

On inverse scattering in electromagnetic field in classical relativistic mechanics at high energies

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Abstract. We consider the multidimensional Newton-Einstein equation in static electromagnetic field

$$\begin{aligned} \dot{p} &= F(x, \dot{x}), \quad F(x, \dot{x}) = -\nabla V(x) + \frac{1}{c} B(x) \dot{x}, \\ p &= \frac{\dot{x}}{\sqrt{1 - \frac{|\dot{x}|^2}{c^2}}}, \quad \dot{p} = \frac{dp}{dt}, \quad \dot{x} = \frac{dx}{dt}, \quad x \in C^1(\mathbb{R}, \mathbb{R}^d), \end{aligned} \quad (*)$$

where $V \in C^2(\mathbb{R}^d, \mathbb{R})$, $B(x)$ is the $d \times d$ real antisymmetric matrix with elements $B_{i,k}(x) = \frac{\partial}{\partial x_i} \mathbf{A}_k(x) - \frac{\partial}{\partial x_k} \mathbf{A}_i(x)$, and $|\partial_x^j \mathbf{A}_i(x)| + |\partial_x^j V(x)| \leq \beta_{|j|} (1 + |x|)^{-(\alpha + |j|)}$ for $x \in \mathbb{R}^d$, $|j| \leq 2$, $i = 1..d$ and some $\alpha > 1$. We give estimates and asymptotics for scattering solutions and scattering data for the equation (*) for the case of small angle scattering. We show that at high energies the velocity valued component of the scattering operator uniquely determines the X-ray transforms $P\nabla V$ and $PB_{i,k}$ for $i, k = 1..d$, $i \neq k$. Applying results on inversion of the X-ray transform P we obtain that for $d \geq 2$ the velocity valued component of the scattering operator at high energies uniquely determines (V, B) . In addition we show that our high energy asymptotics found for the configuration valued component of the scattering operator doesn't determine uniquely V when $d \geq 2$ and B when $d = 2$ but that it uniquely determines B when $d \geq 3$.

1 Introduction

1.1 The Newton-Einstein equation.

Consider the multidimensional Newton-Einstein equation in static electromagnetic field

$$\begin{aligned} \dot{p} &= F(x, \dot{x}), \quad F(x, \dot{x}) = -\nabla V(x) + \frac{1}{c} B(x) \dot{x}, \\ p &= \frac{\dot{x}}{\sqrt{1 - \frac{|\dot{x}|^2}{c^2}}}, \quad \dot{p} = \frac{dp}{dt}, \quad \dot{x} = \frac{dx}{dt}, \quad x \in C^1(\mathbb{R}, \mathbb{R}^d), \end{aligned} \quad (1.1)$$

where $V \in C^2(\mathbb{R}^d, \mathbb{R})$, $B(x)$ is the $d \times d$ real antisymmetric matrix with elements $B_{i,k}(x) = \frac{\partial}{\partial x_i} \mathbf{A}_k(x) - \frac{\partial}{\partial x_k} \mathbf{A}_i(x)$, $\mathbf{A} = (\mathbf{A}_1, \dots, \mathbf{A}_d) \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ and

$$|\partial_x^j \mathbf{A}_i(x)| + |\partial_x^j V(x)| \leq \beta_{|j|} (1 + |x|)^{-(\alpha + |j|)} \quad (1.2)$$

for $x \in \mathbb{R}^d$, $|j| \leq 2$, $i = 1..d$ and some $\alpha > 1$ (here j is the multiindex $j \in (\mathbb{N} \cup \{0\})^d$, $|j| = \sum_{n=1}^d j_n$ and $\beta_{|j|}$ are positive real constants and $B(x)\dot{x} = \left(\sum_{l=1}^d B_{1,l}(x)\dot{x}_l, \dots, \sum_{l=1}^d B_{d,l}(x)\dot{x}_l\right)$). The equation (1.1) is an equation for $x = x(t)$ and is the equation of motion in \mathbb{R}^d of a relativistic particle of mass $m = 1$ and charge $e = 1$ in an external electromagnetic field described by V and \mathbf{A} (see [E] and, for example, Section 17 of [LL2]). In this equation x is the position of the particle, p is its impulse, F is the force acting on the particle, t is the time and c is the speed of light.

For the equation (1.1) the energy

$$E = c^2 \sqrt{1 + \frac{|p(t)|^2}{c^2}} + V(x(t))$$

is an integral of motion. Note that the energy E does not depend on \mathbf{A} because the magnetic force $(1/c)B(x)\dot{x}$ is orthogonal to the velocity \dot{x} of the particle.

1.2 Yajima's results.

Yajima [Y] studied in dimension 3 (without loss of generality for the case of dimension $d \geq 2$) the direct scattering of relativistic particle in an external electromagnetic field described by four vector $(V(x), \mathbf{A}(x))$ where the scalar potential V and the vector potential \mathbf{A} are both rapidly decreasing. We recall results of Yajima [Y] in our case. We denote by B_c the euclidean open ball whose radius is c and whose centre is 0.

Under the conditions (1.2), the following is valid (see [Y]): for any $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \neq 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^d)$ such that

$$x(t) = v_- t + x_- + y_-(t), \quad (1.3)$$

where $\dot{y}_-(t) \rightarrow 0$, $y_-(t) \rightarrow 0$, as $t \rightarrow -\infty$; in addition for almost any $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \neq 0$,

$$x(t) = v_+ t + x_+ + y_+(t), \quad (1.4)$$

where $v_+ \neq 0$, $|v_+| < c$, $v_+ = a(v_-, x_-)$, $x_+ = b(v_-, x_-)$, $\dot{y}_+(t) \rightarrow 0$, $y_+(t) \rightarrow 0$, as $t \rightarrow +\infty$.

The map $S : B_c \times \mathbb{R}^d \rightarrow B_c \times \mathbb{R}^d$ given by the formulas

$$v_+ = a(v_-, x_-), \quad x_+ = b(v_-, x_-) \quad (1.5)$$

is called the scattering map for the equation (1.1). The functions $a(v_-, x_-)$, $b(v_-, x_-)$ are called the scattering data for the equation (1.1).

By $\mathcal{D}(S)$ we denote the domain of definition of S ; by $\mathcal{R}(S)$ we denote the range of S (by definition, if $(v_-, x_-) \in \mathcal{D}(S)$, then $v_- \neq 0$ and $a(v_-, x_-) \neq 0$).

Under the conditions (1.2), the map S has the following simple properties (see [Y]): $\mathcal{D}(S)$ is an open subset of $B_c \times \mathbb{R}^d$ and $\text{Mes}((B_c \times \mathbb{R}^d) \setminus \mathcal{D}(S)) = 0$ for

the Lebesgue measure on $B_c \times \mathbb{R}^d$ induced by the Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$; the map $S : \mathcal{D}(S) \rightarrow \mathcal{R}(S)$ is continuous and preserves the element of volume, $a(v_-, x_-)^2 = v_-^2$.

1.3 A representation of the scattering data.

If $V(x) \equiv 0$ and $B(x) \equiv 0$, then $a(v_-, x_-) = v_-$, $b(v_-, x_-) = x_-$, $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \neq 0$. Therefore for $a(v_-, x_-)$, $b(v_-, x_-)$ we will use the following representation

$$\begin{aligned} a(v_-, x_-) &= v_- + a_{sc}(v_-, x_-) \\ b(v_-, x_-) &= x_- + b_{sc}(v_-, x_-) \end{aligned} \quad (v_-, x_-) \in \mathcal{D}(S). \quad (1.6)$$

We will use the fact that, under the conditions (1.2), the map S is uniquely determined by its restriction to $\mathcal{M}(S) = \mathcal{D}(S) \cap \mathcal{M}$, where

$$\mathcal{M} = \{(v_-, x_-) \in B_c \times \mathbb{R}^d \mid v_- \neq 0, v_- x_- = 0\}.$$

This observation is completely similar to the related observation of [No1], [Jo2] and is based on the fact that if $x(t)$ satisfies (1.1), then $x(t+t_0)$ also satisfies (1.1) for any $t_0 \in \mathbb{R}$.

1.4 X-ray transform.

Consider

$$T\mathbb{S}^{d-1} = \{(\theta, x) \mid \theta \in \mathbb{S}^{d-1}, x \in \mathbb{R}^d, \theta x = 0\},$$

where \mathbb{S}^{d-1} is the unit sphere in \mathbb{R}^d .

Consider the X-ray transform P which maps each function f with the properties

$$f \in C(\mathbb{R}^d, \mathbb{R}^m), \quad |f(x)| = O(|x|^{-\beta}), \quad \text{as } |x| \rightarrow \infty, \quad \text{for some } \beta > 1$$

into a function $Pf \in C(T\mathbb{S}^{d-1}, \mathbb{R}^m)$, where Pf is defined by

$$Pf(\theta, x) = \int_{-\infty}^{+\infty} f(t\theta + x) dt, \quad (\theta, x) \in T\mathbb{S}^{d-1}.$$

Concerning the theory of the X-ray transform, the reader is referred to [GGG], [Na], [No1].

1.5 Main results of the work.

The main results of the present work consist in the small angle scattering estimates and asymptotics for the scattering data a_{sc} and b_{sc} (and scattering solutions) for the equation (1.1) and in application of these asymptotics and estimates to inverse scattering for the equation (1.1) at high energies. Our main results include, in particular, Theorem 1.1, Propositions 1.1, 1.2, formulated below in this subsection and Theorems 3.1, 3.2 given in Section 3.

Theorem 1.1. *Let the conditions (1.2) be valid and $(\theta, x) \in TS^{d-1}$. Let r be a positive constant such that $0 < r \leq 1$, $r < c/\sqrt{2}$. Then*

$$\lim_{\substack{s \rightarrow c \\ s < c}} \frac{s}{\sqrt{1 - \frac{s^2}{c^2}}} a_{sc}(s\theta, x) = \int_{-\infty}^{+\infty} F(\tau\theta + x, c\theta) d\tau, \quad (1.7a)$$

and, in addition,

$$\left| \int_{-\infty}^{+\infty} F(\tau\theta + x, s\theta) d\tau - \frac{s}{\sqrt{1 - \frac{s^2}{c^2}}} a_{sc}(s\theta, x) \right| \leq \frac{C_1}{\sqrt{1 + \frac{s^2}{4(c^2 - s^2)}}} \quad (1.7b)$$

for $s_1 < s < c$, where $C_1 = C_1(c, d, \beta_0, \beta_1, \beta_2, \alpha, |x|, r)$ and $s_1 = s_1(c, d, \beta_1, \beta_2, \alpha, |x|, r)$ are defined in Section 4 (in subsection 4.3);

$$\begin{aligned} \lim_{\substack{s \rightarrow c \\ s < c}} \frac{s^2}{\sqrt{1 - \frac{s^2}{c^2}}} b_{sc}(s\theta, x) &= \int_{-\infty}^0 \int_{-\infty}^{\tau} F(\sigma\theta + x, c\theta) d\sigma d\tau \\ &\quad - \int_0^{+\infty} \int_{\tau}^{+\infty} F(\sigma\theta + x, c\theta) d\sigma d\tau + PV(\theta, x)\theta, \end{aligned} \quad (1.8a)$$

and, in addition,

$$\begin{aligned} \left| \frac{b_{sc}(s\theta, x)}{\sqrt{1 - \frac{s^2}{c^2}}} - \frac{1}{c^2} PV(\theta, x)\theta + \frac{1}{s^2} \int_0^{+\infty} \int_{\tau}^{+\infty} F(\sigma\theta + x, s\theta) d\sigma d\tau \right. \\ \left. - \frac{1}{s^2} \int_{-\infty}^0 \int_{-\infty}^{\tau} F(\sigma\theta + x, s\theta) d\sigma d\tau \right| \leq C_2 \sqrt{1 - \frac{s^2}{c^2}} \end{aligned} \quad (1.8b)$$

for $s_2 < s < c$, and where $C_2 = C_2(c, d, \beta_0, \beta_1, \beta_2, \alpha, |x|, r)$ and $s_2 = s_2(c, d, \beta_1, \beta_2, \alpha, |x|, r)$ are defined in Section 4 (in subsection 4.3).

Theorem 1.1 follows from Theorem 3.1 and Theorem 3.2 given in Section 3.

Remark 1.1. The estimates (1.7b) and (1.8b) (with the explicit values of s_1 , s_2 , C_1 and C_2 given in subsection 4.3) also give the asymptotics of a_{sc} , b_{sc} , when the parameters c , r , α , d , $s > \sqrt{2}r$, θ , x are fixed and the norm β_m decreases to 0 (where $\beta_m = \max(\beta_0, \beta_1, \beta_2)$). Therefore Theorem 1.1 gives also the ‘‘Born approximation’’ for the scattering data at fixed energy when the electromagnetic field is sufficiently weak. The leading term of this approximation is completely similar to the leading term at high energies. Therefore all results of the present work on finding the electromagnetic field (respectively, B) from leading term of the scattering data a_{sc} (respectively, b_{sc}) at high energies remain valid also for finding

the electromagnetic field from a_{sc} (respectively, b_{sc}) at fixed energy in small norm Born approximation. Note also that

$$\left(\frac{1}{c^2} PV(\theta, x)\theta + \frac{1}{s^2} \int_0^{+\infty} \int_{-\infty}^{+\infty} F(\sigma\theta + x, s\theta) d\sigma d\tau - \frac{1}{s^2} \int_{-\infty}^0 \int_{-\infty}^{\tau} F(\sigma\theta + x, s\theta) d\sigma d\tau \right) \star \theta = \left(\frac{1}{c^2} - \frac{1}{s^2} \right) PV(\theta, x),$$

for $0 < s < c$, $(\theta, x) \in T\mathbb{S}^{d-1}$, where \star denotes the usual scalar product on \mathbb{R}^d . Hence from (1.8b) and results on inversion of the X-ray transform (see [R], [GGG], [Na], [No1]) it follows that for $d \geq 2$ the leading term of the ‘‘Born approximation’’ for b_{sc} at fixed energy uniquely determines V .

Consider the vector-functions $w_1(V, \mathbf{A}, \theta, x)$ and $w_2(V, \mathbf{A}, \theta, x)$, $(\theta, x) \in T\mathbb{S}^{d-1}$, arising in the right-hand sides of (1.7a) and (1.8a):

$$w_1(V, \mathbf{A}, \theta, x) = \int_{-\infty}^{+\infty} F(\tau\theta + x, c\theta) d\tau \quad (1.9a)$$

$$w_2(V, \mathbf{A}, \theta, x) = \int_{-\infty}^0 \int_{-\infty}^{\tau} F(\sigma\theta + x, c\theta) d\sigma d\tau - \int_0^{+\infty} \int_{\tau}^{+\infty} F(\sigma\theta + x, c\theta) d\sigma d\tau + PV(\theta, x)\theta. \quad (1.9b)$$

Remark 1.2. Using, in particular, that B is antisymmetric one can see that the vectors $w_1(V, \mathbf{A}, \theta, x)$, $w_2(V, \mathbf{A}, \theta, x)$ are orthogonal to θ for any $(\theta, x) \in T\mathbb{S}^{d-1}$ and any potential (V, \mathbf{A}) satisfying (1.2).

Consider the d -dimensional smooth manifolds

$$\mathcal{V}_{i,k} = \{(\theta, x) \in T\mathbb{S}^{d-1} | \theta_j = 0, j = 1 \dots d, j \neq i, j \neq k\}, \quad (1.11)$$

for $i, k = 1..d$, $i \neq k$.

Proposition 1.1. *Let $(V, \mathbf{A}) \in C^2(\mathbb{R}^d, \mathbb{R}) \times C^2(\mathbb{R}^d, \mathbb{R}^d)$ satisfy (1.2). Then $w_1(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ uniquely determines (V, B) and the following formulas are valid:*

$$P(\nabla V)(\theta, x) = -\frac{1}{2}(w_1(V, \mathbf{A}, \theta, x) + w_1(V, \mathbf{A}, -\theta, x)), \quad (1.12a)$$

for $(\theta, x) \in T\mathbb{S}^{d-1}$, $i, k = 1..d$, $i \neq k$;

$$PB_{i,k}(\theta, x) = \theta_k \frac{1}{2}(w_1(V, \mathbf{A}, \theta, x)_i - w_1(V, \mathbf{A}, -\theta, x)_i) - \theta_i \frac{1}{2}(w_1(V, \mathbf{A}, \theta, x)_k - w_1(V, \mathbf{A}, -\theta, x)_k) \quad (1.12b)$$

for $(\theta, x) \in \mathcal{V}_{i,k}$, $i, k = 1..d$, $i \neq k$.

Proposition 1.2. *Let $(V, \mathbf{A}) \in C^2(\mathbb{R}^d, \mathbb{R}) \times C^2(\mathbb{R}^d, \mathbb{R}^d)$ satisfy (1.2). Then $w_2(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ does not determine uniquely V . For $d = 2$, $w_2(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ does not determine uniquely B . For $d \geq 3$, $w_2(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ uniquely determines B .*

In Section 5 (see Proposition 5.3) we give a formula ((5.10a)) which shows that for $d = 3$, the Fourier transform of the first derivatives of B can be reconstructed from $w_2(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ and we give a formula ((5.10b)) which shows that for $d \geq 4$ the X-ray transform of B can be reconstructed from $w_2(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$.

Proposition 1.1 and Proposition 1.2 are proved in Section 5.

From (1.7a), (1.9a) and (1.12) and inversion formulas for the X-ray transform P for $d \geq 2$ (see [R], [GGG], [Na], [No1]) it follows that a_{sc} determines uniquely ∇V and B at high energies. Moreover for $d \geq 2$ methods of reconstruction of f from Pf (see [R], [GGG], [Na], [No1]) permit to reconstruct ∇V and B from the velocity valued component a of the scattering map at high energies. The formula (1.8a) and Proposition 1.2 show that the first term of the asymptotics of b_{sc} doesn't determine uniquely the potential V when $d \geq 2$ and B when $d = 2$ but that it uniquely determines B when $d \geq 3$. Note that F. Nicoleau paid our attention to the fact that, in addition of Proposition 1.2, the vector function $w_2(V, \theta, x)$, $(\theta, x) \in T\mathbb{S}^{d-1}$, uniquely determines V modulo spherical symmetric potentials when $d \geq 2$, and that $w_2(V, \theta, x)$, $(\theta, x) \in T\mathbb{S}^{d-1}$, uniquely determines B modulo spherical symmetric magnetic fields when $d = 2$.

Remark 1.3. The condition (1.2) in all results and estimates which appear in Introduction and in Sections 2, 3, 4 can be weakened to condition (4.11) given at the end of Section 4.

1.6 Historical remarks.

Note that inverse scattering for the classical multidimensional Newton equation at high energies was first studied by Novikov [No1] without magnetic field. Novikov proved two formulas which link scattering data at high energies to the X-ray transform of $-\nabla V$ and V . Following Novikov's framework [No1], the author generalized these two formulas to the relativistic case without magnetic field in [Jo2]. We shall follow the same way to obtain Theorem 1.1 of the present work. Note that a first version of this paper was available as preprint [Jo1]. Note also that for the classical multidimensional Newton equation in a bounded open strictly convex domain an inverse boundary value problem at high energies was first studied in [GN].

To our knowledge the inverse scattering problem for a particle in electromagnetic field in classical and classical relativistic mechanics was not considered in the literature for the case of nonzero magnetic field B before the preprint [Jo1] of the present article (concerning results given in the literature on this problem for $B \equiv 0$ see [No1], [Jo2] and references therein). However, in quantum mechanics the

inverse scattering problem for a particle in electromagnetic field with $B \neq 0$ was considered, in particular, in [HN], [ER1], [I], [Ju], [ER2], [Ni], [A], [Ha], [WY] (concerning results given in the literature on this problem for $B \equiv 0$ see, in addition, [F], [EW], [No2] and references given in [No2]).

1.7 Structure of the paper.

Further, our paper is organized as follows. In Section 2 we transform the differential equation (1.1) with initial conditions (1.3) into a system of integral equations which takes the form $(y_-, \dot{y}_-) = A_{v_-, x_-}(y_-, \dot{y}_-)$. Then we study A_{v_-, x_-} on a suitable space and we give estimates and contraction estimates about A_{v_-, x_-} (Lemmas 2.1, 2.2, 2.3). In Section 3 we give estimates and asymptotics for the deflection $y_-(t)$ from (1.3) and for scattering data $a_{sc}(v_-, x_-)$, $b_{sc}(v_-, x_-)$ from (1.6) (Theorem 3.1 and Theorem 3.2). From these estimates and asymptotics the two formulas (1.7a) and (1.8a) will follow when the parameters c , β_m , α , d , \hat{v}_- , x_- are fixed and $|v_-|$ increases (where $\beta_{|j|}$, α , d are constants from (1.2), $\beta_m = \max(\beta_0, \beta_1, \beta_2)$; $\hat{v}_- = v_-/|v_-|$). In these cases $\sup |\theta(t)|$ decreases, where $\theta(t)$ denotes the angle between the vectors $\dot{x}(t) = v_- + \dot{y}_-(t)$ and v_- , and we deal with small angle scattering. Note that, under the conditions of Theorem 3.1, without additional assumptions, there is the estimate $\sup |\theta(t)| < \frac{1}{4}\pi$ and we deal with rather small angle scattering (concerning the term “small angle scattering” see [No1] and Section 20 of [LL1]). Theorem 1.1 follows from Theorem 3.1 and Theorem 3.2. In Section 4 we sketch the proof of Lemmas 2.1, 2.2, 2.3 and Theorem 3.2. Section 5 is devoted to Proofs of Proposition 1.1 and Proposition 1.2.

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2 A contraction map

Let us transform the differential equation (1.1) in a system of integral equations. Consider the function $g : \mathbb{R}^d \rightarrow B_c$ defined by

$$g(x) = \frac{x}{\sqrt{1 + \frac{|x|^2}{c^2}}}$$

where $x \in \mathbb{R}^d$. One can see that g has, in particular, the following simple properties:

$$|g(x) - g(y)| \leq \sqrt{d}|x - y|, \text{ for } x, y \in \mathbb{R}^d, \quad (2.1)$$

g is an infinitely smooth diffeomorphism between \mathbb{R}^d and B_c , its inverse is given by

$$\gamma(x) = \frac{x}{\sqrt{1 - \frac{|x|^2}{c^2}}}, x \in B_c.$$

Now, if x satisfies the differential equation (1.1) and the initial conditions (1.3), then x satisfies the system of integral equations

$$x(t) = v_- t + x_- + \int_{-\infty}^t \left[g \left(\gamma(v_-) + \int_{-\infty}^{\tau} F(x(s), \dot{x}(s)) ds \right) - v_- \right] d\tau, \quad (2.2a)$$

$$\dot{x}(t) = g(\gamma(v_-) + \int_{-\infty}^t F(x(s), \dot{x}(s)) ds), \quad (2.2b)$$

where $F(x, \dot{x}) = -\nabla V(x) + \frac{1}{c} B(x) \dot{x}$, $v_- \in B_c \setminus \{0\}$.

For $y_-(t)$ of (1.3) this system takes the form

$$(y_-(t), u_-(t)) = A_{v_-, x_-}(y_-, u_-)(t), \quad (2.3)$$

where $u_-(t) = \dot{y}_-(t)$ and

$$A_{v_-, x_-}(f, h)(t) = (A_{v_-, x_-}^1(f, h)(t), A_{v_-, x_-}^2(f, h)(t))$$

$$A_{v_-, x_-}^1(f, h)(t) = \int_{-\infty}^t \left[g(\gamma(v_-) + \int_{-\infty}^{\tau} F(v_- s + x_- + f(s), v_- + h(s)) ds) - v_- \right] d\tau,$$

$$A_{v_-, x_-}^2(f, h)(t) = g(\gamma(v_-) + \int_{-\infty}^t F(v_- s + x_- + f(s), v_- + h(s)) ds) - v_-,$$

for $v_- \in B_c \setminus \{0\}$.

From (2.3), (1.2), (2.1) (applied on “ x ” = $\gamma(v_-) + \int_{-\infty}^{\tau} F(v_- s + x_- + y_-(s), v_- + \dot{y}_-(s)) ds$ and “ y ” = $\gamma(v_-)$) and $y_-(t) \in C^1(\mathbb{R}, \mathbb{R}^d)$, $|y_-(t)| + |\dot{y}_-(t)| \rightarrow 0$, as $t \rightarrow -\infty$, it follows, in particular, that

$$\begin{aligned} (y_-(t), \dot{y}_-(t)) &\in C(\mathbb{R}, \mathbb{R}^d) \times C(\mathbb{R}, \mathbb{R}^d) \\ \text{and } |\dot{y}_-(t)| &= O(|t|^{-\alpha}), |y_-(t)| = O(|t|^{-\alpha+1}), \text{ as } t \rightarrow -\infty, \end{aligned} \quad (2.4)$$

where $v_- \in B_c \setminus \{0\}$ and x_- are fixed.

Consider the complete metric space

$$M_{T,r} = \{(f, h) \in C(]-\infty, T], \mathbb{R}^d) \times C(]-\infty, T], \mathbb{R}^d) \mid \|(f, h)\|_T \leq r\},$$

$$\text{where } \|(f, h)\|_T = \max \left(\sup_{t \in]-\infty, T]} |h(t)|, \sup_{t \in]-\infty, T]} |f(t) - th(t)| \right) \quad (2.5)$$

(where for $T = +\infty$ we understand $]-\infty, T]$ as $]-\infty, +\infty[$). From (2.4) it follows that, at fixed $T < +\infty$,

$$(y_-(t), \dot{y}_-(t)) \in M_{T,r} \text{ for some } r \text{ depending on } y_-(t) \text{ and } T. \quad (2.6)$$

Let $z_1(c, d, \beta_1, \alpha, r_x, r)$ be defined as the root of the following equation

$$\frac{z_1}{\sqrt{1 - \frac{z_1^2}{c^2}}} - \frac{2^{\alpha+5}\beta_1 d(2 + r/c)}{\alpha(z_1/\sqrt{2} - r)(r_x/\sqrt{2} + 1)^\alpha} = 0, \quad z_1 \in]\sqrt{2}r, c[, \quad (2.7)$$

where r_x and r are some nonnegative numbers such that $0 < r \leq 1$, $r < c/\sqrt{2}$.

Lemma 2.1. *Under the conditions (1.2), the following is valid: if $(f, h) \in M_{T,r}$, $0 < r \leq 1$, $r < c/\sqrt{2}$, $x_- \in \mathbb{R}^d$, $v_- \in B_c$, $|v_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $v_- x_- = 0$, then*

$$\begin{aligned} \|A_{v_-, x_-}(f, h)\|_T &\leq \rho_T(c, d, \beta_1, \alpha, |v_-|, |x_-|, r) & (2.8a) \\ &= \frac{1}{\sqrt{1 + |v_-|^2/(4(c^2 - |v_-|^2))}} \\ &\quad \times \frac{2^{\alpha+2}d\sqrt{d}\beta_1(2 + r/c)(|v_-|/\sqrt{2} + 1 - r)}{(\alpha - 1)(|v_-|/\sqrt{2} - r)^2(1 + |x_-|/\sqrt{2} - (|v_-|/\sqrt{2} - r)T)^{\alpha-1}} \end{aligned}$$

for $T \leq 0$,

$$\begin{aligned} \|A_{v_-, x_-}(f, h)\|_T &\leq \rho(c, d, \beta_1, \alpha, |v_-|, |x_-|, r) & (2.8b) \\ &= \frac{1}{\sqrt{1 + |v_-|^2/(4(c^2 - |v_-|^2))}} \\ &\quad \times \frac{2^{\alpha+3}d\sqrt{d}\beta_1(2 + r/c)(|v_-|/\sqrt{2} + 1 - r)}{(\alpha - 1)(|v_-|/\sqrt{2} - r)^2(1 + |x_-|/\sqrt{2})^{\alpha-1}} \end{aligned}$$

for $T \leq +\infty$; if $(f_1, h_1), (f_2, h_2) \in M_{T,r}$, $0 < r \leq 1$, $r < c/\sqrt{2}$, $|v_-| < c$, $v_- x_- = 0$, $|v_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, then

$$\begin{aligned} \|A_{v_-, x_-}(f_2, h_2) - A_{v_-, x_-}(f_1, h_1)\|_T & & (2.9a) \\ &\leq \lambda_T(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r)\|(f_2 - f_1, h_2 - h_1)\|_T, \end{aligned}$$

$$\begin{aligned} \lambda_T(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r) &= \frac{1}{\sqrt{1 + |v_-|^2/(4(c^2 - |v_-|^2))}} \\ &\quad \times \frac{2^{\alpha+4}d^2\tilde{\beta}(1 + \frac{1}{c})(\frac{|v_-|}{\sqrt{2}} + 1 - r)^2}{(\alpha - 1)(\frac{|v_-|}{\sqrt{2}} - r)^3(1 + \frac{|x_-|}{\sqrt{2}} - (\frac{|v_-|}{\sqrt{2}} - r)T)^{\alpha-1}} \end{aligned}$$

for $T \leq 0$,

$$\|A_{v_-, x_-}(f_2, h_2) - A_{v_-, x_-}(f_1, h_1)\|_T \leq \lambda(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r)\|(f_2 - f_1, h_2 - h_1)\|_T, \quad (2.9b)$$

$$\lambda(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r) = \frac{1}{\sqrt{1 + |v_-|^2/(4(c^2 - |v_-|^2))}} \times \frac{2^{2\alpha+9} 3d^3 \tilde{\beta} (1 + \tilde{\beta}) (1 + 1/c)^3 (|v_-|/\sqrt{2} + 1 - r)^3}{(\alpha - 1)(|v_-|/\sqrt{2} - r)^4 (1 + |x_-|/\sqrt{2})^{\alpha-1}}$$

for $T \leq +\infty$, where $\tilde{\beta} = \max(\beta_1, \beta_2)$.

Note that

$$\begin{aligned} & \max \left(\frac{\rho_T(c, d, \beta_1, \alpha, |v_-|, |x_-|, r)}{r}, \lambda_T(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r) \right) \\ & \leq \mu_T(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r) \\ & = \frac{1}{\sqrt{1 + |v_-|^2/(4(c^2 - |v_-|^2))}} \\ & \quad \times \frac{2^{\alpha+4} d^2 \tilde{\beta} (1 + 1/c) (|v_-|/\sqrt{2} + 1 - r)^2}{r(\alpha - 1)(|v_-|/\sqrt{2} - r)^3 (1 + |x_-|/\sqrt{2} - (|v_-|/\sqrt{2} - r)T)^{\alpha-1}} \end{aligned} \quad (2.10a)$$

for $T \leq 0$,

$$\begin{aligned} & \max \left(\frac{\rho(c, d, \beta_1, \alpha, |v_-|, |x_-|, r)}{r}, \lambda(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r) \right) \\ & \leq \mu(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r) \\ & = \frac{1}{\sqrt{1 + |v_-|^2/(4(c^2 - |v_-|^2))}} \\ & \quad \times \frac{2^{2\alpha+9} 3d^3 \tilde{\beta} (1 + \tilde{\beta}) (1 + 1/c)^3 (|v_-|/\sqrt{2} + 1 - r)^3}{r(\alpha - 1)(|v_-|/\sqrt{2} - r)^4 (1 + |x_-|/\sqrt{2})^{\alpha-1}} \end{aligned} \quad (2.10b)$$

for $T \leq +\infty$, where $\tilde{\beta} = \max(\beta_1, \beta_2)$, $0 < r \leq 1$, $r < c/\sqrt{2}$, $|v_-| < c$, $|v_-| \geq z_1$, $v_- x_- = 0$.

From Lemma 2.1 and the estimates (2.10) we obtain the following result.

Corollary 2.1. *Under the conditions (1.2), $0 < r \leq 1$, $r < c/\sqrt{2}$, $x_- \in \mathbb{R}^d$, $v_- \in B_c$, $|v_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $v_- x_- = 0$, the following result is valid:*

if $\mu_T(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r) < 1$, then A_{v_-, x_-} is a contraction map in $M_{T, r}$ for $T \leq 0$;

if $\mu(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r) < 1$, then A_{v_-, x_-} is a contraction map in $M_{T, r}$ for $T \leq +\infty$.

Taking into account (2.6) and using Lemma 2.1, Corollary 2.1 and the lemma about the contraction maps we will study the solution $(y_-(t), u_-(t))$ of the equation (2.3) in $M_{T, r}$.

We will use also the following results.

Lemma 2.2. *Under the conditions (1.2), $(f, h) \in M_{T,r}$, $0 < r \leq 1$, $r < c/\sqrt{2}$, $x_- \in \mathbb{R}^d$, $v_- \in B_c$, $|v_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $v_- x_- = 0$, the following is valid:*

$$\begin{aligned} |A_{v_-, x_-}^2(f, h)(t)| &\leq \zeta_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t) \\ &= \frac{1}{\sqrt{1 + |v_-|^2/(4(c^2 - |v_-|^2))}} \\ &\quad \times \frac{d\sqrt{d}\beta_1 2^{\alpha+2}(2 + r/c)}{\alpha(|v_-|/\sqrt{2} - r)(1 + |x_-|/\sqrt{2} - (|v_-|/\sqrt{2} - r)t)^\alpha}, \end{aligned} \quad (2.11)$$

$$\begin{aligned} |A_{v_-, x_-}^1(f, h)(t)| &\leq \xi_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t) \\ &= \frac{1}{\sqrt{1 + |v_-|^2/(4(c^2 - |v_-|^2))}} \\ &\quad \times \frac{d\sqrt{d}\beta_1 2^{\alpha+2}(2 + r/c)}{\alpha(\alpha - 1)(\frac{|v_-|}{\sqrt{2}} - r)^2(1 + \frac{|x_-|}{\sqrt{2}} - (\frac{|v_-|}{\sqrt{2}} - r)t)^{\alpha-1}}, \end{aligned} \quad (2.12)$$

for $t \leq T$, $T \leq 0$;

$$A_{v_-, x_-}^1(f, h)(t) = k_{v_-, x_-}(f, h)t + l_{v_-, x_-}(f, h) + H_{v_-, x_-}(f, h)(t), \quad (2.13)$$

where

$$k_{v_-, x_-}(f, h) = g(\gamma(v_-)) + \int_{-\infty}^{+\infty} F(v_-s + x_- + f(s), v_- + h(s)) ds - v_-, \quad (2.14a)$$

$$\begin{aligned} l_{v_-, x_-}(f, h) &= \int_{-\infty}^0 \left[g(\gamma(v_-)) + \int_{-\infty}^\tau F(v_-s + x_- + f(s), v_- + h(s)) ds - v_- \right] d\tau \\ &\quad + \int_0^{+\infty} \left[g(\gamma(v_-)) + \int_{-\infty}^\tau F(v_-s + x_- + f(s), v_- + h(s)) ds \right. \\ &\quad \left. - g(\gamma(v_-)) + \int_{-\infty}^{+\infty} F(v_-s + x_- + f(s), v_- + h(s)) ds \right] d\tau, \end{aligned} \quad (2.14b)$$

$$|k_{v_-, x_-}(f, h)| \leq 2\zeta_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, 0), \quad (2.15a)$$

$$|l_{v_-, x_-}(f, h)| \leq 2\xi_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, 0), \quad (2.15b)$$

$$\begin{aligned} |\dot{H}_{v_-, x_-}(f, h)(t)| &\leq \zeta_+(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t) \\ &= \frac{1}{\sqrt{1 + |v_-|^2/(4(c^2 - |v_-|^2))}\alpha(|v_-|/\sqrt{2} - r)} \end{aligned} \quad (2.16)$$

$$\begin{aligned}
& \times \frac{d\sqrt{d}\beta_1 2^{\alpha+2}(2+r/c)}{(1+|x_-|/\sqrt{2}+(|v_-|/\sqrt{2}-r)t)^\alpha}, \\
|H_{v_-,x_-}(f,h)(t)| & \leq \xi_+(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,t) \\
& = \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}\alpha(\alpha-1)(|v_-|/\sqrt{2}-r)^2} \\
& \times \frac{d\sqrt{d}\beta_1 2^{\alpha+2}(2+r/c)}{(1+|x_-|/\sqrt{2}+(|v_-|/\sqrt{2}-r)t)^{(\alpha-1)}},
\end{aligned} \tag{2.17}$$

for $T = +\infty$, $t \geq 0$.

One can see that Lemma 2.2 gives, in particular, estimates and asymptotics for

$$A_{v_-,x_-}(f,h)(t) = (A_{v_-,x_-}^1(f,h)(t), A_{v_-,x_-}^2(f,h)(t)) \text{ as } t \rightarrow \pm\infty.$$

Lemma 2.3. *Let the conditions (1.2) be valid, $(y_-(t), u_-(t)) \in M_{T,r}$ be a solution of (2.3), $T = +\infty$, $0 < r \leq 1$, $r < c/\sqrt{2}$, $x_- \in \mathbb{R}^d$, $v_- \in B_c$, $|v_-| \geq z_1(c,d,\beta_1,\alpha,|x_-|,r)$, $v_-x_- = 0$, then*

$$\begin{aligned}
|k_{v_-,x_-}(y_-,u_-) - k_{v_-,x_-}(0,0)| & \leq \varepsilon'_a(c,d,\beta_1,\tilde{\beta},\alpha,|v_-|,|x_-|,r) \\
& = \frac{d^2\tilde{\beta}(1+\frac{1}{c})2^{\alpha+5}(|v_-|/\sqrt{2}+1-r)}{\alpha(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2})^\alpha} \\
& \times \frac{\rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}},
\end{aligned} \tag{2.18a}$$

$$\begin{aligned}
\left| \frac{k_{v_-,x_-}(y_-,u_-)}{\sqrt{1-\frac{|v_-|^2}{c^2}}} - \int_{-\infty}^{+\infty} F(x_-+v_-s,v_-) ds \right| & \leq \varepsilon_a(c,d,\beta_1,\tilde{\beta},\alpha,|v_-|,|x_-|,r) \\
& = \frac{2^{\alpha+5}d\sqrt{d}\tilde{\beta}(1+\frac{1}{c})(|v_-|/\sqrt{2}+1-r)}{\alpha(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2})^\alpha} \\
& \times \rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r),
\end{aligned} \tag{2.18b}$$

$$\begin{aligned}
|l_{v_-,x_-}(y_-,u_-) - l_{v_-,x_-}(0,0)| & \leq \varepsilon_b(c,d,\beta_1,\tilde{\beta},\alpha,|v_-|,|x_-|,r) \\
& = \frac{2^{2\alpha+9}d^3\tilde{\beta}(1+\tilde{\beta})3(1+1/c)^3(|v_-|/\sqrt{2}+1-r)^2}{(\alpha-1)(|v_-|/\sqrt{2}-r)^4(1+|x_-|/\sqrt{2})^{\alpha-1}} \\
& \times \frac{\rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}},
\end{aligned} \tag{2.18c}$$

where k_{v_-,x_-} and l_{v_-,x_-} are defined in (2.14) and ρ is defined in (2.8b).

We sketch the proof of Lemmas 2.1, 2.2, 2.3 in Section 4.

3 Small angle scattering

Under the conditions (1.2), for any $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \neq 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^d)$ with the initial conditions (1.3). Consider the function $y_-(t)$ from (1.3). This function describes deflection from free motion.

Using Corollary 2.1 the lemma about contraction maps, and Lemmas 2.2 and 2.3 we obtain the following result.

Theorem 3.1. *Let the conditions (1.2) be valid, $\mu(c, d, \tilde{\beta}, \alpha, |v_-|, |x_-|, r) < 1$, $\tilde{\beta} = \max(\beta_1, \beta_2)$, $0 < r \leq 1$, $r < c/\sqrt{2}$, $x_- \in \mathbb{R}^d$, $v_- \in B_c$, $|v_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $v_- x_- = 0$, where μ is defined by (2.10b) and z_1 is defined by (2.7). Then the deflection $y_-(t)$ has the following properties:*

$$(y_-, \dot{y}_-) \in M_{T,r}, \quad T = +\infty; \quad (3.1)$$

$$|\dot{y}_-(t)| \leq \zeta_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t), \quad (3.2)$$

$$|y_-(t)| \leq \xi_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t) \quad \text{for } t \leq 0; \quad (3.3)$$

$$y_-(t) = a_{sc}(v_-, x_-)t + b_{sc}(v_-, x_-) + h(v_-, x_-, t), \quad (3.4)$$

where

$$\left| a_{sc}(v_-, x_-) - \left[\frac{\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_-s + x_-, v_-) ds}{\sqrt{1 + \frac{|\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_-s + x_-, v_-) ds|^2}{c^2}}} - v_- \right] \right| \leq \varepsilon'_a(c, d, \beta_1, \tilde{\beta}, \alpha, |v_-|, |x_-|, r), \quad (3.5a)$$

$$\left| \frac{a_{sc}(v_-, x_-)}{\sqrt{1 - \frac{|v_-|^2}{c^2}}} - \int_{-\infty}^{+\infty} F(v_-s + x_-, v_-) ds \right| \leq \varepsilon_a(c, d, \beta_1, \tilde{\beta}, \alpha, |v_-|, |x_-|, r), \quad (3.5b)$$

$$|b_{sc}(v_-, x_-) - l_{v_-, x_-}(0, 0)| \leq \varepsilon_b(c, d, \beta_1, \tilde{\beta}, \alpha, |v_-|, |x_-|, r), \quad (3.5c)$$

$$|a_{sc}(v_-, x_-)| \leq 2\zeta_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, 0), \quad (3.6a)$$

$$|b_{sc}(v_-, x_-)| \leq 2\xi_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, 0), \quad (3.6b)$$

$$|\dot{h}(v_-, x_-, t)| \leq \zeta_+(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t), \quad (3.7)$$

$$|h(v_-, x_-, t)| \leq \xi_+(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t), \quad (3.8)$$

for $t \geq 0$, where $l_{v_-, x_-}(0, 0)$ (resp. ε'_a , ε_a , ε_b , ζ_- , ζ_+ , ξ_- and ξ_+) is defined in (2.14b) (resp. (2.18a), (2.18b), (2.18c), (2.11), (2.16), (2.12) and (2.17)).

Let $z = z(c, d, \tilde{\beta}, \alpha, r_x, r)$ and $z_2 = z_2(c, d, \beta_1, \alpha, r_x)$ be defined as the roots of the following equations

$$\mu(c, d, \tilde{\beta}, \alpha, z, r_x, r) = 1, \quad z \in]\sqrt{2}r, c[, \quad (3.9)$$

$$\frac{z_2}{\sqrt{1 - \frac{z_2^2}{c^2}}} - \frac{32\beta_1 d}{\alpha(z_2/\sqrt{2})(1 + r_x/\sqrt{2})^\alpha} = 0, \quad z_2 \in]0, c[, \quad (3.10)$$

where μ is defined by (2.10b), r_x and r are some nonnegative numbers such that $0 < r \leq 1$, $r < c/\sqrt{2}$, and where $\tilde{\beta} = \max(\beta_1, \beta_2)$.

We use the following observations.

(I) Let $0 < r \leq 1, r < c/\sqrt{2}, 0 \leq \sigma$

$$\frac{s_1}{\sqrt{1 - \frac{s_1^2}{c^2}}} - \frac{2^{\alpha+5}\beta_1 d(2 + r/c)}{\alpha(s_1/\sqrt{2} - r)(\sigma/\sqrt{2} + 1)^\alpha} > \frac{s_2}{\sqrt{1 - \frac{s_2^2}{c^2}}} - \frac{2^{\alpha+5}\beta_1 d(2 + r/c)}{\alpha(s_2/\sqrt{2} - r)(\sigma/\sqrt{2} + 1)^\alpha}$$

for $\sqrt{2}r < s_2 < s_1 < c$.

(II) Let $0 < r \leq 1, r < c/\sqrt{2}, \sigma \in]\sqrt{2}r, c[$,

$$\frac{\sigma}{\sqrt{1 - \frac{\sigma^2}{c^2}}} - \frac{2^{\alpha+5}\beta_1 d(2 + r/c)}{\alpha(\sigma/\sqrt{2} - r)(s_1/\sqrt{2} + 1)^\alpha} > \frac{\sigma}{\sqrt{1 - \frac{\sigma^2}{c^2}}} - \frac{2^{\alpha+5}\beta_1 d(2 + r/c)}{\alpha(\sigma/\sqrt{2} - r)(s_2/\sqrt{2} + 1)^\alpha}$$

for $0 \leq s_2 < s_1$.

(III) Let $0 < r \leq 1, r < c/\sqrt{2}, x$ some real nonnegative number, $\tilde{\beta} = \max(\beta_1, \beta_2)$ and $\sqrt{2}r < s < c$ then

$$\mu(c, d, \tilde{\beta}, \alpha, s, |x|, r) < 1 \Leftrightarrow s > z(c, d, \tilde{\beta}, \alpha, |x|, r).$$

Observations (I) and (II) imply that $z_1(c, d, \beta_1, \alpha, s_2, r) > z_1(c, d, \beta_1, \alpha, s_1, r)$ for $\sqrt{2}r < s_2 < s_1 < c$ when c, β_1, α, d, r are fixed.

Theorem 3.1 gives, in particular, estimates for the scattering process and asymptotics for the velocity valued component of the scattering map when $c, \beta_1, \beta_2, \alpha, d, \hat{v}_-, x_-$ are fixed (where $\hat{v}_- = v_-/|v_-|$) and $|v_-|$ increases or, e.g., $c, \beta_1, \beta_2, \alpha, d, v_-, \hat{x}_-$ are fixed and $|x_-|$ increases. In these cases $\sup_{t \in \mathbb{R}} |\theta(t)|$ decreases, where $\theta(t)$ denotes the angle between the vectors $\dot{x}(t) = v_- + \dot{y}_-(t)$ and v_- , and we deal with small angle scattering. Note that already under the conditions of Theorem 3.1, without additional assumptions, there is the estimate $\sup_{t \in \mathbb{R}} |\theta(t)| < \frac{1}{4}\pi$ and we deal with a rather small angle scattering. Theorem 3.1 with (3.5c) will give the asymptotics of the configuration valued component $b(v_-, x_-)$ of the scattering map if we can study the asymptotics of $l_{v_-, x_-}(0, 0)$. This is the subject of Theorem 3.2.

Theorem 3.2. Let $c, d, \beta_0, \beta_1, \alpha, |x|$ be fixed. Then there exists a constant $C = C_{c,d,\beta_0,\beta_1,\alpha,|x|}$ such that

$$\left| \frac{l_{v,x}(0,0)}{\sqrt{1 - \frac{|v|^2}{c^2}}} - \frac{1}{c^2} PV(\hat{v}, x)\hat{v} + \frac{1}{|v|^2} \int_0^{+\infty} \int_{\tau}^{+\infty} F(\sigma\hat{v} + x, v) d\sigma d\tau - \frac{1}{|v|^2} \int_{-\infty}^0 \int_{-\infty}^{\tau} F(\sigma\hat{v} + x, v) d\sigma d\tau \right| \leq C \sqrt{1 - \frac{|v|^2}{c^2}} \quad (3.11)$$

for any $v \in B_c$, $|v| \geq z_2(c, d, \beta_1, \alpha, |x|)$, $vx = 0$, and where $\hat{v} = v/|v|$.

We sketch the proof of Theorem 3.2 in Section 4.

Remark 3.1. Let $c, d, \beta_0, \beta_1, \alpha, |x|$ be fixed and let $s_0 \in]0, c[$ be fixed. Then by very slight modifications to the proof of Theorem 3.2, one can prove that there exists a constant $C = C_{c,d,\beta_0,\beta_1,\alpha,|x|,s_0}$ such that the estimate (3.11) holds for any $v \in B_c$, $|v| \geq \max(s_0, z_2(c, d, \beta_1, \alpha, |x|))$, and $\frac{C}{\max(\beta_0, \beta_1)^2} \rightarrow C_3$ as $\max(\beta_0, \beta_1) \rightarrow 0$, where $C_3 = C_{3c,d,\alpha,|x|,s_0}$ is a positive real constant.

4 About the proof of Lemmas 2.1, 2.2, 2.3 and Theorems 3.2 and 1.1

The way we prove Lemmas 2.1, 2.2, 2.3 and Theorem 3.2 of the present work, is actually exactly the same as the way we prove lemmas 2.1, 2.2, 2.3 and theorem 3.2 of [Jo2].

4.1 Inequalities for F and g .

Before sketching the proof of Lemmas 2.1, 2.2, 2.3 and Theorem 3.2, we shall give some estimates about the growth of g defined by

$$g(x) = \frac{x}{\sqrt{1 + \frac{|x|^2}{c^2}}}, \quad x \in \mathbb{R}^d,$$

and we shall prove Lemma 4.1 given below.

We remind that g has the following simple properties (see [Jo2]):

$$|\nabla g_i(x)|^2 \leq \frac{1}{1 + \frac{|x|^2}{c^2}}, \quad (4.1)$$

$$|g(x) - g(y)| \leq \sqrt{d} \sup_{\varepsilon \in [0,1]} \frac{1}{\sqrt{1 + \frac{|\varepsilon x + (1-\varepsilon)y|^2}{c^2}}} |x - y|, \quad (4.2)$$

$$|\nabla g_i(x) - \nabla g_i(y)| \leq \frac{3\sqrt{d}}{c} \sup_{\varepsilon \in [0,1]} \frac{1}{1 + \frac{|\varepsilon x + (1-\varepsilon)y|^2}{c^2}} |x - y|, \quad (4.3)$$

for $x, y \in \mathbb{R}^d$, $i = 1..d$, and where $g = (g_1, \dots, g_d)$.

Lemma 4.1. *Under the conditions (1.2), the following estimates are valid:*

$$|F(x, y)| \leq 2d\beta_1(1 + |x|)^{-(\alpha+1)}(1 + \frac{1}{c}|y|) \text{ for } x, y \in \mathbb{R}^d, \quad (4.4)$$

$$\begin{aligned} |F(x, y) - F(x', y')| &\leq \frac{1}{c}2d\beta_1 \sup_{\varepsilon \in [0,1]} (1 + |\varepsilon x + (1 - \varepsilon)x'|)^{-(\alpha+1)}|y - y'| \\ &\quad + 2d\sqrt{d}\beta_2 \sup_{\varepsilon \in [0,1]} (1 + |\varepsilon y + (1 - \varepsilon)y'|/c)(1 + |\varepsilon x + (1 - \varepsilon)x'|)^{-(\alpha+2)} \\ &\quad \times |x - x'|, \end{aligned} \quad (4.5)$$

for $x, y, x', y' \in \mathbb{R}^d$.

Let $(f, h), (f_1, h_1), (f_2, h_2) \in M_{T,r}$, $v_- \in B_c \setminus \{0\}$, $v_- x_- = 0$, $|v_-| > \sqrt{2}r$, then

$$|f(s)| \leq (1 + |s|)\|(f, h)\|_T, \quad (4.6)$$

$$|h(s)| \leq \|(f, h)\|_T, \quad (4.7)$$

for $s \leq T$;

$$2(1 + |x_- + v_-s + f(s)|) \geq (1 + |x_-|/\sqrt{2} + (|v_-|/\sqrt{2} - r)|s|), \text{ for } s \leq T, \quad (4.8)$$

$$\left| \int_{-\infty}^t F(v_-s + x_- + f(s), v_- + h(s)) ds \right| \leq \frac{\beta_1 d 2^{\alpha+3} (2 + r/c)}{\alpha (|v_-|/\sqrt{2} - r) (|x_-|/\sqrt{2} + 1)^\alpha}, \quad (4.9)$$

$$\begin{aligned} &\left(1 + \frac{1}{c^2} \left| \gamma(v_-) + \varepsilon_1 \int_{-\infty}^t F(v_-s + x_- + f_1(s), v_- + h_1(s)) ds \right. \right. \\ &\quad \left. \left. + \varepsilon_2 \int_{\sigma}^{\tau} F(v_-s + x_- + f_2(s), v_- + h_2(s)) ds \right|^2 \right)^{-\beta} \\ &\leq \left(1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)} \right)^{-\beta}, \end{aligned} \quad (4.10)$$

for $\tau, t \in [-\infty, T]$, $\sigma \in [-\infty, \tau]$, $\beta > 0$, $-1 \leq \varepsilon_1 \leq 1$, $-1 \leq \varepsilon_2 \leq 1$, $(f_1, h_1), (f_2, h_2) \in M_{T,r}$ and if $|v_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $|v_-| < c$, where γ is defined by

$$\gamma(v) = \frac{v}{\sqrt{1 - |v|^2/c^2}},$$

for $v \in B_c$.

Proof of Lemma 4.1. The estimates (4.4) and (4.5) follows directly from the formula $F(x, y) = -\nabla V(x) + \frac{1}{c}B(x)y$ and $B(x) = [\frac{\partial}{\partial x_j} \mathbf{A}_k(x) - \frac{\partial}{\partial x_k} \mathbf{A}_j(x)]_{j,k=1..d}$ and the

conditions (1.2). The inequalities (4.6), (4.7) and (4.8) follow from the definition of $M_{T,r}$. Using (4.4), (4.8), we obtain (4.9) and using (4.9) and the definition of $z_1(c, d, \beta_1, \alpha, |x_-|, r)$ we obtain (4.10). \square

4.2 Sketch of proofs of Lemmas 2.1, 2.2, 2.3 and Theorem 3.2.

One can prove Lemmas 2.1, 2.2, 2.3 of the present work by repeating the proof of lemmas 2.1, 2.2, 2.3 of [Jo2] and by making the following replacements. First the estimates given in lemmas 4.1, 4.3 of [Jo2] are replaced by the estimates of Lemma 4.1 of the present work. Then, to prove Lemmas 2.1, 2.2, 2.3, we replace $\frac{d}{dt}A_{v_-,x_-}(f)$, $A_{v_-,x_-}(f)$ and $F(v_-s + x_- + f(s))$, for $f \in M_{T,r}$, in the proof of lemmas 2.1, 2.2, 2.3 of [Jo2] by $A_{v_-,x_-}^2(f, h)$, $A_{v_-,x_-}^1(f, h)$ and $F(v_-s + x_- + f(s), v_- + h(s))$ for $(f, h) \in M_{T,r}$. \square

One can prove Theorem 3.2 by repeating the proof of theorem 3.2 of [Jo2] and by making the following replacements. We replace the estimates given in lemmas 4.1, 4.3 of [Jo2] by the estimates of Lemma 4.1 of our present work and we replace $F(\tau\theta + x)$ of the proof of theorem 3.2 of [Jo2] by $F(\tau\theta + x, s\theta)$. \square

4.3 Constants C_1, C_2, s_1, s_2 of Theorem 1.1.

As it was mentioned already in Introduction, Theorem 1.1 follows from Theorem 3.1 and Theorem 3.2. In addition, constants C_1, C_2, s_1, s_2 , which appear in Theorem 1.1, are given explicitly by

$$\begin{aligned} s_1 &= \max(z(c, d, \tilde{\beta}, \alpha, |x|, r), z_1(c, d, \beta_1, \alpha, |x|, r)), \\ s_2 &= \max(z(c, d, \tilde{\beta}, \alpha, |x|, r), z_1(c, d, \beta_1, \alpha, |x|, r), z_2(c, d, \beta_1, \alpha, |x|)), \\ C_1 &= \frac{d^3 \tilde{\beta}^2 2^{2\alpha+9} (1 + \frac{1}{c})^2 c (\frac{c}{\sqrt{2}} + 1 - r)^2}{\alpha(\alpha - 1) (\frac{s_1}{\sqrt{2}} - r)^4 (1 + \frac{|x|}{\sqrt{2}})^{2\alpha-1}}, \\ C_2 &= C + \frac{4d^4 \sqrt{d} \tilde{\beta}^2 (1 + \tilde{\beta}) 2^{3\alpha+15} (1 + 1/c)^4 (\frac{c}{\sqrt{2}} + 1 - r)^3}{(\alpha - 1)^2 (\frac{s_2}{\sqrt{2}} - r)^6 (1 + \frac{|x|}{\sqrt{2}})^{2\alpha-2}}, \end{aligned}$$

where z, z_1, z_2 are defined by (3.9), (2.7), (3.10) and $\tilde{\beta} = \max(\beta_1, \beta_2)$ and where C is either $C_{c,d,\beta_0,\beta_1,\alpha,|x|}$ (the constant of Theorem 3.2) or $C_{c,d,\beta_0,\beta_1,\alpha,|x|,s_0}$ (the constant of Remark 3.1) with $s_0 = \sqrt{2}r$.

4.4 Weakened assumptions.

Let $M_d(\mathbb{R})$ denote the space of $d \times d$ matrix with real elements. Let $V \in C^2(\mathbb{R}^d, \mathbb{R})$ so that:

$$|\partial_x^j V(x)| \leq \beta_{|j|} (1 + |x|)^{-(\alpha+|j|)}, \quad (4.11a)$$

for $|j| \leq 2$ and some $\alpha > 1$ (here j is the multiindex $j \in (\mathbb{N} \cup \{0\})^d$, $|j| = \sum_{n=1}^d j_n$ and $\beta_{|j|}$ are positive real constants). Let $B \in C^1(\mathbb{R}^d, M_d(\mathbb{R}))$ so that:

$$B(x) \text{ is a } d \times d \text{ antisymmetric matrix with real elements } B_{m,n}(x), \quad (4.11b)$$

$$\frac{\partial}{\partial x_i} B_{k,l}(x) + \frac{\partial}{\partial x_l} B_{i,k}(x) + \frac{\partial}{\partial x_k} B_{l,i}(x) = 0, \quad (4.11c)$$

for $x \in \mathbb{R}^d$, for $i, k, l = 1..d$;

$$|\partial_x^j B_{i,k}(x)| \leq \beta_{|j|+1} (1 + |x|)^{-(\alpha+|j|+1)}, \quad (4.11d)$$

for $i, k = 1..d$ and for $|j| \leq 1$.

Let \mathbf{A} be the transversal gauge of B , i.e.

$$\mathbf{A}(x) = - \int_0^1 s B(sx) \cdot x ds. \quad (4.12)$$

Under the conditions (4.11b), (4.11c) and (4.11d), \mathbf{A} satisfies

$$|\mathbf{A}(x)| \leq \beta (1 + |x|)^{-1}, \quad (4.13a)$$

$$B_{i,k}(x) = \frac{\partial}{\partial x_i} \mathbf{A}_k(x) - \frac{\partial}{\partial x_k} \mathbf{A}_i(x). \quad (4.13b)$$

for $x \in \mathbb{R}^d$, $i, k = 1..d$ and some positive real constant β .

If we replace assumptions (1.2) by assumptions (4.11) given above, then the estimates (4.4) and (4.5) still hold. As a consequence, using also Remark 5.2, we obtain that assumptions (1.2) in all results and estimates which appear in Introduction and in Sections 2, 3 can be weakened to assumptions (4.11).

5 Proofs of Proposition 1.1 and Proposition 1.2

We first prove Proposition 1.1.

Let $(\theta, x) \in T\mathbb{S}^{d-1}$. We note that

$$\int_{-\infty}^{+\infty} B(\tau(-\theta) + x)(-\theta) d\tau = - \int_{-\infty}^{+\infty} B(\tau\theta + x)\theta d\tau \quad (5.1a)$$

and we remind that

$$P(\nabla V)(-\theta, x) = P(\nabla V)(\theta, x). \quad (5.1b)$$

Using (1.9a) and (5.1) we obtain formulas (1.12). (To obtain (1.12b) we also use the relation $\theta_i^2 + \theta_k^2 = 1$ for $(\theta, x) \in \mathcal{V}_{i,k}$, $\theta = (\theta_1, \dots, \theta_d)$.) From (1.12a) and results on inversion of the X-ray transform, we obtain that $w_1(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ uniquely determines ∇V and thus it uniquely determines V ((V, \mathbf{A}) satisfies (1.2)). From (1.12b) and results on inversion of the X-ray transform (see the scheme given below) it follows that $w_1(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ uniquely determines B .

Proposition 1.1 is proved. \square

Let $i, k = 1..d$, $i \neq k$. To reconstruct $B_{i,k}$ from $w_1(V, \mathbf{A}, \theta, x)$ given for all $(\theta, x) \in \mathcal{V}_{i,k}$, we give the following scheme which is based on the formula (1.12b) ($\mathcal{V}_{i,k}$ is defined in (1.11)). The formula (1.12b) gives the value of all integrals of $B_{i,k}$ over any straight line of any two-dimensional affine plane Y whose tangent vector space is $Y_{i,k} = \{(x'_1, \dots, x'_d) \in \mathbb{R}^d | x'_j = 0, j \neq i, j \neq k\}$. Now, to reconstruct $B_{i,k}$ at a point $x' \in \mathbb{R}^d$ we consider in \mathbb{R}^d a two-dimensional plane Y containing x' and whose tangent vector space is $Y_{i,k}$. We interpret TS^{d-1} as the set of all rays in \mathbb{R}^d and we consider in TS^{d-1} the subset $TS^1(Y)$ which is the set of all rays lying in Y . Then we restrict $PB_{i,k}$ on $TS^1(Y)$ and reconstruct $B_{i,k}(x')$ from these data using methods of reconstruction of f from Pf for $d = 2$.

To prove Proposition 1.2 we will first prove Proposition 5.1 and Proposition 5.2 given below.

Let $\mathbf{A} \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ and

$$|\partial_x^j \mathbf{A}_i(x)| \leq \beta_{|j|} (1 + |x|)^{-(\alpha + |j|)} \quad (5.2)$$

for $x \in \mathbb{R}^d$, $|j| \leq 2$, $i = 1..d$ and some $\alpha > 1$ (here j is the multiindex $j \in (\mathbb{N} \cup \{0\})^d$, $|j| = \sum_{n=1}^d j_n$ and $\beta_{|j|}$ are positive real constants). We recall that $B(x)$ is the $d \times d$ real antisymmetric matrix with elements

$$B_{i,k}(x) = \frac{\partial}{\partial x_i} \mathbf{A}_k(x) - \frac{\partial}{\partial x_k} \mathbf{A}_i(x) \quad (5.3)$$

for $x \in \mathbb{R}^d$. We define the vector $w_3(\mathbf{A}, \theta, x)$ by

$$w_3(\mathbf{A}, \theta, x) = \int_{-\infty}^0 \int_{-\infty}^{\tau} B(x + \sigma\theta) \theta d\sigma d\tau - \int_0^{+\infty} \int_{\tau}^{+\infty} B(x + \sigma\theta) \theta d\sigma d\tau, \quad (5.4)$$

for any $(\theta, x) \in TS^{d-1}$. We define a function $\tilde{w}_3(\mathbf{A}) : \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ by $\tilde{w}_3(\mathbf{A})(y, x) = |y| w_3(\mathbf{A}, \frac{y}{|y|}, x - \frac{x \cdot y}{|y|^2} y)$ for $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d \setminus \{0\}$. From (5.2) and (5.4), it follows that

$$\tilde{w}_3(\mathbf{A})(y, x) = \int_{-\infty}^{\frac{-xy}{|y|^2}} \int_{-\infty}^{\tau} B(x + \sigma y) y d\sigma d\tau - \int_{\frac{-xy}{|y|^2}}^{+\infty} \int_{\tau}^{+\infty} B(x + \sigma y) y d\sigma d\tau, \quad (5.5)$$

and $\tilde{w}_3(\mathbf{A}) = (\tilde{w}_3(\mathbf{A})_1, \dots, \tilde{w}_3(\mathbf{A})_d) \in C^1(\mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d, \mathbb{R}^d)$.

Proposition 5.1. *Let $\mathbf{A} \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ satisfy (5.2), then B defined by (5.3) satisfies:*

$$\sum_{j=1}^d \theta_j [\theta_k P B_{i,j}(\theta, x) - \theta_i P B_{k,j}(\theta, x)] - P B_{i,k}(\theta, x) = \quad (5.6a)$$

$$\frac{\partial}{\partial x_i} \tilde{w}_3(\mathbf{A})_k(\theta, x) - \frac{\partial}{\partial x_k} \tilde{w}_3(\mathbf{A})_i(\theta, x),$$

$$\sum_{j=1}^d \theta_j [\theta_k P B_{i,j,l}(\theta, x) - \theta_i P B_{k,j,l}(\theta, x)] - P B_{i,k,l}(\theta, x) = \tilde{w}_3(\mathbf{A})_{i,k,l}(\theta, x), \quad (5.6b)$$

for $(\theta, x) \in T\mathbb{S}^{d-1}$, $i, k, l = 1..d$, where P denotes the X-ray transform and where $\tilde{w}_3(\mathbf{A})_{m,n,l}(\theta, x) = \frac{\partial}{\partial x_l} \left(\frac{\partial}{\partial x_m} \tilde{w}_3(\mathbf{A})_n - \frac{\partial}{\partial x_n} \tilde{w}_3(\mathbf{A})_m \right) (\theta, x)$, $B_{m,n,l}(x) = \frac{\partial}{\partial x_l} B_{m,n}(x)$, for $\theta \in \mathbb{S}^{d-1}$, $x \in \mathbb{R}^d$, $m, n = 1..d$.

Remark 5.1. Let $d = 3$, $l = 1..d$. Formula (5.6b) gives in fact

$$\begin{aligned} & \theta \star (-P B_{2,3,l}(\theta, x), P B_{1,3,l}(\theta, x), -P B_{1,2,l}(\theta, x)) = \\ & \theta \star (\tilde{w}_3(\mathbf{A})_{2,3,l}(\theta, x), -\tilde{w}_3(\mathbf{A})_{1,3,l}(\theta, x), \tilde{w}_3(\mathbf{A})_{1,2,l}(\theta, x)), \end{aligned} \quad (5.7)$$

for any $(\theta, x) \in T\mathbb{S}^2$ where \star denotes the usual scalar product on \mathbb{R}^3 .

Proof of Proposition 5.1. Let $i, k, l = 1..d$ be fixed. Under the conditions (5.2), using (5.5) we obtain by straightforward calculations

$$\begin{aligned} & \frac{\partial}{\partial x_k} \tilde{w}_3(\mathbf{A})_i(\theta, x) - \frac{\partial}{\partial x_i} \tilde{w}_3(\mathbf{A})_k(\theta, x) \\ &= -\theta_k \sum_{j=1}^d \theta_j \int_{-\infty}^{+\infty} B_{i,j}(\sigma\theta + x) d\sigma d\tau + \theta_i \sum_{j=1}^d \theta_j \int_{-\infty}^{+\infty} B_{k,j}(\sigma\theta + x) d\sigma d\tau \\ &+ \sum_{j=1}^d \theta_j \left\{ \int_{-\infty}^{-x \star \theta} \int_{-\infty}^{\tau} \left[\frac{\partial}{\partial x_k} B_{i,j}(\sigma\theta + x) - \frac{\partial}{\partial x_i} B_{k,j}(\sigma\theta + x) \right] d\sigma d\tau \right. \\ &\left. - \int_{-x \star \theta}^{+\infty} \int_{\tau}^{+\infty} \left[\frac{\partial}{\partial x_k} B_{i,j}(\sigma\theta + x) - \frac{\partial}{\partial x_i} B_{k,j}(\sigma\theta + x) \right] d\sigma d\tau \right\} \end{aligned} \quad (5.8)$$

for $x \in \mathbb{R}^d$, $\theta \in \mathbb{S}^{d-1}$, $\theta = (\theta_1, \dots, \theta_d)$. Note that

$$\frac{\partial}{\partial x_k} B_{i,j}(x) - \frac{\partial}{\partial x_i} B_{k,j}(x) = \frac{\partial}{\partial x_j} B_{i,k}(x), \quad x \in \mathbb{R}^d, \quad j = 1..d. \quad (5.9)$$

Let $\theta \in \mathbb{S}^{d-1}$ be fixed. Using (5.8), (5.2) and (5.9) we obtain (5.6a). Under conditions (5.2), the function $h_{i,k,\theta}$ which is defined by $h_{i,k,\theta}(x) = \frac{\partial}{\partial x_k} \tilde{w}_3(\mathbf{A})_i(\theta, x) - \frac{\partial}{\partial x_i} \tilde{w}_3(\mathbf{A})_k(\theta, x)$, $x \in \mathbb{R}^d$, satisfies $h_{i,k,\theta} \in C^1(\mathbb{R}^d, \mathbb{R})$ and (5.6b) follows immediately from (5.6a).

Proposition 5.1 is proved. \square

Proposition 5.2. Let $\mathbf{A} \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ satisfy (5.2).

i. if $d = 2$ then $w_3(\mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ does not determine uniquely the magnetic field B defined by (5.3),

ii. if $d \geq 3$ then $w_3(\mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$ uniquely determines the magnetic field B defined by (5.3). In addition the following formulas are valid: if $d = 3$ then

$$(-\mathcal{F}B_{2,3,l}(p), \mathcal{F}B_{1,3,l}(p), -\mathcal{F}B_{1,2,l}(p)) = \quad (5.10a)$$

$$(2\pi)^{-3/2} \sum_{j=1}^2 \left[\theta_p^j \star \left(\int_{\Pi_{\theta_p^j}} e^{-iy \star p} \tilde{w}_3(\mathbf{A})_{2,3,l}(\theta_p^j, y) dy, - \int_{\Pi_{\theta_p^j}} e^{-iy \star p} \tilde{w}_3(\mathbf{A})_{1,3,l}(\theta_p^j, y) dy, \right. \right.$$

$$\left. \int_{\Pi_{\theta_p^j}} e^{-iy \star p} \tilde{w}_3(\mathbf{A})_{1,2,l}(\theta_p^j, y) dy \right) \Big] \theta_p^j,$$

for $p \in \mathbb{R}^3 \setminus \{0\}$ and any orthonormal family $\{\theta_p^1, \theta_p^2\}$ of the plane Π_p , where $\Pi_{p'}$ is the vector plane $\{y \in \mathbb{R}^3 | y \star p' = 0\}$ for $p' \in \mathbb{R}^3 \setminus \{0\}$, and \mathcal{F} denotes the classical Fourier transform on $L^1(\mathbb{R}^3)$ (and where $i = \sqrt{-1}$); if $d \geq 4$ then

$$PB_{j,k}(\theta, x) = \frac{\partial}{\partial x_k} \tilde{w}_3(\mathbf{A})_j(\theta, x) - \frac{\partial}{\partial x_j} \tilde{w}_3(\mathbf{A})_k(\theta, x) \quad (5.10b)$$

for $(\theta, x) \in \tilde{\mathcal{V}}_{j,k}$, where $\tilde{\mathcal{V}}_{j,k}$ is the $(2d-4)$ -dimensional manifold $\{(\theta, x) \in T\mathbb{S}^{d-1} | \theta = (\theta_1, \dots, \theta_d), \theta_j = \theta_k = 0\}$.

Proof of Proposition 5.2. We first prove the item (i). Let $\xi \in C^1(\mathbb{R}^+, \mathbb{R})$ be such that

$$\mathbf{A}(x) = (-x_2 \xi(|x|^2), x_1 \xi(|x|^2)), \quad x \in \mathbb{R}^2,$$

satisfies (5.2) and

$$B(x) = \begin{pmatrix} 0 & 2\xi(|x|^2) + 2|x|^2 \xi'(|x|^2) \\ -2\xi(|x|^2) - 2|x|^2 \xi'(|x|^2) & 0 \end{pmatrix} \neq 0$$

(e.g. $\xi(t) = \frac{1}{(1+t)^\sigma}$, $t \in \mathbb{R}^+$, $\sigma > 1$ or $\xi(t) = e^{-t}$, $t \in \mathbb{R}^+$). Let $(\theta, x) \in T\mathbb{S}^{d-1}$ be fixed. Using $|\sigma\theta + x|^2 = \sigma^2 + |x|^2$ and (5.4), we obtain $w_3(\mathbf{A}, \theta, x) = 0$. The item (i) is proved.

We prove the item (ii). We shall distinguish the case $d = 3$ from the case $d \geq 4$.

First let $d \geq 4$. Let $j, k = 1..d$ be fixed, $j \neq k$. Formula (5.6a) implies (5.10b). Let $x' \in \mathbb{R}^d$. As $d \geq 4$, the dimension of the vector space $H_{j,k} = \{(x_1, \dots, x_d) \in \mathbb{R}^d | x_j = x_k = 0\}$ is greater than or equal to 2. Let Y be any affine plane of \mathbb{R}^d which passes through x' and whose tangent vector space is a subspace of $H_{j,k}$. From (5.10b), it follows that the integral of $B_{j,k}$ over any straight line of Y is known from $\frac{\partial}{\partial x_k} \tilde{w}_3(\mathbf{A})_j(\theta, x) - \frac{\partial}{\partial x_j} \tilde{w}_3(\mathbf{A})_k(\theta, x)$ given for all $(\theta, x) \in \tilde{\mathcal{V}}_{j,k}$. Thus using results on inversion of the X-ray transform P (see [GGG], [Na] and [No1]), we obtain

that $B_{j,k}(y)$, $y \in Y$, can be reconstructed from $\frac{\partial}{\partial x_k} \tilde{w}_3(\mathbf{A})_j(\theta, x) - \frac{\partial}{\partial x_j} \tilde{w}_3(\mathbf{A})_k(\theta, x)$ given for all $(\theta, x) \in \tilde{\mathcal{V}}_{j,k}$ (we remind that $x' \in Y$).

Then let $d = 3$ and let $l = 1..3$ be fixed. Under conditions (5.2), $B_{j,k,l} \in L^1(\mathbb{R}^3)$, for $j, k = 1..3$. Let $p \in \mathbb{R}^3$ be fixed. From (5.7) we obtain

$$\begin{aligned} \theta \star (-\mathcal{F}B_{2,3,l}(p), \mathcal{F}B_{1,3,l}(p), -\mathcal{F}B_{1,2,l}(p)) = & \quad (5.11) \\ (2\pi)^{-3/2} \theta \star \left(\int_{\Pi_\theta} e^{-iy \star p} \tilde{w}_3(\mathbf{A})_{2,3,l}(\theta, y) dy, - \int_{\Pi_\theta} e^{-iy \star p} \tilde{w}_3(\mathbf{A})_{1,3,l}(\theta, y) dy, \right. \\ & \left. \int_{\Pi_\theta} e^{-iy \star p} \tilde{w}_3(\mathbf{A})_{1,2,l}(\theta, y) dy \right) \end{aligned}$$

for any $\theta \in \mathbb{S}^2$, $\theta \star p = 0$. To prove that (5.11) implies (5.10a), we shall use the following

Lemma 5.1. *Under the conditions (5.2), $(-\mathcal{F}B_{2,3,l}(p), \mathcal{F}B_{1,3,l}(p), -\mathcal{F}B_{1,2,l}(p)) \star p = 0$, for $p \in \mathbb{R}^3$.*

Lemma 5.1 and (5.11) imply (5.10a) (see [Jo1] for a proof of Lemma 5.1).

Let $m, n = 1, 2, 3$ $m \neq n$. Using the injectivity of the Fourier transform and (5.10a), we obtain that $B_{m,n,l}$ is uniquely determined by $w_3(\mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$. Since $B_{m,n}$ vanishes at infinity, we deduce that $B_{m,n}$ is uniquely determined by $w_3(\mathbf{A}, \theta, x)$ given for all $(\theta, x) \in T\mathbb{S}^{d-1}$.

Proposition 5.2 is proved. \square

Now we are ready to prove Proposition 1.2. We note that

$$w_3(\mathbf{A}, \theta, x) = \frac{1}{2}(w_2(V, \mathbf{A}, \theta, x) + w_2(V, \mathbf{A}, -\theta, x)), \quad (5.12a)$$

$$\begin{aligned} PV(\theta, x) \theta + \int_0^{+\infty} \int_\tau^{+\infty} \nabla V(s\theta + x) ds d\tau - \int_{-\infty}^0 \int_{-\infty}^\tau \nabla V(s\theta + x) ds d\tau & \quad (5.12b) \\ = \frac{1}{2}(w_2(V, \mathbf{A}, \theta, x) - w_2(V, \mathbf{A}, -\theta, x)), \end{aligned}$$

for $(\theta, x) \in T\mathbb{S}^{d-1}$. The formulas (5.12), Proposition 1.1 of [Jo2] and Proposition 5.2 imply Proposition 1.2. \square

Remark 5.2. If we replace the conditions (5.2) and the formula (5.3) by the conditions (4.11b), (4.11c) and (4.11d), then Propositions 5.1, 5.2 and Lemma 5.1 still hold.

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