1.4) and (1.5) we have

$$|F(x,v)| \leq \beta_1 n(1+|x|)^{-(\alpha+1)}(1+|v|), \text{ for } (x,v)\in\mathbb{R}^n\times\mathbb{R}^n. \quad (A.1)$$

Let

$$I(t)=\frac{1}{2}|x(t)|^2. \quad (A.2)$$

Then using the conservation of energy and equation (1.1) we have

$$\dot I(t) = x(t)\cdot\dot x(t), \quad (A.3)$$

$$\ddot I(t) = 2E-2V(x(t))-x(t)\cdot F(x(t),\dot x(t)), \quad (A.4)$$

for $t\in\mathbb{R}$. Using (A.1) and estimate on V and $|\dot x|(t)=\sqrt{2(E-V(x(t)))}\leq \sqrt{2(E+\beta_0)}$ for $t\in\mathbb{R}$, we obtain that

$$\begin{aligned}\ddot I(t) &\geq 2E-2\beta_0(1+|x(t)|)^{-\alpha}-n\beta_1(1+\sqrt{2(E+\beta_0)})|x(t)|(1+|x(t)|)^{-\alpha-1}\\ &\geq (2\beta_0+n\beta_1)(1+\sqrt{2(E+\beta_0)})(C_E-(1+|x(t)|)^{-\alpha}), \quad (A.5)\end{aligned}$$

for $t\in\mathbb{R}$. Hence we have

$$\ddot I(t)\geq(2\beta_0+n\beta_1)(1+\sqrt{2(E+\beta_0)})\frac{C_E}{2}=E \text{ whenever } |x(t)|\geq R_E. \quad (A.6)$$

Let $t_0=\inf\{t\in[0,T]\mid |x(t)|\geq R_E\}$. Then using (A.2) we have

$$\dot I(t_0)=\lim_{h\to 0^+}\frac{I(t_0)-I(t_0-h)}{h}\geq 0. \quad (A.7)$$

Combining (A.6) and (A.7) we obtain that $|x(t)|\geq R_E$, $\dot I(t)\geq 0$, $\ddot I(t)\geq E$ for $t\in[t_0,+\infty)$. Moreover we have

$$I(t)=I(t_0)+\dot I(t_0)(t-t_0)+\int_{t_0}^t(s-t_0)\ddot I(s)ds\geq\frac{1}{2}R_E^2+\frac{E}{2}(t-t_0)^2, \quad (A.8)$$

for $t\in[t_0,+\infty)$. This proves (2.3). Then using that $|\dot x(t)|\leq\sqrt{2(E+\beta_0)}$ for $t\in\mathbb{R}$ and using (2.3) we have

$$|F(x(\tau),\dot x(\tau))|\leq n\beta_1(1+\sqrt{E}|\tau-T|)^{-(\alpha+1)}(1+\sqrt{2(E+\beta_0)}), \quad (A.9)$$

for $\tau\in[T,+\infty[$. Equation (1.1) then gives

$$x(t)=x_+ + tv_+ + y_+(t), \quad (A.10)$$

for $t \in (0,+\infty)$, where

$$v_+ = \dot{x}(0) + \int_0^{+\infty} F(x(\tau),\dot{x}(\tau))d\tau, \qquad (A.11)$$

$$x_+ = x(0) - \int_0^{+\infty}\int_\sigma^{+\infty} F(x(\tau),\dot{x}(\tau))d\tau d\sigma, \qquad (A.12)$$

$$y_+(t) = \int_t^{+\infty}\int_\sigma^{+\infty} F(x(\tau),\dot{x}(\tau))d\tau d\sigma, \qquad (A.13)$$

for $t \in (0,+\infty)$, where by (A.9) the integrals in (A.11), (A.12) and (A.13) are absolutely convergent ($\alpha > 1$) and $|y_+(t)| + |\dot{y}_+(t)| \to 0$ as $t \to +\infty$. $\square$

Proof Lemma 3.1. Note that for $q \geq \beta$ we have $q^2/(2R^2) > E+\beta_0(1+R)^{-\alpha} \geq E - W(R)$ and $\lim_{r\to+\infty} q^2/(2r^2) = 0 < E = \lim_{r\to+\infty}(E - W(r))$. Hence using (3.3) we obtain $r_{\min,q} \in (R,+\infty)$ and

$$W(r_{\min,q}) + \frac{q^2}{2r_{\min,q}^2} = E, \qquad (A.14)$$

for $q \in [\beta,+\infty)$.

Let $q \in [\beta,+\infty)$. From (A.14) it follows that $q^2/(2r_{\min,q}^2) \leq E + \beta_0$ and then $r_{\min,q} \geq q/\sqrt{2(E+\beta_0)}$. Combining this latter estimate and (A.14) we obtain that

$$2\Big(E - \sup_{r\in\big(\frac{q}{\sqrt{2(E+\beta_0)}},+\infty\big)} W(r)\Big) \leq \frac{q^2}{r_{\min,q}^2} \leq 2\Big(E - \inf_{r\in\big(\frac{q}{\sqrt{2(E+\beta_0)}},+\infty\big)} W(r)\Big). \qquad (A.15)$$

Then using again (3.1) we obtain

$$\sup_{r\in\big(\frac{q}{\sqrt{2(E+\beta_0)}},+\infty\big)} |W(r)| \leq \beta_0 \left(\frac{q}{\sqrt{2(E+\beta_0)}}\right)^{-\alpha}. \qquad (A.16)$$

Combining (A.15) and (A.16) we obtain

$$\frac{q}{\sqrt{2E + 2\beta_0\frac{(2(E+\beta_0))^{\frac{\alpha}{2}}}{q^\alpha}}} \leq r_{\min,q} \leq \frac{q}{\sqrt{2E - 2\beta_0\frac{(2(E+\beta_0))^{\frac{\alpha}{2}}}{q^\alpha}}}. \qquad (A.17)$$

Estimates (3.5) and the asymptotics (3.6) follows from (A.17).

Note that using (3.1) we have

$$rW'(r) < \beta_1 r^{-\alpha} \leq \beta_1 q^{-\alpha}(2(E+\beta_0))^{\frac{\alpha}{2}}, \qquad (A.18)$$

$$2E - 2W(r) > 2E - 2\beta_0 r^{-\alpha} \geq 2E - 2\beta_0 q^{-\alpha}(2(E+\beta_0))^{\frac{\alpha}{2}}, \qquad (A.19)$$

for $r \geq q(2(E+\beta_0))^{-\frac{1}{2}}$ and $q \geq \beta$. From (A.18) and (A.19) we obtain

$$rW'(r) < 2E - 2W(r), \tag{A.20}$$

for $r \geq q(2(E+\beta_0))^{-\frac{1}{2}}$ and $q \geq \beta$. Then consider the function $f \in C^2([\beta,+\infty)\times(R,+\infty),\mathbb{R})$ defined by $f(q,r) = 2E - 2W(r) - \frac{q^2}{r^2}$ for $(q,r) \in [\beta,+\infty)\times(R,+\infty)$ $(\frac{\partial f}{\partial r}(q,r) = -2W'(r) + 2q^2/r^3)$. We have $f(q, r_{\min,q}) = 0$ for $q \geq \beta$ and from the implicit function theorem and (A.20) it follows that $r_{\min,.}$ is a C^2 strictly increasing function from $[\beta,+\infty)$ to $(R,+\infty)$ so that $r_{\min,q}^3 W'(r_{\min,q}) < q^2$ for $q \geq \beta$ and the derivative of $r_{\min,.}$ is given by (3.4). $\square$

Derivation of formulas (3.16). We first make the change of variables "r" $= r_q(t)$ in (3.10) ($dr = \dot r_q(t)dt$ and $\dot r_q(t) = \sqrt{2E - \frac{q^2}{r_q(t)^2} - 2W(r_q(t))}$, see (3.9)) and we obtain

$$g(q) = 2\int_{r_{\min,q}}^{+\infty} \frac{dr}{r^2\sqrt{2E - \frac{q^2}{r^2} - 2W(r)}}, \quad \text{for} \quad q > \beta. \tag{A.21}$$

Performing the change of variables "$r^{-1} = s$" in (A.21) we obtain

$$g(u^{-\frac{1}{2}}) = 2\int_0^{\chi(u)} \frac{ds}{\sqrt{2E - \frac{s^2}{u} - 2W(s^{-1})}}, \quad \text{for} \quad u \in (0,\beta^{-2}). \tag{A.22}$$

Let $\sigma \in (0,\beta^{-2})$. From (A.22) and (3.15) it follows that

$$H(\sigma) = \int_0^{\chi(\sigma)} \left(\int_{\phi(s)}^{\sigma} \frac{du}{\sqrt{\sigma - u}\sqrt{2(E - W(s^{-1}))u - s^2}} \right) ds. \tag{A.23}$$

And performing the change of variables $u = \phi(s) + \varepsilon(\sigma - \phi(s))$ in (A.23) ($du = (\sigma - \phi(s))d\varepsilon$) and using the equality (3.14) we obtain

$$\begin{aligned} \int_{\phi(s)}^{\sigma} \frac{du}{\sqrt{\sigma - u}\sqrt{2(E - W(s^{-1}))u - s^2}} &= \phi(s)^{\frac{1}{2}} s^{-1} \int_{\phi(s)}^{\sigma} \frac{du}{\sqrt{\sigma - u}\sqrt{u - \phi(s)}} \\ = \phi(s)^{\frac{1}{2}} s^{-1} \int_0^1 \frac{d\varepsilon}{\sqrt{\varepsilon}\sqrt{1-\varepsilon}} &= \phi(s)^{\frac{1}{2}} s^{-1}\pi, \end{aligned} \tag{A.24}$$

for $s \in (0, \chi(\beta^{-2}))$ (we used the integral value $\pi = \int_0^1 \frac{d\varepsilon}{\sqrt{\varepsilon}\sqrt{1-\varepsilon}}$). Using (A.23), (A.24) and (3.14) we obtain the first equality in (3.16)

$$H(\sigma) = \pi\int_0^{\chi(\sigma)} \phi(s)^{\frac{1}{2}} s^{-1} ds = \pi\int_0^{\chi(\sigma)} (2(E - W(s^{-1})))^{-\frac{1}{2}} ds. \tag{A.25}$$

From (A.25) it follows that

$$\frac{dH}{d\sigma}(\sigma) = \frac{\pi}{\sqrt{2(E - W(\chi(\sigma)^{-1}))}} \frac{d\chi}{d\sigma}(\sigma). \tag{A.26}$$

Then combining (3.13) and (A.26) we obtain the second equality in (3.16). $\square$

We end this appendix by giving details on the equality $q' = q$ in section 3.2. We keep the notations of section 3.2. We set $u_\theta = \left(\frac{x}{r}\right)^\perp$ and we have

$$r^2 \dot\theta u_\theta = r\dot x - \dot r x = r(v_- + \dot y_-) - \dot r(x_- + tv_- + y_-). \tag{A.27}$$

Using $x_- \cdot v_- = 0$ and $y_- = o(1)$, $\dot y_- = o(t^{-1})$, as $t \to -\infty$, we have

$$\begin{aligned} r(t) &= |x_- + tv_- + y_-| = \left(t^2|v_-|^2 + 2ty_- \cdot v_- + |x_- + y_-|^2\right)^{\frac{1}{2}} \\ &= -t|v_-| + o(1),\ t \to -\infty, \end{aligned} \tag{A.28}$$

$$\begin{aligned} \dot r(t) &= \frac{(v_- + \dot y_-)\cdot(x_- + tv_- + y_-)}{r(t)} = \frac{t|v_-|^2 + o(1)}{-t|v_-| + o(1)} \\ &= -|v_-| + o(t^{-1}),\ t \to -\infty, \end{aligned} \tag{A.29}$$

and we obtain

$$\begin{aligned} r^2\dot\theta u_\theta &= (-|v_-|t + o(1))(v_- + \dot y_-) - (-|v_-| + o(t^{-1}))(x_- + tv_- + y_-) \\ &= |v_-|x_- + o(1),\ t \to -\infty, \end{aligned} \tag{A.30}$$

$$u_\theta = \left(\frac{x(t)}{r(t)}\right)^\perp = \left(\frac{x_- + tv_- + o(1)}{-t|v_-| + o(1)}\right)^\perp = -\widehat{v_-}^\perp + o(1),\ t \to -\infty \tag{A.31}$$

where $\hat w = \frac{w}{|w|}$ for $w \neq 0$. Using (A.30) and (A.31) we obtain

$$r^2\dot\theta u_\theta \cdot \widehat{v_-}^\perp = x_- \cdot v_-^\perp + o(1),\ u_\theta \cdot \widehat{v_-}^\perp = -1 + o(1),\ t \to -\infty, \tag{A.32}$$

which proves that $q' = r^2\dot\theta = -x_- \cdot v_-^\perp$. $\square$

References

[1] N. H. Abel, *Auflösung einer mechanischen Aufgabe*, J. Reine Angew. Math. 1, 153–157 (1826). French transl.: *Résolution d'un problème de mécanique*, Œuvres complètes de Niels Henrik Abel (L. Sylow, S. Lie, eds) vol.1, pp97-101, Grøndahl, Christiana (Oslo), 1881.

[2] M. A. Astaburuaga, C. Fernandez, V. H. Cortés, *The direct and inverse problem in Newtonian Scattering*, Proc. Roy. Soc. Edinburgh Sect. A 118, 119–131 (1991).

[3] N. S. Dairbekov, G. P. Paternain, P. Stefanov and G. Uhlmann, *The boundary rigidity problem in the presence of a magnetic field*. Adv. Math. 216, 535–609 (2007).

[4] A. Einstein, *Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen*, Jahrbuch der Radioaktivität und Elektronik 4, 411–462 (1907).

[5] O. B. Firsov, *Determination of the force acting between atoms via differential effective elastic cross section*, Zh. ksper. Teoret. Fiz 24 (1953), 279–283 (Russian); see also Problem 7 of 18 of L. D. Landau, E. M. Lifschitz,*Mechanics*, Pergamon Press Oxford, 1960.

[6] H. Funke and Yu. Ratis, *An inverse problem for classical relativistic mechanics*, Inverse Problems 6:(2), 13–16 (1990).

[7] M. L. Gerver, N. S. Nadirashvili, *Inverse problem of mechanics at high energies*, Comput. Seismology 15, 118–125 (1983) (Russian).

[8] A. Jollivet, *On inverse scattering in electromagnetic field in classical relativistic mechanics at high energies*, Asymptot. Anal. 55:(1& 2), 103–123 (2007).

[9] A. Jollivet, *On inverse problems in electromagnetic field in classical mechanics at fixed energy*, J. Geom. Anal. 17:(2), 275–320 (2007).

[10] A. Jollivet, *On inverse scattering at high energies for the multidimensional nonrelativistic Newton equation in electromagnetic field*, J. Inverse Ill-Posed Probl. 17:5, 441–476 (2009).

[11] J. B. Keller, *Inverse problems*, Amer. Math. Monthly 83, 107–118, (1976).

[12] J. B. Keller, I. Kay and J. Shmoys, *Determination of the potential from scattering data*, Phys. Rev. 102 (1956), 557–559.

[13] L. D. Landau, E. M. Lifschitz, The Classical Theory of Fields Pergamon Press New York, 1971.

[14] M. Loss, B. Thaller, *Scattering of particles by long-range magnetic fields*, Ann. Physics 176, 159–180 (1987).

[15] R. G. Novikov, *Small angle scattering and X-ray transform in classical mechanics*, Ark. Mat. 37, 141–169 (1999).

[16] B. Simon, *Wave operators for classical particle scattering*. Comm. Math. Phys. 23, 37–48 (1971).

[17] K. Yajima, *Classical scattering for relativistic particles*, J. Fac. Sci., Univ. Tokyo, Sect. I A, 29, 599–611 (1982).

A. Jollivet
Laboratoire de Physique Théorique et Modélisation,
CNRS UMR 8089/Université de Cergy-Pontoise
95302 Cergy-Pontoise, France
e-mail: alexandre.jollivet@u-cergy.fr