Inverse Problems **23** (2007) 231–242 doi:10.1088/0266-5611/23/1/012

On inverse problems for the multidimensional relativistic Newton equation at fixed energy

Alexandre Jollivet

Laboratoire de Mathématiques Jean Leray (UMR 6629), Université de Nantes, F-44322, Nantes Cedex 03, BP 92208, France

E-mail: jollivet@math.univ-nantes.fr

Received 30 June 2006, in final form 21 November 2006
Published 12 January 2007
Online at stacks.iop.org/IP/23/231

Abstract
In this paper, we consider inverse scattering and inverse boundary value problems at sufficiently large and fixed energy for the multidimensional relativistic Newton equation with an external potential $V, V \in C^2$. Using known results, we obtain, in particular, theorems of uniqueness.

1. Introduction

1.1. Relativistic Newton equation

Consider the Newton equation in the relativistic case (that is the Newton–Einstein equation) in an open subset Ω of $\mathbb{R}^n$, $n \geqslant 2$,

$$\dot{p} = -\nabla V(x), \qquad p = \frac{\dot{x}}{\sqrt{1 - \frac{|\dot{x}|^2}{c^2}}}, \qquad \dot{p} = \frac{\mathrm{d}p}{\mathrm{d}t}, \qquad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}, \tag{1.1}$$

where $V \in C^2(\bar{\Omega}, \mathbb{R})$ (i.e. there exists $\tilde{V} \in C^2(\mathbb{R}^n, \mathbb{R})$ such that $\tilde{V}$ restricted to $\bar{\Omega}$ is equal to V) and $x = x(t)$ is a C^1 function with values in Ω.

By $\|V\|_{C^2}$ we denote the supremum of the set $\left\{\left|\partial_x^j V(x)\right| \middle| x \in \Omega,\ j = (j_1, \ldots, j_n) \in (\mathbb{N} \cup \{0\})^n,\ \sum_{i=1}^n j_i \leqslant 2\right\}$.

Equation (1.1) is an equation for $x = x(t)$ and is the equation of motion of a relativistic particle of mass $m = 1$ in an external scalar potential V (see [E] and, for example, section 17 of [LL]). The potential V can be, for example, an electric potential or a gravitational potential. In this equation x is the position of the particle, p is its impulse, t is the time and c is the speed of light.

For equation (1.1) the energy

$$E = c^2\sqrt{1 + \frac{|p(t)|^2}{c^2}} + V(x(t))$$

is an integral of motion. We denote by B_c the Euclidean open ball whose radius is c and whose centre is 0.

In this paper we consider equation (1.1) in two situations. We study equation (1.1) when

$$\begin{aligned}&\Omega = D \text{ where } D \text{ is a bounded strictly convex (in the strong sense)}\\&\text{open domain of } \mathbb{R}^n,\ n \geqslant 2, \text{ with } C^2 \text{ boundary.}\end{aligned} \tag{1.2a}$$

And we study equation (1.1) when

$$\Omega = \mathbb{R}^n \text{ and } \left|\partial_x^j V(x)\right| \leqslant \beta_{|j|}(1+|x|)^{-\alpha-|j|}, \qquad x \in \mathbb{R}^n, \quad n \geqslant 2, \tag{1.2b}$$

for $|j| \leqslant 2$ and some $\alpha > 1$ (here j is the multi-index $j \in (\mathbb{N} \cup \{0\})^n$, $|j| = \sum_{i=1}^n j_i$ and $\beta_{|j|}$ are positive real constants).

For equation (1.1) under condition (1.2*a*), we consider boundary data. For equation (1.1) under condition (1.2*b*), we consider scattering data.

1.2. Boundary data

For equation (1.1) under condition (1.2*a*), one can prove that at sufficiently large energy E (i.e. $E > E(\|V\|_{C^2}, D)$), the solutions x of energy E have the following properties (see subsections 3.1, 3.2 and 3.3 of section 3):

$$\begin{aligned}&\text{for each solution } x(t) \text{ there are } t_1, t_2 \in \mathbb{R}, t_1 < t_2, \text{ such that}\\&x \in C^3([t_1,t_2],\mathbb{R}^n), x(t_1), x(t_2) \in \partial D, x(t) \in D \text{ for } t \in]t_1,t_2[,\\&x(s_1) \neq x(s_2) \text{ for } s_1, s_2 \in [t_1,t_2], s_1 \neq s_2;\end{aligned} \tag{1.3}$$

$$\begin{aligned}&\text{for any two distinct points } q_0, q \in \bar{D}, \text{ there is one and only one solution}\\&x(t) = x(t,E,q_0,q) \text{ such that } x(0) = q_0, x(s) = q \text{ for some } s > 0.\end{aligned} \tag{1.4}$$

Let (q_0, q) be two distinct points of ∂D. By $s(E, q_0, q)$ we denote the time at which $x(t, E, q_0, q)$ reaches q. By $k(E, q_0, q)$ we denote the velocity vector $\dot{x}(s(E, q_0, q), E, q_0, q)$. We consider $k(E, q_0, q)$, $q_0, q \in \partial D, q_0 \neq q$, as the boundary value data.

1.3. Scattering data

For equation (1.1) under condition (1.2*b*), the following is valid (see [Y]): for any $(v_-, x_-) \in B_c \times \mathbb{R}^n, v_- \neq 0$, equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = v_- t + x_- + y_-(t), \tag{1.5}$$

where $\dot{y}_-(t) \to 0$, $y_-(t) \to 0$, as $t \to -\infty$; in addition for almost any $(v_-, x_-) \in B_c \times \mathbb{R}^n$, $v_- \neq 0$,

$$x(t) = v_+ t + x_+ + y_+(t), \tag{1.6}$$

where $v_+ \neq 0, |v_+| < c, v_+ = a(v_-, x_-), x_+ = b(v_-, x_-), \dot{y}_+(t) \to 0, y_+(t) \to 0$, as $t \to +\infty$.

For an energy $E > c^2$, the map $S_E : \mathbb{S}_E \times \mathbb{R}^n \to \mathbb{S}_E \times \mathbb{R}^n$ (where $\mathbb{S}_E = \{v \in B_c \| v| = c\sqrt{1-\left(\frac{c^2}{E}\right)^2}\}$) given by the formulae

$$v_+ = a(v_-, x_-), \qquad x_+ = b(v_-, x_-) \tag{1.7}$$

is called the scattering map at fixed energy E for equation (1.1) under condition (1.2*b*). By $\mathcal{D}(S_E)$ we denote the domain of definition of S_E. The data $a(v_-, x_-), b(v_-, x_-)$ for $(v_-, x_-) \in \mathcal{D}(S_E)$ are called the scattering data at fixed energy E for equation (1.1) under condition (1.2*b*).

1.4. Inverse scattering and boundary value problems

In the present paper, we consider the following inverse boundary value problem at fixed energy for equation (1.1) under condition (1.2*a*):

Problem 1. Given $k(E, q_0, q)$ for all $(q_0, q) \in \partial D \times \partial D$, $q_0 \neq q$, at fixed sufficiently large energy E, find V.

The main results of the present work include the following theorem of uniqueness for problem 1.

Theorem 1.1. *At fixed $E > E(\|V\|_{C^2}, D)$, the boundary data $k(E, q, q_0)$, $(q_0, q) \in \partial D \times \partial D$, $q_0 \neq q$, uniquely determine V.*

Theorem 1.1 follows from a reduction of problem 1 to the problem of determining an isotropic Riemannian metric from its hodograph by means of the 'relativistic' Maupertuis principle, from the 'stability' property of the simple Riemannian metrics, and from theorem 3.1 (see section 3).

In the present paper, we also consider the following inverse scattering problem at fixed energy for equation (1.1) under condition (1.2*b*):

Problem 2. Given S_E at fixed energy E, find V.

The main results of the present work include the following theorem of uniqueness for problem 2.

Theorem 1.2. *Let $\lambda \in \mathbb{R}^+$ and let D be a bounded strictly convex (in the strong sense) open domain of $\mathbb{R}^n$, $n \geqslant 2$, with C^2 boundary. Let $V_1, V_2 \in C_0^2(\mathbb{R}^n, \mathbb{R})$, $\max(\|V_1\|_{C^2}, \|V_2\|_{C^2}) \leqslant \lambda$, and $supp(V_1) \cup supp(V_2) \subseteq D$. Let S_E^i be the scattering map at fixed energy E subordinate to V_i for $i = 1, 2$. There exists a nonnegative real constant $E(\lambda, D)$ such that for any $E > E(\lambda, D)$, $V_1 \equiv V_2$ if and only if $S_E^1 \equiv S_E^2$.*

Theorem 1.2 follows from theorem 1.1 and proposition 2.1.

Remark 1.1. Note that for $V \in C_0^2(\mathbb{R}^n, \mathbb{R})$, if $E < c^2 + \sup\{V(x) | x \in \mathbb{R}^n\}$ then S_E does not determine uniquely V.

Note also that reducing problem 1 to the problem of determining an isotropic Riemannian metric from its hodograph, one can give also stability estimates for problem 1 under the assumptions of theorem 1.1.

Note that before the present work it was not clear whether problem 1 is reduced to some known problem of the integral geometry or leads to some new problem in this domain. For example, for the case of the multidimensional (relativistic or nonrelativistic) Newton equation in an electromagnetic field a natural generalization of problem 1 is not reduced already, in general, to some known problem, see [J3]. The main results of [J3] include, in particular, generalizations of theorems 1.1 and 1.2 to this much more complicated 'electromagnetic' case.

1.5. Historical remarks

An inverse boundary value problem at fixed energy and at high energies was studied in [GN] for the multidimensional nonrelativistic Newton equation in a bounded open strictly convex domain. In [GN] results of uniqueness and stability for the inverse boundary value problem at fixed energy are derived from the classical Maupertuis principle and from results for the problem of determining an isotropic Riemannian metric from its hodograph (for this geometrical problem, see [MR, B, BG]). Theorem 1.1 of the present work is a generalization to the relativistic case of the related result of [GN].

Novikov [N2] studied inverse scattering for the nonrelativistic multidimensional Newton equation. Novikov [N2] gave, in particular, a connection between the inverse scattering problem at fixed energy and Gerver–Nadirashvili's inverse boundary value problem at fixed energy. Theorem 1.2 of the present work is a generalization of theorem 5.2 of [N2] to the relativistic case.

Inverse scattering at high energies for the relativistic multidimensional Newton equation was studied by the author (see [J1, J2]).

As regards analogues of theorems 1.1, 1.2 and proposition 2.1 for nonrelativistic quantum mechanics, see [N1, NSU, N3] and further references therein. As regards an analogue of theorem 1.2 for relativistic quantum mechanics, see [I]. As regards results given in the literature on inverse scattering in quantum mechanics at high energy limit, see references given in [J2].

1.6. Structure of the paper

The paper is organized as follows. In section 2, we give some properties of boundary data and scattering data and we connect the inverse scattering problem at fixed energy to the inverse boundary value problem at fixed energy. In section 3 we give, actually, a proof of theorem 1.1 (based on the 'relativistic' Maupertuis principle, the 'stability' property of the simple Riemannian metrics and on theorem 3.1).

2. Scattering data and boundary value data

2.1. Properties of the boundary value data

Let D be a bounded strictly convex (in the strong sense) open domain of $\mathbb{R}^n$, $n \geqslant 2$, with C^2 boundary.

At fixed sufficiently large E (i.e. $E > E(\|V\|_{C^2}, D) \geqslant c^2 + \sup_{x\in\bar{D}} V(x)$) solutions $x(t)$ of equation (1.1) under condition (1.2*a*) have the following properties (see subsections 3.1, 3.2 and 3.3 of section 3):

$$
\begin{aligned}
&\text{for each solution } x(t) \text{ there are } t_1, t_2 \in \mathbb{R},\ t_1 < t_2, \text{ such that}\\
&x \in C^3([t_1, t_2], \mathbb{R}^n),\ x(t_1), x(t_2) \in \partial D,\ x(t) \in D \text{ for } t \in]t_1, t_2[,\\
&x(s_1) \neq x(s_2) \text{ for } s_1, s_2 \in [t_1, t_2],\ s_1 \neq s_2,\ \dot{x}(t_1)N(x(t_1)) < 0\\
&\text{and } \dot{x}(t_2)N(x(t_2)) > 0, \text{ where } N(x(t_i)) \text{ is the unit outward}\\
&\text{normal vector of } \partial D \text{ at } x(t_i) \text{ for } i = 1, 2;
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
&\text{for any two points } q_0, q \in \bar{D},\ q \neq q_0, \text{ there is one and only one solution}\\
&x(t) = x(t, E, q_0, q) \text{ such that } x(0) = q_0,\ x(s) = q \text{ for some } s > 0;\\
&\dot{x}(0, E, q_0, q) \in C^1((\bar{D} \times \bar{D})\backslash\bar{G}, \mathbb{R}^n), \text{ where } \bar{G} \text{ is the diagonal in } \bar{D} \times \bar{D},
\end{aligned}
\tag{2.2}
$$

(where by '$\dot{x}(0, E, q_0, q) \in C^1((\bar{D} \times \bar{D})\backslash\bar{G}, \mathbb{R}^n)$' we mean that there exists an open neighbourhood Ω of $\bar{D}$ such that $\dot{x}(0, E, q_0, q)$ is the restriction to $(\bar{D} \times \bar{D})\backslash\bar{G}$ of a function which belongs to $C^1((\Omega \times \Omega)\backslash\Delta)$ where Δ is the diagonal of $\Omega \times \Omega$). Let $E > E(\|V\|_{C^2}, D)$.

Consider the solution $x(t, E, q_0, q)$ from (2.2) for $q_0, q \in \partial D, q_0 \neq q$. We recall that $s = s(E, q_0, q)$ is the root of the equation

$$x(s, E, q_0, q) = q, \qquad s > 0,$$

and we recall that $k(E, q_0, q) = \dot{x}(s(E, q_0, q), E, q_0, q)$. We consider $k(E, q_0, q), q_0, q \in \partial D, q_0 \neq q$ as the boundary value data.

Let $k_0(E, q_0, q) = \dot{x}(0, E, q_0, q)$. Note that

$$\begin{aligned} &k_0(E, q_0, q) = -k(E, q, q_0), \\ &|k_0(E, q_0, q)| = c\sqrt{1 - \left(\frac{E - V(q_0)}{c^2}\right)^{-2}}, \end{aligned} \tag{2.3}$$

for $E > E(\|V\|_{C^2}, D)$ and $(q, q_0) \in (\partial D \times \partial D)\backslash\partial G$.

2.2. Boundary data for the nonrelativistic case

If one considers the nonrelativistic Newton equation in D instead of equation (1.1) under condition (1.2*a*), one obtains the existence of a constant $E'(\|V\|_{C^2}, D)$ such that the solutions $x(t)$ of the nonrelativistic Newton equation with energy $E = \frac{1}{2}|\dot{x}(t)|^2 + V(x(t)), E > E'(\|V\|_{C^2}, D)$, also have properties (2.1) and (2.2) (see [GN]). Hence one can define the time $s'(E, q_0, q)$ and the vector $k'(E, q_0, q)$ for $E > E'(\|V\|_{C^2}, D), (q_0, q) \in (\partial D \times \partial D)\backslash\partial G$, as were defined $s(E, q_0, q), k(E, q_0, q)$ for $E > E(\|V\|_{C^2}, D), (q_0, q) \in (\partial D \times \partial D)\backslash\partial G$. In [GN], $s'(E, q_0, q)$ and $k'(E, q_0, q)$ for $E > E'(\|V\|_{C^2}, D), (q_0, q) \in (\partial D \times \partial D)\backslash\partial G$, were taken as boundary value data for the multidimensional nonrelativistic Newton equation and [GN] obtains, in particular, that $s'(E, q_0, q)$ given for all $E > E'(\|V\|_{C^2}, D), (q_0, q) \in (\partial D \times \partial D)\backslash\partial G$ uniquely determines V and that $k'(E, q_0, q)$ given for all $(q_0, q) \in (\partial D \times \partial D)\backslash\partial G$ uniquely determines V on $\bar{D}$ at fixed energy $E > E'(\|V\|_{C^2}, D)$.

2.3. Properties of the scattering operator

For equation (1.1) under condition (1.2*b*), the following is valid (see [Y]): for any $(v_-, x_-) \in B_c \times \mathbb{R}^n, v_- \neq 0$, equation (1.1) under condition (1.2*b*) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = v_- t + x_- + y_-(t), \tag{2.4}$$

where $\dot{y}_-(t) \to 0, y_-(t) \to 0$, as $t \to -\infty$; in addition for almost any $(v_-, x_-) \in B_c \times \mathbb{R}^n$, $v_- \neq 0$,

$$x(t) = v_+ t + x_+ + y_+(t), \tag{2.5}$$

where $v_+ \neq 0, |v_+| < c, v_+ = a(v_-, x_-), x_+ = b(v_-, x_-), \dot{y}_+(t) \to 0, y_+(t) \to 0$, as $t \to +\infty$.

The map $S : B_c \times \mathbb{R}^n \to B_c \times \mathbb{R}^n$ given by the formulae

$$v_+ = a(v_-, x_-), \qquad x_+ = b(v_-, x_-) \tag{2.6}$$

is called the scattering map for equation (1.1) under condition (1.2*b*). The functions $a(v_-, x_-)$, $b(v_-, x_-)$ are called the scattering data for equation (1.1) under condition (1.2*b*).

By $\mathcal{D}(S)$ we denote the domain of definition of S; by $\mathcal{R}(S)$ we denote the range of S (by definition, if $(v_-, x_-) \in \mathcal{D}(S)$, then $v_- \neq 0$ and $a(v_-, x_-) \neq 0$).

The map S has the following simple properties (see [Y]): for any $(v, x) \in B_c \times \mathbb{R}^n$, $(v, x) \in \mathcal{D}(S)$ if and only if $(-v, x) \in \mathcal{R}(S)$; $\mathcal{D}(S)$ is an open set of $B_c \times \mathbb{R}^n$ and $\mathrm{Mes}((B_c \times \mathbb{R}^n)\backslash\mathcal{D}(S)) = 0$ for the Lebesgue measure on $B_c \times \mathbb{R}^n$ induced by the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^n$; the map $S : \mathcal{D}(S) \to \mathcal{R}(S)$ is continuous and preserves the element of volume; for any $(v, x) \in \mathcal{D}(S)$, $a(v, x)^2 = v^2$.

The map S restricted to

$$\Sigma_E = \left\{ (v_-, x_-) \in B_c \times \mathbb{R}^n \| v_-| = c\sqrt{1 - \left(\frac{c^2}{E}\right)^2} \right\}$$

is the scattering operator at fixed energy E and is denoted by S_E.

We will use the fact that the map S is uniquely determined by its restriction to $\mathcal{M}(S) = \mathcal{D}(S) \cap \mathcal{M}$, where

$$\mathcal{M} = \{(v_-, x_-) \in B_c \times \mathbb{R}^n | v_- \neq 0, v_- x_- = 0\}.$$

This observation is completely similar to the related observation of [N2, J1] and is based on the fact that if $x(t)$ satisfies (1.1), then $x(t + t_0)$ also satisfies (1.1) for any $t_0 \in \mathbb{R}$. In particular, the map S at fixed energy E is uniquely determined by its restriction to $\mathcal{M}_E(S) = \mathcal{D}(S) \cap \mathcal{M}_E$, where $\mathcal{M}_E = \Sigma_E \cap \mathcal{M}$.

2.4. Inverse scattering problem and inverse boundary value problem

Assume that

$$V \in C_0^2(\bar{D}, \mathbb{R}). \tag{2.7}$$

We consider equation (1.1) under condition (1.2*a*) and equation (1.1) under condition (1.2*b*). We shall connect the boundary value data $k(E, q, q_0)$ for $E > E(\|V\|_{C^2}, D)$ and $(q, q_0) \in (\partial D \times \partial D) \backslash \partial G$, to the scattering data a, b.

Proposition 2.1. *Let $E > E(\|V\|_{C^2}, D)$. Under condition (2.7), the following statement is valid: $s(E, q_0, q), k(E, q_0, q)$ given for all $(q, q_0) \in (\partial D \times \partial D) \backslash \partial G$, are determined uniquely by the scattering data $a(v_-, x_-), b(v_-, x_-)$ given for all $(v_-, x_-) \in \mathcal{M}_E(S)$. The converse statement holds: $s(E, q_0, q), k(E, q_0, q)$ given for all $(q, q_0) \in (\partial D \times \partial D) \backslash \partial G$, determine uniquely the scattering data $a(v_-, x_-), b(v_-, x_-)$ for all $(v_-, x_-) \in \mathcal{M}_E(S)$.*

Proof of proposition 2.1. First of all we introduce functions χ, τ_- and τ_+ dependent on D. For $(v, x) \in \mathbb{R}^n \backslash \{0\} \times \mathbb{R}^n$, $\chi(v, x)$ denotes the nonnegative number of points contained in the intersection of ∂D with the straight line parametrized by $\mathbb{R} \to \mathbb{R}^n$, $t \mapsto tv + x$. As D is a strictly convex (in the strong sense) open domain of $\mathbb{R}^n$ with C^2 boundary, $\chi(v, x) \leqslant 2$ for all $v, x \in \mathbb{R}^n$, $v \neq 0$.

Let $(v, x) \in \mathbb{R}^n \backslash \{0\} \times \mathbb{R}^n$. Assume that $\chi(v, x) \geqslant 1$. The real $\tau_-(v, x)$ denotes the smallest real number t such that $\tau_-(v, x)v + x \in \partial D$, and the real $\tau_+(v, x)$ denotes the greatest real number t such that $\tau_+(v, x)v + x \in \partial D$ (if $\chi(v, x) = 1$ then $\tau_-(v, x) = \tau_+(v, x)$).

Direct statement. Let $(q_0, q) \in (\partial D \times \partial D) \backslash \partial G$. Under conditions (2.7) and from (2.1) and (2.2), it follows that there exists a unique $(v_-, x_-) \in \mathcal{M}_E(S)$ such that

$$\chi(v_-, x_-) = 2,$$

$$q_0 = x_- + \tau_-(v_-, x_-)v_-,$$

$$q = b(v_-, x_-) + \tau_+(a(v_-, x_-), b(v_-, x_-))a(v_-, x_-).$$

In addition, $s(E, q_0, q) = \tau_+(a(v_-, x_-), b(v_-, x_-)) - \tau_-(v_-, x_-)$ and $k(E, q_0, q) = a(v_-, x_-)$.

Converse statement. Let $(v_-, x_-) \in \mathcal{M}_E(S)$. Under conditions (2.7), if $\chi(v_-, x_-) \leqslant 1$ then $(a(v_-, x_-), b(v_-, x_-)) = (v_-, x_-)$.

Assume that $\chi(v_-, x_-) = 2$. Let

$$q_0 = x_- + \tau_-(v_-, x_-)v_-.$$

From (2.1) and (2.2) it follows that there is one and only one solution of the equation

$$-k(E, q, q_0) = v_-, \qquad q \in \partial D, \quad q \neq q_0. \tag{2.8}$$

We denote by $q(v_-, x_-)$ the unique solution of (2.8). Hence we obtain

$$\begin{aligned} a(v_-, x_-) &= k(E, q_0, q(v_-, x_-)), \\ b(v_-, x_-) &= q(v_-, x_-) - k(E, q_0, q(v_-, x_-)) \\ &\quad \times (s(E, q_0, q(v_-, x_-)) + \tau_-(v_-, x_-)). \end{aligned}$$

Proposition 2.1 is proved. □

For a more complete discussion about connection between scattering data and boundary value data, see [N2] considering the nonrelativistic Newton equation.

3. Inverse boundary value problem

In this section, problem 1 of the introduction is studied. In subsection 3.2, we relate solutions of the Newton–Einstein equation (1.1) at fixed energy E in D with geodesics of some isotropic Riemannian metric $r_{V,E}(x)|\mathrm{d}x|$ (dependent on V and E) in D, see lemma 3.1. In subsection 3.3, we show that the metric $r_{V,E}(x)|\mathrm{d}x|$ is simple in D if $E \geqslant E(\|V\|_{C^2}, D)$. To this end we use a general 'stability' property of the simple Riemannian metrics, see [SU]. We have also a direct proof of the simplicity of $r_{V,E}(x)|\mathrm{d}x|$ in D if $E \geqslant E(\|V\|_{C^2}, D)$ but this proof is rather long (and we do not reproduce it in the present paper). Finally, in subsection 3.4, developing [GN], we reduce problem 1 to the problem of determining an isotropic Riemannian metric from its hodograph and obtain theorem 3.1.

3.1. Hamiltonian system

Let $E > c^2 + \sup_{x\in\bar{D}} V(x)$. Take $\tilde{V} \in C^2(\mathbb{R}^n, \mathbb{R})$ such that $\tilde{V}_{|\bar{D}} \equiv V$. We shall still denote $\tilde{V}$ by V. Take an open neighbourhood Ω of $\bar{D}$ such that $E > c^2 + \sup_{x\in\Omega} V(x)$. Equation (1.1) in Ω is the Euler–Lagrange equation for the Lagrangian L defined by $L(\dot{x}, x) = -c^2\sqrt{1 - \frac{\dot{x}^2}{c^2}} - V(x)$, $\dot{x} \in B_c$ and $x \in \Omega$. The Hamiltonian H associated to the Lagrangian L by Legendre's transform (with respect to $\dot{x}$) is $H(p, x) = c^2\sqrt{1 + \frac{p^2}{c^2}} + V(x)$ where $p \in \mathbb{R}^n$ and $x \in \Omega$. Then equation (1.1) in Ω is equivalent to the Hamilton's equation

$$\dot{x} = \frac{\partial H}{\partial p}(p, x), \qquad \dot{p} = -\frac{\partial H}{\partial x}(p, x). \tag{3.1}$$

3.2. Maupertuis principle

In this subsection we apply the Maupertuis principle to the Hamiltonian system (3.1).

Let $(p(t), x(t))$, $t \in [t_1, t_2]$, be a solution of (3.1). Let $\gamma(t) = (p(t), x(t), t)$, $t \in [t_1, t_2]$. Then

$$\begin{aligned} &\gamma \text{ is a critical point of the functional } J \text{ defined by} \\ &J(\gamma') = \int_{\gamma'} p\,\mathrm{d}x - H(p, x)\,\mathrm{d}t \text{ on the set of the } C^1 \text{ functions} \\ &\gamma' : [t_1, t_2] \to \mathbb{R} \times \Omega \times [t_1, t_2],\ t \mapsto (p'(t), x'(t), t) \\ &\text{with boundary conditions } x'(t_1) = x(t_1) \text{ and } x'(t_2) = x(t_2). \end{aligned} \tag{3.2}$$

Let Σ denote the $(2n-1)$-dimensional smooth manifold $\{(p,x)\in\mathbb{R}^n\times\Omega | H(p,x)=E\}$. From (3.2), it follows that

$$\begin{aligned}&\text{for any } (p(t),x(t)),\ t\in[t_1,t_2], \text{ solution of (3.1) with energy } E\\ &\text{and for any strictly increasing } C^1 \text{ function } \phi \text{ from some closed interval}\\ &[t_-,t_+] \text{ of } \mathbb{R} \text{ onto } [t_1,t_2], \text{ the } C^1 \text{ map } \bar{\gamma} \text{ defined by}\\ &\bar{\gamma}(t)=(p(\phi(t)),x(\phi(t))),\ t\in[t_-,t_+], \text{ is a critical point for the functional}\\ &\bar{J} \text{ defined by } \bar{J}(\gamma')=\int_{\gamma'} p\,\mathrm{d}x \text{ on the set of the } C^1 \text{ functions}\\ &\gamma':[t_-,t_+]\to\Sigma,\ t\mapsto(p'(t),x'(t)) \text{ with boundary conditions}\\ &x'(t_-)=x(t_1) \text{ and } x'(t_+)=x(t_2).\end{aligned}\tag{3.3}$$

Let $y\in C^2([t_1,t_2],\Omega)$ be such that $\dot{y}(t)\neq 0, t\in[t_1,t_2]$. Let ϕ_y be the strictly increasing C^1 function from $[t_1,t_+]$ $(t_+>t_1)$ onto $[t_1,t_2]$ defined by $\phi_y(t_1)=t_1$ and $H\big(\frac{\partial L}{\partial\dot{x}}(\dot{\phi}(t)\dot{y}(\phi(t)),y(\phi(t))),y(\phi(t))\big)=E,\ t\in[t_1,t_+]$, i.e. ϕ_y is the function which satisfies the ordinary differential equation $\dot{\phi}_y(t)=c^2\frac{\sqrt{1-(\frac{E-V(y(\phi_y(t)))}{c^2})^{-2}}}{|\dot{y}(\phi_y(t))|}$, $t\in[t_1,t_+]$, with initial datum $\phi_y(t_1)=t_1$. Let $\bar{\gamma}(t)=\big(\frac{\partial L}{\partial\dot{x}}(\dot{\phi}(\phi^{-1}(t))\dot{y}(t),y(t)),\ y(t)\big), t\in[t_1,t_2]$. Then, $\bar{J}(\bar{\gamma}(t))=\int_{t_1}^{t_+}\frac{|\dot{\phi}(t)\dot{y}(\phi(t))|}{\sqrt{1-\frac{\dot{y}(\phi(t))^2\dot{\phi}(t)^2}{c^2}}}|\dot{\phi}(t)\dot{y}(\phi(t))|\,\mathrm{d}t$. Hence, using that $H(\bar{\gamma}(t))=E, t\in[t_1,t_2]$, we obtain that

$$\bar{J}(\bar{\gamma}(t))=\int_{t_1}^{t_+}r_{V,E}(y(\phi(t))|\dot{y}(\phi(t))|\dot{\phi}(t)\,\mathrm{d}t=\int_{t_1}^{t_2}r_{V,E}(y(t))|\dot{y}(t)|\,\mathrm{d}t,\tag{3.4}$$

where $r_{V,E}(x)=c\sqrt{\left(\frac{E-V(x)}{c^2}\right)^2-1},\ x\in\Omega$.

From (3.3) and (3.4), it follows that if $x(t), t\in[t_1,t_2]$, is a solution of (1.1) in Ω with energy E, then $x(t)$ is a critical point of the functional $l(y)=\int_{t_1}^{t_2}r_{V,E}(y(t))|\dot{y}(t)|\,\mathrm{d}t$ defined on the set of the functions $y\in C^1([t_1,t_2],\Omega)$ with boundary conditions $y(t_1)=x(t_1)$ and $y(t_2)=x(t_2)$ (Maupertuis principle). As $l(y)$ is the Riemannian length of the curve parametrized by $y\in C^1([t_1,t_2],\Omega)$ for the Riemannian metric $r_{V,E}(x)|\mathrm{d}x|$ in Ω, one obtains that if $x(t), t\in[t_1,t_2]$, is a solution of (1.1) with energy E, then $x(t)$ composed with its parametrization by arclength (for the Riemannian metric $r_{V,E}(x)|\mathrm{d}x|$ in Ω) gives a geodesic of the Riemannian metric $r_{V,E}(x)|\mathrm{d}x|$ in Ω.

For any solution $x:[0,t_+]\to\Omega$ of equation (1.1) in Ω with energy E, the parametrization by arclength of $x(t)$ is given by the strictly increasing C^2 function ψ_x from $[0,t'_+]$ $(t'_+>0)$ onto $[0,t_+]$ defined by the ordinary differential equation $\dot{\psi}_x(t)=\frac{\frac{E-V(x(\psi_x(t))}{c^4}}{\left(\frac{E-V(x(\psi_x(t)))}{c^2}\right)^2-1}$ with initial datum $\psi_x(0)=0$.

Applying Maupertuis principle we obtained the following lemma.

Lemma 3.1. *Under the assumption* $E>c^2+\sup_{x\in\Omega}V(x)$ *the following statement is valid: for any solution* $x:[0,t_+]\to\Omega$ *of equation (1.1) in* Ω *with energy* E*, the map* $y:[0,t'_+]\to\Omega$ *defined by* $y(t)=x(\psi_x(t)), t\in[0,t'_+]$*, is a geodesic of the Riemannian metric* $r_{V,E}(y)|\mathrm{d}y|$ *in* Ω *which satisfies* $r_{V,E}(y)|\dot{y}|\equiv 1$.

We obtain, in particular, that trajectories $\{x(t)\}$ of the multidimensional relativistic Newton equation in Ω with energy E coincide with the geodesics of Riemannian metric $r_{V,E}(x)|\mathrm{d}x|$ in Ω where $|\mathrm{d}x|$ is the canonical Euclidean metric on Ω. (In connection with the Maupertuis principle and analogue of lemma 3.1 for the Newton equation in the nonrelativistic case, see for example section 45 of [A].)

3.3. Simple metrics

We recall the definition of a simple metric g in a bounded open domain U of $\mathbb{R}^n$ with C^2 boundary (denoted by ∂U) (see for example [SU]).

Let U be a bounded open domain of $\mathbb{R}^n$ with C^2 boundary (denoted by ∂U) and let g be a C^2 Riemannian metric in $\bar{U}$. For $x \in \partial U$ the second fundamental form Π (with respect to g) of the boundary at x is defined on the tangent space $T_x(\partial U)$ of ∂U at x by the formula

$$\Pi(\zeta) = g_x(\nabla_\zeta N(x), \zeta),$$

where $\zeta \in T_x(\partial U)$ and $N(x)$ denotes the unit outward normal vector to the boundary at x ($g_x(N(x), N(x)) = 1$), and where ∇N denotes the covariant derivative of the vector field N with respect to the Levi-Civita connection of the metric g.

We say that g is *simple* in $\bar{U}$, if the second fundamental form is positive definite at every point $x \in \partial U$ and every two points $x, y \in \bar{U}$ are joined by a unique geodesic smoothly depending on x and y. The latter means that the mapping $\exp_x : \exp_x^{-1}(\bar{U}) \subseteq T_x\bar{U} \to \bar{U}$ is a diffeomorphism for any $x \in \bar{U}$, where $\exp_x(v)$ denotes the point which is reached at time 1 by the geodesic in $\bar{U}$ which starts at x with the velocity v at time 0 ($T_x\bar{U}$ denotes the tangent space of $\bar{U}$ at the point x).

As was mentioned in [SU], if a Riemannian metric g is close enough to a fixed simple metric g_0 in $C^2(\bar{U})$, then g is also simple.

Here, as D is assumed to be a bounded strictly convex (in the strong sense) open domain of $\mathbb{R}^n$ with C^2 boundary, it follows that the Euclidean metric $|\mathrm{d}x|$ is simple in $\bar{D}$. Hence, from the fact mentioned in [SU], it follows that there exists $E(\|V\|_{C^2}, D)$ such that for $E > E(\|V\|_{C^2}, D)$ the metric $\frac{c r_{V,E}(x)}{E}|\mathrm{d}x| = c^2\sqrt{\left(\frac{1-\frac{V(x)}{E}}{c^2}\right)^2 - \frac{1}{E^2}}|\mathrm{d}x|$ is also simple in $\bar{D}$.

Hence for $E > E(\|V\|_{C^2}, D)$ the metric $r_{V,E}(x)|\mathrm{d}x|$ is simple in $\bar{D}$.

Then one can consider properties (2.1) and (2.2) as consequences of lemma 3.1 and the fact that the metric $r_{V,E}(x)|\mathrm{d}x|$ is simple in $\bar{D}$.

Let $l_{V,E}$ denote the distance on $\bar{D}$ induced by the Riemannian metric $r_{V,E}(x)|\mathrm{d}x|$.

3.4. Properties of $l_{V,E}$ at fixed and sufficiently large energy E

Let $E > E(\|V\|_{C^2}, D)$. From properties (2.1) and (2.2) (or from the fact that $r_{V,E}(x)|\mathrm{d}x|$ is simple), it follows that

$$l_{V,E} \in C(\bar{D} \times \bar{D}, \mathbb{R}), \tag{3.5}$$

$$l_{V,E} \in C^2((\bar{D} \times \bar{D})\backslash\bar{G}, \mathbb{R}), \tag{3.6}$$

$$\max\left(\left|\frac{\partial l_{V,E}}{\partial x_i}(\zeta, x)\right|, \left|\frac{\partial l_{V,E}}{\partial \zeta_i}(\zeta, x)\right|\right) \leqslant C_1, \tag{3.7}$$

$$\left|\frac{\partial^2 l_{V,E}}{\partial \zeta_i \partial x_j}(\zeta, x)\right| \leqslant \frac{C_2}{|\zeta - x|}, \tag{3.8}$$

for $(\zeta, x) \in (\bar{D}\times\bar{D})\backslash\bar{G}$, $\zeta = (\zeta_1, \ldots, \zeta_n)$, $x = (x_1, \ldots, x_n)$, and $i = 1, \ldots, n$, $j = 1, \ldots, n$, and where C_1 and C_2 depend on V and D; the map $\nu_{V,E} : \partial D \times D \to \mathbb{S}^{n-1}$, defined by

$$\nu_{V,E}(\zeta, x) = \frac{-1}{r_{V,E}(x)}\left(\frac{\partial l_{V,E}}{\partial x_1}(\zeta, x), \ldots, \frac{\partial l_{V,E}}{\partial x_n}(\zeta, x)\right) \tag{3.9}$$

has the following properties:

$$\nu_{V,E} \in C^1(\partial D \times D, \mathbb{S}^{n-1}), \tag{3.10a}$$

$$\begin{array}{l}\text{the map } \nu_{V,E,x} : \partial D \to \mathbb{S}^{n-1}, \zeta \to \nu_{V,E}(\zeta, x) \\ \text{is a } C^1 \text{ orientation preserving diffeomorphism from } \partial D \text{ onto } \mathbb{S}^{n-1}\end{array} \tag{3.10b}$$

for $x \in D$ (where we choose the canonical orientation of $\mathbb{S}^{n-1}$ and the orientation of ∂D given by the canonical orientation of $\mathbb{R}^n$ and the unit outward normal vector),

$$\nu_{V,E}(\zeta, x) = \frac{k_0(E, x, \zeta)}{|k_0(E, x, \zeta)|} = -\frac{k(E, \zeta, x)}{|k(E, \zeta, x)|}, \tag{3.10c}$$

for $(\zeta, x) \in \partial D \times D$. Note that from (2.3), (3.9) and (3.10c) one obtains

$$\left(\frac{\partial l_{V,E}}{\partial x_1}(\zeta, x), \ldots, \frac{\partial l_{V,E}}{\partial x_n}(\zeta, x)\right) = \frac{k(E, \zeta, x)}{\sqrt{1 - \frac{k(E,\zeta,x)^2}{c^2}}}, \tag{3.11}$$

for $(\zeta, x) \in \partial D \times D$.

3.5. Determination of an isotropic Riemannian metric

We consider the following geometrical problem:

at fixed energy $E > E(\|V\|_{C^2}, D)$, does $l_{V,E}(\zeta, x)$, given for all $(\zeta, x) \in \partial D \times \partial D$, determine uniquely $r_{V,E}$ on $\bar{D}$?

Muhometov–Romanov [MR], Beylkin [B] and Bernstein–Gerver [BG] study the question of determining an isotropic Riemannian metric from its hodograph. Results in [B] and [BG] are obtained with smoothness conditions that are too strong so that one could apply these results to our problem. Therefore, for sake of consistency, we give results (lemma 3.2 and theorem 3.1) that already appear with stronger smoothness conditions in [B] and [BG].

We denote by $\omega_{0,V}$ the $n-1$ differential form on $\partial D \times D$ obtained in the following manner:

–for $x \in D$, let $\omega_{V,x}$ be the pull-back of ω_0 by $\nu_{V,E,x}$ where ω_0 denotes the canonical orientation form on $\mathbb{S}^{n-1}$ (i.e. $\omega_0(\zeta)(v_1, \ldots, v_{n-1}) = \det(\zeta, v_1, \ldots, v_{n-1})$, for $\zeta \in \mathbb{S}^{n-1}$ and $v_1, \ldots, v_{n-1} \in T_\zeta \mathbb{S}^{n-1}$),

–for $(\zeta, x) \in \partial D \times D$ and for any $v_1, \ldots, v_{n-1} \in T_{(\zeta,x)}(\partial D \times D)$,

$$\omega_{0,V}(\zeta, x)(v_1, \ldots, v_{n-1}) = \omega_{V,x}(\zeta)(\sigma'_{(\zeta,x)}(v_1), \ldots, \sigma'_{(\zeta,x)}(v_{n-1})),$$

where $\sigma : \partial D \times D \to \partial D,\ (\zeta', x') \mapsto \zeta'$, and $\sigma'_{(\zeta,x)}$ denotes the derivative (linear part) of σ at (ζ, x).

From smoothness of $\nu_{V,E}, \sigma$ and ω_0, it follows that $\omega_{0,V}$ is a continuous $n-1$ form on $\partial D \times D$.

Now let $\lambda \in \mathbb{R}^+$ and $V_1, V_2 \in C^2(\bar{D}, \mathbb{R})$ such that $\max(\|V_1\|_{C^2}, \|V_2\|_{C^2}) \leqslant \lambda$. Let $E > E(\lambda, D)$.

Consider the differential forms Φ_0 on $(\partial D \times \partial D)\backslash \bar{G}$ and Φ_1 on $(\partial D \times \bar{D})\backslash \bar{G}$ defined by

$$\begin{aligned}\Phi_0(\zeta, x) = -(-1)^{\frac{n(n+1)}{2}} d_x(l_{V_2,E} - l_{V_1,E})(\zeta, x) \wedge d_\zeta(l_{V_2,E} - l_{V_1,E})(\zeta, x) \\ \wedge \sum_{p+q=n-2} (dd_\zeta l_{V_1,E}(\zeta, x))^p \wedge (dd_\zeta l_{V_2,E}(\zeta, x))^q,\end{aligned} \tag{3.12}$$

for $(\zeta, x) \in (\partial D \times \partial D)\backslash\bar{G}$, where $d = d_\zeta + d_x$,

$$\Phi_1(\zeta, x) = -(-1)^{\frac{n(n-1)}{2}} \left[d_x l_{V_1,E}(\zeta, x) \wedge (dd_\zeta l_{V_1,E}(\zeta, x))^{n-1} + d_x l_{V_2,E}(\zeta, x) \wedge (dd_\zeta l_{V_2,E}(\zeta, x))^{n-1} - d_x l_{V_1,E}(\zeta, x) \wedge (dd_\zeta l_{V_2,E}(\zeta, x))^{n-1} - d_x l_{V_2,E}(\zeta, x) \wedge (dd_\zeta l_{V_1,E}(\zeta, x))^{n-1}\right], \tag{3.13}$$

for $(\zeta, x) \in (\partial D \times \bar{D})\backslash\bar{G}$, where $d = d_\zeta + d_x$.

From (3.6)–(3.8), it follows that Φ_0 is continuous on $(\partial D \times \partial D)\backslash\bar{G}$ and integrable on $\partial D \times \partial D$ and Φ_1 is continuous on $(\partial D \times \bar{D})\backslash\bar{G}$ and integrable on $\partial D \times \bar{D}$.

Lemma 3.2. *Let $\lambda \in \mathbb{R}^+$ and $E > E(\lambda, D)$. Let $V_1, V_2 \in C^2(\bar{D}, \mathbb{R})$ such that* $\max(\|V_1\|_{C^2}, \|V_2\|_{C^2}) \leqslant \lambda$. *The following equalities are valid:*

$$\int_{\partial D\times\partial D} \Phi_0 = \int_{\partial D\times\bar{D}} \Phi_1; \tag{3.14}$$

$$\frac{1}{(n-1)!}\Phi_1(\zeta, x) = \Big(r_{V_1,E}(x)^n \omega_{0,V_1}(\zeta, x) + r_{V_2,E}(x)^n \omega_{0,V_2}(\zeta, x) - \big\langle \nabla_x l_{V_1,E}(\zeta, x), \nabla_x l_{V_2,E}(\zeta, x)\big\rangle \times \big(r_{V_1,E}(x)^{n-2}\omega_{0,V_1}(\zeta, x) + r_{V_2,E}(x)^{n-2}\omega_{0,V_2}(\zeta, x)\big)\Big) \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n, \tag{3.15}$$

for $(\zeta, x) \in \partial D \times D$, where $\nabla_x l_{V_i,E}(\zeta, x) = \big(\frac{\partial l_{V_i,E}}{\partial x_1}(\zeta, x), \ldots, \frac{\partial l_{V_i,E}}{\partial x_n}(\zeta, x)\big)$ for $(\zeta, x) \in \partial D \times D$ and $i = 1, 2$, and where $\langle ., .\rangle$ denotes the usual scalar product on $\mathbb{R}^n$.

Equality (3.14) follows from regularization and Stokes' formula. Using lemma 3.2, we obtain the following theorem of uniqueness and stability.

Theorem 3.1. *Let $\lambda \in \mathbb{R}^+$ and $E > E(\lambda, D)$. Let $V_1, V_2 \in C^2(\bar{D}, \mathbb{R})$ such that* $\max(\|V_1\|_{C^2}, \|V_2\|_{C^2}) \leqslant \lambda$. *The following estimate is valid:*

$$\int_D \big(r_{V_1,E}(x) - r_{V_2,E}(x)\big)\big(r_{V_1,E}(x)^{n-1} - r_{V_2,E}(x)^{n-1}\big)\,\mathrm{d}x \leqslant \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}(n-1)!}\int_{\partial D\times\partial D} \Phi_0. \tag{3.16}$$

Note that for $V_1, V_2 \in C^3$ and $\partial D \in C^\infty$ theorem 3.1 follows directly from the stability estimate of [B] and [BG] for the problem of determining an isotropic Riemannian metric from its hodograph. Similar remarks are also valid for lemma 3.2. For $V_1, V_2 \in C^2$ and $\partial D \in C^2$ an estimate similar to (3.16) follows directly from a stability estimate of [MR] for the problem of determining an isotropic Riemannian metric from its hodograph. As we have followed Gerver–Nadirashvili's framework [GN], we have chosen to extend the related results obtained namely in [BG] and [B] to the case of less smooth metrics.

Note that theorem 3.1 is a generalization to the relativistic case of theorem 4 of [GN].

References

[A] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)

[BG] Bernstein I N and Gerver M L 1980 A condition for distinguishing metrics from hodographs *Comput. Seismol.* **13** 50–73 (Russian)

[B] Beylkin G 1979 Stability and uniqueness of the solution of the inverse kinematic problem of seismology in higher dimensions *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **84** 3–6 (Russian)
Beylkin G 1983 *J. Soviet Math.* **21** 251–4 (Engl. Transl.)

[E] Einstein A 1907 Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen *Jahrbuch der Radioaktivität und Elektronik* **4** 411–62

[GN] Gerver M L and Nadirashvili N S 1983 Inverse problem of mechanics at high energies *Comput. Seismol.* **15** 118–25 (Russian)

[I] Isozaki H 1997 Inverse scattering theory for Dirac operators *Ann. Inst. H Poincaré Phys. Théor.* **66** 237–70

[J1] Jollivet A 2006 On inverse scattering for the multidimensional relativistic Newton equation at high energies *J. Math. Phys.* **47** 062902

[J2] Jollivet A 2005 On inverse scattering in electromagnetic field in classical relativistic mechanics at high energies *Preprint* math-ph/0506008

[J3] Jollivet A 2006 On inverse problems in electromagnetic field in classical mechanics at fixed energy In preparation

[LL] Landau L D and Lifschitz E M 1971 *The Classical Theory of Fields* (New York: Pergamon)

[MR] Muhometov R G and Romanov V G 1978 On the problem of determining an isotropic Riemannian metric in n-dimensional space *Dokl. Akad. Nauk SSSR* **243** 41–4 (Russian)
Muhometov R G and Romanov V G 1978 *Sov. Math. Dokl.* **19** 1330–3 (Engl. Transl.)

[NSU] Nachman A, Sylvester J and Uhlmann G 1988 An n-dimensional Borg–Levinson theorem *Commun. Math. Phys.* **115** 595–605

[N1] Novikov R G 1988 Multidimensional inverse spectral problem for the equation $-\Delta\psi + (V(x) - Eu(x))\psi = 0$ *Funkt. Anal. Prilozhen* **22**(4) 11–22 (in Russian)
Novikov R G 1988 *Funct. Anal. Appl.* **22** 263–72 (Engl. Transl.)

[N2] Novikov R G 1999 Small angle scattering and x-ray transform in classical mechanics *Ark. Mat.* **37** 141–69

[N3] Novikov R G 2005 The $\bar{\partial}$-approach to approximate inverse scattering at fixed energy in three dimensions *IMRP Int. Math. Res. Pap.* **6** 287–349

[SU] Stefanov P and Uhlmann G 2005 Boundary rigidity and stability for generic simple metrics *J. Am. Math. Soc.* **18** 975–1003

[Y] Yajima K 1982 Classical scattering for relativistic particles *J. Fac. Sci. Univ. Tokyo 1A* **29** 599–611