Inverse scattering at high energies for the multidimensional Newton equation in a long range potential

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July 12, 2014

Abstract

We define scattering data for the Newton equation in a potential $V \in C^2(\mathbb{R}^n, \mathbb{R})$, $n \geq 2$, that decays at infinity like $r^{-\alpha}$ for some $\alpha \in (0, 1]$. We provide estimates on the scattering solutions and scattering data and we prove, in particular, that the scattering data at high energies uniquely determine the short range part of the potential up to the knowledge of the long range tail of the potential. The Born approximation at fixed energy of the scattering data is also considered. We then change the definition of the scattering data and consider also inverse scattering in other asymptotic regimes. These results were obtained by developing the inverse scattering approach of [Novikov, 1999].

1 Introduction

Consider the multidimensional Newton equation in an external static force F deriving from a scalar potential V:

$$\ddot{x}(t) = F(x(t)) = -\nabla V(x(t)), \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, $\dot{x}(t) = \frac{\mathrm{d}x}{\mathrm{d}t}(t)$, $n \geq 2$.

When $n = 3$ then equation (1.1) is the equation of motion of a nonrelativistic particle of mass $m = 1$ and charge $e = 1$ in an external and static

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electric (or gravitational) field described by V (see [11]) where x denotes the position of the particle, $\dot{x}$ denotes its velocity and $\ddot{x}$ denotes its acceleration and t denotes the time.

We also assume throughout this paper that V satisfies the following conditions

$$F = F^l + F^s, \tag{1.2}$$

where $F^l := -\nabla V^l$, $F^s := -\nabla V^s$ and $(V^l, V^s) \in (C^2(\mathbb{R}^n, \mathbb{R}))^2$, and where V^l satisfies the following long range assumptions

$$|\partial_x^j V^l(x)| \leq \beta^l_{|j|}(1 + |x|)^{-(\alpha+|j|)}, \tag{1.3}$$

and V^s satisfies the following short range assumptions

$$|\partial_x^j V^s(x)| \leq \beta^s_{|j|+1}(1 + |x|)^{-(\alpha+1+|j|)}, \tag{1.4}$$

for $x \in \mathbb{R}^n$ and $|j| \leq 2$ and for some $\alpha \in (0, 1]$ (here j is the multiindex $j \in (\mathbb{N} \cup \{0\})^n, |j| = \sum_{m=1}^n j_m$, and β^l_m and $\beta^s_{m'}$ are positive real constants for $m = 0, 1, 2$ and for $m' = 1, 2, 3$). Note that the assumption $0 < \alpha \leq 1$ includes the decay rate of a Coulombian potential at infinity. Indeed for a Coulombian potential $V(x) = \frac{1}{|x|}$, estimates (1.3) are satisfied uniformly for $|x| > \varepsilon$ and $\alpha = 1$ for any $\varepsilon > 0$. Although our potentials (V^l, V^s) are assumed to be C^2 on the entire space $\mathbb{R}^n$, our present work may provide interesting results even in presence of singularities for the potentials (V^l, V^s).

For equation (1.1) the energy

$$E = \frac{1}{2}|\dot{x}(t)|^2 + V(x(t)) \tag{1.5}$$

is an integral of motion.

For $\sigma > 0$ we denote by $\mathcal{B}(0, \sigma)$ the Euclidean open ball of center 0 and radius σ, $\mathcal{B}(0, \sigma) = \{y \in \mathbb{R}^n \mid |y| < \sigma\}$, and we denote by $\overline{\mathcal{B}(0, \sigma)} = \{y \in \mathbb{R}^n \mid |y| \leq \sigma\}$ its closure. We set $\mu := \sqrt{\frac{2^4 n \max(\beta^l_1, \beta^l_2)}{\alpha}}$. Under conditions (1.3) the following is valid (see Section 4 for a proof): for any $v \in \mathbb{R}^n \backslash \mathcal{B}(0, \mu)$, there exists a unique solution $z_\pm(v, .)$ of the equation

$$\ddot{z}(t) = F^l(z(t)),\ t \in \mathbb{R}, \tag{1.6}$$

so that

$$\dot{z}_\pm(v, t) - v = o(1),\ \text{as } t \to \pm\infty,\ z_\pm(v, 0) = 0, \tag{1.7}$$

and

$$|z_\pm(v, t) - tv| \leq \frac{2^{\frac{3}{2}} n^{\frac{1}{2}} \beta^l_1}{\alpha|v|}|t| \text{ for } t \in \mathbb{R}. \tag{1.8}$$

When $F^l \equiv 0$ then $z_\pm(v,t) = tv$ for any $(t,v) \in \mathbb{R} \times \mathbb{R}^n$, $v \neq 0$.

Then under conditions (1.3) and (1.4), the following is valid: for any $(v_-, x_-) \in \mathbb{R}^n \backslash \mathcal{B}(0,\mu) \times \mathbb{R}^n$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = z_-(v_-, t) + x_- + y_-(t), \tag{1.9}$$

where $|\dot{y}_-(t)| + |y_-(t)| \to 0$, as $t \to -\infty$; in addition for almost any $(v_-, x_-) \in \mathbb{R}^n \backslash \mathcal{B}(0,\mu) \times \mathbb{R}^n$,

$$x(t) = z_+(v_+, t) + x_+ + y_+(t), \tag{1.10}$$

for a unique $(v_+, x_+) \in \mathbb{R}^n \times \mathbb{R}^n$, where $|v_+| = |v_-| \geq \mu$ by conservation of the energy (1.5), and where $v_+ =: a(v_-, x_-)$, $x_+ =: b(v_-, x_-)$, and $|\dot{y}_+(t)| + |y_+(t)| \to 0$, as $t \to +\infty$. A solution x of (1.1) that satisfies (1.9) and (1.10) for some (v_-, x_-), $v_- \neq 0$, is called a scattering solution.

We call the map $S : (\mathbb{R}^n \backslash \mathcal{B}(0,\mu)) \times \mathbb{R}^n \to (\mathbb{R}^n \backslash \mathcal{B}(0,\mu)) \times \mathbb{R}^n$ given by the formulas

$$v_+ = a(v_-, x_-),\ x_+ = b(v_-, x_-), \tag{1.11}$$

the scattering map for the equation (1.1). In addition, $a(v_-, x_-)$, $b(v_-, x_-)$ are called the scattering data for the equation (1.1), and we define

$$a_{sc}(v_-, x_-) = a(v_-, x_-) - v_-, \quad b_{sc}(v_-, x_-) = b(v_-, x_-) - x_-. \tag{1.12}$$

By $\mathcal{D}(S)$ we denote the set of definition of S. Under the conditions (1.3) and (1.4) the map $S : \mathcal{D}(S) \to (\mathbb{R}^n \backslash \mathcal{B}(0,\mu)) \times \mathbb{R}^n$ is continuous, and $\mathrm{Mes}(((\mathbb{R}^n \backslash \mathcal{B}(0,\mu)) \times \mathbb{R}^n) \backslash \mathcal{D}(S)) = 0$ for the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^n$. In addition the map S is uniquely determined by its restriction to $\mathcal{M}(S) = \mathcal{D}(S) \cap \mathcal{M}$, where $\mathcal{M} = \{(v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n \mid v_- \neq 0, v_- \cdot x_- = 0\}$. Indeed if $x(t)$ is a solution of (1.1) then $x(t+t_0)$ is also a solution of (1.1) for any $t_0 \in \mathbb{R}$, and we also have $z_\pm(v, t+t_0) - z_\pm(v,t) - t_0 v = \int_t^{t_0+t} \int_{\pm\infty}^{\sigma} F^l(z_\pm(v,\tau)) d\tau d\sigma$ which gives $|z_\pm(v, t+t_0) - z_\pm(v,t) - t_0 v| + |\dot{z}_\pm(v, t+t_0) - \dot{z}_\pm(v,t)| \to 0$ as $t \to \pm\infty$ for any $(v, t_0) \in (\mathbb{R}^n \backslash \mathcal{B}(0,\mu)) \times \mathbb{R}$. We then obtain

$$a(v_-, x_- + t_0 v_-) = a(v_-, x_-),\ b(v_-, x_- + t_0 v_-) = b(v_-, x_-) + t_0 a(v_-, x_-), \tag{1.13}$$

for any $t_0 \in \mathbb{R}$ and for $(v_-, x_-) \in \mathcal{D}(S)$.

For the forward classical scattering theory we refer the reader to [1, 7, 17, 8, 16, 2, 3] and references therein. Our definition of the scattering map is derived from constructions given in [8, 3].

One can imagine the following experimental setting that allows to measure the scattering data without knowing the potential V inside a (a priori bounded) region of interest. First choose a potential V^l that generates the same long range effects as V does. Then compute the solutions $z_\pm(v,.)$ of equation (1.6). Then for a fixed $(v_-,x_-)\in(\mathbb{R}^n\backslash\mathcal{B}(0,\mu))\times\mathbb{R}^n$ send a particle far away from the region of interest with a trajectory asymptotic to $x_-+z_-(v_-,.)$ at large and negative times. When the particle escapes any bounded region of the space at finite time, then detect the particle and find $S(v_-,x_-)=(v_+,x_+)$ so that the trajectory of the particle is asymptotic to $x_++z_+(v_+,.)$ at large and positive times far away from the bounded region of interest.

In this paper we consider the following inverse scattering problem for equation (1.1):

$$\text{Given } S \text{ and given the long range tail } F^l \text{ of the force } F, \text{ find } F^s. \tag{1.14}$$

The main results of the present work consist in estimates and asymptotics for the scattering data (a_{sc},b_{sc}) and scattering solutions for the equation (1.1) and in application of these asymptotics and estimates to the inverse scattering problem (1.14) at high energies. Our main results include, in particular, Theorem 1.1 and Corollary 1.2 given below that provide the high energies asymptotics of the scattering data and the Born approximation of the scattering data at fixed energy respectively.

Consider

$$T\mathbb{S}^{n-1}:=\{(\theta,x)\in\mathbb{S}^{n-1}\times\mathbb{R}^n \mid \theta\cdot x=0\},$$

and for any $m\in\mathbb{N}$ consider the x-ray transform P defined by

$$Pf(\theta,x):=\int_{-\infty}^{+\infty}f(t\theta+x)dt$$

for any function $f\in C(\mathbb{R}^n,\mathbb{R}^m)$ so that $|f(x)|=O(|x|^{-\tilde\beta})$ as $|x|\to+\infty$ for some $\tilde\beta>1$. For $(\sigma,\tilde\beta,r,\tilde\alpha)\in(0,+\infty)^2\times(0,1)\times(0,1]$, let $\rho_0=\rho_0(\sigma,r,\tilde\beta,\tilde\alpha)$ be defined as the root of the equation

$$1=\frac{6\tilde\beta n(\sigma+1)}{\tilde\alpha r(\frac{\rho_0}{2^{\frac{3}{2}}}-r)(1-r)^{\tilde\alpha+2}}\big(1+\frac{1}{\frac{\rho_0}{2^{\frac{3}{2}}}-r}\big),\ \rho_0>2^{\frac{3}{2}}r. \tag{1.15}$$

Then we have the following results.

Theorem 1.1. *Let* $(\theta,x)\in T\mathbb{S}^{n-1}$. *Under conditions* (1.3) *and* (1.4) *the following limits are valid*

$$\lim_{\rho\to+\infty}\rho a_{sc}(\rho\theta,x) = PF^l(\theta,x)+PF^s(\theta,x), \tag{1.16}$$

$$\lim_{\rho\to+\infty}\rho^2\theta\cdot\Big(b_{sc}(\rho\theta,x)-W(\rho\theta,x)\Big) = -PV^s(\theta,x), \tag{1.17}$$

where

$$W(v,x) := \int_{-\infty}^{0}\int_{-\infty}^{\sigma}\big(F^l(z_-(v,\tau)+x)-F^l(z_-(v,\tau))\big)d\tau d\sigma \quad (1.18)$$
$$-\int_{0}^{+\infty}\int_{\sigma}^{+\infty}\big(F^l(z_+(a(v,x),\tau)+x)-F^l(z_+(a(v,x),\tau))\big)d\tau d\sigma,$$

for $(v,x)\in\mathcal{D}(S)$.

In addition,

$$\Big|\rho a_{sc}(\rho\theta,x)-\int_{-\infty}^{+\infty}F(\tau\theta+x)d\tau\Big| \le \frac{8n^2(|x|+2)\beta^2\rho\big(1+\frac{1}{2^{-\frac{3}{2}}\rho-r}\big)^2}{\alpha^2(1-r)^{2\alpha+3}(2^{-\frac{3}{2}}\rho-r)^2}, \quad (1.19)$$

$$\Big|\rho^2\theta\cdot\big(b_{sc}(\rho\theta,x)-W(\rho\theta,x)\big)+PV^s(\theta,x)\Big| \le \frac{11n^2(|x|+2)\beta^2\rho^2\big(1+\frac{1}{2^{-\frac{3}{2}}\rho-r}\big)^2}{\alpha^2(\alpha+1)(1-r)^{2\alpha+2}(2^{-\frac{3}{2}}\rho-r)^3}, \quad (1.20)$$

for $(r,(\theta,x))\in(0,1)\times T\mathbb{S}^{n-1}$ *and for* $\rho>\rho_0(|x|,r,\beta,\alpha)$, *where* $\beta=\max(\beta_1^l,\beta_2^l,\beta_2^s,\beta_3^s)$.

Note that the vector W defined by (1.18) is known from the scattering data and from F^l. Then from (1.16) (resp. (1.17)) and inversion formulas for the X-ray transform for $n\ge 2$ (see [15, 6, 12, 13]) it follows that F^s can be reconstructed from a_{sc} (resp. b_{sc}).

Theorem 1.1 provides results on the asymptotics of the scattering data at fixed energy $E=\frac{1}{2}$ when the external force is weaker and weaker. In other words Theorem 1.1 also provides the Born approximation of the scattering data at fixed energy $E=\frac{1}{2}$. More precisely, consider the Newton equation (1.1) in the external potential $V^\gamma := \gamma^2 V$ where γ is a positive coupling constant and where $V\in C^2(\mathbb{R}^n,\mathbb{R})$ satisfies (1.3) and (1.4). We denote by $(a_{sc}^\gamma, b_{sc}^\gamma)$ (resp. (a_{sc},b_{sc})) the scattering data related to V^γ (resp. V), and we denote by $\mathcal{D}(S)$ the set of definition of the scattering map related to V. Then we have

$$a_{sc}^\gamma(\theta,x)=\gamma a_{sc}(\frac{\theta}{\gamma},x),\ \ b_{sc}^\gamma(\theta,x)=b_{sc}(\frac{\theta}{\gamma},x), \quad (1.21)$$

for $(\theta,x)\in T\mathbb{S}^{n-1}$ so that $(\frac{\theta}{\gamma},x)\in\mathcal{D}(S)$. Formulas (1.21) are derived from the observation that $x(\gamma t)$ is a solution of the Newton equation in the potential V^γ whenever $x(t)$ is a solution of the Newton equation in V. Then Theorem 1.1 is rewritten as follows.

Corollary 1.2. *Let* $(\theta, x) \in T\mathbb{S}^{n-1}$. *We have*

$$\lim_{\gamma \to 0^+} \gamma^{-2} a_{sc}^{\gamma}(\theta, x) = PF^l(\theta, x) + PF^s(\theta, x), \quad (1.22)$$

$$\lim_{\gamma \to 0^+} \gamma^{-2} \theta \cdot \Big(b_{sc}^{\gamma}(\theta, x) - W(\frac{\theta}{\gamma}, x) \Big) = -PV^s(\theta, x). \quad (1.23)$$

In addition,

$$\big| \gamma^{-2} a_{sc}^{\gamma}(\theta, x) - \int_{-\infty}^{+\infty} F(\tau\theta + x) d\tau \big| \le \gamma \frac{8n^2(|x| + 2)\beta^2 \big(1 + \frac{\gamma}{2^{-\frac{3}{2}} - r\gamma}\big)^2}{\alpha^2 (1-r)^{2\alpha+3} (2^{-\frac{3}{2}} - r\gamma)^2}, \quad (1.24)$$

$$\big| \gamma^{-2} \theta \cdot \big(b_{sc}^{\gamma}(\theta, x) - W(\frac{\theta}{\gamma}, x) \big) + PV^s(\theta, x) \big| \le \gamma \frac{11n^2(|x| + 2)\beta^2 \big(1 + \frac{\gamma}{2^{-\frac{3}{2}} - r\gamma}\big)^2}{\alpha^2(\alpha+1)(1-r)^{2\alpha+2}(2^{-\frac{3}{2}} - r\gamma)^3}, \quad (1.25)$$

for $(r, (\theta, x)) \in (0, 1) \times T\mathbb{S}^{n-1}$ *and for* $\gamma < \rho_0(|x|, r, \beta, \alpha)^{-1}$, *where* W *is defined by* (1.18).

From formulas (1.22) and (1.23) and from inversion formulas of the x-ray transform for $n \ge 2$ it follows that F^s can be reconstructed from the Born approximation at fixed energy $E = \frac{1}{2}$ of the scattering data a_{sc}^{γ} and b_{sc}^{γ}.

Theorem 1.1 is a generalization of [13, formulas (4.8a), (4.8b), (4.9a) and (4.9b)] where inverse scattering for the classical multidimensional Newton equation was studied in the short range case ($F^l \equiv 0$). We develop Novikov's framework [13] to obtain our results. Note that results [13, formulas (4.8b) and (4.9b)] also provide the approximation of the scattering data $(a_{sc}(v_-, x_-), b_{sc}(v_-, x_-))$ for the short range case ($F^l \equiv 0$) when the parameters α, n, v_- and β are fixed and $|x_-| \to +\infty$. Such an asymptotic regime is not covered by Theorem 1.1. Therefore we shall modify in Section 3 the definition of the scattering map in order to study these modified scattering data in the following three asymptotic regimes: at high energies, Born approximation at fixed energy, and when the parameters α, n, v_- and β are fixed and $|x_-| \to +\infty$.

For inverse scattering at fixed energy for the multidimensional Newton equation, see for example [9] and references therein.

For the inverse scattering problem in quantum mechanics for the Schrödinger equation, see for example [5], [4], [14] and references given in [14].

Our paper is organized as follows. In Section 2 we transform the differential equation (1.1) with initial conditions (1.9) into an integral equation which takes the form $y_- = A(y_-)$. Then we study the nonlinear integral operator A on a suitable space (Lemma 2.1) and we give estimates for the deflection $y_-(t)$ in (1.9) and for the scattering data $a_{sc}(v_-, x_-), b_{sc}(v_-, x_-)$ (Theorem 2.3). Then we prove Theorem 1.1. Note that we work with small angle scattering compared to the dynamics generated by F^l with respect to the "free" solutions $z_-(v_-, t)$: In particular, the angle between the vectors $\dot{x}(t) = \dot{z}_-(v_-, t) + \dot{y}_-(t)$ and $\dot{z}_-(v_-, t)$ goes to zero when the parameters β, α, n, $v_-/|v_-|$, x_- are fixed and $|v_-|$ increases. We also provide similar results when one replaces the "free" solutions $z_-(v, .)$ by some other functions "$z_{-,N}(v, .)$" that may be easier to compute numerically (Formulas (2.38) and (2.39)). In Section 3 we change the definition of the scattering map so that one can obtain for the modified scattering data $(\tilde{a}_{sc}(v_-, x_-), \tilde{b}_{sc}(v_-, x_-))$ their approximation at high energies, or their Born approximation at fixed energy, or their approximation when the parameters α, n, v_- and β are fixed and $|x_-| \to +\infty$ (Theorem 3.3, Corollary 3.4 and formulas (3.38) and (3.39)). Sections 4, 5 and 6 are devoted to proofs of our Theorem 2.3 and Lemmas 2.1 and 2.2.

2 Scattering solutions

2.1 Integral equation

For the rest of this Section we set $\beta_2 := \max(\beta_2^l, \beta_2^s)$.

Let $(v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n$, $v_- \cdot x_- = 0$ and $|v_-| \geq \mu$. Then the function y_- in (1.9) satisfies the integral equation $y_- = A(y_-)$ where

$$A(f)(t) = \int_{-\infty}^{t} \int_{-\infty}^{\sigma} \Big(F(z_-(v_-, \tau) + x_- + f(\tau)) - F^l(z_-(v_-, \tau)) \Big) d\tau d\sigma \quad (2.1)$$

for $t \in \mathbb{R}$ and for $f \in C(\mathbb{R}, \mathbb{R}^n)$, $\sup_{(-\infty,0]} |f| < \infty$. Under conditions (1.3) and (1.4) we have $A(f) \in C^2(\mathbb{R}, \mathbb{R}^n)$ for $f \in C(\mathbb{R}, \mathbb{R}^n)$ so that $\sup_{(-\infty,0]} |f| < \infty$.

For $r > 0$ we introduce the following complete metric space M_r defined by

$$M_r = \{f \in C(\mathbb{R}, \mathbb{R}^n) \mid \sup_{(-\infty,0]} |f| + \sup_{t \in [0,+\infty)} \Big(\frac{|f(t)|}{1 + |t|} \Big) \leq r\}, \quad (2.2)$$

and endowed with the norm $\|.\|$ where $\|f\| = \sup_{(-\infty,0]} |f| + \sup_{t\in[0,+\infty)} \Big(\frac{|f(t)|}{1+|t|} \Big)$. Then we have the following estimate and contraction estimate for the map A restricted to M_r.

Lemma 2.1. *Let* $(v_-, x_-) \in (\mathbb{R}^n \backslash \mathcal{B}(0, \mu)) \times \mathbb{R}^n$, $v_- \cdot x_- = 0$, *and let* $r \in$

$(0, \min(\frac{|v_-|}{2^{\frac{3}{2}}}, 1))$. *Then the following estimates are valid:*

$$\begin{aligned} \|A(f)\| &\leq \lambda_0(n, \alpha, \beta_2, |x_-|, |v_-|, r) \qquad (2.3)\\ &:= \frac{\beta_2(n(2|x_-|+r)+2\sqrt{n})}{(\frac{|v_-|}{2^{\frac{3}{2}}}-r)(1-r)^\alpha}\Big(\frac{2}{\alpha(\frac{|v_-|}{2^{\frac{3}{2}}}-r)}+\frac{1}{(\alpha+1)(1-r)}\Big), \end{aligned}$$

and

$$\|A(f_1)-A(f_2)\| \leq \lambda(n, \alpha, \beta_2, \beta_3^s, |x_-|, |v_-|, r)\|f_1-f_2\|, \qquad (2.4)$$

$$\begin{aligned} \lambda(n, \alpha, \beta_2, \beta_3^s, |x_-|, |v_-|, r) &:= \frac{2n}{\alpha(\frac{|v_-|}{2^{\frac{3}{2}}}-r)(1-r+\frac{|x_-|}{\sqrt{2}})^\alpha}\Big(\beta_2+\frac{\beta_3^s}{1-r+\frac{|x_-|}{\sqrt{2}}}\Big)\\ &\times\Big(\frac{1}{1-r+\frac{|x_-|}{\sqrt{2}}}+\frac{1}{\frac{|v_-|}{2^{\frac{3}{2}}}-r}\Big), \end{aligned}$$

for $(f, f_1, f_2) \in M_r^3$.

A proof of Lemma 2.1 is given in Section 5.
We also need the following result.

Lemma 2.2. *Let* $(v_-, x_-) \in (\mathbb{R}^n \backslash \mathcal{B}(0,\mu)) \times \mathbb{R}^n$, $v_- \cdot x_- = 0$, *and let* $r \in (0, \min(\frac{|v_-|}{2^{\frac{3}{2}}}, 1))$. *When* $y_- \in M_r$ *is a fixed point for the map* A *then* $z_-(v_-, .)+x_-+y_-$ *is a scattering solution for equation* (1.1) *and*

$$z_-(v_-, t)+x_-+y_-(t) = z_+(a(v_-, x_-), t)+b(v_-, x_-)+y_+(t), \qquad (2.5)$$

for $t \geq 0$, *where*

$$\begin{aligned} a(v_-, x_-) &:= v_- + \int_{-\infty}^{+\infty} F(z_-(v_-, \tau)+x_-+y_-(\tau))d\tau, \qquad (2.6)\\ b(v_-, x_-) &:= x_- + l(y_-)+l_1+l_2(y_-), \qquad (2.7) \end{aligned}$$

$$y_+(t) := \int_t^{+\infty}\int_\sigma^{+\infty}\big(F(z_-(v_-, \tau)+x_-+y_-(\tau))-F^l(z_+(a(v_-, x_-), \tau))\big)d\tau d\sigma, \qquad (2.8)$$

for $t \geq 0$, *and where*

$$\begin{aligned} l(y_-) &:= \int_{-\infty}^0\int_{-\infty}^\sigma \big(F\big(z_-(v_-, \tau)+x_-+y_-(\tau)\big)-F^l\big(z_-(v_-, \tau)\big)\big)\, d\tau d\sigma\\ &-\int_0^{+\infty}\int_\sigma^{+\infty} F^s\big(z_-(v_-, \tau)+x_-+y_-(\tau)\big)d\tau d\sigma, \qquad (2.9) \end{aligned}$$

$$l_1 := -\int_0^{+\infty}\int_\sigma^{+\infty}\big(F^l(z_+(a(v_-,x_-),\tau)+x_-)-F^l(z_+(a(v_-,x_-),\tau))\big)d\tau d\sigma, \tag{2.10}$$

$$l_2(y_-) := -\int_0^{+\infty}\int_\sigma^{+\infty}\big(F^l(z_-(v_-,\tau)+x_-+y_-(\tau))-F^l(z_+(a(v_-,x_-),\tau)+x_-)\big)d\tau d\sigma, \tag{2.11}$$

for $t \geq 0$.

Lemma 2.2 is proved in Section 4. Note that l_1 is known from the scattering data and the knowledge of F^l.

2.2 Estimates on the scattering solutions

In this Section our main results consist in estimates and asymptotics for the scattering data (a_{sc}, b_{sc}) and scattering solutions for the equation (1.1).

Theorem 2.3. *Under the assumptions of Lemma 2.2 the following estimates are valid*

$$|\dot y_-(t)| \leq \frac{\beta_2\big(n(|x_-|+r)+\sqrt{n}\big)}{(\alpha+1)\big(\frac{|v_-|}{2\sqrt2}-r\big)\Big(1-r+|t|\big(\frac{|v_-|}{2\sqrt2}-r\big)\Big)^{\alpha+1}}, \tag{2.12}$$

$$|y_-(t)| \leq \frac{\beta_2(n(|x_-|+r)+\sqrt{n})}{\alpha(\alpha+1)\big(\frac{|v_-|}{2\sqrt2}-r\big)^2\Big(1-r+|t|\big(\frac{|v_-|}{2\sqrt2}-r\big)\Big)^{\alpha}}, \tag{2.13}$$

for $t \leq 0$. *In addition*

$$|a_{sc}(v_-,x_-)| \leq \frac{2n^{\frac12}}{\big(\frac{|v_-|}{2\sqrt2}-r\big)(1+\frac{|x_-|}{\sqrt2}-r)^\alpha}\Big(\frac{\beta_1^l}{\alpha}+\frac{\beta_2}{(\alpha+1)(1+\frac{|x_-|}{\sqrt2}-r)}\Big). \tag{2.14}$$

$$|l(y_-)| \leq \frac{\beta_2 n^{\frac12}(n^{\frac12}(|x_-|+r)+2)}{\alpha(\alpha+1)(1-r)^\alpha\big(\frac{|v_-|}{2\sqrt2}-r\big)^2}, \tag{2.15}$$

and

$$\Big|a_{sc}(v_-,x_-)-\int_{-\infty}^{+\infty}F(z_-(v_-,\tau)+x_-)d\tau\Big|$$
$$\leq \frac{4\max(\beta_2,\beta_3^s)^2 n^{\frac32}(n^{\frac12}(2|x_-|+r)+2)}{\alpha^2(\frac{|v_-|}{2^{\frac32}}-r)^2(1-r)^{2\alpha+3}}\Big(\frac{1}{\frac{|v_-|}{2^{\frac32}}-r}+1\Big)^2, \tag{2.16}$$

$$|l(y_-)-l(0)| \leq \frac{4\max(\beta_2,\beta_3^s)^2 n^{\frac32}(n^{\frac12}(2|x_-|+r)+2)\big(\frac{1}{\frac{|v_-|}{2^{\frac32}}-r}+1\big)^2}{\alpha^2(\alpha+1)(\frac{|v_-|}{2^{\frac32}}-r)^3(1-r)^{2\alpha+2}}. \tag{2.17}$$

In addition when

$$\frac{6n\max(\beta_1^l,\beta_2)}{\alpha\big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\big)^2(1-r)^{\alpha+1}}\le 1, \tag{2.18}$$

then

$$|l_1| \le \frac{8\beta_2 n|x_-|}{\alpha(\alpha+1)|v_-|^2}, \tag{2.19}$$

$$|l_2(y_-)| \le \frac{2n^{\frac{3}{2}}\beta_2^2(n^{\frac{1}{2}}(3|x_-|+r)+3)}{\alpha^2(\alpha+1)^2(1-r)^{2\alpha}(\frac{|v_-|}{2^{\frac{3}{2}}}-r)^4}, \tag{2.20}$$

$$|y_+(t)| \le \frac{4n\beta_2(1+|x_-|)}{\alpha(\alpha+1)(\frac{|v_-|}{2^{\frac{3}{2}}}-r)^2(1-r+\frac{|x_-|}{\sqrt{2}}+t(\frac{|v_-|}{2^{\frac{3}{2}}}-r))^\alpha}, \tag{2.21}$$

for $t\ge 0$.

A proof of Theorem 2.3 is given in Section 6. We now prove Theorem 1.1 combining Theorem 2.3, Lemma 2.1 and estimate (1.8).

2.3 Proof of Theorem 1.1

Let $(v_-,x_-)\in\mathbb{R}^n\times\mathbb{R}^n$, $v_-\cdot x_-=0$ and $|v_-|\ge\mu$. We first prove estimates (2.25) and (2.27) given below. We use the following estimate (2.22)

$$|x_-+\eta v_-\tau+(1-\eta)z_-(v_-,\tau)|\ge|x_-+\tau v_-|-|z_-(v_-,\tau)-\tau v_-|$$
$$\ge \frac{|x_-|}{\sqrt{2}}+\Big(\frac{|v_-|}{\sqrt{2}}-\frac{2^{\frac{3}{2}}n^{\frac{1}{2}}\beta_1^l}{\alpha|v_-|}\Big)|\tau|\ge\frac{|x_-|}{\sqrt{2}}+|\tau|\frac{|v_-|}{2^{\frac{3}{2}}}, \tag{2.22}$$

for $\eta\in(0,1)$ and $\tau\in\mathbb{R}$ (we used (1.8) and the inequality $|v_-|\ge\mu$). Then from (1.3), (1.4), (2.22) and (1.8) it follows that

$$|F^s(z_-(v_-,\tau)+x_-)-F^s(\tau v_-+x_-)| \le \sup_{\eta\in(0,1)}\frac{n\beta_3^s|z_-(v_-,\tau)-\tau v_-|}{(1+|x_-+\eta v_-+(1-\eta)z_-(v_-,\tau)|)^{\alpha+3}}$$
$$\le \frac{2^{\frac{3}{2}}n^{\frac{3}{2}}\beta_3^s\beta_1^l|\tau|}{\alpha|v_-|(1+\frac{|x_-|}{\sqrt{2}}+|\tau|\frac{|v_-|}{2^{\frac{3}{2}}})^{\alpha+3}}, \tag{2.23}$$

for $\tau\in\mathbb{R}$. Similarly

$$|F^l(z_-(v_-,\tau)+x_-)-F^l(\tau v_-+x_-)|\le\frac{2^{\frac{3}{2}}n^{\frac{3}{2}}|\tau|\beta_2\beta_1^l}{\alpha|v_-|(1+\frac{|x_-|}{\sqrt{2}}+|\tau|\frac{|v_-|}{2^{\frac{3}{2}}})^{\alpha+2}}, \tag{2.24}$$

for $\tau \in \mathbb{R}$. Then using (2.23) and (2.24) we have

$$\Big|\int_{-\infty}^{+\infty} F(z_-(v_-,\tau)+x_-)d\tau - \int_{-\infty}^{+\infty} F(\tau v_- + x_-)d\tau\Big|$$

$$\le \frac{2^{\frac{5}{2}} n^{\frac{3}{2}} \max(\beta_2, \beta_3^s)\beta_1^l}{\alpha|v_-|}\int_{-\infty}^{+\infty} \frac{|\tau| d\tau}{(1+\frac{|x_-|}{\sqrt{2}}+\frac{|v_-|}{2^{\frac{3}{2}}}|\tau|)^{\alpha+2}} \le \frac{2^{\frac{13}{2}} n^{\frac{3}{2}} \max(\beta_2,\beta_3^s)\beta_1^l}{\alpha^2|v_-|^3(1+\frac{|x_-|}{\sqrt{2}})^{\alpha}}. \tag{2.25}$$

Set

$$\Delta_1 = \int_{-\infty}^{0}\int_{-\infty}^{\sigma} F^s\big(z_-(v_-,\tau)+x_-\big)d\tau d\sigma - \int_0^{+\infty}\int_{\sigma}^{+\infty} F^s\big(z_-(v_-,\tau)+x_-\big)d\tau d\sigma. \tag{2.26}$$

Then using (2.23) we have

$$\big|\Delta_1 - \int_{-\infty}^{0}\int_{-\infty}^{\sigma} F^s\big(\tau v_- + x_-\big)d\tau d\sigma + \int_0^{+\infty}\int_{\sigma}^{+\infty} F^s\big(\tau v_- + x_-\big)d\tau d\sigma\big|$$

$$\le \frac{2^{\frac{5}{2}} n^{\frac{3}{2}} \beta_1^l \beta_3^s}{\alpha|v_-|}\int_{-\infty}^{0}\int_{-\infty}^{\sigma} \frac{|\tau| d\tau d\sigma}{(1+\frac{|x_-|}{\sqrt{2}}+\frac{|v_-|}{2\sqrt{2}}|\tau|)^{\alpha+3}} \le \frac{2^7 n^{\frac{3}{2}} \beta_1^l\beta_3^s}{\alpha^2(\alpha+1)|v_-|^4(1+\frac{|x_-|}{\sqrt{2}})^{\alpha}}. \tag{2.27}$$

Let $r \in (0, \min(\frac{|v_-|}{2^{\frac{3}{2}}}, 1))$. Note that

$$\max\big(\frac{\lambda_0}{r}, \lambda, \frac{6n\max(\beta_1^l,\beta_2)}{\alpha\big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\big)^2(1-r)^{\alpha+1}}\big) \le \frac{6\beta n(|x_-|+1)}{\alpha r(\frac{|v_-|}{2^{\frac{3}{2}}}-r)(1-r)^{\alpha+2}}\big(1+\frac{1}{\frac{|v_-|}{2^{\frac{3}{2}}}-r}\big), \tag{2.28}$$

where λ_0 and λ are defined by (2.3) and (2.4) respectively. Assume that $|v_-| > \rho_0(|x_-|, r, \beta, \alpha)$ where ρ_0 is the root of the equation (1.15). Then from (1.15) and Lemma 2.1 it follows that A has a unique fixed point in M_r denoted by y_-. Then adding (2.16) and (2.25) we obtain (1.19). Note also that

$$l(0) = \int_{-\infty}^{0}\int_{-\infty}^{\sigma}\big(F^l\big(z_-(v_-,\tau)+x_-\big) - F^l\big(z_-(v_-,\tau)\big)\big)\,d\tau d\sigma$$
$$+\Delta_1.$$

Hence adding (2.27), (2.20) and (2.17) we obtain (1.20). Theorem 1.1 is proved. □

2.4 Motivations for changing the definition of the scattering map

For a solution x at a nonzero energy for equation (1.1) we say that it is a scattering solution when there exists $\varepsilon > 0$ so that $1 + |x(t)| \geq \varepsilon(1 + |t|)$ for $t \in \mathbb{R}$ (see [3]). In the Introduction and in the previous subsections we choose to parametrize the scattering solutions of equation (1.1) by the solutions $z_\pm(v, .)$ of the equation (1.6) (see the asymptotic behaviors (1.9) and (1.10)), and then to formulate the inverse scattering problem (1.14) using this parametrization. To compute the "free" solutions $z_\pm(v, .)$ one has to integrate equation (1.6). For some cases solving (1.6) leads to simple exact formulas (see [11, Section 15] when F^l is a Coulombian force). In general one may choose to approximate the solutions $z_\pm(v, .)$ by the functions $z_{\pm,N+1}(v, .)$ defined below. In general the approximating functions $z_{\pm,N+1}(v, .)$ are easier to compute numerically and/or analytically, and in this Subsection we use these approximations to obtain an other formulation of the inverse scattering problem and to mention results similar to Theorem 1.1 and to those given in the previous subsections.

Assume without loss of generality that $\alpha \notin \{\frac{1}{m} \mid m \in \mathbb{N},\ m > 0\}$ and set $N = \lfloor \alpha^{-1} \rfloor$ the integer part of α^{-1}. Then let $v \in \mathbb{R}^n \backslash \mathcal{B}(0, \mu)$. Proceeding from [8] we define by induction

$$z_{\pm,0}(v, t) = tv,\ z_{\pm,m+1}(v, t) = tv + \int_0^t \int_{\pm\infty}^\sigma F^l(z_{\pm,m}(v, \tau))d\tau d\sigma, \tag{2.29}$$

for $t \in \mathbb{R}$ and for $m = 0 \dots N$. Then one can prove the following estimates by induction (see Section 4 and see also [8])

$$|z_{\pm,m}(v, t) - tv| \leq \frac{2^{\frac{3}{2}} n^{\frac{1}{2}} \beta_1^l}{\alpha |v|} |t|,\ t \in \mathbb{R},\ m = 1 \dots N + 1, \tag{2.30}$$

$$|z_{\pm,m+1}(v, t) - z_{\pm,m}(v, t)| \leq \frac{2^{m+1} n^{m+\frac{1}{2}} (\beta_2^l)^m \beta_1^l \Big(\big(1 + |t| \frac{|v|}{\sqrt{2}}\big)^{1-(m+1)\alpha} - 1 \Big)}{\alpha^{m+1} |v|^{2m+2} \Pi_{j=1}^{m+1} (1 - j\alpha) j}, \tag{2.31}$$

for $m = 0 \dots N - 1$ and for $\pm t \geq 0$,

$$|z_{\pm,N+1}(v, t) - z_{\pm,N}(v, t)| \leq \frac{2^{N+1} n^{N+\frac{1}{2}} (\beta_2^l)^N \beta_1^l}{\alpha^{N+1} |v|^{2N+2} \Pi_{j=1}^{N+1} j |1 - j\alpha|}, \tag{2.32}$$

for $\pm t \geq 0$, and

$$|z_{\pm,m+1}(v, t) - z_{\pm,m}(v, t)| \leq \frac{2^{2m+\frac{3}{2}} n^{m+\frac{1}{2}} (\beta_2^l)^m \beta_1^l}{\alpha^{m+1} |v|^{2m+1}} |t|, \tag{2.33}$$

for $t\in\mathbb{R}$ and for $m=1\dots N$. For $(v_-,x_-)\in\mathbb{R}^n\times\mathbb{R}^n$, $|v_-|\ge\mu$, there exists a unique solution $x(t)$ of equation (1.1) so that

$$x(t)=x_-+z_{-,N+1}(v_-,t)+y_-(t),\ t\in\mathbb{R}\text{ and }\lim_{t\to-\infty}(|y_-(t)|+|\dot y_-(t)|)=0. \tag{2.34}$$

In addition when the solution x in (2.34) is a scattering solution then there exists a unique $(v_+,x_+)\in\mathbb{R}^n\times\mathbb{R}^n$, $|v_+|=|v_-|$ so that

$$x(t)=x_++z_{+,N+1}(v_+,t)+y_+(t),\ t\in\mathbb{R}\text{ and }\lim_{t\to+\infty}(|y_+(t)|+|\dot y_+(t)|)=0. \tag{2.35}$$

In that case we define the scattering data $(a_N(v_-,x_-),b_N(v_-,x_-)):=(v_+,x_+)$, and we consider the inverse scattering problem

$$\text{Given } (a_N,b_N) \text{ and given the long range tail } F^l \text{ of the force } F,\ \text{find } F^s. \tag{2.36}$$

The function y_- in (2.34) satisfies the following integral equation $y_-=A_N(y_-)$ where

$$A_N(f)(t)=\int_{-\infty}^t\int_{-\infty}^\sigma\Big(F(z_{-,N+1}(v_-,\tau)+x_-+f(\tau))-F^l(z_{-,N}(v_-,\tau))\Big)d\tau d\sigma \tag{2.37}$$

for $t\in\mathbb{R}$ and for $f\in C(\mathbb{R},\mathbb{R}^n)$, $\sup_{(-\infty,0]}|f|<\infty$. Then with appropriate changes in the proof of Lemma 2.1 we can study the operator A_N restricted to M_r and we can obtain estimate and contraction estimate similar to (2.3) and (2.4). We also obtain the analog of Lemma 2.2 by appropriate change in its proof, and the decomposition (2.5) remains valid by replacing a, b and $z_-(v_-,\tau)+x_-+y_-(\tau)$ by a_N, b_N and $z_{-,N+1}(v_-,\tau)+x_-+y_-(\tau)$ in (2.5)–(2.11), and by replacing z_+ and $F^l\big(z_-(v_-,\tau)\big)$ by $z_{+,N+1}$ and $F^l\big(z_{-,N}(v_-,\tau)\big)$ in (2.5) and (2.9), and by replacing z_+ by $z_{+,N}$ in (2.8), (2.10) and (2.11). An analog of Theorem 2.3 can be proved for the scattering solutions and scattering data (a_N,b_N). Set $a_{sc,N}(v_-,x_-):=a_N(v_-,x_-)-v_-$ and $b_{sc,N}(v_-,x_-):=b_N(v_-,x_-)-x_-$. Finally the following high energies limits are valid. Let $(\theta,x)\in T\mathbb{S}^{n-1}$, then

$$\lim_{\rho\to+\infty}\rho a_{sc,N}(\rho\theta,x) = PF^l(\theta,x)+PF^s(\theta,x), \tag{2.38}$$

$$\lim_{\rho\to+\infty}\rho^2\theta\cdot\Big(b_{sc,N}-W_N\Big)(\rho\theta,x) = -PV^s(\theta,x), \tag{2.39}$$

where

$$W_N(v,x) := \int_{-\infty}^0\int_{-\infty}^\sigma\big(F^l(z_{-,N}(v,\tau)+x)-F^l(z_{-,N}(v,\tau))\big)d\tau d\sigma \tag{2.40}$$
$$-\int_0^{+\infty}\int_\sigma^{+\infty}\big(F^l(z_{+,N}(a_N(v,x),\tau)+x)-F^l(z_{+,N}(a_N(v,x),\tau))\big)d\tau d\sigma,$$

for $(v,x)\in\mathbb{R}^n\times\mathbb{R}^n$, $v\cdot x=0$ and $|v|>C$ for some constant C. The vector W_N defined by (2.40) is known from the scattering data and from F^l.

For the Problem (2.36), from (2.38) (resp. (2.39)) and inversion formulas for the X-ray transform for $n\geq 2$ (see [15, 6, 12, 13]) it follows that F^s can be reconstructed from $a_{sc,N}$ (resp. $b_{sc,N}$).

The Born approximation of $a_{sc,N}$, $b_{sc,N}$ at fixed energy can also be derived from (2.38) and (2.39) by using the same observations that leaded to Corollary 1.2.

The limits (2.38) and (2.39) follow from estimates similar to (1.19) and (1.20). However these estimates do not provide the asymptotics of the scattering data $(a_{sc,N},b_{sc,N})$ when the parameters α, n, v_- and β are fixed and $|x_-|\to+\infty$. Motivated by this disadvantage, we introduce a new family of "free" solutions $z_\pm(v,x,.)$ of equation (1.6) that will be used for parametrizing some unbounded solutions of the Newton equation (1.1) at nonzero energy and for measuring their deviation. Those free solutions $z_\pm(v,x,.)$ have the properties that $\lim_{t\to\pm\infty}\dot z_\pm(v,x,t)=v$ and $z_\pm(v,x,0)=x$. In addition, for $(v_-,x_-)\in\mathbb{R}^n\times\mathbb{R}^n$, $v_-\cdot x_-=0$, so that the energy $E=\frac{1}{2}|v_-|^2$ is sufficiently high, then there exists a unique solution $x\in C^2(\mathbb{R},\mathbb{R}^n)$ of equation (1.1) such that

$$x(t)=z_-(v_-,x_-,t)+y_-(t),\tag{2.41}$$

where $|\dot y_-(t)|+|y_-(t)|\to 0$, as $t\to-\infty$. Such a solution also satisfies:

$$x(t)=z_+(\tilde a,\tilde b,t)+y_+(t),\tag{2.42}$$

where $|\dot y_+(t)|+|y_+(t)|\to 0$, as $t\to+\infty$ for a unique $(\tilde a,\tilde b)\in\mathbb{R}^n\times\mathbb{R}^n$. The map $\tilde S$ defined by $\tilde S(v_-,x_-)=(\tilde a,\tilde b)$ at sufficiently high energies is our modified scattering map.

In the next section, we provide more details on its definition, and we study its asymptotics in the three regimes: at high energies, its Born approximation at fixed energy, and at fixed energy when the "impact parameter" $|x_-|\to+\infty$.

3 A modified scattering map

First let us introduce the new family mentioned above of "free" solutions $z_\pm(v,x,.)$ of equation (1.6). We set

$$\tilde\mu(\sigma):=\sqrt{\frac{2^5 n\max(\beta_1^l,\beta_2^l)}{\alpha(\frac{1}{2}+\frac{\sigma}{2^{\frac{3}{2}}})^\alpha}},\ \text{for }\sigma\geq 0.\tag{3.1}$$

Let $(v,x)\in(\mathbb{R}^n)^2$ and let $(w,h)\in\overline{\mathcal{B}(v,\frac{|v|}{4\sqrt{2}})}\times\overline{\mathcal{B}(0,\frac{1}{2}+\frac{|x|}{2^{\frac{3}{2}}})}$ so that $v\cdot x=0$ and $|v|\geq\tilde{\mu}(|x|)$. Then there exists a unique solution $z_\pm(w,x+h,.)$ of the equation (1.6) so that

$$\dot{z}_\pm(w,x+h,t)-w=o(1),\ \text{as } t\to\pm\infty,\ z_\pm(w,x+h,0)=x+h, \tag{3.2}$$

and

$$|z_\pm(w,x+h,t)-x-h-tw|\leq\frac{2^{\frac{5}{2}}n^{\frac{1}{2}}\beta_1^l}{\alpha|v|(\frac{1}{2}+\frac{|x|}{2\sqrt{2}})^\alpha}|t|, \tag{3.3}$$

for $t\in\mathbb{R}$. Under conditions (1.3) and (1.4), the following is valid: for any $(v_-,x_-)\in\mathbb{R}^n\backslash\{0\}\times\mathbb{R}^n$ so that $|v_-|\geq\tilde{\mu}(|x_-|)$ and $v_-\cdot x_-=0$, then the equation (1.1) has a unique solution $x\in C^2(\mathbb{R},\mathbb{R}^n)$ such that

$$x(t)=z_-(v_-,x_-,t)+y_-(t), \tag{3.4}$$

where $|\dot{y}_-(t)|+|y_-(t)|\to 0$, as $t\to-\infty$.

In addition the function y_- in (3.4) satisfies the integral equation $y_-=\mathcal{A}(y_-)$ where

$$\mathcal{A}(f)(t)=\int_{-\infty}^t\int_{-\infty}^\sigma\Big(F(z_-(v_-,x_-,\tau)+f(\tau))-F^l(z_-(v_-,x_-,\tau))\Big)d\tau d\sigma \tag{3.5}$$

for $t\in\mathbb{R}$ and for $f\in C(\mathbb{R},\mathbb{R}^n)$, $\sup_{(-\infty,0]}|f|<\infty$. We study the map $\mathcal{A}$ defined by (3.5) on the metric space M_r defined by (2.2). Set

$$\tilde{k}(v_-,x_-,f)=v_-+\int_{-\infty}^{+\infty}F\big(z_-(v_-,x_-,\tau)+f(\tau)\big)d\tau, \tag{3.6}$$

for $f\in M_r$. For the rest of the section we set $\beta_2=\max(\beta_2^l,\beta_2^s)$.

Lemma 3.1. *Let $(v_-,x_-)\in\mathbb{R}^n\times\mathbb{R}^n$, $v_-\cdot x_-=0$, $|v_-|\geq\tilde{\mu}(|x_-|)$, and let $r\in(0,\min(\frac{|v_-|}{2^{\frac{3}{2}}},1+\frac{|x_-|}{\sqrt{2}}))$. Then the following estimates are valid:*

$$\begin{aligned}\|\mathcal{A}(f)\| &\leq \tilde{\lambda}_0(n,\alpha,\beta_2,|x_-|,|v_-|,r) \qquad (3.7)\\ &= \frac{2\beta_2 n^{\frac{1}{2}}(n^{\frac{1}{2}}r+1)}{(\frac{|v_-|}{2^{\frac{3}{2}}}-r)(1-r+\frac{|x_-|}{\sqrt{2}})^\alpha}\Big(\frac{1}{(\alpha+1)(1-r+\frac{|x_-|}{\sqrt{2}})}+\frac{2}{\alpha(\frac{|v_-|}{2^{\frac{3}{2}}}-r)}\Big),\end{aligned}$$

$$\|\mathcal{A}(f_1)-\mathcal{A}(f_2)\|\leq\lambda(n,\alpha,\beta_2,\beta_3^s,|x_-|,|v_-|,r)\|f_1-f_2\|, \tag{3.8}$$

and

$$|\tilde{k}(v_-,x_-,f)-v_-|\leq\frac{2n^{\frac{1}{2}}}{\big(\frac{|v_-|}{2\sqrt{2}}-r\big)(1+\frac{|x_-|}{\sqrt{2}}-r)^\alpha}\Big(\frac{\beta_1^l}{\alpha}+\frac{\beta_2}{(\alpha+1)(1+\frac{|x_-|}{\sqrt{2}}-r)}\Big), \tag{3.9}$$

for $(f,f_1,f_2)\in M_r^3$, where λ is defined in (2.4).

For $(\sigma,\tilde\beta,r,\tilde\alpha)\in(0,+\infty)^3\times(0,1]$, $r<\frac{1}{2}+\frac{\sigma}{2^{\frac{3}{2}}}$, let $\tilde\rho_0=\tilde\rho_0(\sigma,r,\tilde\beta,\tilde\alpha)$ be defined as the root of the equation

$$1=\frac{20\tilde\beta n(1+r)}{\tilde\alpha r(\frac{\tilde\rho_0}{2^{\frac{3}{2}}}-r)(\frac{1}{2}-r+\frac{\sigma}{2^{\frac{3}{2}}})^{\tilde\alpha}}\big(1+\frac{1}{\frac{\tilde\rho_0}{2^{\frac{3}{2}}}-r}\big),\ \tilde\rho_0>2^{\frac{3}{2}}r. \tag{3.10}$$

Let $(v_-,x_-)\in\mathbb{R}^n\times\mathbb{R}^n$, $v_-\cdot x_-=0$, and let $r\in(0,\frac{1}{2}+\frac{|x_-|}{2^{\frac{3}{2}}})$ so that $|v_-|>\tilde\rho_0(|x_-|,r,\beta,\alpha)$, where $\beta=\max(\beta_1^l,\beta_2,\beta_3^s)$. Then from Lemma 3.1 the map $\mathcal{A}$ is a contraction in M_r and we denote by y_- its unique fixed point, and from (3.9) one can consider the free solution $z_+(\tilde a(v_-,x_-),x_-,.)$ where

$$\tilde a(v_-,x_-)=\tilde k(v_-,x_-,y_-). \tag{3.11}$$

Then $z_-(v_-,x_-,.)+y_-$ is a scattering solution of (1.1) (in the sense given in Section 2.4) and the following decomposition is valid

$$z_-(v_-,x_-,t)+y_-(t)=z_+(\tilde a(v_-,x_-),x_-+h,t)+\big(\mathcal{G}_{v_-,x_-}(h)-h\big)+H(v_-,x_-,y_-,h,t), \tag{3.12}$$

for $t\ge 0$, where

$$\begin{aligned}\mathcal{G}_{v_-,x_-}(h) &= \mathcal{A}(y_-)(0)-H(v_-,x_-,y_-,h,0), && (3.13)\\ H(v_-,x_-,y_-,h,t) &= \int_t^{+\infty}\int_\sigma^{+\infty}\Big(F\big(z_-(v_-,x_-,\tau)+y_-(\tau)\big)\\ &\quad -F^l\big(z_+(\tilde a(v_-,x_-),x_-+h,\tau)\big)\Big)d\tau d\sigma, && (3.14)\end{aligned}$$

for $t\ge 0$ and for $|h|\le\frac{1}{2}+\frac{|x_-|}{2\sqrt{2}}$. Lemma 3.2 provides an estimate and a contraction estimate on the map $\mathcal{G}_{v_-,x_-}$.

Lemma 3.2. *The following estimates are valid:*

$$\begin{aligned}|\mathcal{G}_{v_-,x_-}(h)| &\le \frac{\beta_2(6(nr+\sqrt{n})+n(1+\frac{|x_-|}{\sqrt{2}}))}{2\alpha(\alpha+1)\big(\frac{|v_-|}{2\sqrt{2}}-r\big)^2\big(\frac{1}{2}+\frac{|x_-|}{2\sqrt{2}}-r\big)^\alpha} && (3.15)\\ &\le \frac{1}{4}+\frac{|x_-|}{10\sqrt{2}}, && (3.16)\end{aligned}$$

$$|\mathcal{G}_{v_-,x_-}(h)-\mathcal{G}_{v_-,x_-}(h')|\le\frac{16n\beta_2^l|h-h'|}{\alpha(\alpha+1)|v_-|^2\big(\frac{1}{2}+\frac{|x_-|}{2\sqrt{2}}\big)^\alpha}\le\frac{|h-h'|}{10}, \tag{3.17}$$

for $(h,h')\in\mathbb{R}^n\times\mathbb{R}^n$, $|h'|\le|h|\le\frac{1}{4}+\frac{|x_-|}{2^{\frac{5}{2}}}$.

We denote by $\tilde{b}_{sc}(v_-,x_-)$ the unique fixed point of $\mathcal{G}_{v_-,x_-}$ in $\overline{\mathcal{B}}(0,\frac{1}{4}+\frac{|x_-|}{2^{\frac{5}{2}}})$, and we set $\tilde{b}(v_-,x_-):=x_-+\tilde{b}_{sc}(v_-,x_-)$ and $\tilde{a}_{sc}(v_-,x_-):=\tilde{a}(v_-,x_-)-v_-$. The decomposition (3.12) becomes

$$z_-(v_-,x_-,t)+y_-(t)=z_+(\tilde{a}(v_-,x_-),\tilde{b}(v_-,x_-),t)+y_+(t), \quad (3.18)$$

$$y_+(t)=H(v_-,x_-,y_-,\tilde{b}_{sc}(v_-,x_-),t), \quad (3.19)$$

for $t\geq 0$.

The map $\tilde{S}$ defined by $\tilde{S}(v_-,x_-)=(\tilde{a}(v_-,x_-),\tilde{b}(v_-,x_-))$ on the set

$$\tilde{\mathcal{M}} = \{(v_-,x_-)\in\mathbb{R}^n\times\mathbb{R}^n \mid v_-\cdot x_-=0,\ |v_-|>\tilde{\rho}_0(|x_-|,r,\beta,\alpha) \text{ for some } r\in(0,\frac{1}{2}+\frac{|x_-|}{2^{\frac{3}{2}}})\}$$

is our modified scattering map.

Links between $(\tilde{a},\tilde{b})$ and the scattering data (a,b) defined in Section 2 are given by the 2 following formulas provided that $|v_-|\geq\mu$ where μ is the constant defined in the Introduction:

$$\tilde{a}=a\big(v_-,\lim_{t\to-\infty}(z_-(v_-,x_-,t)-z_-(v_-,t))\big), \quad (3.20)$$

$$b\big(v_-,\lim_{t\to-\infty}(z_-(v_-,x_-,t)-z_-(v_-,t))\big)=\lim_{t\to+\infty}\Big(z_+(\tilde{a},\tilde{b},t)-z_+(\tilde{a},t)\Big). \quad (3.21)$$

We shortened $\tilde{a}(v_-,x_-)$ and $\tilde{b}(v_-,x_-)$ to $\tilde{a}$ and $\tilde{b}$ in (3.20) and (3.21).

The inverse scattering problem for equation (1.1) can now be formulated as follows

$$\text{Given } (\tilde{a}_{sc},\tilde{b}_{sc}) \text{ and } F^l,\ \text{find } F^s. \quad (3.22)$$

Theorem 3.3 and Corollary 3.4 provide results similar to Theorems 2.3 and 1.1.

Theorem 3.3. *Let $(v_-,x_-)\in\mathbb{R}^n\times\mathbb{R}^n$, $v_-\cdot x_-=0$, and let $r\in(0,\frac{1}{2}+\frac{|x_-|}{2^{\frac{3}{2}}})$, $|v_-|>\tilde{\rho}_0(|x_-|,r,\beta,\alpha)$. Then the following estimates are valid:*

$$|\dot{y}_-(t)| \leq \frac{\beta_2\big(nr+\sqrt{n}\big)}{(\alpha+1)\big(\frac{|v_-|}{2\sqrt{2}}-r\big)\Big(1+\frac{|x_-|}{\sqrt{2}}-r+|t|\big(\frac{|v_-|}{2\sqrt{2}}-r\big)\Big)^{\alpha+1}}, \quad (3.23)$$

$$|y_-(t)| \leq \frac{\beta_2\big(nr+\sqrt{n}\big)}{\alpha(\alpha+1)\big(\frac{|v_-|}{2\sqrt{2}}-r\big)^2\Big(1+\frac{|x_-|}{\sqrt{2}}-r+|t|\big(\frac{|v_-|}{2\sqrt{2}}-r\big)\Big)^{\alpha}}, \quad (3.24)$$

for $t \le 0$; and

$$|\tilde{a}_{sc}(v_-,x_-)| \;\le\; \frac{6\sqrt{n}\max(\beta_1^l,\beta_2)}{\alpha\big(\frac{|v_-|}{2\sqrt{2}}-r\big)\big(1+\frac{|x_-|}{\sqrt{2}}-r\big)^{\alpha}}, \tag{3.25}$$

$$|\tilde{b}_{sc}(v_-,x_-)| \;\le\; \frac{4\beta_2(nr+\sqrt{n})}{\alpha(\alpha+1)\big(\frac{|v_-|}{2\sqrt{2}}-r\big)^2\big(\frac{1}{2}+\frac{|x_-|}{2\sqrt{2}}-r\big)^{\alpha}}, \tag{3.26}$$

$$|y_+(t)| \le \frac{2\beta_2\sqrt{n}}{\alpha(\alpha+1)(\frac{|v_-|}{2^{\frac{3}{2}}}-r)^2(\frac{1}{2}+\frac{|x_-|}{2\sqrt{2}}-r+t(\frac{|v_-|}{2^{\frac{3}{2}}}-r))^{\alpha}}, \tag{3.27}$$

for $t \ge 0$. In addition

$$\Big|\tilde{a}_{sc}(v_-,x_-)-\tilde{W}(v_-,x_-)-\int_{-\infty}^{+\infty}F^s(\tau v_-+x_-)d\tau\Big| \le \frac{2n^{\frac{3}{2}}(n^{\frac{1}{2}}r+1)\beta^2\big(3+\frac{2}{2^{\frac{3}{2}}|v_-|-r}\big)^2}{\alpha(\alpha+1)(\frac{|v_-|}{2^{\frac{3}{2}}}-r)^2(1-r+\frac{|x_-|}{\sqrt{2}})^{2\alpha+1}}, \tag{3.28}$$

$$\big|\tilde{b}_{sc}(v_-,x_-)-\int_{-\infty}^{0}\int_{-\infty}^{\sigma}F^s(\tau v_-+x_-)d\tau d\sigma+\int_0^{+\infty}\int_{\sigma}^{+\infty}F^s(\tau v_-+x_-)d\tau d\sigma\big|$$
$$\le \frac{2n^{\frac{3}{2}}(n^{\frac{1}{2}}r+1)\beta^2\big(3+\frac{2}{2^{-\frac{3}{2}}|v_-|-r}\big)^2}{\alpha^2(\alpha+1)(\frac{|v_-|}{2^{\frac{3}{2}}}-r)^3(\frac{1}{2}-r+\frac{|x_-|}{2^{\frac{3}{2}}})^{2\alpha}}, \tag{3.29}$$

where

$$\tilde{W}(v_-,x_-) := \int_{-\infty}^{0}F^l(z_-(v_-,x_-,\tau))d\tau + \int_0^{+\infty}F^l(z_+(\tilde{a}(v_-,x_-),x_-,\tau))d\tau. \tag{3.30}$$

Note that $\tilde{W}$ is known from the modified scattering data and from F^l.

Corollary 3.4. *Let $(\theta,x)\in T\mathbb{S}^{n-1}$. Under conditions* (1.3) *and* (1.4) *the following limits are valid:*

$$\lim_{\rho\to+\infty}\rho(\tilde{a}_{sc}(\rho\theta,x)-\tilde{W}(\rho\theta,x)) = PF^s(\theta,x), \tag{3.31}$$

$$\lim_{\rho\to+\infty}\rho^2\theta\cdot\tilde{b}_{sc}(\rho\theta,x) = -PV^s(\theta,x), \tag{3.32}$$

In addition,

$$\rho\big(\tilde{a}_{sc}(\rho\theta,\eta x)-\tilde{W}(\rho\theta,\eta x)\big) = PF^s(\theta,\eta x)+O(\eta^{-2\alpha-1}), \tag{3.33}$$

$$\rho^2\theta\cdot\tilde{b}_{sc}(\rho\theta,\eta x) = -PV^s(\theta,\eta x)+O(\eta^{-2\alpha}), \tag{3.34}$$

for $\rho>\tilde{\rho}_0(\eta|x|,r,\beta,\alpha)$, as $\eta\to+\infty$.

Formulas (3.31) and (3.32) prove that F^s can be reconstructed from the high energies asymptotics of the modified scattering data. For Problem (3.22) this implies that F^s can be reconstructed from $\tilde{a}_{sc}$ and $\tilde{b}_{sc}$.

Formulas (3.33) and (3.34) also provide the asymptotics of $(\tilde{a}_{sc}(v,x), \tilde{b}_{sc}(v,x))$ when the parameters α, n, v and β are fixed and the "impact parameter" $|x| \to +\infty$ $(v \cdot x = 0)$.

A result similar to Corollary 1.2 also holds for the Born approximation at fixed energy of the modified scattering data. Such a result is derived from Corollary 3.4 and from estimates (3.28) and (3.29) in the same way Corollary 1.2 was derived from Theorem 1.1. And we obtain that F^s can be reconstructed from the Born approximation of the modified scattering data at fixed energy.

Now assume that α^{-1} is not an integer and denote by N its integer part. One may also approximate the solutions $z_\pm(v,x,.)$ by the functions $z_{\pm,N+1}(v,x,.)$ defined below that may be easier to compute. Proceeding from [8] we define by induction

$$z_{\pm,0}(v,x,t) = x + tv, \;\; z_{\pm,m+1}(v,x,t) = x + tv + \int_0^t \int_{\pm\infty}^{\sigma} F^l(z_{\pm,m}(v,x,\tau))d\tau d\sigma, \tag{3.35}$$

for $(t,v,x) \in \mathbb{R} \times (\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}^n$ and for $m = 0 \ldots N$. Then we can repeat the study done above. For $(v_-, x_-) \in \mathbb{R}^n \times \mathbb{R}^n$, $v_- \cdot x_- = 0$, $|v_-| \geq \tilde{\mu}(|x_-|)$, there exists a unique solution $x(t)$ of equation (1.1) so that

$$x(t) = z_{-,N+1}(v_-, x_-, t) + y_-(t), \;\; t \in \mathbb{R} \text{ and } \lim_{t \to -\infty}(|y_-(t)| + |\dot{y}_-(t)|) = 0. \tag{3.36}$$

The function y_- in (3.36) satisfies the following integral equation $y_- = \mathcal{A}_N(y_-)$ where

$$\mathcal{A}_N(f)(t) = \int_{-\infty}^t \int_{-\infty}^{\sigma} \Big(F(z_{-,N+1}(v_-, x_-, \tau) + f(\tau)) - F^l(z_{-,N}(v_-, x_-, \tau))\Big) d\tau d\sigma \tag{3.37}$$

for $t \in \mathbb{R}$ and for $f \in C(\mathbb{R}, \mathbb{R}^n)$, $\sup_{(-\infty,0]} |f| < \infty$. Then estimates similar to (3.7) and (3.8) can be obtained for the operator $\mathcal{A}_N$ restricted to M_r. A decomposition similar to (3.12) holds, and this defines a new map $\mathcal{G}_{v_-,x_-}$. An analog of Lemma 3.2 can be proved, which allows to define the scattering data $\tilde{b}_N$. Then an analog of Theorem 3.3 can be proved for the scattering solutions and scattering data $(\tilde{a}_N, \tilde{b}_N)$, and the following high energies limits are valid

$$\lim_{\rho \to +\infty} \rho(\tilde{a}_{sc,N}(\rho\theta, x) - \tilde{W}_N(\rho\theta, x)) = PF^s(\theta, x), \tag{3.38}$$

$$\lim_{\rho \to +\infty} \rho^2 \theta \cdot \tilde{b}_{sc,N}(\rho\theta, x) = -PV^s(\theta, x), \tag{3.39}$$

for $(\theta, x) \in T\mathbb{S}^{n-1}$, where

$$\tilde{W}_N(v,x) := \int_{-\infty}^{0} F^l(z_{-,N}(v,x,\tau))d\tau + \int_0^{+\infty} F^l(z_{+,N}(\tilde{a}_N(v,x),x,\tau))d\tau. \tag{3.40}$$

for $(v,x) \in \mathbb{R}^n \times \mathbb{R}^n$, $v \cdot x = 0$, $|v| > C$ for some constant C. The vector $\tilde{W}_N$ defined by (3.40) is known from the scattering data and from F^l. From (3.38) (resp. (3.39)) it follows that F^s can be reconstructed from $\tilde{a}_{sc,N}$ (resp. $\tilde{b}_{sc,N}$).

The Born approximation of $\tilde{a}_{sc,N}$, $\tilde{b}_{sc,N}$ at fixed energy can also be derived from (3.38) and (3.39) by using the same observations that leaded to Corollary 1.2.

The limits (3.38) and (3.39) follow from estimates similar to (3.28) and (3.29) that also give the first leading term of the asymptotics of the scattering data $(\tilde{a}_{sc,N}, \tilde{b}_{sc,N})$ when the parameters α, n, ρ, θ and β are fixed and $|x| \to +\infty$.

We refer the reader to the preprint [10] for a proof of Lemmas 3.1, 3.2, and Theorem 3.3 and Corollary 3.4.

4 Proof of Lemma 2.2

Before proving Lemma 2.2 we give a proof of the existence and uniqueness of the "free" solution z_+. Similarly one can prove the existence and uniqueness of z_- or just use the relation "$z_-(v,t) = z_+(-v,-t)$").

Proof. Set $C' = \frac{2^{\frac{3}{2}} n^{\frac{1}{2}} \beta_1^l}{\alpha |v|}$. Let $\mathcal{V}$ be the complete metric space defined by

$$\mathcal{V} := \{g \in C(\mathbb{R}, \mathbb{R}^n) \mid |g(t)| \le C'|t| \text{ for } t \in \mathbb{R}\},$$

endowed with the following norm $\|g\|_{\mathcal{V}} := \sup_{t \in \mathbb{R}\setminus\{0\}} \left|\frac{g(t)}{t}\right|$. We consider the integral equation

$$G_+ f(t) = -\int_0^t \int_\sigma^{+\infty} F^l(\tau v + f(\tau)) d\tau d\sigma, \ t \in \mathbb{R}, \tag{4.1}$$

for $f \in \mathcal{V}$. First note that

$$|\tau v + f(\tau)| \ge \big(|v| - C'\big)|\tau| \ge \frac{|v|}{\sqrt{2}}|\tau|, \tag{4.2}$$

for $\tau \in \mathbb{R}$ and $f \in \mathcal{V}$ (we used that $C' \le \frac{|v|}{4\sqrt{2}}$). Using (1.3) we obtain that

$$|G_+ f(t)| \le \sqrt{n}\beta_1^l \int_{-|t|}^{0} \int_\sigma^{+\infty} \big(1 + \frac{|v|}{\sqrt{2}}|\tau|\big)^{-\alpha-1} d\tau d\sigma \le C'|t|, \tag{4.3}$$

for $t \in \mathbb{R}$ and $f \in \mathcal{V}$. Now let $(f_1, f_2) \in \mathcal{V}^2$. Then using (1.3) we have

$$|F^l(\tau v + f_1(\tau)) - F^l(\tau v + f_2(\tau))| \leq \sup_{\varepsilon \in (0,1)} \frac{n\beta_2^l |f_1 - f_2|(\tau)}{(1 + |\tau v + (\varepsilon f_1 + (1-\varepsilon) f_2)(\tau)|)^{\alpha+2}},$$

for $\tau \in \mathbb{R}$. Hence using also (4.2) we have

$$\begin{aligned} |G_+ f_1(t) - G_+ f_2(t)| &\leq n\beta_2^l \|f_1 - f_2\|_{\mathcal{V}} \int_{-|t|}^0 \int_\sigma^{+\infty} \frac{|\tau| d\tau d\sigma}{\left(1 + \frac{|v|}{\sqrt{2}}|\tau|\right)^{\alpha+2}} \\ &\leq \frac{4n\beta_2^l |t| \|f_1 - f_2\|_{\mathcal{V}}}{\alpha |v|^2} \leq 4^{-1} \|f_1 - f_2\|_{\mathcal{V}} |t|, \quad (4.4) \end{aligned}$$

for $t \in \mathbb{R}$ (we used that $|v| \geq \mu$). From (4.3) and (4.4) it follows that the operator G_+ is a contraction map from $\mathcal{V}$ to $\mathcal{V}$. Set $z_+(v,t) = tv + f_v(t)$ for $t \in \mathbb{R}$, where f_v denotes the unique fixed point of G_+ in $\mathcal{V}$. Then $z_+(v,.)$ satisfies (1.6), (1.7) and (1.8). $\square$

For the proof of Lemma 2.2 we recall the following standard result (see also [8, Lemma II.2]). For sake of consistency we provide a proof of Lemma 4.1 at the end of this Section.

Lemma 4.1. *Let $x(t)$ be a solution of equation* (1.1) *and let $z(t)$ be a solution of equation* (1.6). *Assume that there exists a vector $v \in \mathbb{R}^n$, $v \neq 0$, so that*

$$\lim_{t \to +\infty} \dot{z}(t) = \lim_{t \to +\infty} \dot{x}(t) = v. \quad (4.5)$$

Then

$$\sup_{(0,+\infty)} |x - z| < \infty. \quad (4.6)$$

Proof of Lemma 2.2. We need the following preliminary estimate (4.8). Using (2.2) we have for $\tau \in \mathbb{R}$ and $f \in M_r$,

$$|f(\tau)| \leq r|\tau| + r. \quad (4.7)$$

Hence

$$\begin{aligned} |z_-(v_-, \tau) + x_- + f(\tau)| &\geq |x_- + \tau v_-| - |z_-(v_-, \tau) - \tau v_-| - |f(\tau)| \\ &\geq \frac{|x_-|}{\sqrt{2}} - r + |\tau| \Big(\frac{|v_-|}{2\sqrt{2}} - r \Big). \quad (4.8) \end{aligned}$$

We used (1.8) and the inequality $|x_- + \tau v_-| \geq \frac{|x_-|}{\sqrt{2}} + |\tau| \frac{|v_-|}{\sqrt{2}}$ $(x_- \cdot v_- = 0)$ and (4.7) and we used the condition $|v_-| \geq \mu$. Hence the integral $\int_{-\infty}^{+\infty} F(z_-(v_-, \tau) +$

$x_- + f(\tau))d\tau$ is absolutely convergent for any $f \in M_r$. And when $y_- \in M_r$ is a fixed point for A then $z_-(v_-,.) + x_- + y_-$ satisfies equation (1.1) (see (2.1)) and $\dot z_-(v_-,t) + \dot y_-(t) = v_- + \int_{-\infty}^t F(z_-(v_-,\tau) + x_- + y_-(\tau))d\tau \to a(v_-,x_-)$ as $t \to +\infty$, where $a(v_-,x_-)$ is defined in (2.6). Then from Lemma 4.1 it follows that $\sup_{t\in(0,+\infty)} |z_-(v_-,t)+x_-+y_-(t)-z_+(a(v_-,x_-),t)| < +\infty$. Using this latter estimate and (1.3) and (1.4), and using (4.8) we obtain that the integrals on the right hand sides of (2.8) and (2.11) are absolutely convergent. Then the decomposition (2.5) follows from the equality $A(y_-) = y_-$ and (2.1) and straightforward computations. □

Proof of Lemma 4.1. We set $\delta(t) = x(t) - z(t)$ for $t \geq 0$. Property (4.5) shows that there exists $\varepsilon > 0$ so that

$$1 + |\eta x(t) + (1-\eta)z(t)| \geq \varepsilon(1+t), \text{ for } t \geq 0 \text{ and } \eta \in [0,1]. \tag{4.9}$$

Then from equations (1.1) and (1.6) it follows that

$$\delta(t) = \delta(0) - \int_0^t \int_\sigma^{+\infty} F^s(x(\tau))d\tau d\sigma - \int_0^t \int_\sigma^{+\infty} (F^l(x(\tau)) - F^l(z(\tau)))d\tau d\sigma, \tag{4.10}$$

for $t \geq 0$, where the integrals on the right hand side of (4.10) are absolutely convergent (see (4.9) and (1.3) and (1.4)). Note that

$$\int_0^t \int_\sigma^{+\infty} |F^s(x(\tau))|d\tau d\sigma \leq \sqrt{n}\beta_2\varepsilon^{-\alpha-2} \int_0^t \int_\sigma^{+\infty} \frac{d\tau d\sigma}{(1+\tau)^{\alpha+2}} \leq \frac{\sqrt{n}\beta_2}{\alpha(\alpha+1)\varepsilon^{\alpha+2}}, \tag{4.11}$$

for $t \geq 0$. Hence using (4.10) we obtain

$$|\delta(t)| \leq C_0 + \int_0^t \int_\sigma^{+\infty} |F^l(x(\tau)) - F^l(z(\tau))|d\tau d\sigma, \text{ for } t \geq 0, \tag{4.12}$$

where $C_0 = |\delta(0)| + \frac{\sqrt{n}\beta_2}{\alpha(\alpha+1)\varepsilon^{\alpha+2}}$.

One may assume without loss of generality that $\alpha \neq \frac{1}{m}$ for any $m \in \mathbb{N}$. Otherwise replace α by some $\alpha' \in (0,\alpha)$ so that $\alpha' \neq \frac{1}{m'}$ for any $m' \in \mathbb{N}$. Then

$$\begin{aligned}\int_0^t \int_\sigma^{+\infty} \Big(|F^l(x(\tau))| + |F^l(z(\tau)|\Big)d\tau d\sigma &\leq 2\sqrt{n}\beta_1^l\varepsilon^{-\alpha-1} \int_0^t \int_\sigma^{+\infty} \frac{d\tau d\sigma}{(1+\tau)^{\alpha+1}} \\ &\leq \frac{2\sqrt{n}\beta_1^l}{\alpha(1-\alpha)\varepsilon^{\alpha+1}}(1+t)^{1-\alpha}, \end{aligned} \tag{4.13}$$

for $t \geq 0$. Using also (4.12) we obtain that there exist positive constants C_1 and C_1' so that $|\delta(t)| \leq C_1 + C_1' t^{1-\alpha}$ for $t \geq 0$. Now using (4.12), the growth properties of F^l (1.3) and (4.9) we obtain

$$|\delta(t)| \leq C_0 + n\beta_2\varepsilon^{-\alpha-2}\int_0^t\int_\sigma^{+\infty}\frac{|\delta(\tau)|}{(1+\tau)^{\alpha+2}}d\tau d\sigma, \tag{4.14}$$

for $t \geq 0$. Then using (4.14) we prove by induction the following: For any $m = 1\dots\lfloor\alpha^{-1}\rfloor$ there exist positive constants C_m and C'_m so that $|\delta(t)| \leq C_m + C'_m t^{1-m\alpha}$ for $t \geq 0$. Combining again this latter estimate for $m = \lfloor\alpha^{-1}\rfloor$ and the estimate (4.14) we obtain

$$\begin{aligned}|\delta(t)| &\leq C_0 + n\beta_2\varepsilon^{-\alpha-2}\int_0^t\int_\sigma^{+\infty}\frac{C_m + C'_m\tau^{1-m\alpha}}{(1+\tau)^{\alpha+2}}d\tau d\sigma\\ &\leq C_0 + \frac{n\beta_2 C_m}{\alpha(\alpha+1)\varepsilon^{\alpha+2}} + \frac{C'_m n\beta_2}{(m+1)\alpha((m+1)\alpha-1)\varepsilon^{\alpha+2}},\end{aligned}$$

for $t \geq 0$ ($(m+1)\alpha - 1 > 0$), which proves the lemma. □

5 Proof of Lemma 2.1

Proof. We first prove (2.3). We need the following estimates for $A(f)(t)$ (5.5) and (5.9). Using (1.3), (1.4) and (4.8) we obtain

$$|F^s(z_-(v_-,\tau) + x_- + f(\tau))| \leq \frac{\beta_2\sqrt{n}}{\Big(1 + \frac{|x_-|}{\sqrt{2}} - r + |\tau|\big(\frac{|v_-|}{2\sqrt{2}} - r\big)\Big)^{\alpha+2}}, \tag{5.1}$$

and

$$|F^l(z_-(v_-,\tau) + x_- + f(\tau)) - F^l(z_-(v_-,\tau))| \leq \frac{\beta_2 n(|x_-| + |f(\tau)|)}{\Big(1 - r + |\tau|\big(\frac{|v_-|}{2\sqrt{2}} - r\big)\Big)^{\alpha+2}}, \tag{5.2}$$

for $\tau \in \mathbb{R}$. In addition from (2.1) it follows that

$$\dot{A}(f)(t) = \int_{-\infty}^t \Big(F(z_-(v_-,\tau) + x_- + f(\tau)) - F^l(z_-(v_-,\tau))\Big)d\tau,\ t \in \mathbb{R}. \tag{5.3}$$

Therefore combining (5.1), (5.2) (with $|f(\tau)| \leq \sup_{(-\infty,0)}|f|$ for $\tau \leq 0$) and (5.3) we obtain

$$|\dot{A}(f)(t)| \leq \beta_2\int_{-\infty}^t\frac{\sqrt{n} + n(|x_-| + |f(\tau)|)}{\Big(1 - r + |\tau|\big(\frac{|v_-|}{2\sqrt{2}} - r\big)\Big)^{\alpha+2}}d\tau \leq \frac{\beta_2\big(n(|x_-| + \sup_{(-\infty,0)}|f|) + \sqrt{n}\big)}{(\alpha+1)\big(\frac{|v_-|}{2\sqrt{2}} - r\big)\Big(1 - r + |t|\big(\frac{|v_-|}{2\sqrt{2}} - r\big)\Big)^{\alpha+1}}, \tag{5.4}$$

for $t \le 0$. Then integrating over $(-\infty, t)$ we obtain

$$|A(f)(t)| \le \frac{\beta_2(n(|x_-| + \sup_{(-\infty,0)}|f|) + \sqrt{n})}{\alpha(\alpha+1)\big(\frac{|v_-|}{2\sqrt{2}} - r\big)^2\Big(1 - r + |t|\big(\frac{|v_-|}{2\sqrt{2}} - r\big)\Big)^{\alpha}}, \tag{5.5}$$

for $t \le 0$.

Now let $t \ge 0$. Using (5.3) we have

$$\begin{aligned}\dot{A}(f)(t) &= \dot{A}(f)(0) + \int_0^t \big(F^l(z_-(v_-,\tau) + x_- + f(\tau)) - F^l(z_-(v_-,\tau))\big)d\tau \\ &\quad + \int_0^t F^s(z_-(v_-,\tau) + x_- + f(\tau))d\tau. \end{aligned} \tag{5.6}$$

Hence from (5.4) (with "$t = 0$"), (5.1) and (5.2) (with $|f(\tau)| \le (1+\tau)\sup_{s\in(0+\infty)} \frac{|f(s)|}{1+s}$ for $\tau \ge 0$) it follows that

$$\begin{aligned}|\dot{A}(f)(t)| &\le \frac{\beta_2\big(n(|x_-| + \sup_{(-\infty,0)}|f|) + \sqrt{n}\big)}{(\alpha+1)\big(\frac{|v_-|}{2\sqrt{2}} - r\big)\big(1-r\big)^{\alpha+1}} \\ &+\beta_2 \int_0^t \frac{n^{\frac{1}{2}} + n|x_-| + n(1+\tau)\sup_{s\in(0,+\infty)}\frac{|f(s)|}{1+s}}{\Big(1 - r + |\tau|\big(\frac{|v_-|}{2\sqrt{2}} - r\big)\Big)^{\alpha+2}}d\tau \\ &\le \frac{\beta_2\big(2n|x_-| + n\|f\| + 2\sqrt{n}\big)}{(\alpha+1)\big(\frac{|v_-|}{2\sqrt{2}} - r\big)\Big(1-r\Big)^{\alpha+1}} + \frac{n\beta_2 \sup_{s\in(0,+\infty)}\frac{|f(s)|}{1+s}}{\alpha\big(\frac{|v_-|}{2\sqrt{2}} - r\big)^2(1-r)^{\alpha}}. \end{aligned} \tag{5.7}$$

We also have $A(f)(t) = A(f)(0) + \int_0^t \dot{A}(f)(s)ds$ and

$$\sup_{t\in(0,+\infty)} \frac{|A(f)(t)|}{1+t} \le \max\big(|A(f)(0)|, \sup_{(0,+\infty)} |\dot{A}(f)|\big). \tag{5.8}$$

Then we use (5.5) at $t = 0$ and (5.7), and we obtain

$$\sup_{t\in(0,+\infty)} \frac{|A(f)(t)|}{1+t} \le \frac{\beta_2(n(2|x_-| + \|f\|) + 2\sqrt{n})}{\big(\frac{|v_-|}{2\sqrt{2}} - r\big)\Big(1-r\Big)^{\alpha}}\Big(\frac{1}{(\alpha+1)(1-r)} + \frac{1}{\alpha\big(\frac{|v_-|}{2\sqrt{2}} - r\big)}\Big). \tag{5.9}$$

Then (2.3) follows from (5.5) and (5.9) and the estimate $\|f\| \le r$.

It remains to prove (2.4). Estimate (2.4) will follow from (5.14) and (5.20) given below. Let $(f_1, f_2) \in M_r^2$. Using (1.3), (1.4) and (4.8) we obtain

$$|F^l(z_-(v_-,\tau) + x_- + f_1(\tau)) - F^l(z_-(v_-,\tau) + x_- + f_2(\tau))| \le \frac{\beta_2 n|f_1 - f_2|(\tau)}{\big(1 + \frac{|x_-|}{\sqrt{2}} - r + |\tau|\big(\frac{|v_-|}{2\sqrt{2}} - r\big)\big)^{\alpha+2}}, \tag{5.10}$$

$$|F^s(z_-(v_-,\tau)+x_-+f_1(\tau))-F^s(z_-(v_-,\tau)+x_-+f_2(\tau))| \leq \frac{\beta_3^s n|f_1-f_2|(\tau)}{\big(1+\frac{|x_-|}{\sqrt{2}}-r+|\tau|\big(\frac{|v_-|}{2\sqrt{2}}-r\big)\big)^{\alpha+3}}, \tag{5.11}$$

for $\tau \in \mathbb{R}$. In addition from (5.3) it follows that

$$\begin{aligned}
&\dot{A}(f_1)(t)-\dot{A}(f_2)(t) \\
&= \int_{-\infty}^{t} \big(F^l(z_-(v_-,\tau)+x_-+f_1(\tau))-F^l(z_-(v_-,\tau)+x_-+f_2(\tau))\big)d\tau \\
&+ \int_{-\infty}^{t} \Big(F^s(z_-(v_-,\tau)+x_-+f_1(\tau))-F^s(z_-(v_-,\tau)+x_-+f_2(\tau))\Big)d\tau. \qquad (5.12)
\end{aligned}$$

Hence we integrate in the τ variable over the interval $(-\infty,t)$, $t \leq 0$, both sides of (5.10) and (5.11) where we use the inequality $|f_1-f_2|(\tau) \leq \sup_{(-\infty,0)}|f_1-f_2|$, and we obtain

$$\begin{aligned}
|\dot{A}(f_1)(t)-\dot{A}(f_2)(t)| \ \leq\ & \frac{n\sup_{(-\infty,0]}|f_1-f_2|}{\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big(1+\frac{|x_-|}{\sqrt{2}}-r+|t|\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big)^{\alpha+1}}\Bigg[\frac{\beta_2}{(\alpha+1)} \\
&+ \frac{\beta_3^s}{(\alpha+2)\Big(1+\frac{|x_-|}{\sqrt{2}}-r+|t|\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big)}\Bigg], \qquad (5.13)
\end{aligned}$$

for $t \leq 0$. Then we integrate in the t variable both sides of (5.13) and we obtain

$$\begin{aligned}
|A(f_1)(t)-A(f_2)(t)| \ \leq\ & \frac{n\sup_{(-\infty,0]}|f_1-f_2|}{(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)^2\Big(1+\frac{|x_-|}{\sqrt{2}}-r+|t|\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big)^{\alpha}}\Bigg[\frac{\beta_2}{\alpha} \\
&+ \frac{\beta_3^s}{(\alpha+2)\Big(1+\frac{|x_-|}{\sqrt{2}}-r+|t|\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big)}\Bigg], \qquad (5.14)
\end{aligned}$$

for $t \leq 0$.

From (5.6) it follows that

$$\begin{aligned}
&\dot{A}(f_1)(t)-\dot{A}(f_2)(t) = \dot{A}(f_1)(0)-\dot{A}(f_2)(0) \qquad (5.15)\\
&+ \int_0^t \big(F^l(z_-(v_-,\tau)+x_-+f_1(\tau))-F^l(z_-(v_-,\tau)+x_-+f_2(\tau))\big)d\tau \\
&+ \int_0^t \Big(F^s(z_-(v_-,\tau)+x_-+f_1(\tau))-F^s(z_-(v_-,\tau)+x_-+f_2(\tau))\Big)d\tau.
\end{aligned}$$

We have

$$\Big|\int_0^t \Big(F^s(z_-(v_-,\tau)+x_-+f_1(\tau)) - F^s(z_-(v_-,\tau)+x_-+f_2(\tau))\Big)d\tau\Big|$$

$$\le n\beta_3^s \sup_{s\in[0,+\infty)} \frac{|(f_1-f_2)(s)|}{1+s}\int_0^t \frac{1+\tau}{\Big(1+\frac{|x_-|}{\sqrt{2}}-r+|\tau|\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big)^{\alpha+3}}d\tau$$

$$\le \frac{n\beta_3^s \sup_{s\in[0,+\infty)}\frac{|(f_1-f_2)(s)|}{1+s}}{\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)(1+\frac{|x_-|}{\sqrt{2}}-r)^{\alpha+1}}\left[\frac{1}{(\alpha+2)(1+\frac{|x_-|}{\sqrt{2}}-r)}+\frac{1}{(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)}\right], \tag{5.16}$$

and

$$\Big|\int_0^t \Big(F^l(z_-(v_-,\tau)+x_-+f_1(\tau)) - F^l(z_-(v_-,\tau)+x_-+f_2(\tau))\Big)d\tau\Big|$$

$$\le \frac{n\beta_2 \sup_{s\in[0,+\infty)}\frac{|(f_1-f_2)(s)|}{1+s}}{\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)(1+\frac{|x_-|}{\sqrt{2}}-r)^{\alpha}}\left[\frac{1}{(\alpha+1)(1+\frac{|x_-|}{\sqrt{2}}-r)}+\frac{1}{\alpha\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)}\right]. \tag{5.17}$$

Hence using also (5.13) (for "$t=0$") we obtain

$$|\dot{A}(f_1)(t)-\dot{A}(f_2)(t)| \le \frac{n\|f_1-f_2\|}{\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big(1+\frac{|x_-|}{\sqrt{2}}-r\Big)^{\alpha+1}}\left[\frac{\beta_2}{(\alpha+1)}+\frac{\beta_3^s}{(\alpha+2)\Big(1+\frac{|x_-|}{\sqrt{2}}-r\Big)}\right]$$
$$+\frac{n\sup_{s\in[0,+\infty)}\frac{|(f_1-f_2)(s)|}{1+s}}{\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)^2(1+\frac{|x_-|}{\sqrt{2}}-r)^{\alpha}}\left[\frac{\beta_3^s}{(\alpha+1)(1+\frac{|x_-|}{\sqrt{2}}-r)}+\frac{\beta_2}{\alpha}\right]. \tag{5.18}$$

Next similarly to (5.8) we have

$$\sup_{t\in(0,+\infty)}\frac{|A(f_1)(t)-A(f_2)(t)|}{1+t} \le \max\big(|A(f_1)(0)-A(f_2)(0)|, \sup_{(0,+\infty)}|\dot{A}(f_1)-\dot{A}(f_2)|\big). \tag{5.19}$$

Then we use (5.14) at $t=0$, and we use (5.18), and we obtain

$$\sup_{t\in(0,+\infty)}\frac{|A(f_1)(t)-A(f_2)(t)|}{1+t} \le \frac{n\|f_1-f_2\|}{\alpha(\frac{|v_-|}{2\sqrt{2}}-r)(1+\frac{|x_-|}{\sqrt{2}}-r)^{\alpha}}\big(\beta_2+\frac{\beta_3^s}{1+\frac{|x_-|}{\sqrt{2}}-r}\big)$$
$$\times\big(\frac{1}{1+\frac{|x_-|}{\sqrt{2}}-r}+\frac{1}{\frac{|v_-|}{2\sqrt{2}}-r}\big). \tag{5.20}$$

□

6 Proof of Theorem 2.3

In this Section we shorten $a(v_-,x_-)$, $a_{sc}(v_-,x_-)$ to a and a_{sc}.

The estimate (2.12) (resp. (2.13)) follows from the assumption $y_- = A(y_-)$, and $\sup_{(-\infty,0)}|y_-| \le r$ and the estimate (5.4) (resp. (5.5)). Using (1.3) and (4.8) we obtain

$$|F^l(z_-(v_-,\tau)+x_-+y_-(\tau))| \le \frac{\beta_1^l\sqrt{n}}{\big(1+\frac{|x_-|}{\sqrt{2}}-r+|\tau|\big(\frac{|v_-|}{2\sqrt{2}}-r\big)\big)^{\alpha+1}}, \tag{6.1}$$

for $\tau\in\mathbb{R}$. Using (2.6), (5.1) and (6.1) we obtain

$$|a_{sc}| \le \int_{-\infty}^{+\infty}\frac{\beta_1^l n^{\frac12}d\tau}{(1+\frac{|x_-|}{\sqrt{2}}-r+\big(\frac{|v_-|}{2\sqrt{2}}-r\big)|\tau|)^{\alpha+1}}+\int_{-\infty}^{+\infty}\frac{\beta_2 n^{\frac12}d\tau}{(1+\frac{|x_-|}{\sqrt{2}}-r+\big(\frac{|v_-|}{2\sqrt{2}}-r\big)|\tau|)^{\alpha+2}},$$

which proves (2.14).

Then using (2.9), (5.1) and (5.2) we have

$$\begin{aligned}
|l(y_-)| &\le 2\beta_2 n^{\frac12}\int_{-\infty}^{0}\int_{-\infty}^{\sigma}\Big(1+\frac{|x_-|}{\sqrt{2}}-r+\big(\frac{|v_-|}{2\sqrt{2}}-r\big)|\tau|\Big)^{-\alpha-2}d\tau d\sigma\\
&\quad+\beta_2 n\int_{-\infty}^{0}\int_{-\infty}^{\sigma}\frac{|x_-|+\sup_{(-\infty,0)}|y_-|}{\big(1-r+\big(\frac{|v_-|}{2\sqrt{2}}-r\big)|\tau|\big)^{\alpha+2}}d\tau d\sigma\\
&\le \frac{2\beta_2 n^{\frac12}}{\alpha(\alpha+1)\big(\frac{|v_-|}{2\sqrt{2}}-r\big)^2\big(1+\frac{|x_-|}{\sqrt{2}}-r\big)^{\alpha}}+\frac{\beta_2 n(|x_-|+\sup_{(-\infty,0)}|y_-|)}{\alpha(\alpha+1)\big(\frac{|v_-|}{2\sqrt{2}}-r\big)^2(1-r)^{\alpha}}\\
&\le \frac{\beta_2(n|x_-|+2n^{\frac12}+n\sup_{(-\infty,0)}|y_-|)}{\alpha(\alpha+1)(1-r)^{\alpha}\big(\frac{|v_-|}{2\sqrt{2}}-r\big)^2}.
\end{aligned}$$

This estimate with the estimate $\sup_{(-\infty,0)}|y_-|\le r$ proves (2.15). Now using (2.6) and (5.18) we obtain that

$$\begin{aligned}
&|a_{sc}-\int_{-\infty}^{+\infty}F(z_-(v_-,\tau)+x_-)d\tau| = \lim_{t\to+\infty}|\dot{A}(y_-)(t)-\dot{A}(0)(t)|\\
&\le \frac{n\|y_-\|}{\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big(1+\frac{|x_-|}{\sqrt{2}}-r\Big)^{\alpha+1}}\left[\frac{\beta_2}{(\alpha+1)}+\frac{\beta_3^s}{(\alpha+2)\Big(1+\frac{|x_-|}{\sqrt{2}}-r\Big)}\right]\\
&\quad+\frac{n\sup_{s\in[0,+\infty)}\frac{|y_-(s)|}{1+s}}{\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)^2(1+\frac{|x_-|}{\sqrt{2}}-r)^{\alpha}}\left[\frac{\beta_3^s}{(\alpha+1)(1+\frac{|x_-|}{\sqrt{2}}-r)}+\frac{\beta_2}{\alpha}\right].
\end{aligned} \tag{6.2}$$

Then note that $\|y_-\| = \|A(y_-)\|$ is bounded by the right hand side of (2.3). Hence combining this latter bound on $\|y_-\|$ and (6.2) and the estimate $1 + \frac{|x_-|}{\sqrt{2}} - r \geq 1 - r$ we obtain (2.16).

Using (2.9), (5.10) and (5.11) we obtain

$$
\begin{aligned}
& |l(y_-) - l(0)| \\
\leq\ & \int_{-\infty}^{0}\int_{-\infty}^{\sigma} \left|F^l\big(z_-(v_-,\tau) + x_- + y_-(\tau)\big) - F^l\big(z_-(v_-,\tau) + x_-\big)\right| d\tau d\sigma \\
& + \int_{-\infty}^{0}\int_{-\infty}^{\sigma} \left|F^s\big(z_-(v_-,\tau) + x_- + y_-(\tau)\big) - F^s\big(z_-(v_-,\tau) + x_-\big)\right| d\tau d\sigma \\
& + \int_{0}^{+\infty}\int_{\sigma}^{+\infty} |F^s\big(z_-(v_-,\tau) + x_- + y_-(\tau)\big) - F^s\big(z_-(v_-,\tau) + x_-\big)| d\tau d\sigma \\
\leq\ & \frac{n}{(\alpha+1)\big(\frac{|v_-|}{2\sqrt{2}} - r\big)^2 (1 + \frac{|x_-|}{\sqrt{2}} - r)^\alpha}\Big(\frac{\beta_2 \sup_{s\in(-\infty,0)} |y_-(s)|}{\alpha} + \frac{\beta_3^s \|y_-\|}{(\alpha+2)\big(1 + \frac{|x_-|}{\sqrt{2}} - r\big)} \\
& + \frac{\beta_3^s \sup_{s\in(-\infty,0)} \frac{|y_-(s)|}{1+s}}{\alpha\big(\frac{|v_-|}{2\sqrt{2}} - r\big)}\Big).
\end{aligned} \tag{6.3}
$$

Then $\|y_-\|$ is bounded by the right hand side of (2.3), and combining this latter bound on $\|y_-\|$ and (6.3) and the estimate $1 + \frac{|x_-|}{\sqrt{2}} - r \geq 1 - r$ we obtain (2.17).

It remains to prove (2.19), (2.20) and (2.21). From (2.14) it follows that

$$
|a_{sc}| \leq \frac{4n^{\frac{1}{2}} \max(\beta_1^l, \beta_2)}{\alpha\big(\frac{|v_-|}{2\sqrt{2}} - r\big)(1-r)^{\alpha+1}}. \tag{6.4}
$$

In addition using the identity $|a(v_-,x_-)| = |v_-|$ that follows from the conservation of energy, we have

$$
|z_+(a,t) - ta| \leq \frac{2^{\frac{3}{2}} n^{\frac{1}{2}} \beta_1^l}{\alpha |v_-|}|t|,\ \ t \in \mathbb{R}. \tag{6.5}
$$

Then we use (4.7) for "$f = y_-$", and we use (6.4), (6.5) and (1.8), and we obtain

$$
\begin{aligned}
& |x_- + \eta(z_-(v_-,t) + y_-(t)) + (1-\eta) z_+(a,t)| \\
\geq\ & |x_- + tv_-| - |y_-(t)| - |z_-(v_-,t) - v_- t| - |z_+(a,t) - ta| - |t||a_{sc}| \\
\geq\ & \frac{|x_-|}{\sqrt{2}} - r + |t|\left(\frac{|v_-|}{\sqrt{2}} - \frac{2^{\frac{5}{2}} n^{\frac{1}{2}} \beta_1^l}{\alpha |v_-|} - \frac{4\max(\beta_1^l, \beta_2) n^{\frac{1}{2}}}{\alpha\big(\frac{|v_-|}{2\sqrt{2}} - r\big)(1-r)^{\alpha+1}} - r\right)
\end{aligned}
$$

$$\geq \frac{|x_-|}{\sqrt{2}}-r+|t|\left(\frac{|v_-|}{\sqrt{2}}-\frac{6\max(\beta_1^l,\beta_2)n^{\frac{1}{2}}}{\alpha\big(\frac{|v_-|}{2\sqrt{2}}-r\big)(1-r)^{\alpha+1}}-r\right)\geq \frac{|x_-|}{\sqrt{2}}-r+|t|(\frac{|v_-|}{2\sqrt{2}}-r), \tag{6.6}$$

for $\eta \in (0,1)$ and $t \in \mathbb{R}$ (we used (2.18) and we used the estimate $|x_-+tv_-| \geq \frac{|x_-|}{\sqrt{2}} + |t|\frac{|v_-|}{\sqrt{2}}$ that follows from $x_- \cdot v_- = 0$). Similarly

$$|\eta x_- + z_+(a,t)| \geq \frac{|v_-|}{2\sqrt{2}}|t|, \text{ for } (\eta,t) \in (0,1)\times\mathbb{R}. \tag{6.7}$$

From (2.10), (1.3) and (6.7) it follows that

$$\begin{aligned}|l_1| &\leq \beta_2 n|x_-|\int_0^{+\infty}\int_\sigma^{+\infty}\sup_{\eta\in(0,1)}(1+|\eta x_- + z_+(a,\tau)|)^{-\alpha-2}d\tau d\sigma\\ &\leq \beta_2 n|x_-|\int_0^{+\infty}\int_\sigma^{+\infty}\big(1+\frac{|v_-|}{2\sqrt{2}}|\tau|\big)^{-\alpha-2}d\tau d\sigma,\end{aligned}$$

which gives (2.19). Using (2.11), (1.3) and (6.6) we obtain

$$|l_2(y_-)| \leq n\beta_2\int_0^{+\infty}\int_\sigma^{+\infty}\frac{|z_-(v_-,\tau)+y_-(\tau)-z_+(a,\tau)|d\tau d\sigma}{\big(1+\frac{|x_-|}{\sqrt{2}}-r+|\tau|(\frac{|v_-|}{2\sqrt{2}}-r)\big)^{\alpha+2}}. \tag{6.8}$$

Then using (2.5) and (2.7) we have

$$|z_-(v_-,\tau)+y_-(\tau)-z_+(a,\tau)| \leq |l(y_-)|+|l_1|+|l_2(y_-)|+|y_+(\tau)|, \tag{6.9}$$

for $\tau \in (0,+\infty)$. Combining (6.8) and (6.9) we obtain

$$(1-\varepsilon(0))|l_2(y_-)| \leq \varepsilon(0)(|l(y_-)|+|l_1|+\sup_{(0,+\infty)}|y_+|), \tag{6.10}$$

where

$$\varepsilon(t) := \frac{n\beta_2}{\alpha(\alpha+1)\big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\big)^2\big(1+\frac{|x_-|}{\sqrt{2}}-r+t\big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\big)\big)^{\alpha}}, \tag{6.11}$$

for $t \geq 0$. From (2.8) and (5.1) it follows that

$$\begin{aligned}|y_+(t)| &\leq n^{-\frac{1}{2}}\varepsilon(t) \qquad (6.12)\\ &+\int_t^{+\infty}\int_\sigma^{+\infty}\big|F^l(z_-(v_-,\tau)+x_-+y_-(\tau))-F^l(z_+(a,\tau))\big|d\tau d\sigma,\end{aligned}$$

for $t \geq 0$. Then similarly to (6.10) we have

$$(1-\varepsilon(t))\sup_{(t,+\infty)}|y_+| \leq \varepsilon(t)\big(n^{-\frac{1}{2}}+|x_-|+|l_2(y_-)|+|l(y_-)|+|l_1|\big), \tag{6.13}$$

for $t \geq 0$. Using (2.18) and (6.11) we have

$$\sup_{t\in(0,+\infty)} \varepsilon(t) = \varepsilon(0) \leq 6^{-1}. \tag{6.14}$$

Then multiplying (6.10) by $(1-\varepsilon(0))$ and using (6.13) for $t=0$ we obtain

$$(1-2\varepsilon(0))|l_2(y_-)| \leq \varepsilon(0)^2(n^{-\frac{1}{2}}+|x_-|)+\varepsilon(0)(|l(y_-)|+|l_1|). \tag{6.15}$$

Using (6.15), (6.13) and (6.14) we have

$$|l_2(y_-)| \leq 2\varepsilon(0)((n^{-\frac{1}{2}}+|x_-|)\varepsilon(0)+|l(y_-)|+|l_1|), \tag{6.16}$$

$$\sup_{(t,+\infty)} |y_+| \leq 2\varepsilon(t)\big(n^{-\frac{1}{2}}+|x_-|+|l_2(y_-)|+|l(y_-)|+|l_1|\big), \tag{6.17}$$

for $t \geq 0$. Then (2.20) follows from (6.11) (for $t=0$), (2.15) and (2.19). Then (2.21) follows from (2.20), (2.15), (2.19) (combined with (2.18)) and (6.11). □

Acknowledgments

The author would like to thank R. G. Novikov and the anonymous referee for comments on previous versions of this paper.

References

[1] J. M. Cook, *Banach algebras and asymptotic mechanics*. 1967 Cargèse Lectures in Theoretical Physics: Application of Mathematics to Problems in Theoretical Physics (Cargèse, 1965) pp. 209–245 Gordon and Breach Science Publ., New York.

[2] J. Dereziński, *Large Time Behavior of Classical N-Body Systems*, Comm. Math. Phys. 148, 503–520 (1992).

[3] J. Dereziński, C. Gérard, Scattering theory of classical and quantum N-particle systems, Springer-Verlag, Berlin Heidelberg, 1997.

[4] V. Enss, R. Weder, *Inverse potential scattering: a geometrical approach*. Mathematical Quantum Theory II: Schrödinger operators (Feldman, J., Froese, R. and Rosen, L., eds.), CRM Proc. Lecture Notes 8, Amer. Math. Soc., Providence, R.I., 1995.

[5] L. D. Faddeev, *Uniqueness of solution of the inverse scattering problem*, Vestnik. Leningrad. Univ. 11:(7), 126–130 (1956).

[6] I. M. Gel'fand, S. G. Gindikin, M. I. Graev, *Integral geometry in affine and projective spaces*, Itogi Nauki i Tekhniki, Sovr. Prob. Mat. 16, 53-226 (1980) (Russian).

[7] W. Hunziker, *The S-Matrix in Classical Mechanics*, Comm. math. Phys. 8, 282-299 (1968).

[8] I. W. Herbst, *Classical scattering with long range forces*, Comm. Math. Phys. 35, 193–214 (1974).

[9] A. Jollivet, *On inverse scattering at fixed energy for the multidimensional Newton equation in a non-compactly supported field*, J. Inverse Ill-Posed Probl. 21:(6), 713734 (2013).

[10] A. Jollivet, *Inverse scattering at high energies for the multidimensional Newton equation in a long range potential*, preprint 2013, arXiv:1306.3638.

[11] L. D. Landau, E. M. Lifschitz, Mechanics. Pergamon Press Oxford, 1960.

[12] F. Natterer, The Mathematics of Computerized Tomography. Stuttgart: Teubner and Chichester: Wiley, 1986.

[13] R. G. Novikov, *Small angle scattering and X-ray transform in classical mechanics*, Ark. Mat. 37, 141–169 (1999).

[14] R. G. Novikov, *The $\overline{\partial}$-approach to approximate inverse scattering at fixed energy in three dimensions*, IMRP Int. Math. Res. Pap. 6, 287–349 (2005).

[15] J. Radon, *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*. Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Nat. K1 69, 262-277 (1917).

[16] M. Reed, B. Simon, Methods of modern mathematical physics. III. Scattering theory. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1979. xv+463 pp.

[17] B. Simon, *Wave operators for classical particle scattering*. Comm. Math. Phys. 23, 37–48 (1971).