On inverse scattering at high energies for the multidimensional nonrelativistic Newton equation in electromagnetic field

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Abstract. We consider the multidimensional nonrelativistic Newton equation in a static electromagnetic field

$$\ddot{x} = F(x, \dot{x}),\ F(x, \dot{x}) := -\nabla V(x) + B(x)\dot{x},\ \dot{x} = \frac{dx}{dt},\ x \in C^2(\mathbb{R}, \mathbb{R}^n), \qquad (*)$$

where $V \in C^2(\mathbb{R}^n, \mathbb{R})$, $B(x)$ is the $n \times n$ real antisymmetric matrix with elements $B_{i,k}(x)$, $B_{i,k} \in C^1(\mathbb{R}^n, \mathbb{R})$ (and B satisfies the closure condition), and $|\partial_x^{j_1} V(x)| + |\partial_x^{j_2} B_{i,k}(x)| \le \beta_{|j_1|}(1 + |x|)^{-(\alpha+|j_1|)}$ for $x \in \mathbb{R}^n$, $1 \le |j_1| \le 2$, $0 \le |j_2| \le 1$, $|j_2| = |j_1| - 1$, $i, k = 1 \ldots n$ and some $\alpha > 1$. We give estimates and asymptotics for scattering solutions and scattering data for the equation $(*)$ for the case of small angle scattering. We show that at high energies the velocity valued component of the scattering operator uniquely determines the X-ray transforms $P\nabla V$ and $PB_{i,k}$ (on sufficiently rich sets of straight lines). Applying results on inversion of the X-ray transform P we obtain that for $n \ge 2$ the velocity valued component of the scattering operator at high energies uniquely determines $(\nabla V, B)$. We also consider the problem of recovering $(\nabla V, B)$ from our high energies asymptotics found for the configuration valued component of the scattering operator. Results of the present work were obtained by developing the inverse scattering approach of [R. Novikov, 1999] for $(*)$ with $B \equiv 0$ and of [Jollivet, 2005] for the relativistic version of $(*)$. We emphasize that there is an interesting difference in asymptotics for scattering solutions and scattering data for $(*)$ on the one hand and for its relativistic version on the other.

1 Introduction

1.1 *The nonrelativistic Newton equation*

Consider the multidimensional nonrelativistic Newton equation in an external static electromagnetic field:

$$\ddot{x}(t) = F(x(t), \dot{x}(t)) := -\nabla V(x(t)) + B(x(t))\dot{x}(t), \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, $\dot{x}(t) = \frac{\mathrm{d}x}{\mathrm{d}t}(t)$, and $V \in C^2(\mathbb{R}^n, \mathbb{R})$ and for any $x \in \mathbb{R}^n$, $B(x)$ is a $n \times n$ antisymmetric matrix with elements $B_{i,k}(x)$, $B_{i,k} \in C^1(\mathbb{R}^n, \mathbb{R})$, which satisfy

$$\frac{\partial B_{i,k}}{\partial x_l}(x) + \frac{\partial B_{l,i}}{\partial x_k}(x) + \frac{\partial B_{k,l}}{\partial x_i}(x) = 0, \tag{1.2}$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and for $l, i, k = 1 \ldots n$.

For $n = 3$, the equation (1.1) is the equation in $\mathbb{R}^n$ of motion of a nonrelativistic particle of mass $m = 1$ and charge $e = 1$ in an external and static electromagnetic field described by (V, B) (see, for example, Section 17 of [LL2]). For the electromagnetic field the function V is an electric potential and B is the magnetic field. In this equation (1.1), x denotes the position of the particle, $\dot{x}$ denotes its velocity, $\ddot{x}$ denotes its acceleration and t denotes the time.

For equation (1.1) the energy

$$E = \frac{1}{2}|\dot{x}(t)|^2 + V(x(t)) \tag{1.3}$$

is an integral of motion. Note that the energy E does not depend on B because the magnetic force $B(x)\dot{x}$ is orthogonal to the velocity $\dot{x}$ of the particle.

We assume that the electromagnetic coefficients V and B are of short range on $\mathbb{R}^n$. More precisely we assume that (V, B) satisfies the following conditions

$$|\partial_x^{j_1} V(x)| \leq \beta_{|j_1|}(1 + |x|)^{-\alpha-|j_1|},\ x \in \mathbb{R}^n, \tag{1.4}$$

$$|\partial_x^{j_2} B_{i,k}(x)| \leq \beta_{|j_2|+1}(1 + |x|)^{-\alpha-1-|j_2|},\ x \in \mathbb{R}^n, \tag{1.5}$$

for $|j_1| \leq 2, |j_2| \leq 1$, $i, k = 1 \ldots n$ and some $\alpha > 1$ (here j_l is the multiindex $j_l = (j_{l,1}, \ldots, j_{l,n}) \in (\mathbb{N} \cup \{0\})^n$, $|j_l| = \sum_{k=1}^n j_{l,k}$ and $\beta_{|j_l|}$ are positive real constants). For example, conditions (1.4) and (1.5) are fulfilled for some real constants β_i, $i = 1, 2, 3$ when V and B are smooth and compactly supported on $\mathbb{R}^n$. Note also that the assumptions (1.4), (1.5) are actually similar to the assumptions of the works [HN], [ER], [Ni], [Ar] and [WY], where inverse scattering is considered for quantum non relativistic particle in electromagnetic field (or by other words for the Schrödinger equation in electromagnetic field).

1.2 *Scattering data*

Under conditions (1.4)–(1.5), the following is valid (see, for example, [S] where classical scattering of particles in a short-range electric field is studied, and see [LT] where classical scattering of particles in a long-range magnetic field is studied): for any $(v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n,\ v_- \neq 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = v_- t + x_- + y_-(t), \tag{1.6}$$

where $\dot{y}_-(t) \to 0,\ y_-(t) \to 0,$ as $t \to -\infty$; in addition for almost any $(v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n,\ v_- \neq 0,$

$$x(t) = v_+ t + x_+ + y_+(t), \tag{1.7}$$

where $v_+ \neq 0,\ v_+ = a(v_-, x_-),\ x_+ = b(v_-, x_-),\ \dot{y}_+(t) \to 0,\ \ y_+(t) \to 0,$ as $t \to +\infty$.

The map $S : (\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}^n \to (\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}^n$ given by the formulas

$$v_+ = a(v_-, x_-),\ x_+ = b(v_-, x_-), \tag{1.8}$$

is called the scattering map for the equation (1.1). In addition, $a(v_-, x_-)$, $b(v_-, x_-)$ are called the scattering data for the equation (1.1).

By $\mathcal{D}(S)$ we denote the set of definition of S ; by $\mathcal{R}(S)$ we denote the range of S (by definition, if $(v_-, x_-) \in \mathcal{D}(S)$, then $v_- \neq 0$ and $a(v_-, x_-) \neq 0$).

Under the conditions (1.4)–(1.5), the map S has the following simple properties: $\mathcal{D}(S)$ is an open set of $\mathbb{R}^n \times \mathbb{R}^n$ and $\mathrm{Mes}((\mathbb{R}^n \times \mathbb{R}^n) \backslash \mathcal{D}(S)) = 0$ for the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^n$; the map $S : \mathcal{D}(S) \to \mathcal{R}(S)$ is continuous and preserves the element of volume ; $a(v_-, x_-)^2 = v_-^2$. (This latter property is seen to be a consequence of the conservation of energy (1.3) by combining the short range of V (1.4) for $j_1 = (0, \ldots, 0)$, and the asymptotics (1.6), (1.7) for $(v_-, x_-) \in \mathcal{D}(S)$.)

If $V(x) \equiv 0$ and $B(x) \equiv 0$, then $a(v_-, x_-) = v_-,\ b(v_-, x_-) = x_-,\ (v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n,\ v_- \neq 0$. Therefore for $a(v_-, x_-),\ b(v_-, x_-)$ we will use the following representation

$$\begin{aligned} a(v_-, x_-) &= v_- + a_{sc}(v_-, x_-), \\ b(v_-, x_-) &= x_- + b_{sc}(v_-, x_-), \end{aligned} \qquad (v_-, x_-) \in \mathcal{D}(S). \tag{1.9}$$

We will use the fact that, under the conditions (1.4)–(1.5), the map S is uniquely determined by its restriction to $\mathcal{M}(S) = \mathcal{D}(S) \cap \mathcal{M}$, where

$$\mathcal{M} = \{(v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n | v_- \neq 0, v_- x_- = 0\}.$$

This observation is based on the fact that if $x(t)$ satisfies equation (1.1), then $x(t + t_0)$ also satisfies (1.1) for any $t_0 \in \mathbb{R}$.

1.3 *X-ray transform*

Consider

$$T\mathbb{S}^{n-1} = \{(\theta, x) | \theta \in \mathbb{S}^{n-1}, x \in \mathbb{R}^n, \theta x = 0\},$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$.

Consider the X-ray transform P which maps each function f with the properties

$$f \in C(\mathbb{R}^n, \mathbb{R}^m),\ |f(x)| = O(|x|^{-\beta}),\ \text{as } |x| \to \infty,\ \text{for some } \beta > 1,$$

into a function $Pf \in C(T\mathbb{S}^{n-1}, \mathbb{R}^m)$ where Pf is defined by

$$Pf(\theta, x) = \int_{-\infty}^{+\infty} f(t\theta + x)dt,\ (\theta, x) \in T\mathbb{S}^{n-1}.$$

Concerning the theory of the X-ray transform, the reader is referred to [R], [GGG], [Na] and [No1].

1.4 *Main results of the work*

The main results of the present work consist in the small angle scattering estimates and asymptotics for the scattering data a_{sc} and b_{sc} (and scattering solutions) for the equation (1.1) and in application of these asymptotics and estimates to inverse scattering for the equation (1.1) at high energies. Our main results include, in particular, Theorem 1.1, Proposition 1.1 formulated below and Theorem 3.1 given in Section 3.1.

Theorem 1.1. *Under conditions* (1.4)–(1.5)*, we have*

$$\begin{aligned} \lim_{s\to+\infty} a_{sc}(s\theta, x) &= W_{1,1}(B, \theta, x) \\ &= \int_{-\infty}^{+\infty} B(\tau\theta + x)\theta d\tau, \end{aligned} \tag{1.10}$$

and

$$\lim_{s\to+\infty} s\left(a_{sc}(s\theta, x) - W_{1,1}(B, \theta, x)\right) = W_{1,2}(V, B, \theta, x) \tag{1.11}$$

$$= -P(\nabla V)(\theta, x) + \int_{-\infty}^{+\infty} B(\tau\theta + x)(\int_{-\infty}^{\tau} B(\sigma\theta + x)\theta d\sigma)d\tau$$

$$+ \sum_{k=1}^{n} \theta_k \left(\Omega_{1,1,k}(\theta, x), \ldots, \Omega_{1,n,k}(\theta, x)\right)$$

for $(\theta,x)\in T\mathbb{S}^{n-1}$, $\theta=(\theta_1,\ldots,\theta_n)$, *where*

$$\Omega_{1,i,k}=\int_{-\infty}^{+\infty}\nabla B_{i,k}(x+\tau\theta)\circ\left(\int_{-\infty}^{\tau}\int_{-\infty}^{\sigma}B(\eta\theta+x)\theta d\eta d\sigma\right)d\tau$$

for $i,k=1\ldots n$ *(*$\circ$ *denotes the usual scalar product on* $\mathbb{R}^n$*); in addition, we have*

$$\lim_{s\to+\infty}sb_{sc}(s\theta,x)=W_{2,1}(B,\theta,x) \tag{1.12}$$
$$=\int_{-\infty}^{0}\int_{-\infty}^{\tau}B(\sigma\theta+x)\theta d\sigma d\tau-\int_{0}^{+\infty}\int_{\tau}^{+\infty}B(\sigma\theta+x)\theta d\sigma d\tau,$$

$$\lim_{s\to+\infty}s\left(sb_{sc}(s\theta,x)-W_{2,1}(B,\theta,x)\right)=W_{2,2}(V,B,\theta,x) \tag{1.13}$$
$$=\int_{-\infty}^{0}\int_{-\infty}^{\tau}(-\nabla V(\sigma\theta+x))d\sigma d\tau-\int_{0}^{+\infty}\int_{\tau}^{+\infty}(-\nabla V(\sigma\theta+x))d\sigma d\tau$$
$$+\int_{-\infty}^{0}\int_{-\infty}^{\tau}B(\sigma\theta+x)\left(\int_{-\infty}^{\sigma}B(\eta\theta+x)\theta d\eta\right)d\sigma d\tau$$
$$-\int_{0}^{+\infty}\int_{\tau}^{+\infty}B(\sigma\theta+x)\left(\int_{-\infty}^{\sigma}B(\eta\theta+x)\theta d\eta\right)d\sigma d\tau$$
$$+\sum_{k=1}^{n}\theta_k\left(\Omega_{2,1,k}(\theta,x),\ldots,\Omega_{2,n,k}(\theta,x)\right)$$

for $(\theta,x)\in T\mathbb{S}^{n-1}$, $\theta=(\theta_1,\ldots,\theta_n)$, *where*

$$\begin{aligned}\Omega_{2,i,k}(\theta,x) \;=\; &\int_{-\infty}^{0}\int_{-\infty}^{\tau}\nabla B_{i,k}(\sigma\theta+x)\circ\left(\int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1}B(\eta_2\theta+x)\theta d\eta_2 d\eta_1\right)d\sigma d\tau\\ &-\int_{0}^{+\infty}\int_{\tau}^{+\infty}\nabla B_{i,k}(\sigma\theta+x)\circ\left(\int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1}B(\eta_2\theta+x)\theta d\eta_2 d\eta_1\right)d\sigma d\tau\end{aligned}$$

for $i,k=1\ldots n$ *(*$\circ$ *denotes the usual scalar product on* $\mathbb{R}^n$*).*

Theorem 1.1 gives the first two leading terms of the high energies asymptotics of the scattering data. Theorem 1.1 follows from Theorem 3.1. Theorem 1.1 is proved in Section 3.2.

Note that Theorem 3.1 (see (3.16) and (3.18)) also gives the asymptotics of a_{sc}, b_{sc}, when the parameters α, n, $s > 0$, θ, x are fixed and the norm β_m decreases to 0 (where $\beta_m = \max(\beta_0, \beta_1, \beta_2)$), that is Theorem 3.1 also gives the "Born approximation" for the scattering data at fixed energy when the electromagnetic field is sufficiently weak (see Section 3.3).

Proposition 1.1. *Under conditions* (1.4)–(1.5)*, the following statements are valid:*

(i) $W_{1,1}(B, \theta, x)$ *given for all* $(\theta, x) \in T\mathbb{S}^{n-1}$*, uniquely determines* B*;*

(ii) $W_{1,1}(B, \theta, x)$, $W_{1,2}(V, B, \theta, x)$ *given for all* $(\theta, x) \in T\mathbb{S}^{n-1}$*, uniquely determine* (V, B)*;*

(iii) if $n \geq 3$ $W_{2,1}(B, \theta, x)$, $W_{2,2}(V, B, \theta, x)$ *given for all* $(\theta, x) \in T\mathbb{S}^{n-1}$*, uniquely determine* (V, B)*;*

(iv) if $n = 2$*, then* V *and* B *are not uniquely determined by* $W_{2,1}(B, \theta, x)$, $W_{2,2}(V, B, \theta, x)$ *given for all* $(\theta, x) \in T\mathbb{S}^{n-1}$.

Proposition 1.1 is proved in Section 3.5.

Using (1.10), (1.11) and Proposition 1.1 (*i*), (*ii*) we obtain that a_{sc} determines uniquely ∇V and B at high energies. Actually for $n \geq 2$ methods of reconstruction of f from Pf permit to reconstruct ∇V and B from the velocity valued component a of the scattering map at high energies (see Sections 3.4, 3.5). The formulas (1.12), (1.13) and Proposition 1.1 (*iii*), (*iv*) show that the first two leading terms of the high energies asymptotics of b_{sc} do not determine uniquely (V, B) when $n = 2$ but that they uniquely determine (V, B) when $n \geq 3$. Actually, (V, B) can be reconstructed from the first two leading terms of the high energies asymptotics of b_{sc} when $n \geq 3$ (see Sections 3.4, 3.5).

1.5 *Historical remarks*

Note that inverse scattering for the classical multidimensional nonrelativistic Newton equation at high energies was first studied by Novikov [No1] for $B \equiv 0$. Novikov proved, in particular, two formulas which link scattering data at high energies to the X-ray transform of $-\nabla V$ and V. These formulas are generalized by formulas (1.11), (1.13) of the present work for the case $B \not\equiv 0$.

Developing Novikov's approach [No1], the author also studied the inverse scattering for the relativistic multidimensional Newton equation at high energies for $B \equiv 0$ [Jo1] and for $B \not\equiv 0$ [Jo2]. In the relativistic case in contrast with (1.10)–(1.11), the following formula was obtained in [Jo2, Theorem 1.1]:

$$\lim_{\substack{s\to c\\ s<c}} \frac{s}{\sqrt{1-\frac{s^2}{c^2}}} a^r_{sc}(s\theta,x) = W^r_1(V,B,\theta,x) \tag{1.14}$$

$$:= -P(\nabla V)(\theta,x) + \int_{-\infty}^{+\infty} B(\tau\theta+x)\theta d\tau,$$

for $(\theta,x)\in T\mathbb{S}^{n-1}$ where $c>0$ denotes the speed of light. In particular, one can see that in the relativistic case the leading high energies term of a^r_{sc} depends on the both magnetic and electric fields B and ∇V, whereas in the nonrelativistic case the leading high energies term of a_{sc} depends on the magnetic field B only (and is independent of the electric one). This difference is quite interesting in our opinion.

A similar difference occurs in quantim mechanics between inverse scattering at high energies for the nonrelativistic case (see [Ni], [Ar]) and inverse scattering at high energies for the relativistic case (see [I], [Ju], [Ha]).

Note also that for the classical multidimensional nonrelativistic Newton equation in a bounded open strictly convex domain an inverse boundary value problem at high energies was first studied in [GN].

Concerning the inverse problems for the classical multidimensional nonrelativistic Newton equation at fixed energy, we refer the reader to [GN], [No1], [Jo3], [DPSU] and references given in [No1], [Jo3].

Concerning the inverse problem for (1.1) in the one-dimensional case, we can mention the works [Ab], [K], [AFC].

Concerning the inverse scattering problem for a particle in electromagnetic field (with $B\not\equiv 0$ or $B\equiv 0$) in quantim mechanics, see [HN], [I], [Ju], [ER], [Ni], [Ar], [Ha], [WY] and [No2] and references given in [Jo2].

1.6 *Structure of the paper*

Further, our paper is organized as follows. In Section 2 we transform the differential equation (1.1) with initial conditions (1.6) into a system of integral equations which takes the form $(y_-,\dot y_-)=A_{v_-,x_-}(y_-,\dot y_-)$. Then we study A_{v_-,x_-} on a suitable space and we give estimates for A_{v_-,x_-} and for $(A_{v_-,x_-})^2$, and, in particular, contraction estimates for $(A_{v_-,x_-})^2$ (Lemmas 2.1, 2.2, 2.3, 2.4). In Section 3.1 we give estimates and asymptotics for the deflection $y_-(t)$ from (1.6) and for scattering data $a_{sc}(v_-,x_-)$, $b_{sc}(v_-,x_-)$ from (1.9) (Theorem 3.1). From these estimates and asymptotics the four formulas (1.10)–(1.13) will follow when the parameters β_m, α, n, $\hat v_-$, x_- are fixed and $|v_-|$ increases (where $\beta_{|j|}$, α, n are constants from (1.4)-(1.5), $\beta_m=\max(\beta_0,\beta_1,\beta_2)$; $\hat v_-=v_-/|v_-|$) (see the proof of Theorem 1.1 given in Section 3.2). In this case $\sup|\vartheta(t)|$ decreases, where $\vartheta(t)$ denotes the angle between the vectors $\dot x(t)=v_-+\dot y_-(t)$

and v_-, and we deal with small angle scattering. Note that, under the conditions of Theorem 3.1, without additional assumptions, there is the estimate $\sup|\vartheta(t)| < \frac{1}{4}\pi$ and we deal with rather small angle scattering (concerning the term "small angle scattering" see [No1] and Section 20 of [LL1]). In Section 3.3 we also consider the "Born approximation" of the scattering data at fixed energy. In Section 3.4 we remind some results on inversion of the X-ray transform P and on reconstruction of a function f from its X-ray transform Pf. In Section 3.5 we prove Proposition 1.1. In Section 4 we prove Lemmas 2.1, 2.2, 2.3. In Section 5, we prove Lemma 2.4.

2 Contraction maps

If x satisfies the differential equation (1.1) and the initial conditions (1.6), then x satisfies the system of integral equations

$$x(t) = tv_- + x_- + \int_{-\infty}^{t}\int_{-\infty}^{\tau} F(x(s),\dot x(s))dsd\tau, \tag{2.1}$$

$$\dot x(t) = v_- + \int_{-\infty}^{t} F(x(s),\dot x(s))ds, \tag{2.2}$$

for $t \in \mathbb{R}$, where $F(x,\dot x) = -\nabla V(x) + B(x)\dot x$, $v_- \in \mathbb{R}^n\backslash\{0\}$.

For $y_-(t)$ of (1.6) this system takes the form

$$(y_-, u_-) = A_{v_-,x_-}(y_-,u_-)(t), \tag{2.3}$$

where $u_-(t) = \dot y_-(t)$ and

$$A_{v_-,x_-}(f,h)(t) = (A^1_{v_-,x_-}(f,h)(t), A^2_{v_-,x_-}(f,h)(t)), \tag{2.4}$$

$$A^1_{v_-,x_-}(f,h)(t) = \int_{-\infty}^{t} A^2_{v_-,x_-}(f,h)(\tau)d\tau, \tag{2.5}$$

$$A^2_{v_-,x_-}(f,h)(t) = \int_{-\infty}^{t} F(x_- + \tau v_- + f(\tau), v_- + h(\tau))d\tau, \tag{2.6}$$

for $v_- \in \mathbb{R}^n\backslash\{0\}$.

From (2.3), (1.4)–(1.5) and $y_-(t) \in C^1(\mathbb{R},\mathbb{R}^n)$, $|y_-(t)| + |\dot y_-(t)| \to 0$, as $t \to -\infty$, it follows, in particular, that

$$\begin{aligned}&(y_-(t),\dot y_-(t)) \in C(\mathbb{R},\mathbb{R}^n)\times C(\mathbb{R},\mathbb{R}^n)\\ &\text{and } |\dot y_-(t)| = O(|t|^{-\alpha}),\ |y_-(t)| = O(|t|^{-\alpha+1}),\ \text{as } t \to -\infty,\end{aligned} \tag{2.7}$$

where $v_- \neq 0$ and x_- are fixed.

For nonnegative real numbers R and r, consider the complete metric space

$$M_{T,R,r} = \{(f,h) \in C(]-\infty,T],\mathbb{R}^n) \times C(]-\infty,T],\mathbb{R}^n) \mid \sup_{t\in]-\infty,T]} |f(t)-th(t)| \le r, \ \sup_{t\in]-\infty,T]} |h(t)| \le R\}, \tag{2.8}$$

with the norm $\|.\|_{\infty,T}$ defined by

$$\|(f,h)\|_{\infty,T} = \max\left(\sup_{t\in]-\infty,T]} |f(t)-th(t)|, \sup_{t\in]-\infty,T]} |h(t)|\right), \tag{2.9}$$

where $T \in]-\infty,+\infty]$ (if $T=+\infty$, $]-\infty,T]$ must be replaced by $]-\infty,+\infty[$)). From (2.7) it follows that

$$(y_-(t),\dot{y}_-(t)) \in M_{T,R,r} \text{ for some } R \text{ and } r \text{ depending on } y_-(t) \text{ and } T. \tag{2.10}$$

Lemma 2.1. *Let $R>0$, $0<r\le 1$ and let $(v_-,x_-)\in\mathbb{R}^n\times\mathbb{R}^n$ be such that $|v_-|>\sqrt{2}R$, $v_-x_-=0$. Then under conditions* (1.4)–(1.5)*, the following estimates are valid :*

$$\sup_{t\in]-\infty,T]} |A^2_{v_-,x_-}(f,h)(t)| \le \rho_{T,2}(n,\beta_1,\alpha,|v_-|,|x_-|,R) = \frac{2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v_-|+\sqrt{n}R)}{\alpha(\frac{|v_-|}{\sqrt{2}}-R)(1+\frac{|x_-|}{\sqrt{2}}+(\frac{|v_-|}{\sqrt{2}}-R)|T|)^{\alpha}}, \tag{2.11}$$

$$\sup_{t\in]-\infty,T]} |A^1_{v_-,x_-}(f,h)(t)-tA^2_{v_-,x_-}(f,h)(t)| \le \rho_{T,1}(n,\beta_1,\alpha,|v_-|,|x_-|,R) = \frac{2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v_-|+\sqrt{n}R)}{(\alpha-1)(\frac{|v_-|}{\sqrt{2}}-R)^2(1+\frac{|x_-|}{\sqrt{2}}+(\frac{|v_-|}{\sqrt{2}}-R)|T|)^{\alpha-1}}, \tag{2.12}$$

for $T\le 0$ and $(f,h)\in M_{T,R,r}$;

$$\sup_{t\in]-\infty,T]} |A^2_{v_-,x_-}(f,h)(t)| \le \rho_2(n,\beta_1,\alpha,|v_-|,|x_-|,R) = \frac{2^{\alpha+2}\beta_1\sqrt{n}(1+\sqrt{n}|v_-|+\sqrt{n}R)}{\alpha(\frac{|v_-|}{\sqrt{2}}-R)(1+\frac{|x_-|}{\sqrt{2}})^{\alpha}}, \tag{2.13}$$

$$\sup_{t\in]-\infty,T]} |A^1_{v_-,x_-}(f,h)(t)-tA^2_{v_-,x_-}(f,h)(t)| \le \rho_1(n,\beta_1,\alpha,|v_-|,|x_-|,R) = \frac{2^{\alpha+2}\beta_1\sqrt{n}(1+\sqrt{n}|v_-|+\sqrt{n}R)}{\alpha(\alpha-1)(\frac{|v_-|}{\sqrt{2}}-R)^2(1+\frac{|x_-|}{\sqrt{2}})^{\alpha-1}}. \tag{2.14}$$

for $T\ge 0$ and $(f,h)\in M_{T,R,r}$.

Remark 2.1. Note that for fixed n, β_1, α, $|x_-|$, R, we have

$$\rho_1(n,\beta_1,\alpha,|v_-|,|x_-|,R)\to 0, \text{ as } |v_-|\to+\infty; \tag{2.15}$$

$$\rho_2(n,\beta_1,\alpha,|v_-|,|x_-|,R)\to \frac{2^{\alpha+2}\sqrt{2}\beta_1 n}{\alpha(1+\frac{|x_-|}{\sqrt{2}})^{\alpha}}, \text{ as } |v_-|\to+\infty \tag{2.16}$$

(we used (2.13)–(2.14)).

Lemma 2.2. *Let $R>0$, $0<r\le 1$ and let $(v_-,x_-)\in\mathbb{R}^n\times\mathbb{R}^n$ be such that $|v_-|>\sqrt{2}R$, $v_-x_-=0$. Then under conditions* (1.4)–(1.5)*, for (f_1,h_1), $(f_2,h_2)\in M_{T,R,r}$, the following contraction estimates are valid :*

$$\sup_{t\in]-\infty,T]}|A^2_{v_-,x_-}(f_1,h_1)(t)-A^2_{v_-,x_-}(f_2,h_2)(t)|\le\lambda_{4,T}\sup_{t\in]-\infty,T]}|h_1(t)-h_2(t)|$$
$$+\lambda_{3,T}\sup_{t\in]-\infty,T]}|f_1(t)-f_2(t)-t(h_1(t)-h_2(t))| \tag{2.17}$$

$$\sup_{t\in]-\infty,T]}|\left(A^1_{v_-,x_-}(f_1,h_1)-A^1_{v_-,x_-}(f_2,h_2)\right)(t)$$
$$-t\left(A^2_{v_-,x_-}(f_1,h_1)-A^2_{v_-,x_-}(f_2,h_2)\right)(t)|\le\lambda_{2,T}\sup_{t\in]-\infty,T]}|h_1(t)-h_2(t)|$$
$$+\lambda_{1,T}\sup_{t\in]-\infty,T]}|f_1(t)-f_2(t)-t(h_1(t)-h_2(t))|, \tag{2.18}$$

for $T\le 0$, where $\lambda_{1,T}$, $\lambda_{2,T}$, $\lambda_{3,T}$ and $\lambda_{4,T}$ are given below by formulas (2.21)–(2.24)*;*

$$\sup_{t\in]-\infty,T]}|A^2_{v_-,x_-}(f_1,h_1)(t)-A^2_{v_-,x_-}(f_2,h_2)(t)|\le\lambda_4\sup_{t\in]-\infty,T]}|h_1(t)-h_2(t)|$$
$$+\lambda_3\sup_{t\in]-\infty,T]}|f_1(t)-f_2(t)-t(h_1(t)-h_2(t))|, \tag{2.19}$$

$$\sup_{t\in]-\infty,T]}|\left(A^1_{v_-,x_-}(f_1,h_1)-A^1_{v_-,x_-}(f_2,h_2)\right)(t)$$
$$-t\left(A^2_{v_-,x_-}(f_1,h_1)-A^2_{v_-,x_-}(f_2,h_2)\right)(t)|\le\lambda_2\sup_{t\in]-\infty,T]}|h_1(t)-h_2(t)|$$
$$+\lambda_1\sup_{t\in]-\infty,T]}|f_1(t)-f_2(t)-t(h_1(t)-h_2(t))|, \tag{2.20}$$

for $T\ge 0$, where λ_1, λ_2, λ_3 and λ_4 are given below by formulas (2.25).

Lemmas 2.1, 2.2 are proved in Section 4.

Let $R > 0$, $0 < r \le 1$ and let $(v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n$ be such that $|v_-| > \sqrt{2}R$. For $T \le 0$, real constants $\lambda_{i,T}$ for $i = 1\ldots 4$, which appear in estimates (2.17)–(2.18) given in Lemma 2.2, are defined by the following formulas:

$$\lambda_{1,T}(n,\beta_2,\alpha,|v_-|,|x_-|,R) = \frac{2^{\alpha+2}n\beta_2(1+\sqrt{n}|v_-|+\sqrt{n}R)}{\alpha(\frac{|v_-|}{\sqrt{2}}-R)^2(1+\frac{|x_-|}{\sqrt{2}}+(\frac{|v_-|}{\sqrt{2}}-R)|T|)^{\alpha}}, \quad (2.21)$$

$$\lambda_{2,T}(n,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,R) = 2^{\alpha+1}n\frac{\beta_1(\frac{|v_-|}{\sqrt{2}}-R)+2\beta_2(1+\sqrt{n}|v_-|+\sqrt{n}R)}{(\alpha-1)(\frac{|v_-|}{\sqrt{2}}-R)^3(1+\frac{|x_-|}{\sqrt{2}}+(\frac{|v_-|}{\sqrt{2}}-R)|T|)^{\alpha-1}}, \quad (2.22)$$

$$\lambda_{3,T}(n,\beta_2,\alpha,|v_-|,|x_-|,R) = \frac{2^{\alpha+2}n\beta_2(1+\sqrt{n}|v_-|+\sqrt{n}R)}{(\alpha+1)(\frac{|v_-|}{\sqrt{2}}-R)(1+\frac{|x_-|}{\sqrt{2}}+(\frac{|v_-|}{\sqrt{2}}-R)|T|)^{\alpha+1}}, \quad (2.23)$$

$$\lambda_{4,T}(n,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,R) = 2^{\alpha+1}n\frac{\beta_1(\frac{|v_-|}{\sqrt{2}}-R)+2\beta_2(1+\sqrt{n}|v_-|+\sqrt{n}R)}{\alpha(\frac{|v_-|}{\sqrt{2}}-R)^2(1+\frac{|x_-|}{\sqrt{2}}+(\frac{|v_-|}{\sqrt{2}}-R)|T|)^{\alpha}}, \quad (2.24)$$

Real constants λ_i for $i = 1\ldots 4$, which appear in estimates (2.19)-(2.20) given in Lemma 2.2, are defined by the following formulas:

$$\lambda_1 = \frac{2\lambda_{1,0}}{\alpha+1},\ \lambda_2 = \frac{2\lambda_{2,0}}{\alpha},\ \lambda_3 = 2\lambda_{3,0},\ \lambda_4 = 2\lambda_{4,0}. \quad (2.25)$$

We define real number $\lambda_T(n,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,R)$ by

$$\begin{aligned}\lambda_T \ &= \ \max(\lambda_{1,T}\lambda_{3,T}+\lambda_{2,T}\lambda_{3,T}+\lambda_{3,T}\lambda_{4,T}+\lambda_{4,T}^2, \quad (2.26)\\ &\quad\ \lambda_{1,T}^2+\lambda_{1,T}\lambda_{2,T}+\lambda_{2,T}\lambda_{4,T}+\lambda_{2,T}\lambda_{3,T}),\end{aligned}$$

for $T \le 0$; we define positive real number $\lambda(n,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,R)$ by

$$\begin{aligned}\lambda \ &= \ \max(\lambda_1\lambda_3+\lambda_2\lambda_3+\lambda_3\lambda_4+\lambda_4^2, \quad (2.27)\\ &\quad\ \lambda_1^2+\lambda_1\lambda_2+\lambda_2\lambda_4+\lambda_2\lambda_3).\end{aligned}$$

Remark 2.2. Note that for fixed n, β_1, β_2, α, $|x_-|$, R, T, we have

$$\begin{aligned}&\lambda_T(n,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,R) = O(|v_-|^{-1}),\ \text{as } |v_-| \to +\infty;\\ &\lambda(n,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,R) = O(|v_-|^{-1}),\ \text{as } |v_-| \to +\infty.\end{aligned} \quad (2.28)$$

Taking into account Lemma 2.1, Lemma 2.2, we obtain the following Corollary 2.1.

Corollary 2.1. *Let $R > 0$, $0 < r \le 1$ and let $(v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n$ be such that $|v_-| > \sqrt{2}R$, $v_-x_- = 0$. Then under conditions* (1.4)–(1.5), *the following statements are valid :*

(i) for $T \le 0$*, if* $\max(\frac{\rho_{T,1}}{r}, \frac{\rho_{T,2}}{R}) \le 1$ *then* $(A_{v_-,x_-})^2$ *is a map from* $M_{T,R,r}$ *into* $M_{T,R,r}$ *and* $(A_{v_-,x_-})^2$ *satisfies the following inequality*

$$\|(A_{v_-,x_-})^2(f_1,h_1) - (A_{v_-,x_-})^2(f_2,h_2)\|_{\infty,T} \le \lambda_T \|(f_1 - f_2, h_1 - h_2)\|_{\infty,T}, \tag{2.29}$$

for (f_1,h_1), $(f_2,h_2) \in M_{T,R,r}$ *;*

(ii) if $\max(\frac{\rho_1}{r}, \frac{\rho_2}{R}) \le 1$ *then for* $T = +\infty$, $(A_{v_-,x_-})^2$ *is a map from* $M_{T,R,r}$ *into* $M_{T,R,r}$ *and* $(A_{v_-,x_-})^2$ *satisfies the following inequality*

$$\|(A_{v_-,x_-})^2(f_1,h_1) - (A_{v_-,x_-})^2(f_2,h_2)\|_{\infty,T} \le \lambda \|(f_1 - f_2, h_1 - h_2)\|_{\infty,T}, \tag{2.30}$$

for (f_1,h_1), $(f_2,h_2) \in M_{T,R,r}$.

(Constants $\rho_{T,1}$, $\rho_{T,2}$, ρ_1, ρ_2, λ_T *and* λ *are respectively defined by* (2.12), (2.11), (2.14), (2.13), (2.26) *and* (2.27)*.)*

Taking into account (2.10) and using Lemmas 2.1, 2.2, Corollary 2.1 (see also (3.2)–(3.4)) and the lemma about the contraction maps we will study the solution $(y_-(t), u_-(t))$ of the equation (2.3) in $M_{T,R,r}$.

We will use also the following results (Lemmas 2.3, 2.4).

Lemma 2.3. *Let conditions* (1.4)–(1.5) *be valid. Let* $R > 0$, $0 < r \le 1$ *and let* $(v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n$ *be such that* $|v_-| > \sqrt{2}R$, $v_- x_- = 0$*. Assume* $\max(\frac{\rho_2}{R}, \frac{\rho_1}{r}) \le 1$ *where* ρ_1, ρ_2 *are respectively defined by* (2.14), (2.13)*. Then the following statements are valid :*

$$[\big(A_{v_-,x_-}\big)^2]_1(f,h)(t) = k_{v_-,x_-}(f,h)t + l_{v_-,x_-}(f,h) + H_{v_-,x_-}(f,h)(t), t \ge 0, \tag{2.31}$$

where $\big(A_{v_-,x_-}\big)^2 = ([\big(A_{v_-,x_-}\big)^2]_1, [\big(A_{v_-,x_-}\big)^2]_2)$ *and*

$$k_{v_-,x_-}(f,h) = \int\limits_{-\infty}^{+\infty} F\left(x_- + sv_- + A^1_{v_-,x_-}(f,h)(s), v_- + A^2_{v_-,x_-}(f,h)(s)\right) ds, \tag{2.32}$$

$$l_{v_-,x_-}(f,h) = \int\limits_{-\infty}^{0}\int\limits_{-\infty}^{s} F\left(x_- + \tau v_- + A^1_{v_-,x_-}(f,h)(\tau), v_- + A^2_{v_-,x_-}(f,h)(\tau)\right) d\tau ds$$

$$- \int\limits_{0}^{+\infty}\int\limits_{s}^{+\infty} F\left(x_- + \tau v_- + A^1_{v_-,x_-}(f,h)(\tau), v_- + A^2_{v_-,x_-}(f,h)(\tau)\right) d\tau ds \tag{2.33}$$

$$H_{v_-,x_-}(f,h)(t) = \int\limits_t^{+\infty}\int\limits_\tau^{+\infty} F\left(x_- + \tau v_- + A^1_{v_-,x_-}(f,h)(\tau), v_- + A^2_{v_-,x_-}(f,h)(\tau)\right) d\tau ds, \tag{2.34}$$

for $t \geq 0$, $(f,h) \in M_{T,R,r}$, $T = +\infty$. *In addition, the following estimates are valid :*

$$\begin{aligned}
|k_{v_-,x_-}(f,h)| &\leq \rho_2(n,\beta_1,\alpha,|v_-|,|x_-|,R), & (2.35)\\
|l_{v_-,x_-}(f,h)| &\leq \rho_1(n,\beta_1,\alpha,|v_-|,|x_-|,R), & (2.36)\\
|\dot H_{v_-,x_-}(f,h)(t)| &\leq \zeta(n,\beta_1,\alpha,|v_-|,|x_-|,R,t) & (2.37)\\
&= \frac{2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v_-|+\sqrt{n}R)}{\alpha(\frac{|v_-|}{\sqrt{2}}-R)(1+\frac{|x_-|}{\sqrt{2}}+(\frac{|v_-|}{\sqrt{2}}-R)t)^\alpha},
\end{aligned}$$

$$\begin{aligned}
|H_{v_-,x_-}(f,h)(t)| &\leq \xi(n,\beta_1,\alpha,|v_-|,|x_-|,R,t) & (2.38)\\
&= \frac{2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v_-|+\sqrt{n}R)}{\alpha(\alpha-1)(\frac{|v_-|}{\sqrt{2}}-R)^2(1+\frac{|x_-|}{\sqrt{2}}+(\frac{|v_-|}{\sqrt{2}}-R)t)^{\alpha-1}},
\end{aligned}$$

for $t \geq 0$, $(f,h) \in M_{T,R,r}$, $T = +\infty$; *in addition, for* $(f,h) \in M_{T,R,r}$, $T = +\infty$, *such that* $(f,h) = A_{v_-,x_-}(f,h)$, *we have*

$$\begin{aligned}
|k_{v_-,x_-}(f,h) - k_{v_-,x_-}(0,0)| &\leq \delta_{1,1}(n,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,R) & (2.39)\\
&= (\lambda_2\lambda_3+\lambda_4^2)\rho_2 + (\lambda_1\lambda_3+\lambda_3\lambda_4)\rho_1,\\
|l_{v_-,x_-}(f,h) - l_{v_-,x_-}(0,0)| &\leq \delta_{2,1}(n,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,R) & (2.40)\\
&= (\lambda_1\lambda_2+\lambda_2\lambda_4)\rho_2 + (\lambda_1^2+\lambda_2\lambda_3)\rho_1,
\end{aligned}$$

where λ_1, λ_2, λ_3, λ_4 *are defined by* (2.25).

Lemma 2.3 is proved in Section 4.

Remark 2.3. Note that for fixed n, β_1, β_2, α, $|x_-|$, R, we have

$$\begin{aligned}
\delta_{1,1}(n,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,R) &= O(|v_-|^{-2}), \text{ as } |v_-| \to +\infty,\\
\delta_{2,1}(n,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,R) &= O(|v_-|^{-3}), \text{ as } |v_-| \to +\infty,
\end{aligned}$$

where $\delta_{i,1}$, $i = 1,2$, are defined by (2.39)–(2.40) (we used (2.25) and (2.13)–(2.14)).

Lemma 2.4. *Let conditions* (1.4)–(1.5) *be valid. Let* $R > 0$, $0 < r \leq 1$ *and let* $(v_-,x_-) \in \mathbb{R}^n \times \mathbb{R}^n$ *be such that* $|v_-| > \sqrt{2}R$, $v_-x_- = 0$. *Assume* $\max(\frac{\rho_2}{R},\frac{\rho_1}{r}) \leq 1$ *where* ρ_1, ρ_2 *are respectively defined by* (2.14), (2.13). *Then* *the following statements are valid:*

$$|k_{v_-,x_-}(0,0) - w_{1,v_-,x_-}| \le \delta_{1,2}(n,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,R) \tag{2.41}$$
$$= \frac{2^{\alpha+4}\sqrt{2}n^3(1+\sqrt{n}|v_-|)(2\alpha^2+\alpha-2)\beta_1(\beta_1+2\beta_2+\beta_1\beta_2)}{(\alpha-1)\alpha(\alpha+1)\frac{|v_-|}{\sqrt{2}}(\frac{|v_-|}{\sqrt{2}}-R)^2(1+\frac{|x_-|}{\sqrt{2}})^{2\alpha}},$$
$$|l_{v_-,x_-}(0,0) - w_{2,v_-,x_-}| \le \delta_{2,2}(n,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,R) \tag{2.42}$$
$$= \frac{2^{\alpha+5}n^3(2\alpha+4)\beta_1(2\beta_2+\beta_1+\beta_1\beta_2)(1+\sqrt{n}|v_-|)}{(\alpha-1)\alpha^2(\alpha+1)\frac{|v_-|}{\sqrt{2}}(\frac{|v_-|}{\sqrt{2}}-R)^3(1+\frac{|x_-|}{\sqrt{2}})^{2\alpha-1}},$$

where w_{1,v_-,x_-} and w_{2,v_-,x_-} are defined below by (2.43) *and* (2.45).

Lemma 2.4 is proved in Section 5.

Let $(v_-,x_-)\in\mathbb{R}^n\times\mathbb{R}^n$, $v_-\neq 0$. Let $\hat{v}_- = \frac{v_-}{|v_-|}$, $\hat{v}_- = (\hat{v}_-^1,\ldots,\hat{v}_-^n)$. Then vectors w_{1,v_-,x_-} and w_{2,v_-,x_-}, which appear in (2.41) and (2.42), are defined by

$$\begin{aligned} w_{1,v_-,x_-} &= \int_{-\infty}^{+\infty} B(\tau\hat{v}_-+x_-)\hat{v}_-d\tau - \frac{1}{|v_-|}P(\nabla V)(\hat{v}_-,x_-)d\tau \\ &+\frac{1}{|v_-|}\int_{-\infty}^{+\infty} B(\tau\hat{v}_-+x_-)\left(\int_{-\infty}^{\tau} B(\sigma\hat{v}_-+x_-)\hat{v}_-d\sigma\right)d\tau \\ &+\frac{1}{|v_-|}\sum_{k=1}^{n}\hat{v}_-^k(\Omega_{3,1,k}(v_-,x_-),\ldots,\Omega_{3,n,k}(v_-,x_-)), \end{aligned} \tag{2.43}$$

where

$$\begin{aligned} \Omega_{3,i,k}(v_-,x_-) &= \int_{-\infty}^{+\infty}\int_0^1 \nabla B_{i,k}\left(\tau\hat{v}_-+x_-+\frac{\varepsilon}{|v_-|}\int_{-\infty}^{\tau}\int_{-\infty}^{\sigma} B(\eta\hat{v}_-+x_-)\hat{v}_-d\eta d\sigma\right) \\ &\circ\left(\int_{-\infty}^{\tau}\int_{-\infty}^{\sigma} B(\eta\hat{v}_-+x_-)\hat{v}_-d\eta d\sigma\right)d\varepsilon d\tau, \end{aligned} \tag{2.44}$$

for $i, k = 1 \ldots n$ ($\circ$ denotes the usual scalar product on $\mathbb{R}^n$), and

$$w_{2,v_-,x_-} = \frac{1}{|v_-|}\left(\int_{-\infty}^{0}\int_{-\infty}^{\tau} B(\sigma\hat{v}_- + x_-)\hat{v}_- d\sigma d\tau - \int_{0}^{+\infty}\int_{\tau}^{+\infty} B(\sigma\hat{v}_- + x_-)\hat{v}_- d\sigma d\tau\right)$$

$$+\frac{1}{|v_-|^2}\left[\int_{-\infty}^{0}\int_{-\infty}^{\tau} B(\sigma\hat{v}_- + x_-)\left(\int_{-\infty}^{\sigma} B(\eta\hat{v}_- + x_-)\hat{v}_- d\eta\right) d\sigma d\tau \right. \tag{2.45}$$

$$\left. - \int_{0}^{+\infty}\int_{\tau}^{+\infty} B(\sigma\hat{v}_- + x_-)\left(\int_{-\infty}^{\sigma} B(\eta\hat{v}_- + x_-)\hat{v}_- d\eta\right) d\sigma d\tau\right]$$

$$+\frac{1}{|v_-|^2}\sum_{k=1}^{n} \hat{v}_-^k (\Omega_{4,1,k}(v_-, x_-), \ldots, \Omega_{4,n,k}(v_-, x_-))$$

$$+\frac{1}{|v_-|^2}\left[\int_{-\infty}^{0}\int_{-\infty}^{\tau} (-\nabla V(\sigma\hat{v}_- + x_-)) d\sigma d\tau - \int_{0}^{+\infty}\int_{\tau}^{+\infty} (-\nabla V)(\sigma\hat{v}_- + x_-) d\sigma d\tau\right],$$

where

$$\Omega_{4,i,k}(v_-, x_-) = \tag{2.46}$$

$$\int_{-\infty}^{0}\int_{-\infty}^{\tau}\int_{0}^{1} \nabla B_{i,k}\left(\sigma\hat{v}_- + x_- + \frac{\varepsilon}{|v_-|}\int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1} B(\eta_2\hat{v}_- + x_-)\hat{v}_- d\eta_2 d\eta_1\right)$$

$$\circ \left(\int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1} B(\eta_2\hat{v}_- + x_-)\hat{v}_- d\eta_2 d\eta_1\right) d\varepsilon d\sigma d\tau$$

$$- \int_{0}^{+\infty}\int_{\tau}^{+\infty}\int_{0}^{1} \nabla B_{i,k}\left(\sigma\hat{v}_- + x_- + \frac{\varepsilon}{|v_-|}\int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1} B(\eta_2\hat{v}_- + x_-)\hat{v}_- d\eta_2 d\eta_1\right)$$

$$\circ \left(\int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1} B(\eta_2\hat{v}_- + x_-)\hat{v}_- d\eta_2 d\eta_1\right) d\varepsilon d\sigma d\tau$$

for $i, k = 1 \ldots n$ ($\circ$ denotes the usual scalar product on $\mathbb{R}^n$).

3 Small angle scattering and inverse scattering

3.1 *Small angle scattering.* Let the constants from (1.4)–(1.5) ($\beta_{|j|}$, α and n) and $r \in]0,1]$ be fixed, and let r_x be a nonnegative real number and let R be a positive number such that

$$R > \frac{2^{\alpha+2}\sqrt{2}\beta_1 n}{\alpha(1+\frac{r_x}{\sqrt{2}})^\alpha} \tag{3.1}$$

(see (2.16)). Consider the real numbers $z_1 = z_1(n, \beta_1, \alpha, R, r_x)$, $z_2 = z_2(n, \beta_1, \alpha, R, r, r_x)$ and $z_3 = z_3(n, \beta_1, \beta_2, \alpha, R, r_x)$ defined as the roots of the following equations

$$\frac{\rho_1(n, \beta_1, \alpha, z_1, r_x, R)}{r} = 1, \ z_1 > \sqrt{2}R, \tag{3.2}$$

$$\frac{\rho_2(n, \beta_1, \alpha, z_2, r_x, R)}{R} = 1, \ z_2 > \sqrt{2}R, \tag{3.3}$$

$$\lambda(n, \beta_1, \beta_2, \alpha, z_3, r_x, R) = 1, \ z_3 > \sqrt{2}R, \tag{3.4}$$

where ρ_1, ρ_2 and λ are respectively defined by (2.14), (2.13) and (2.27).

Note that from (2.14), (2.13) and (2.27) it follows that

$$\rho_1(n, \beta_1, \alpha, s_1, r_x, R) > \rho_1(n, \beta_1, \alpha, s_2, r_x, R), \text{ for } \sqrt{2}R < s_1 < s_2, \tag{3.5}$$

$$\rho_2(n, \beta_1, \alpha, s_1, r_x, R) > \rho_2(n, \beta_1, \alpha, s_2, r_x, R), \text{ for } \sqrt{2}R < s_1 < s_2, \tag{3.6}$$

$$\lambda(n, \beta_1, \beta_2, \alpha, s_1, r_x, R) > \lambda(n, \beta_1, \beta_2, \alpha, s_2, r_x, R), \text{ for } \sqrt{2}R < s_1 < s_2; \tag{3.7}$$

in addition

$$\frac{\rho_2(n, \beta_1, \alpha, s, r_x, R_1)}{R_1} > \frac{\rho_2(n, \beta_1, \alpha, s, r_x, R_2)}{R_2}, \text{ for } 0 < R_1 < R_2 < \frac{s}{\sqrt{2}}. \tag{3.8}$$

As it was already mentioned in Introduction, under the conditions (1.4)–(1.5), for any $(v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n$, $v_- \neq 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ with the initial conditions (1.6). Consider the function $y_-(t)$ from (1.6). This function describes deflection from free motion. Using Corollary 2.1, the lemma about contraction maps and estimate (2.11) of Lemma 2.1, and using Lemmas 2.3, 2.4 and the definition of z_1, z_2, z_3 given by (3.2)–(3.4), we obtain the following result.

Theorem 3.1. *Let conditions* (1.4)–(1.5) *be valid. Let $x_- \in \mathbb{R}^n$ and let $0 < r \le 1$. Let $R > 0$ and $v_- \in \mathbb{R}^n$ be such that R satisfies* (3.1) *(with "r_x"$= |x_-|$)*

and $|v_-| \geq \max(z_1, z_2)$, $|v_-| > z_3$, $v_- x_- = 0$, *where* $z_1 = z_1(n, \beta_1, \alpha, R, |x_-|)$, $z_2 = z_2(n, \beta_1, \alpha, R, r, |x_-|)$ *and* $z_3 = z_3(n, \beta_1, \beta_2, \alpha, R, |x_-|)$ *are respectively defined by* (3.2), (3.3) *and* (3.4). *Then the deflection* $y_-(t)$ *has the following properties:*

$$(y_-, \dot{y}_-) \in M_{T,R,r} \text{ for } T = +\infty; \tag{3.9}$$

$$|\dot{y}_-(t)| \leq \frac{2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v_-|+\sqrt{n}R)}{\alpha(\frac{|v_-|}{\sqrt{2}}-R)(1+\frac{|x_-|}{\sqrt{2}}+(\frac{|v_-|}{\sqrt{2}}-R)|t|)^{\alpha}}, \tag{3.10}$$

$$|y_-(t)| \leq \frac{2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v|+\sqrt{n}R)}{\alpha(\alpha-1)(\frac{|v|}{\sqrt{2}}-R)^2(1+\frac{|x|}{\sqrt{2}}+(\frac{|v_-|}{\sqrt{2}}-R)|t|)^{\alpha-1}}, \tag{3.11}$$

for $t \leq 0$; *in addition,* $(v_-, x_-) \in \mathcal{D}(S)$ *and*

$$y_-(t) = t a_{sc}(v_-, x_-) + b_{sc}(v_-, x_-) + y_+(t), \tag{3.12}$$

where

$$|y_+(t)| \leq \xi(n, \beta_1, \alpha, |v_-|, |x_-|, R, t), \tag{3.13}$$

$$|\dot{y}_+(t)| \leq \zeta(n, \beta_1, \alpha, |v_-|, |x_-|, R, t), \tag{3.14}$$

$$|a_{sc}(v_-, x_-)| \leq \rho_2(n, \beta_1, \alpha, |v_-|, |x_-|, R), \tag{3.15}$$

$$|a_{sc}(v_-, x_-) - w_{1,v_-,x_-}| \leq \delta_{1,1} + \delta_{1,2}, \tag{3.16}$$

$$|b_{sc}(v_-, x_-)| \leq \rho_1(n, \beta_1, \alpha, |v_-|, |x_-|, R), \tag{3.17}$$

$$|b_{sc}(v_-, x_-) - w_{2,v_-,x_-}| \leq \delta_{2,1} + \delta_{2,2}, \tag{3.18}$$

for $t \geq 0$, *where* ξ, ζ, ρ_2, ρ_1, $\delta_{1,1}$, $\delta_{1,2}$, $\delta_{2,1}$, $\delta_{2,2}$, w_{1,v_-,x_-} *and* w_{2,v_-,x_-} *are respectively defined by* (2.38), (2.37), (2.13), (2.14), (2.39), (2.41), (2.40), (2.42), (2.43) *and* (2.45) *(and where* $\mathcal{D}(S)$ *denotes the set of definition of the scattering map* S*).*

Theorem 3.1 gives, in particular, estimates for the scattering process and asymptotics for the scattering data in the following three cases: when β_1, β_2, n, $\hat{v}_-$, x_- are fixed (where $\hat{v}_- = \frac{v_-}{|v_-|}$) and $|v_-|$ increases or , e.g. β_1, β_2, n, v_-, $\hat{x}_-$ are fixed and $|x_-|$ increases, or when the parameters α, n, v_-, x_- are fixed and the norm β_m decreases to 0 (where $\beta_m = \max(\beta_0, \beta_1, \beta_2)$). In the first case, the asymptotics of the scattering data is explicitly given in Theorem 1.1 which is proved in Section 3.2. In the third case, the asymptotics of the scattering data is considered in Section 3.3 and is the “Born approximation” for the scattering data at fixed energy when the electromagnetic field is sufficiently weak.

In this three cases $\sup_{t\in\mathbb{R}} |\vartheta(t)|$ decreases, where $\vartheta(t)$ denotes the angle between the vectors $\dot{x}(t) = v_- + \dot{y}_-(t)$ and v_-, and we deal with small angle scattering. Note that already under the conditions of Theorem 3.1, without additional assumptions, there is the estimate $\sup_{t\in\mathbb{R}}|\vartheta(t)| < \frac{1}{4}\pi$ and we deal with a rather small angle scattering.

Using Theorem 3.1 we can also obtain asymptotics and estimates for small angle scattering for functions which are expressed through $a(v_-,x_-)$ and $b(v_-,x_-)$ (e.g. see [No1] for the time delay for the case $B\equiv 0$).

3.2 *Proof of Theorem 1.1.* Let $(\theta,x)\in T\mathbb{S}^{n-1}$ and let $R>\frac{2^{\alpha+2}\sqrt{2}\beta_1 n}{\alpha(1+\frac{|x|}{\sqrt{2}})^{\alpha}}$. Replacing "$(v_-,x_-)$" by $(s\theta,x)$ in Theorem 3.1, we obtain

$$(s\theta,x)\in\mathcal{D}(S), \tag{3.19}$$

$$|a_{sc}(s\theta,x)-w_{1,s\theta,x}|\le(\delta_{1,1}+\delta_{1,2})(n,\beta_1,\beta_2,\alpha,s,|x|,R), \tag{3.20}$$

$$|b_{sc}(s\theta,x)-w_{2,s\theta,x}|\le(\delta_{2,1}+\delta_{2,2})(n,\beta_1,\beta_2,\alpha,s,|x|,R), \tag{3.21}$$

for $s\in]\sqrt{2}R,+\infty[$, $s>\max\left(z_1(n,\beta_1,\alpha,R,|x|),z_2(n,\beta_1,\alpha,R,r,|x|)\right)$ and $s>z_3(n,\beta_1,\beta_2,\alpha,R,|x|)$.

We prove (1.10) and (1.12). Replacing "(v_-,x_-)" by $(s\theta,x)$ in (2.43) and (2.45) ("$|v_-|$"$=s$, "$\hat{v}_-$"$=\theta$) and using (1.4)–(1.5), we obtain

$$\lim_{s\to+\infty} w_{1,s\theta,x} = W_{1,1}(B,\theta,x), \tag{3.22}$$

$$\lim_{s\to+\infty} s w_{2,s\theta,x} = W_{2,1}(B,\theta,x). \tag{3.23}$$

Combining (3.20), (3.22), definition of $\delta_{1,1}$ (2.39) and definition $\delta_{1,2}$ (2.41) we obtain (1.10). Similarly combining (3.21), (3.23), definition of $\delta_{2,1}$ (2.40) and definition $\delta_{2,2}$ (2.42) we obtain (1.12).

We prove (1.11) and (1.13). Replacing "(v_-,x_-)" by $(s\theta,x)$ in (2.43) and (2.45) ("$|v_-|$"$=s$, "$\hat{v}_-$"$=\theta$) and using (1.4)–(1.5) and Lebesgue dominated convergence theorem, we obtain

$$\lim_{s\to+\infty} s\left(w_{1,s\theta,x}-W_{1,1}(B,\theta,x)\right) = W_{1,2}(V,B,\theta,x), \tag{3.24}$$

$$\lim_{s\to+\infty} s^2\left(w_{2,s\theta,x}-W_{2,1}(B,\theta,x)\right) = W_{2,2}(V,B,\theta,x). \tag{3.25}$$

Combining (3.20), (3.24), definition of $\delta_{1,1}$ (2.39) and definition $\delta_{1,2}$ (2.41) we obtain (1.11). Similarly combining (3.21), (3.25), definition of $\delta_{2,1}$ (2.40) and definition $\delta_{2,2}$ (2.42) we obtain (1.13). □

3.3 *The "Born approximation" for the scattering data at fixed energy.* The estimates (3.16) and (3.18) also give the asymptotics of a_{sc}, b_{sc}, when the parameters R, r, α, n, $|v_-|>\sqrt{2}R$, x_- are fixed and the norm β_m decreases to 0 (where $\beta_m = \max(\beta_0, \beta_1, \beta_2)$). Therefore Theorem 3.1 gives also the "Born approximation" for the scattering data at fixed energy when the electromagnetic field is sufficiently weak.

Let the parameters R, r, α, n, $s > \sqrt{2}R$, be fixed. Note that for fixed $(\theta, x) \in T\mathbb{S}^{n-1}$, from (2.43), (2.45), it follows that

$$\tilde{w}_{i,s\theta,x} - w_{i,s\theta,x} = O(\beta_m^2), \text{ as } \beta_m \to 0, \text{ for } i = 1, 2. \tag{3.26}$$

where vectors $\tilde{w}_{1,s\theta,x}$, $\tilde{w}_{2,s\theta,x}$, are defined by

$$\tilde{w}_{1,s\theta,x} = \int_{-\infty}^{+\infty} B(\tau\theta + x)\theta d\tau - \frac{1}{s} P(\nabla V)(\theta, x), \tag{3.27}$$

$$\begin{aligned} \tilde{w}_{2,s\theta,x} \quad &= \quad \frac{1}{s}\left(\int_{-\infty}^{0}\int_{-\infty}^{\tau} B(\sigma\theta + x)\theta d\sigma d\tau - \int_{0}^{+\infty}\int_{\tau}^{+\infty} B(\sigma\theta + x)\theta d\sigma d\tau\right) \\ &+ \frac{1}{s^2}\left[\int_{-\infty}^{0}\int_{-\infty}^{\tau}(-\nabla V(\sigma\theta + x))d\sigma d\tau - \int_{0}^{+\infty}\int_{\tau}^{+\infty}(-\nabla V)(\sigma\theta + x)d\sigma d\tau\right]. \end{aligned} \tag{3.28}$$

From (3.26) and (3.16), it follows that the leading term of the "Born approximation" for $a_{sc}(s\theta, x)$, $(\theta, x) \in T\mathbb{S}^{n-1}$, at fixed energy, is given by $\tilde{w}_{1,s\theta,x}$.

From (3.26) and (3.18), it follows that the leading term of the "Born approximation" for $b_{sc}(s\theta, x)$, $(\theta, x) \in T\mathbb{S}^{n-1}$, at fixed energy is given by $\tilde{w}_{2,s\theta,x}$.

Note that

$$P(\nabla V)(\theta, x) = -\frac{s}{2}(\tilde{w}_{1,s\theta,x} + \tilde{w}_{1,s(-\theta),x}), \tag{3.29}$$

$$W_{1,1}(B, \theta, x) = \frac{1}{2}(\tilde{w}_{1,s\theta,x} - \tilde{w}_{1,s(-\theta),x}), \tag{3.30}$$

$$\int_{-\infty}^{0}\int_{-\infty}^{\tau} B(\sigma\theta + x)\theta d\sigma d\tau - \int_{0}^{+\infty}\int_{\tau}^{+\infty} B(\sigma\theta + x)\theta d\sigma d\tau = \frac{s}{2}(\tilde{w}_{2,s\theta,x} + \tilde{w}_{2,s(-\theta),x}), \tag{3.31}$$

$$\int_{-\infty}^{0}\int_{-\infty}^{\tau}(-\nabla V)(\sigma\theta + x)d\sigma d\tau - \int_{0}^{+\infty}\int_{\tau}^{+\infty}(-\nabla V)(\sigma\theta + x)d\sigma d\tau = \frac{s^2}{2}(\tilde{w}_{2,s\theta,x} - \tilde{w}_{2,s(-\theta),x}), \tag{3.32}$$

for $s > 0$, $(\theta, x) \in T\mathbb{S}^{n-1}$ where $W_{1,1}(B, \theta, x)$ is defined by (1.10).

Using (3.29), (3.30), formula (3.37) given below and results on inversion of the X-ray transform (see Section 3.4), we obtain that for $n \geq 2$ the electromagnetic field (V, B) can be reconstructed from the leading term $\tilde{w}_{1,s\theta,x}$ of the "Born approximation" for a_{sc} at fixed energy. We can also prove that V for $n \geq 2$ can be reconstructed from the leading term $\tilde{w}_{2,s\theta,x}$ of the "Born approximation" for b_{sc} at fixed energy (see (3.32), (3.39)). For $n \geq 3$, B can be reconstructed from the leading term $\tilde{w}_{2,s\theta,x}$ of the "Born approximation" for b_{sc} at fixed energy (see (3.31) and [Jo2]). For $n = 2$ the leading term $\tilde{w}_{2,s\theta,x}$ of the "Born approximation" for b_{sc} at fixed energy does not determine uniquely B (see (3.31) and, for example, [Jo2]).

3.4 *Inversion of the X-ray transform.* We remind some results on inversion of the X-ray transform P and on reconstruction of a function f from its X-ray transform Pf when $n \geq 2$.

Let $n \geq 2$ and let $f \in C^1(\mathbb{R}^n, \mathbb{R})$ be such that

$$|\partial_x^j f(x)| \leq \gamma_{|j|}(1 + |x|)^{-1-\varepsilon-|j|}, \tag{3.33}$$

for $x \in \mathbb{R}^n$ and for $j \in \mathbb{N}^n \cup \{0\}$, $|j| \leq 1$, where ε is some positive constant and γ_l, $l = 0, 1$, are nonnegative real constants.

Then for the reconstruction of f from Pf when $n = 2$ there are, in particular, the following formulas [No1]

$$f(x) = -\frac{\partial}{\partial x_1} I_2(x) + \frac{\partial}{\partial x_2} I_1(x), \tag{3.34}$$

$$I_j(x) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{S}^1} \theta_j \left(\text{p.v.} \int_{-\infty}^{+\infty} \frac{g(\theta, q)}{x\theta^\perp - q} dq\right) d\theta, \ j = 1, 2, \tag{3.35}$$

$$g(\theta, q) = Pf(\theta, q\theta^\perp), \tag{3.36}$$

for $x \in \mathbb{R}^2$, where $\theta = (\theta_1, \theta_2)$, $\theta^\perp = (-\theta_2, \theta_1)$ and $d\theta$ denotes the standard Euclidean measure on $\mathbb{S}^1$.

Let $n \geq 3$. We reduce the reconstruction of f from Pf to the 2-dimensional case. To reconstruct f at a point $x' \in \mathbb{R}^n$ we consider in $\mathbb{R}^n$ a two-dimensional plane Y containing x'. We consider in $T\mathbb{S}^{n-1}$ the subset $T\mathbb{S}^1(Y)$ which is the set of all rays lying in Y. We restrict Pf on $T\mathbb{S}^1(Y)$ and reconstruct $f(x')$ from these data using for example the method based on formulas (3.34)–(3.36) for the reconstruction of a function from its X-ray transform in the 2-dimensional case.

3.5 *Proof of Proposition 1.1.* Now we prove Proposition 1.1 that deals with the reconstruction of the force field from the high energies asymptotics we found for the scattering data.

Let $1 \leq i < k \leq n$. Note that using the definition of $W_{1,1}$ (1.10) we obtain

$$PB_{i,k}(\theta, x) = \theta_k W_{1,1}(B, \theta, x)_i - \theta_i W_{1,1}(B, \theta, x)_k \tag{3.37}$$

for $(\theta, x) \in \mathcal{V}_{i,k}$, $i, k = 1 \dots n$, where $\mathcal{V}_{i,k}$ is the n-dimensional smooth manifold defined by

$$\mathcal{V}_{i,k} = \{(\theta, x) \in T\mathbb{S}^{n-1} | \theta_j = 0, j = 1 \dots n, j \neq i, j \neq k\}, \tag{3.38}$$

for $i, k = 1 \dots n$, $i \neq k$. (To obtain (3.37) we use the relation $\theta_i^2 + \theta_k^2 = 1$ for $(\theta, x) \in \mathcal{V}_{i,k}$, $\theta = (\theta_1, \dots, \theta_n)$.) Therefore formula (3.37) and (3.34)–(3.36) (with "f"$= B_{i,k}$) prove that $B_{i,k}$ can be reconstructed from $W_{1,1}(B, \theta, x)$, for $(\theta, x) \in T\mathbb{S}^1$ when $n = 2$. Let $n \geq 3$ and let $x' \in \mathbb{R}^n$. Consider the plane $Y_{x'} := \{x' + \mu_1 e_i + \mu_2 e_k \mid (\mu_1, \mu_2) \in \mathbb{R}^2\}$ where $(e_1, \dots, e_n)$ denotes the canonical basis of $\mathbb{R}^n$. Then using (3.37) we obtain that $PB_{i,k}$ is known on all the rays lying on $Y_{x'}$. Therefore using the scheme given in Section 3.4 for the reconstruction of a function from its X-ray transform, we obtain that $B_{i,k}(x')$ can be reconstructed from $W_{1,1}(B, \theta, x)$ given for all $(\theta, x) \in \mathcal{V}_{i,k}$.

The first item of Proposition 1.1 is proved.

Knowing B from $W_{1,1}(B, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{n-1}$ and using (1.11), we obtain that $P(\nabla V)$ is known from $W_{1,2}(V, B, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{n-1}$. Using again the results of Section 3.4, we obtain that ∇V can be reconstructed from $W_{1,1}(B, \theta, x)$ and $W_{1,2}(V, B, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{n-1}$. Item ii is proved.

We prove the third item. We assume that $n \geq 3$. The magnetic field B can be reconstructed from the vector $W_{2,1}(B, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{n-1}$ (see [Jo2]). As B is now known and $W_{2,2}(V, B, \theta, x)$ is given for all $(\theta, x) \in T\mathbb{S}^{n-1}$, from (1.13) it follows that

$$-PV(\theta, x) = \left(\int_{-\infty}^{0} \int_{-\infty}^{\tau} (-\nabla V(\sigma\theta + x)) d\sigma d\tau - \int_{0}^{+\infty} \int_{\tau}^{+\infty} (-\nabla V(\sigma\theta + x)) d\sigma d\tau \right) \circ \theta \tag{3.39}$$

is known for all $(\theta, x) \in T\mathbb{S}^{n-1}$, where $\circ$ denotes the usual scalar product on $\mathbb{R}^n$. Hence using results that are reminded in Section 3.4, we obtain that for $n \geq 3$, (V, B) can be reconstructed from $W_{2,1}(B, \theta, x)$, $W_{2,2}(V, B, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{n-1}$.

We prove the fourth item. Assume that $n = 2$. We shall prove the existence of spherical symmetric magnetic fields B_1 and B_2 satisfying (1.5) and the existence of a spherical symmetric potential V satisfying (1.4) such that $B_1 \not\equiv B_2$, $V \not\equiv 0$ and

$$W_{2,2}(V, B_1, \theta, x) = W_{2,2}(0, B_2, \theta, x), \tag{3.40}$$

for all $(\theta, x) \in T\mathbb{S}^{n-1}$. Note that if B is a spherical symmetric magnetic field satisfying (1.5), then from (1.12) it follows that $W_{2,1}(B, \theta, x) = 0$ for $(\theta, x) \in T\mathbb{S}^{n-1}$.

We denote by $C_0^\infty(\mathbb{R}^l, \mathbb{R})$ the space of infinitely smooth and compactly supported function from $\mathbb{R}^l$ to $\mathbb{R}$, where $l \geq 1$. Let $\chi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ be such that

$$\chi \not\equiv 0,\ \mathrm{supp}\chi \subseteq]0, 1[,\ \chi(x) \geq 0 \text{ for all } x \in \mathbb{R}. \tag{3.41}$$

Consider the even functions $\tilde{f}_i \in C_0^\infty(\mathbb{R}, \mathbb{R})$, $i = 1, 2$, given by the following formulas

$$\tilde{f}_i(q) = \chi(q) + \chi(-q) + \epsilon_i\chi(q - 4) + \epsilon_i\chi(-4 - q),\ \text{for } q \in \mathbb{R}, \tag{3.42}$$

where $\epsilon_1 = 1$ and $\epsilon_2 = -1$. Note that using (3.41)–(3.42) we obtain

$$\tilde{f}_1^{\,2} \equiv \tilde{f}_2^{\,2}. \tag{3.43}$$

Using the Gelfand–Graev–Helgason range characterization of the X-ray transform on the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ (see [GG], [He]), we obtain that there exists an unique function $B_{1,2}^i \in \mathcal{S}(\mathbb{R}^2)$ such that

$$PB_{1,2}^i(\theta, q\theta^\perp) = \tilde{f}_i(q),\ \text{for all } \theta \in \mathbb{S}^1,\ q \in \mathbb{R}. \tag{3.44}$$

Note that $B_{1,2}^i \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ since its X-ray transform is compactly supported on $T\mathbb{S}^2$ (see support results going back to [C], [He] for the classical 2-dimensional X-ray transform).

From (3.44), it follows that for $i = 1, 2$, the Fourier transform $\mathcal{F}B_{1,2}^i$ of the function $B_{1,2}^i$ is given by

$$\mathcal{F}B_{1,2}^i(p) = \int_{-\infty}^{+\infty} e^{-i|p|q} PB_{1,2}^i(\hat{p}^\perp, q\hat{p})dq = \int_{-\infty}^{+\infty} e^{-i|p|q}\tilde{f}_i(q)dq,$$

for $p \in \mathbb{R}^2$, $p \neq 0$, $\hat{p} = \frac{p}{|p|}$ and where $\theta^\perp = (\theta_2, -\theta_1)$ for $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1$. Hence for $i = 1, 2$, the Fourier transform $\mathcal{F}B_{1,2}^i$ is spherical symmetric. Therefore for $i = 1, 2$, $B_{1,2}^i$ is spherical symmetric and we put

$$B_{1,2}^i(x) = f_i(|x|^2) \tag{3.45}$$

for any $x \in \mathbb{R}^2$. We consider the infinitely smooth and compactly supported magnetic fields B_i, $i = 1, 2$, defined by

$$B_i(x) = f_i(|x|^2)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.46}$$

From (3.45), (3.44), (3.41)–(3.42), it follows that $B_1 \not\equiv B_2$. We also consider the potential $V \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ defined by

$$
\begin{aligned}
PV(\theta, q\theta^\perp) = &-\Bigg(\int_{-\infty}^{0}\int_{-\infty}^{\tau} f_1(\sigma^2+q^2)\left(\int_{-\infty}^{\sigma} f_1(\eta^2+q^2)d\eta\right) d\sigma d\tau\\
&- \int_{0}^{+\infty}\int_{\tau}^{+\infty} f_1(\sigma^2+q^2)\left(\int_{-\infty}^{\sigma} f_1(\eta^2+q^2)d\eta\right) d\sigma d\tau\Bigg) \qquad (3.47)\\
&+ \int_{-\infty}^{0}\int_{-\infty}^{\tau} f_2(\sigma^2+q^2)\left(\int_{-\infty}^{\sigma} f_2(\eta^2+q^2)d\eta\right) d\sigma d\tau\\
&- \int_{0}^{+\infty}\int_{\tau}^{+\infty} f_2(\sigma^2+q^2)\left(\int_{-\infty}^{\sigma} f_2(\eta^2+q^2)d\eta\right) d\sigma d\tau,
\end{aligned}
$$

for all $\theta \in \mathbb{S}^1,\ q \in \mathbb{R}$.

We shall prove (3.48) and (3.50). From (1.13), (3.39) and (3.46)–(3.47), it follows that

$$
W_{2,2}(V, B_1, \theta, q\theta^\perp) \circ \theta = W_{2,2}(0, B_2, \theta, q\theta^\perp) \circ \theta \qquad (3.48)
$$

for $q \in \mathbb{R}$, $\theta \in \mathbb{S}^1$ ($\theta = (\theta_1, \theta_2)$, $\theta^\perp = (\theta_2, -\theta_1)$).

From (3.47) it follows that V is spherical symmetric. Hence using also (1.13) and (3.46) we obtain

$$
\begin{aligned}
W_{2,2}(V, B_1, \theta, q\theta^\perp)\circ\theta^\perp = &\ 2q \int_{-\infty}^{0}\int_{-\infty}^{\tau} \frac{df_1}{ds}(s)_{|s=\sigma^2+q^2} \left(\int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1} f_1(\eta_2^2+q^2)d\eta_2 d\eta_1\right) d\sigma d\tau\\
&-2q \int_{0}^{+\infty}\int_{\tau}^{+\infty} \frac{df_1}{ds}(s)_{|s=\sigma^2+q^2} \left(\int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1} f_1(\eta_2^2+q^2)d\eta_2 d\eta_1\right) d\sigma d\tau
\end{aligned}
$$

for $\theta \in \mathbb{S}^1$, $q \in \mathbb{R}$. Let $\theta \in \mathbb{S}^1$ and $q \in \mathbb{R}$. Integrating by parts (we remind that f_1 is compactly supported), we obtain

$$
\begin{aligned}
W_{2,2}(V, B_1, \theta, q\theta^\perp) \circ \theta^\perp = &-2q \int_{-\infty}^{0} \tau \frac{df_1}{ds}(s)_{|s=\tau^2+q^2} \left(\int_{-\infty}^{\tau}\int_{-\infty}^{\eta_1} f_1(\eta_2^2+q^2)d\eta_2 d\eta_1\right) d\tau\\
&-2q \int_{0}^{+\infty} \tau \frac{df_1}{ds}(s)_{|s=\tau^2+q^2} \left(\int_{-\infty}^{\tau}\int_{-\infty}^{\eta_1} f_1(\eta_2^2+q^2)d\eta_2 d\eta_1\right) d\tau
\end{aligned}
$$

$$
= q \int_{-\infty}^{0} f_1(\tau^2+q^2)\left(\int_{-\infty}^{\tau} f_1(\eta^2+q^2)d\eta\right) d\tau + q \int_{0}^{+\infty} f_1(\tau^2+q^2)\left(\int_{-\infty}^{\tau} f_1(\eta^2+q^2)d\eta\right) d\tau
$$

$$= 2q \left(\int_0^{+\infty} f_1(\tau^2 + q^2) d\tau \right)^2 = \frac{q}{2} \tilde{f}_1(q)^2 \tag{3.49}$$

(we used the equality $\frac{d}{d\tau} f_1(\tau^2 + q^2) = 2\tau \frac{d}{ds} f_1(s)_{|s=\tau^2+q^2}$). Using (3.43), (3.45) and (3.49), we obtain

$$W_{2,2}(V, B_1, \theta, q\theta^\perp) \circ \theta^\perp = W_{2,2}(0, B_2, \theta, q\theta^\perp) \circ \theta^\perp. \tag{3.50}$$

Formulas (3.48) and (3.50) prove that $W_{2,2}(V, B_1, \theta, x) = W_{2,2}(0, B_2, \theta, x)$ for all $(\theta, x) \in T\mathbb{S}^1$.

Now it remains to prove that $V \not\equiv 0$. Using first polar coordinates and then using (3.44)–(3.45), we obtain that

$$\int_0^{+\infty} f_i(s) ds = 2 \int_0^{+\infty} r f_i(r^2) dr = \frac{1}{\pi} \int_{\mathbb{R}^2} f_i(|x|^2) dx = \frac{1}{\pi} \int_{-\infty}^{+\infty} \tilde{f}_i(q) dq, \ i = 1, 2. \tag{3.51}$$

Note that $\int_{-\infty}^{+\infty} \tilde{f}_2(q) dq = 0$ and $\int_{-\infty}^{+\infty} \tilde{f}_1(q) dq = 4 \int_{-\infty}^{+\infty} \chi(q) dq > 0$ (we used (3.41), (3.42)). Therefore from (3.51) it follows that

$$\left(\int_0^{+\infty} f_1(s) ds \right)^2 \neq \left(\int_0^{+\infty} f_2(s) ds \right)^2. \tag{3.52}$$

Note that for any $q \in \mathbb{R}$, $i = 1, 2$,

$$\begin{aligned} &\int_{-\infty}^0 \int_{-\infty}^\tau f_i(\sigma^2 + q^2) \left(\int_{-\infty}^\sigma f_i(\eta^2 + q^2) d\eta \right) d\sigma d\tau \\ &- \int_0^{+\infty} \int_\tau^{+\infty} f_i(\sigma^2 + q^2) \left(\int_{-\infty}^\sigma f_i(\eta^2 + q^2) d\eta \right) d\sigma d\tau \\ &= - \int_0^{+\infty} \int_\tau^{+\infty} f_i(\sigma^2 + q^2) \left(\int_{-\sigma}^\sigma f_i(\eta^2 + q^2) d\eta \right) d\sigma d\tau. \end{aligned} \tag{3.53}$$

Assume that $V \equiv 0$, i.e.

$$\int_0^{+\infty} \int_\tau^{+\infty} f_1(\sigma^2+q^2) \left(\int_{-\sigma}^\sigma f_1(\eta^2 + q^2) d\eta \right) d\sigma d\tau = \int_0^{+\infty} \int_\tau^{+\infty} f_2(\sigma^2+q^2) \left(\int_{-\sigma}^\sigma f_2(\eta^2 + q^2) d\eta \right) d\sigma d\tau \tag{3.54}$$

for all $q \in \mathbb{R}$ (we used (3.47), (3.53)).

For $i = 1, 2$, we consider the bounded function $F_i \in C^1([0, +\infty[, \mathbb{R})$ defined by

$$F_i(s) = - \int_s^{+\infty} f_i(t) dt, \text{ for } s \in \mathbb{R}. \tag{3.55}$$

Let $q \in \mathbb{R}$. Note that by integrating by parts, we obtain

$$\int_0^{+\infty}\int_\tau^{+\infty} f_i(\sigma^2+q^2)\left(\int_{-\sigma}^{\sigma} f_i(\eta^2+q^2)d\eta\right)d\sigma d\tau = \int_0^{+\infty}\tau f_i(\tau^2+q^2)\left(\int_{-\tau}^{\tau} f_i(\eta^2+q^2)d\eta\right)d\tau$$

$$= -\int_0^{+\infty} F_i(\tau^2+q^2)f_i(\tau^2+q^2)d\tau \tag{3.56}$$

for $i = 1, 2$ (we used the equality $\frac{d}{d\tau}F_i(\tau^2+q^2) = 2\tau f_i(\tau^2+q^2)$).

From (3.56) and (3.54) and inversion of the X-ray transform (put $g_i(x) = F_i(|x|^2)f_i(|x|^2)$, $x \in \mathbb{R}^2$, then $Pg_i(\theta, x) = \int_{-\infty}^{+\infty} F_i(\tau^2+x^2)f_i(\tau^2+x^2)d\tau$ for $(\theta, x) \in T\mathbb{S}^1$), it follows that

$$F_1(s)f_1(s) = F_2(s)f_2(s), \text{ for } s \in [0, +\infty[. \tag{3.57}$$

Using also (3.55) ($F_i(s) \to 0$ as $s \to +\infty$) and using the equality $2F_1(s)f_1(s) = \frac{dF_1^2}{ds}(s)$, $s \in \mathbb{R}$, we obtain that $F_1^2 \equiv F_2^2$. We obtain, in particular, $F_1(0)^2 = F_2(0)^2$, which with (3.55) contradicts (3.52).

Proposition 1.1 is proved. □

Remark 3.1. Note that there do not exist nontrivial spherical symmetric magnetic fields satisfying (1.5) (and (1.2)) in dimension $n \geq 3$.

Note also that using (1.13) we obtain

$$W_{2,2}(V, B, \theta, x) = W_{2,2}(V, -B, \theta, x)$$

for $(\theta, x) \in T\mathbb{S}^{n-1}$ and for (V, B) satisfying (1.4)–(1.5).

4 Proof of Lemmas 2.1, 2.2, 2.3

Throughout this Section, we omit index $_-$ for v_- and x_-.

4.1 *Preliminary estimates*

First we prove the following Lemma.

Lemma 4.1. *Let $(v, x) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $vx = 0$ and $|v| > \sqrt{2}R$. Let $T \in]-\infty, +\infty]$ and let r be a positive real number such that $r \leq 1$. Then*

$$|f(t)| \leq R|t| + r, \tag{4.1}$$

$$|h(t)| \leq R, \tag{4.2}$$

$$1 + |x + tv + f(t)| \geq \frac{1}{2}\left(1 + \frac{|x|}{\sqrt{2}} + (\frac{|v|}{\sqrt{2}} - R)|t|\right), \tag{4.3}$$

$$|v + h(t)| \leq |v| + R, \tag{4.4}$$

for any $(f,h) \in M_{T,R,r}$ *and* $t \le T$. Under the conditions (1.4)–(1.5), we have

$$|F(x,v)| \le \beta_1\sqrt{n}(1+\sqrt{n}|v|)(1+|x|)^{-\alpha-1}, \tag{4.5}$$
$$|F(x,v)-F(x',v')| \le n\beta_1 \sup_{\varepsilon\in[0,1]}(1+|x+\varepsilon(x'-x)|)^{-\alpha-1}|v-v'| \tag{4.6}$$
$$+n\beta_2|x-x'|\sup_{\varepsilon\in[0,1]}(1+|x+\varepsilon(x'-x)|)^{-\alpha-2}(1+\sqrt{n}|v+\varepsilon(v'-v)|),$$

for $x, x', v, v' \in \mathbb{R}^n$.

Proof of Lemma 4.1. Estimates (4.1) and (4.2) follow immediatly from (2.8). Estimate (4.4) follows from (4.2). Let $(f,h) \in M_{T,R,r}$ and $t \le T$. As $v \circ x = 0$, we obtain

$$|x+tv| \ge \frac{|x|}{\sqrt{2}} + |t|\frac{|v|}{\sqrt{2}}. \tag{4.7}$$

From (4.1), (4.7), it follows that

$$2(1+|x+tv+f(t)|) \ge 2+|x+tv+f(t)|)$$
$$\ge 2+|x+tv|-R|t|-r \ge 2-r+\frac{|x|}{\sqrt{2}}+|t|(\frac{|v|}{\sqrt{2}}-R). \tag{4.8}$$

Then estimate (4.3) follows from (4.8) and the estimate $r \le 1$.

Estimates (4.5)-(4.6) follow from conditions (1.4)-(1.5). □

4.2 *Proof of Lemma 2.1*

Let $(v,x) \in \mathbb{R}^n \times \mathbb{R}^n$ be fixed such that $v \circ x = 0$ and $|v| > \sqrt{2}R$. Let r be a positive number such that $r \le 1$.

Let $(f,h) \in M_{T,R,r}$. From (2.6), (4.5), (4.3) and (4.4), it follows that

$$|A^2_{v,x}(f,h)(t)| \le \beta_1\sqrt{n}\int_{-\infty}^{t}(1+\sqrt{n}|v+h(\tau)|)(1+|x+\tau v+f(\tau)|)^{-\alpha-1}d\tau$$

$$\le 2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v|+\sqrt{n}R)\int_{-\infty}^{t}(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|\tau|)^{-\alpha-1}d\tau, \tag{4.9}$$

for $t \le T$. Hence we obtain the following estimates

$$|A^2_{v,x}(f,h)(t)| \le \frac{2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v|+\sqrt{n}R)}{\alpha(\frac{|v|}{\sqrt{2}}-R)(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|t|)^{\alpha}}, \tag{4.10}$$

for $t \le 0$, $t \le T$;

$$|A^2_{v,x}(f,h)(t)| \le \frac{2^{\alpha+2}\beta_1\sqrt{n}(1+\sqrt{n}|v|+\sqrt{n}R)}{\alpha(\frac{|v|}{\sqrt{2}}-R)(1+\frac{|x|}{\sqrt{2}})^{\alpha}}, \tag{4.11}$$

for $t \geq 0$, $t \leq T$. Estimates (4.10)-(4.11) prove (2.11) and (2.13).

From (2.5) and (4.10), it follows that

$$|t||A^2_{v,x}(f,h)(t)| \leq \frac{2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v|+\sqrt{n}R)}{\alpha(\frac{|v|}{\sqrt{2}}-R)^2(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|t|)^{\alpha-1}}, \tag{4.12}$$

$$|A^1_{v,x}(f,h)(t)| \leq \frac{2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v|+\sqrt{n}R)}{\alpha(\alpha-1)(\frac{|v|}{\sqrt{2}}-R)^2(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|t|)^{\alpha-1}}, \tag{4.13}$$

for $t \leq 0$, $t \leq T$. Hence from (4.12) and (4.13), it follows that

$$|A^1_{v,x}(f,h)(t) - tA^2_{v,x}(f,h)(t)| \leq \frac{2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v|+\sqrt{n}R)}{(\alpha-1)(\frac{|v|}{\sqrt{2}}-R)^2(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|t|)^{\alpha-1}} \tag{4.14}$$

for $t \leq 0$, $t \leq T$.

Let $t \geq 0$ and $t \leq T$. Then from (2.5) and (2.6), it follows that

$$A^1_{v,x}(f,h)(t) - tA^2_{v,x}(f,h)(t) = A^1_{v,x}(f,h)(0) - \int_0^t\int_\tau^t F(x+\sigma v+f(\sigma), v+h(\sigma))d\sigma d\tau. \tag{4.15}$$

Using (4.5), (4.3) and (4.4), we obtain

$$\begin{aligned}
&\left|\int_0^t\int_\tau^t F(x+\tau v+f(\tau), v+h(\tau))d\sigma d\tau\right| \leq \\
&\beta_1\sqrt{n}\int_0^t\int_\tau^t (1+\sqrt{n}|v+h(\sigma)|)(1+|x+\sigma v+f(\sigma)|)^{-\alpha-1}d\sigma d\tau \\
&\leq \frac{2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v|+\sqrt{n}R)}{\alpha(\alpha-1)(\frac{|v|}{\sqrt{2}}-R)^2(1+\frac{|x|}{\sqrt{2}})^{\alpha-1}}.
\end{aligned} \tag{4.16}$$

From (4.13), it follows that

$$|A^1_{v,x}(f,h)(0)| \leq \frac{2^{\alpha+1}\beta_1\sqrt{n}(1+\sqrt{n}|v|+\sqrt{n}R)}{\alpha(\alpha-1)(\frac{|v|}{\sqrt{2}}-R)^2(1+\frac{|x|}{\sqrt{2}})^{\alpha-1}}. \tag{4.17}$$

From (4.15)–(4.17), it follows that

$$|A^1_{v,x}(f,h)(t) - tA^2_{v,x}(f,h)(t)| \leq \frac{2^{\alpha+2}\beta_1\sqrt{n}(1+\sqrt{n}|v|+\sqrt{n}R)}{\alpha(\alpha-1)(\frac{|v|}{\sqrt{2}}-R)^2(1+\frac{|x|}{\sqrt{2}})^{\alpha-1}}. \tag{4.18}$$

Estimates (4.14), (4.18) prove (2.12), (2.14). Lemma 2.1 is proved. □

4.3 *Proof of Lemma 2.2*

Let (f_1,h_1), $(f_2,h_2) \in M_{T,R,r}$. From (2.6) and (4.6), it follows that

$$|A^2_{v,x}(f_1,h_1)(t) - A^2_{v,x}(f_2,h_2)(t)| \leq \tag{4.19}$$
$$n\beta_1 \int\limits_{-\infty}^{t} \sup_{\varepsilon\in[0,1]} (1+|x+vt+\varepsilon f_1(\tau)+(1-\varepsilon)f_2(\tau)|)^{-\alpha-1}|h_2(\tau)-h_1(\tau)|d\tau$$
$$+n\beta_2 \int\limits_{-\infty}^{t} |f_1(\tau)-f_2(\tau)| \sup_{\varepsilon\in[0,1]} (1+|x+vt+\varepsilon f_1(\tau)+(1-\varepsilon)f_2(\tau)|)^{-\alpha-2}$$
$$\times(1+\sqrt{n}|v|+\sqrt{n}|h_1(\tau)+\varepsilon(h_2(\tau)-h_1(\tau))|)d\tau,$$

for $t \leq T$. Note that

$$|h_2(\tau)-h_1(\tau)| \leq \sup_{\sigma\in]-\infty,T]} |h_2(\sigma)-h_1(\sigma)|, \tag{4.20}$$
$$|f_2(\tau)-f_1(\tau)| \leq \sup_{\sigma\in]-\infty,T]} |f_2(\sigma)-f_1(\sigma)-\sigma(h_1(\sigma)-h_2(\sigma))| + |\tau| \sup_{\sigma\in]-\infty,T]} |h_2(\sigma)-h_1(\sigma)|, \tag{4.21}$$

for $\tau \in]-\infty,T]$.

From (4.19)-(4.21), (4.3) and (4.4), it follows that

$$|A^2_{v,x}(f_1,h_1)(t) - A^2_{v,x}(f_2,h_2)(t)|$$
$$\leq 2^{\alpha+1}n\left[\beta_1 \int\limits_{-\infty}^{t} (1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|\tau|)^{-\alpha-1}d\tau + 2\beta_2(1+\sqrt{n}|v|+\sqrt{n}R)\right.$$
$$\left.\times\int\limits_{-\infty}^{t}(1+\frac{|x|}{\sqrt{2}}+(\frac{v}{\sqrt{2}}-R)|\tau|)^{-\alpha-2}|\tau|d\tau\right] \sup_{\sigma\in]-\infty,T]} |h_2(\sigma)-h_1(\sigma)|$$
$$+2^{\alpha+2}n\beta_2(1+\sqrt{n}|v|+\sqrt{n}R) \int\limits_{-\infty}^{t}(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R|\tau|))^{-\alpha-2}d\tau$$
$$\times \sup_{\sigma\in]-\infty,T]} |f_2(\sigma)-f_1(\sigma)-\sigma(h_1(\sigma)-h_2(\sigma))| \tag{4.22}$$

(we also use the convexity of $M_{T,R,r}$, in order to estimate, for example, $|h_1(\tau)+\varepsilon(h_2(\tau)-h_1(\tau))|$ for $\tau \in]-\infty,T]$ and $\varepsilon \in [0,1]$). Hence we obtain the following estimates

$$|A^2_{v,x}(f_1,h_1)(t) - A^2_{v,x}(f_2,h_2)(t)| \le$$
$$2^{\alpha+1} n \frac{\beta_1(\frac{|v|}{\sqrt{2}} - R) + 2\beta_2(1+\sqrt{n}|v| + \sqrt{n}R)}{\alpha(\frac{|v|}{\sqrt{2}} - R)^2(1 + \frac{|x|}{\sqrt{2}} + (\frac{|v|}{\sqrt{2}} - R)|t|)^\alpha} \sup_{\sigma\in]-\infty,T]} |h_2(\sigma) - h_1(\sigma)|$$
$$+ \frac{2^{\alpha+2} n\beta_2(1+\sqrt{n}|v| + \sqrt{n}R)}{(\alpha+1)(\frac{|v|}{\sqrt{2}} - R)(1 + \frac{|x|}{\sqrt{2}} + (\frac{|v|}{\sqrt{2}} - R)|t|)^{\alpha+1}}$$
$$\times \sup_{\sigma\in]-\infty,T]} |f_2(\sigma) - f_1(\sigma) - \sigma(h_1(\sigma) - h_2(\sigma))|, \qquad (4.23)$$

for $t \le 0$, $t \le T$;

$$|A^2_{v,x}(f_1,h_1)(t) - A^2_{v,x}(f_2,h_2)(t)| \le$$
$$2^{\alpha+2} n \frac{\beta_1(\frac{|v|}{\sqrt{2}} - R) + 2\beta_2(1+\sqrt{n}|v| + \sqrt{n}R)}{\alpha(\frac{|v|}{\sqrt{2}} - R)^2(1 + \frac{|x|}{\sqrt{2}})^\alpha} \sup_{\sigma\in]-\infty,T]} |h_2(\sigma) - h_1(\sigma)|$$
$$+ \frac{2^{\alpha+3} n\beta_2(1+\sqrt{n}|v| + \sqrt{n}R)}{(\alpha+1)(\frac{|v|}{\sqrt{2}} - R)(1 + \frac{|x|}{\sqrt{2}})^{\alpha+1}}$$
$$\times \sup_{\sigma\in]-\infty,T]} |f_2(\sigma) - f_1(\sigma) - \sigma(h_1(\sigma) - h_2(\sigma))|, \qquad (4.24)$$

for $t \ge 0$, $t \le T$. Estimates (4.23), (4.24) prove (2.17), (2.19).

From (4.23), it follows that

$$|A^1_{v,x}(f_1,h_1)(t) - A^1_{v,x}(f_2,h_2)(t)| \le$$
$$2^{\alpha+1} n \frac{\beta_1(\frac{|v|}{\sqrt{2}} - R) + 2\beta_2(1+\sqrt{n}|v| + \sqrt{n}R)}{\alpha(\alpha-1)(\frac{|v|}{\sqrt{2}} - R)^3(1 + \frac{|x|}{\sqrt{2}} + (\frac{|v|}{\sqrt{2}} - R)|t|)^{\alpha-1}} \sup_{\sigma\in]-\infty,T]} |h_2(\sigma) - h_1(\sigma)|$$
$$+ \frac{2^{\alpha+2} n\beta_2(1+\sqrt{n}|v| + \sqrt{n}R)}{(\alpha+1)\alpha(\frac{|v|}{\sqrt{2}} - R)^2(1 + \frac{|x|}{\sqrt{2}} + (\frac{|v|}{\sqrt{2}} - R)|t|)^{\alpha}}$$
$$\times \sup_{\sigma\in]-\infty,T]} |f_2(\sigma) - f_1(\sigma) - \sigma(h_1(\sigma) - h_2(\sigma))|, \qquad (4.25)$$

$$|t||A^2_{v,x}(f_1,h_1)(t) - A^2_{v,x}(f_2,h_2)(t)| \le$$
$$2^{\alpha+1} n \frac{\beta_1(\frac{|v|}{\sqrt{2}} - R) + 2\beta_2(1+\sqrt{n}|v| + \sqrt{n}R)}{\alpha(\frac{|v|}{\sqrt{2}} - R)^3(1 + \frac{|x|}{\sqrt{2}} + (\frac{|v|}{\sqrt{2}} - R)|t|)^{\alpha-1}} \sup_{\sigma\in]-\infty,T]} |h_2(\sigma) - h_1(\sigma)|$$
$$+ \frac{2^{\alpha+2} n\beta_2(1+\sqrt{n}|v| + \sqrt{n}R)}{(\alpha+1)(\frac{|v|}{\sqrt{2}} - R)^2(1 + \frac{|x|}{\sqrt{2}} + (\frac{|v|}{\sqrt{2}} - R)|t|)^{\alpha}}$$
$$\times \sup_{\sigma\in]-\infty,T]} |f_2(\sigma) - f_1(\sigma) - \sigma(h_1(\sigma) - h_2(\sigma))|, \qquad (4.26)$$

for $t \le 0$, $t \le T$. Using (4.25)-(4.26), we obtain

$$|A^1_{v,x}(f_1,h_1)(t) - A^1_{v,x}(f_2,h_2)(t) - t(A^2_{v,x}(f_1,h_1)(t) - A^2_{v,x}(f_2,h_2)(t))| \le$$
$$2^{\alpha+1} n \frac{\beta_1(\frac{|v|}{\sqrt{2}}-R)+2\beta_2(1+\sqrt{n}|v|+\sqrt{n}R)}{(\alpha-1)(\frac{|v|}{\sqrt{2}}-R)^3(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|t|)^{\alpha-1}} \sup_{\sigma\in]-\infty,T]} |h_2(\sigma)-h_1(\sigma)| \quad (4.27)$$
$$+\frac{2^{\alpha+2} n\beta_2(1+\sqrt{n}|v|+\sqrt{n}R)}{\alpha(\frac{|v|}{\sqrt{2}}-R)^2(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|t|)^{\alpha}} \sup_{\sigma\in]-\infty,T]} |f_2(\sigma)-f_1(\sigma)-\sigma(h_1(\sigma)-h_2(\sigma))|,$$

for $t \le 0$, $t \le T$. Estimate (2.18) follows from (4.27).

From (4.15), it follows that

$$|A^1_{v,x}(f_1,h_1)(t) - A^1_{v,x}(f_2,h_2)(t) - t(A^2_{v,x}(f_1,h_1)(t) - A^2_{v,x}(f_2,h_2)(t))|$$
$$\le |A^1_{v,x}(f_1,h_1)(0) - A^1_{v,x}(f_2,h_2)(0)| \quad (4.28)$$
$$+\int_0^t\int_\tau^t |F(x+\tau v+f_1(\tau), v+h_1(\tau)) - F(x+\tau v+f_2(\tau), v+h_2(\tau))| d\sigma d\tau$$

for $t \ge 0$, $t \le T$. Using (4.25), we obtain

$$|A^1_{v,x}(f_1,h_1)(0) - A^1_{v,x}(f_2,h_2)(0)| \le \frac{\lambda_2}{2} \sup_{\sigma\in]-\infty,T]} |h_2(\sigma)-h_1(\sigma)|$$
$$+\frac{\lambda_1}{2} \sup_{\sigma\in]-\infty,T]} |f_2(\sigma)-f_1(\sigma)-\sigma(h_1(\sigma)-h_2(\sigma))|, \quad (4.29)$$

where $\lambda_2 = \lambda_2(n,\beta_1,\beta_2,\alpha,|v|,|x|,R)$ and $\lambda_1 = \lambda_1(n,\beta_2,\alpha,|v|,|x|,R)$ are defined by (2.25). From (4.6), (4.3), (4.4) and (4.20)-(4.21), it follows that

$$\int_0^t \int_\tau^t |F(x+sv+f_1(s), v+h_1(s)) - F(x+sv+f_2(s), v+h_2(s))| ds d\tau$$

$$\le \frac{\lambda_2}{2} \sup_{\sigma\in]-\infty,T]} |h_2(\sigma)-h_1(\sigma)| + \frac{\lambda_1}{2} \sup_{\sigma\in]-\infty,T]} |f_2(\sigma)-f_1(\sigma)-\sigma(h_1(\sigma)-h_2(\sigma))|, \quad (4.30)$$

for $t \ge 0$, $t \le T$.

From (4.28)–(4.30), we obtain

$$|A^1_{v,x}(f_1,h_1)(t) - A^1_{v,x}(f_2,h_2)(t) - t(A^2_{v,x}(f_1,h_1)(t) - A^2_{v,x}(f_2,h_2)(t))|$$

$$\le \lambda_2 \sup_{\sigma\in]-\infty,T]} |h_2(\sigma)-h_1(\sigma)| + \lambda_1 \sup_{\sigma\in]-\infty,T]} |f_2(\sigma)-f_1(\sigma)-\sigma(h_1(\sigma)-h_2(\sigma))|, \quad (4.31)$$

for $t \ge 0$, $t \le T$. Estimate (2.20) follows from (4.31).

Lemma 2.2 is proved. □

4.4 *Proof of Lemma 2.3*

Note that using (4.15) and (2.6) we obtain

$$A^1_{v,x}(f,h)(t) = \int_{-\infty}^{0}\int_{-\infty}^{\tau} F(x+\sigma v+f(\sigma), v+h(\sigma))d\sigma d\tau \tag{4.32}$$

$$-\int_{0}^{+\infty}\int_{\tau}^{+\infty} F(x+\sigma v+f(\sigma), v+h(\sigma))d\sigma d\tau$$

$$+t\int_{-\infty}^{+\infty} F(x+\tau v+f(\tau), v+h(\tau))d\tau + \int_{t}^{+\infty}\int_{\tau}^{+\infty} F(x+\sigma v+f(\sigma), v+h(\sigma))d\sigma d\tau$$

for $t \in \mathbb{R}$ and $(f,h) \in M_{T,R,r}$, $T = +\infty$.

Let $T = +\infty$. As $\max(\frac{\rho_2}{R}, \frac{\rho_1}{r}) \le 1$, using Corollary 2.1 we obtain

$$A_{v,x}(f',h') \in M_{T,R,r}, \text{ for any } (f',h') \in M_{T,R,r}. \tag{4.33}$$

Let $(f,h) \in M_{T,R,r}$. Using (4.33) and replacing (f,h) by $A_{v,x}(f,h)$ in (4.32), we obtain (2.31). Estimates (2.35)–(2.38) follow from (2.32)–(2.34), (4.33), (4.5), (4.3) and (4.4) (where we replace (f,h) by $A_{v,x}(f,h)$). Note that from (2.32) and (2.33) (and (1.4)–(1.5)) it follows that

$$k_{v,x}(f',h') = \lim_{t\to+\infty} A^2_{v,x}(A_{v,x}(f',h'))(t), \tag{4.34}$$

$$l_{v,x}(f',h') = \lim_{t\to+\infty} A^1_{v,x}(A_{v,x}(f',h'))(t) - tA^2_{v,x}(A_{v,x}(f',h'))(t), \tag{4.35}$$

for any $(f',h') \in M_{T,R,r}$.

We prove (2.39). The proof of (2.40) is similar to the proof of (2.39). Using (4.34), (4.33) and applying (2.19) ("$(f_1,h_1) = A_{v,x}(f,h)$" and "$(f_2,h_2) = A_{v,x}(0,0)$"), we obtain

$$|k_{v,x}(f,h) - k_{v,x}(0,0)| \le \lambda_4 \sup_{t\in]-\infty,+\infty[} |A^2_{v_-,x_-}(f,h)(t) - A^2_{v_-,x_-}(0,0)(t)|$$

$$+\lambda_3 \sup_{t\in]-\infty,+\infty[} |\left(A^1_{v_-,x_-}(f,h) - A^1_{v_-,x_-}(0,0)\right)(t)$$

$$-t\left(A^2_{v_-,x_-}(f,h) - A^2_{v_-,x_-}(0,0)\right)(t)|. \tag{4.36}$$

Using (4.36), (2.19)–(2.20), we obtain

$$|k_{v_-,x_-}(f,h) - k_{v_-,x_-}(0,0)| \le (\lambda_2\lambda_3 + \lambda_4^2) \sup_{t\in]-\infty,+\infty[} |h(t)|$$

$$+(\lambda_1\lambda_3 + \lambda_3\lambda_4) \sup_{t\in]-\infty,+\infty[} |f(t) - th(t)|. \tag{4.37}$$

Assume that $(f,h) = A_{v,x}(f,h)$. Then from (2.13)–(2.14), it follows that

$$|h(t)| \leq \rho_2 \text{ and } |f(t) - th(t)| \leq \rho_1 \text{ for } t \in \mathbb{R}.$$

These two latter estimates with (4.37) prove (2.39). $\square$

5 Proof of Lemma 2.4

Throughout this Section, we omit index $_-$ for v_- and x_-.

We shall prove (2.41). Note that using changes of variables and the equality $B_{i,k}(x+\sigma v+\omega) = B_{i,k}(x+\sigma v) + \int_0^1 \nabla B_{i,k}(x+\sigma v+\varepsilon\omega)\circ\omega d\varepsilon$ for $\sigma \in \mathbb{R}$, $\omega \in \mathbb{R}^n$ (where $\circ$ denotes the usual scalar product on $\mathbb{R}^n$), we obtain

$$\begin{aligned} w_{1,v,x} &= -\int_{-\infty}^{+\infty}\nabla V(sv+x)ds + \int_{-\infty}^{+\infty}B(sv+x)\left(\int_{-\infty}^{s} B(\tau v+x)vd\tau\right)ds \\ &+ \int_{-\infty}^{+\infty}B\left(sv+x+\int_{-\infty}^{s}\int_{-\infty}^{\tau} B(\sigma v+x)vd\sigma d\tau\right)vds. \end{aligned}$$

Therefore, from (2.32) and (2.6) (we remind that $F(x,v) = -\nabla V(x) + B(x)v$) it follows that

$$|k_{v,x}(0,0) - w_{1,v,x}| \leq \sum_{i=1}^{4}\Delta_{1,i}, \tag{5.1}$$

where

$$\Delta_{1,1} = \int_{-\infty}^{+\infty}\left|\nabla V(sv+x+A^1_{v,x}(0,0)(s)) - \nabla V(sv+x)\right|ds, \tag{5.2}$$

$$\Delta_{1,2} = \int_{-\infty}^{+\infty}\left|B(sv+x+A^1_{v,x}(0,0)(s))\left(\int_{-\infty}^{s}\nabla V(\tau v+x)d\tau\right)\right|ds, \tag{5.3}$$

$$\Delta_{1,3} = \int_{-\infty}^{+\infty}\left|\big(B(sv+x+A^1_{v,x}(0,0)(s)) - B(sv+x)\big)\left(\int_{-\infty}^{s}B(\tau v+x)vd\tau\right)\right|ds, \tag{5.4}$$

$$\Delta_{1,4} = \int_{-\infty}^{+\infty} \Bigg| \Bigg(B(sv + x + A^1_{v,x}(0,0)(s)) - B\left(sv + x + \int_{-\infty}^{s}\int_{-\infty}^{\tau} B(\sigma v + x) v d\sigma d\tau \right)\Bigg) v \Bigg| ds. \tag{5.5}$$

We shall estimate each $\Delta_{1,i}$, $i = 1\ldots 4$. First note that using Corollary 2.1 and the inequality $\max(\frac{\rho_2}{R}, \frac{\rho_1}{r}) \leq 1$ we obtain, in particular,

$$A_{v,x}(0,0) \in M_{T,R,r},\ T = +\infty. \tag{5.6}$$

Note also that using (1.4)–(1.5) and the estimate $|x + \sigma v| \geq \frac{|x|}{\sqrt{2}} + |\sigma|\frac{|v|}{\sqrt{2}}$, $\sigma \in \mathbb{R}$ (we remind that $x \circ v = 0$), we obtain

$$|\nabla V(\sigma v + x)| \leq \beta_1 \sqrt{n}(1 + \frac{|x|}{\sqrt{2}} + |\sigma|\frac{|v|}{\sqrt{2}})^{-\alpha-1}, \tag{5.7}$$

$$|B(\sigma v + x)v| \leq \beta_1 n |v|(1 + \frac{|x|}{\sqrt{2}} + |\sigma|\frac{|v|}{\sqrt{2}})^{-\alpha-1}, \tag{5.8}$$

for $\sigma \in \mathbb{R}$.

We remind that

$$\begin{aligned} A^1_{v,x}(0,0)(s) &= \int_{-\infty}^{s}\int_{-\infty}^{\tau} (-\nabla V)(\sigma v + s) d\sigma d\tau \\ &\quad + \int_{-\infty}^{s}\int_{-\infty}^{\tau} B(\sigma v + x) v d\sigma d\tau, \end{aligned} \tag{5.9}$$

for $s \in \mathbb{R}$.

We shall use the following estimate (5.10): from (5.7)-(5.9), it follows that

$$\begin{aligned} |A^1_{v,x}(0,0)(s)| &\leq \beta_1\sqrt{n}(1 + \sqrt{n}|v|) \int_{-\infty}^{s}\int_{-\infty}^{\tau} (1 + \frac{|x|}{\sqrt{2}} + |\sigma|\frac{|v|}{\sqrt{2}})^{-\alpha-1} d\sigma d\tau \\ &\leq \beta_1\sqrt{n}(1 + \sqrt{n}|v|) \left(\int_{-\infty}^{0}\int_{-\infty}^{\tau} (1 + \frac{|x|}{\sqrt{2}} + |\sigma|\frac{|v|}{\sqrt{2}})^{-\alpha-1} d\sigma d\tau \right. \\ &\quad \left. + |s| \int_{-\infty}^{+\infty} (1 + \frac{|x|}{\sqrt{2}} + |\tau|\frac{|v|}{\sqrt{2}})^{-\alpha-1} d\tau \right) \\ &\leq \frac{2\beta_1\sqrt{n}(1 + \sqrt{n}|v|)}{\alpha(\alpha - 1)|v|^2(1 + \frac{|x|}{\sqrt{2}})^{\alpha-1}} + |s| \frac{2\sqrt{2}\beta_1\sqrt{n}(1 + \sqrt{n}|v|)}{\alpha|v|(1 + \frac{|x|}{\sqrt{2}})^{\alpha}}, \end{aligned} \tag{5.10}$$

for $s \in \mathbb{R}$.

Using (1.4), (5.6) and (4.3), we obtain

$$\begin{aligned} &\left|\nabla V(\sigma v + x + A^1_{v,x}(0,0)(\sigma)) - \nabla V(\sigma v + x)\right| \\ &\le 2^{\alpha+2}\beta_2 n(1 + \frac{|x|}{\sqrt{2}} + (\frac{|v|}{\sqrt{2}} - R)|\sigma|)^{-\alpha-2}|A^1_{v,x}(0,0)(\sigma)|, \end{aligned} \tag{5.11}$$

for all $\sigma \in \mathbb{R}$.

From (5.2), (5.10), (5.11), it follows that

$$\begin{aligned} \Delta_{1,1} &\le n2^{\alpha+2}\beta_2 \int_{-\infty}^{+\infty} (1 + \frac{|x|}{\sqrt{2}} + (\frac{|v|}{\sqrt{2}} - R)|s|)^{-\alpha-2}|A^1_{v,x}(0,0)(s)|ds \\ &\le \frac{2^{\alpha+3}n^{3/2}(2\alpha^2 + \alpha - 2)\beta_1\beta_2(1 + \sqrt{n}|v|)}{(\alpha-1)\alpha^2(\alpha+1)\frac{|v|}{\sqrt{2}}(\frac{|v|}{\sqrt{2}} - R)^2(1 + \frac{|x|}{\sqrt{2}})^{2\alpha}}. \end{aligned} \tag{5.12}$$

Similarly, by using (5.4) (and by using (1.5) instead of (1.4)) and (5.8) we obtain

$$\begin{aligned} \Delta_{1,3} &\le 2^{\alpha+2}\beta_1\beta_2 n^{\frac{5}{2}} \int_{-\infty}^{+\infty} (1 + (\frac{|v|}{\sqrt{2}} - R)|s| + \frac{|x|}{\sqrt{2}})^{-\alpha-2}|A^1_{v,x}(0,0)(s)| \\ &\quad \times \left(\int_{-\infty}^{s} (1 + \frac{|v|}{\sqrt{2}}|\tau| + \frac{|x|}{\sqrt{2}})^{-\alpha-1}|v|d\tau\right) ds \\ &\le \frac{2^{\alpha+4}\sqrt{2}(2\alpha^2 + \alpha - 2)n^3\beta_1^2\beta_2(1 + \sqrt{n}|v|)}{(\alpha-1)\alpha^3(\alpha+1)\frac{|v|}{\sqrt{2}}(\frac{|v|}{\sqrt{2}} - R)^2(1 + \frac{|x|}{\sqrt{2}})^{3\alpha}}. \end{aligned} \tag{5.13}$$

Using (5.3), (1.4)-(1.5), (5.6) and (4.3), we obtain

$$\begin{aligned} \Delta_{1,2} &\le \beta_1^2 n^{3/2}2^{\alpha+1} \int_{-\infty}^{+\infty} (1 + \frac{|x|}{\sqrt{2}} + (\frac{|v|}{\sqrt{2}} - R)|s|)^{-\alpha-1} \\ &\quad \times \left(\int_{-\infty}^{s} (1 + \frac{|x|}{\sqrt{2}} + \frac{|v|}{\sqrt{2}}|\tau|)^{-\alpha-1}d\tau\right) ds \\ &\le \frac{n^{3/2}2^{\alpha+3}\beta_1^2}{\alpha^2\frac{|v|}{\sqrt{2}}(\frac{|v|}{\sqrt{2}} - R)(1 + \frac{|x|}{\sqrt{2}})^{2\alpha}}. \end{aligned}$$

Using (5.5), (5.9), growth property of the elements of B (1.5), and using

(1.4) and the assumption $\max(\frac{\rho_1}{r},\frac{\rho_2}{R})\le 1$ and (4.3), we obtain

$$\begin{aligned}\Delta_{1,4} &\le |v|n^{3/2}2^{\alpha+2}\beta_2\int_{-\infty}^{+\infty}(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|s|)^{-\alpha-2}\\ &\quad\times\left(\int_{-\infty}^{s}\int_{-\infty}^{\tau}|\nabla V(x+\sigma v)|\,d\sigma d\tau\right)ds\\ &\le |v|n^{2}2^{\alpha+2}\beta_1\beta_2\int_{-\infty}^{+\infty}(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|s|)^{-\alpha-2}\\ &\quad\times\left(\int_{-\infty}^{s}\int_{-\infty}^{\tau}(1+\frac{|x|}{\sqrt{2}}+\frac{|v|}{\sqrt{2}}|\sigma|)^{-\alpha-1}d\sigma d\tau\right)ds\\ &\le \frac{n^2 2^{\alpha+3}\sqrt{2}(2\alpha^2+\alpha-2)\beta_1\beta_2}{\alpha^2(\alpha+1)(\alpha-1)(\frac{|v|}{\sqrt{2}}-R)^2(1+\frac{|x|}{\sqrt{2}})^{2\alpha}}.\end{aligned}\tag{5.14}$$

Estimate (2.41) follows from (5.1) and (5.12)-(5.14).

We shall prove (2.42). Note that using changes of variables and the equality $B_{i,k}(x+\sigma v+\omega)=B_{i,k}(x+\sigma v)+\int_0^1\nabla B_{i,k}(x+\sigma v+\varepsilon\omega)\circ\omega d\varepsilon$ for $\sigma\in\mathbb{R}$, $\omega\in\mathbb{R}^n$ (where $\circ$ denotes the usual scalar product on $\mathbb{R}^n$), we obtain

$$\begin{aligned}w_{2,v,x} &= \int_{-\infty}^{0}\int_{-\infty}^{s}B\left(\sigma v+x+\int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1}B(\eta_2 v+x)vd\eta_2 d\eta_1\right)vd\sigma ds\\ &\quad-\int_{0}^{+\infty}\int_{s}^{+\infty}B\left(\sigma v+x+\int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1}B(\eta_2 v+x)vd\eta_2 d\eta_1\right)vd\sigma ds\\ &\quad+\int_{-\infty}^{0}\int_{-\infty}^{\tau}B(\sigma v+x)\left(\int_{-\infty}^{\sigma}B(\eta v+x)vd\eta\right)d\sigma d\tau\\ &\quad-\int_{0}^{+\infty}\int_{\tau}^{+\infty}B(\sigma v+x)\left(\int_{-\infty}^{\sigma}B(\eta v+x)vd\eta\right)d\sigma d\tau\\ &\quad+\int_{-\infty}^{0}\int_{-\infty}^{s}(-\nabla V(\sigma v+x))d\sigma ds-\int_{0}^{+\infty}\int_{s}^{+\infty}(-\nabla V)(\sigma v+x)d\sigma ds,\end{aligned}$$

Therefore, from (2.33) and (2.6) (we remind that $F(x,v)=-\nabla V(x)+B(x)v$) it follows that

$$|l_{v,x}(0,0)-w_{2,v,x}|\le\sum_{i=1}^{6}\Delta_{2,i},\tag{5.15}$$

where

$$\Delta_{2,1} = \int_{-\infty}^{0}\int_{-\infty}^{s}\left|\nabla V(\sigma v + x + A^1_{v,x}(0,0)(\sigma)) - \nabla V(\sigma v + x)\right| d\sigma ds, \tag{5.16}$$

$$\Delta_{2,2} = \int_{0}^{+\infty}\int_{s}^{+\infty}\left|\nabla V(\sigma v + x + A^1_{v,x}(0,0)(\sigma)) - \nabla V(\sigma v + x)\right| d\sigma ds, \tag{5.17}$$

$$\begin{aligned}\Delta_{2,3} \;\;=\;\; & \Bigg|\int_{-\infty}^{0}\int_{-\infty}^{\tau} B(\sigma v + x + A^1_{v,x}(0,0)(\sigma)) \left(\int_{-\infty}^{\sigma}\nabla V(\eta v + x) d\eta\right) d\sigma d\tau \\ & - \int_{0}^{+\infty}\int_{\tau}^{+\infty} B(\sigma v + x + A^1_{v,x}(0,0)(\sigma)) \left(\int_{-\infty}^{\sigma}\nabla V(\eta v + x) d\eta\right) d\sigma d\tau\Bigg|,\end{aligned} \tag{5.18}$$

$$\begin{aligned}\Delta_{2,4} = & \Bigg|\int_{-\infty}^{0}\int_{-\infty}^{\tau}\big(B(\sigma v + x + A^1_{v,x}(0,0)(\sigma)) - B(\sigma v + x)\big)\left(\int_{-\infty}^{\sigma} B(\eta v + x) v d\eta\right) d\sigma d\tau \\ & - \int_{0}^{+\infty}\int_{\tau}^{+\infty}\big(B(\sigma v + x + A^1_{v,x}(0,0)(\sigma)) - B(\sigma v + x)\big)\left(\int_{-\infty}^{\sigma} B(\eta v + x) v d\eta\right) d\sigma d\tau\Bigg|,\end{aligned} \tag{5.19}$$

$$\begin{aligned}\Delta_{2,5} \;\;=\;\; & \Bigg|\int_{-\infty}^{0}\int_{-\infty}^{\tau}\Bigg(B(\sigma v + x + A^1_{v,x}(0,0)(\sigma)) \\ & - B\left(\sigma v + x + \int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1} B(\eta_2 v + x) v d\eta_2 d\eta_1\right)\Bigg) v d\sigma d\tau\Bigg|,\end{aligned} \tag{5.20}$$

$$\begin{aligned}\Delta_{2,6} \;\;=\;\; & \Bigg|\int_{0}^{+\infty}\int_{\tau}^{+\infty}\Bigg(B(\sigma v + x + A^1_{v,x}(0,0)(\sigma)) \\ & - B\left(\sigma v + x + \int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1} B(\eta_2 v + x) v d\eta_2 d\eta_1\right)\Bigg) v d\sigma d\tau\Bigg|.\end{aligned} \tag{5.21}$$

We shall estimate each $\Delta_{2,i}$ for $i = 1\ldots 6$. From (5.16), (5.17), (5.10) and (5.11) it follows that

$$\max(\Delta_{2,1},\Delta_{2,2}) \le \frac{\beta_1\beta_2 n^{3/2}2^{\alpha+2}(2\alpha+3)(1+\sqrt{n}|v|)}{(\alpha-1)\alpha^2(\alpha+1)\frac{|v|}{\sqrt{2}}(\frac{|v|}{\sqrt{2}}-R)^3(1+\frac{|x|}{\sqrt{2}})^{2\alpha-1}}. \qquad (5.22)$$

Using (1.4), (1.5), (5.6), (4.3) and (5.7) we obtain that

$$\begin{aligned}&\left|B(\sigma v + x + A^1_{v,x}(0,0)(\sigma))\int_{-\infty}^{\sigma}(-\nabla V)(\eta v + x)d\eta\right|\\ &\le n^{3/2}2^{\alpha+1}\beta_1^2(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|\sigma|)^{-\alpha-1}\\ &\times\int_{-\infty}^{\sigma}(1+\frac{|x|}{\sqrt{2}}+\frac{|v|}{\sqrt{2}}|\eta|)^{-\alpha-1}d\eta \qquad (5.23)\end{aligned}$$

for all $\sigma \in \mathbb{R}$. From (5.23) and (5.18) it follows that

$$\Delta_{2,3} \le \frac{3n^{3/2}\beta_1^2 2^{\alpha+1}}{(\alpha-1)\alpha^2\frac{|v|}{\sqrt{2}}(\frac{|v|}{\sqrt{2}}-R)^2(1+\frac{|x|}{\sqrt{2}})^{2\alpha-1}}. \qquad (5.24)$$

Using growth properties of B (1.5), (5.6), (4.3), and (5.8) we obtain

$$\begin{aligned}&\left|\left(B(\sigma v + x + A^1_{v,x}(0,0)(\sigma)) - B(\sigma v + x)\right)\left(\int_{-\infty}^{\sigma}B(\eta v + x)v d\eta\right)\right|\\ &\le n^{5/2}\beta_1\beta_2 2^{\alpha+2}|v|(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|\sigma|)^{-\alpha-2}|A^1_{v,x}(0,0)(\sigma)|\\ &\times\int_{-\infty}^{\sigma}(1+\frac{|x|}{\sqrt{2}}+\frac{|v|}{\sqrt{2}}|\eta|)^{-\alpha-1}d\eta,\end{aligned}$$

for all $\sigma \in \mathbb{R}$.

Using also (5.10), we obtain

$$\Delta_{2,4} \le \frac{3n^3(2\alpha+3)\beta_1^2\beta_2 2^{\alpha+2}\sqrt{2}(1+\sqrt{n}|v|)}{(\alpha-1)\alpha^3(\alpha+1)\frac{|v|}{\sqrt{2}}(\frac{|v|}{\sqrt{2}}-R)^3(1+\frac{|x|}{\sqrt{2}})^{3\alpha-1}}. \qquad (5.25)$$

From growth property of B (1.5), and from the inequality $\max(\frac{\rho_1}{r},\frac{\rho_2}{R}) \le 1$, (5.9), (1.5), (4.3) and (5.7), it follows that

$$\left|\left(B(\sigma v + x + A^1_{v,x}(0,0)(\sigma)) - B(\sigma v + x + \int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1}B(\eta_2 v + x)v d\eta_2 d\eta_1)\right)v\right|$$

$$\leq \; n^{3/2}\beta_2 2^{\alpha+2}|v|(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|\sigma|)^{-\alpha-2}\left|\int\limits_{-\infty}^{\sigma}\int\limits_{-\infty}^{\eta_1}\nabla V(\eta_2 v+x)d\eta_2 d\eta_1\right|$$

$$\leq \; n^2\beta_1\beta_2 2^{\alpha+2}|v|(1+\frac{|x|}{\sqrt{2}}+(\frac{|v|}{\sqrt{2}}-R)|\sigma|)^{-\alpha-2}$$

$$\times\int_{-\infty}^{\sigma}\int_{-\infty}^{\eta_1}(1+\frac{|x|}{\sqrt{2}}+\frac{|v|}{\sqrt{2}}|\eta_2|)^{-\alpha-1}d\eta_2 d\eta_1, \qquad (5.26)$$

for all $\sigma\in\mathbb{R}$. Therefore by using (5.20)-(5.21) we obtain

$$\Delta_{2,5}\leq\frac{2^{\alpha+2}\sqrt{2}n^2\beta_1\beta_2}{(\alpha-1)\alpha^2(\alpha+1)\frac{|v|}{\sqrt{2}}(\frac{|v|}{\sqrt{2}}-R)^2(1+\frac{|x|}{\sqrt{2}})^{2\alpha-1}}; \qquad (5.27)$$

and

$$\Delta_{2,6}\leq\frac{2^{\alpha+2}\sqrt{2}n^2(2\alpha+3)\beta_1\beta_2}{(\alpha-1)\alpha^2(\alpha+1)(\frac{|v|}{\sqrt{2}}-R)^3(1+\frac{|x|}{\sqrt{2}})^{2\alpha-1}}. \qquad (5.28)$$

Estimate (2.42) follows from (5.15), (5.22), (5.24), (5.25), (5.27) and (5.28). □

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