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On an inverse problem for the Steklov spectrum of a Riemannian surface

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Abstract. The paper is devoted to the inverse problem of reconstruction, up to a natural gauge transform, of a smooth simply connected Riemannian surface with nonempty boundary from its Steklov spectrum. We demonstrate that the problem has two other equivalent forms: (1) the problem of recovering a positive function on the unit circle from the eigenvalue spectrum of some operator and (2) the problem of recovering an immersion of the unit disk to the Euclidean plane from the corresponding Steklov spectrum. The latter problem is a natural generalization of the classical problem of recovering a planar domain from its Steklov spectrum. We also give qualitative statements on the Steklov spectrum for two classes of Riemannian surfaces.

1. Introduction

We start with posing the problem in an arbitrary dimension, although the two-dimensional case is discussed only in the main part of the paper.

Throughout the paper, the term "smooth" is used as a synonym of "C^∞-smooth". Let (M,g) be a compact connected smooth Riemannian manifold of dimension n with the nonempty boundary ∂M. *The Laplace – Beltrami operator* is defined in local coordinates by

$$\Delta_g = (\det g)^{-1/2} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left((\det g)^{1/2} g^{ij} \frac{\partial}{\partial x_j}\right),$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $\det g = \det(g_{ij})$. *The Dirichlet-to-Neumann operator* (the DN map)

$$\Lambda_g : C^\infty(\partial M) \to C^\infty(\partial M)$$

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is defined by $\Lambda_g f = \frac{\partial u}{\partial \nu}\big|_{\partial M}$, where ν is the unit outward normal to the boundary and u is the solution to the Dirichlet problem

$$\begin{cases} \Delta_g u = 0 \\ u|_{\partial M} = f \end{cases}.$$

As well known, Λ_g is a non-negative self-dual pseudodifferential operator of order 1. Therefore it has a discrete eigenvalue spectrum

$$\mathfrak{S}(M, \Lambda_g) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_k \to \infty\},$$

each eigenvalue is repeated according to its multiplicity, which is called *the Steklov spectrum* of (M, g). Steklov [**15**] introduced the spectrum in the case when M is a domain in $\mathbb{R}^n$ and g is the Euclidean metric.

For isoperimetric inequalities on the Steklov eigenvalues and for bounds on their multiplicities we refer the reader to the recent papers [**2, 4, 6, 8**] and references therein.

The inverse problem for the Steklov spectrum is posed as follows:

Problem 1.1. To which extent is a compact connected Riemannian manifold (M, g) with a nonempty boundary determined by the Steklov spectrum $\mathfrak{S}(M, \Lambda_g)$?

Isometric Riemannian manifolds have coincident Steklov spectra. This is the first obvious ambiguity in Problem 1.1: one has to recover a Riemannian manifold up to an isometry.

Let us use the term “a Riemannian surface” for a two-dimensional Riemannian manifold. The Laplace – Beltrami operator on a Riemannian surface (M, g) possesses the following conformal invariance: $\Delta_{\rho^2 g} = \rho^{-2}\Delta_g$ for any function $0 < \rho \in C^\infty(M)$. This implies $\Lambda_{\rho^2 g} = \Lambda_g$ if $\rho|_{\partial M} = 1$. In particular, $\mathfrak{S}(M, \Lambda_{\rho^2 g}) = \mathfrak{S}(M, \Lambda_g)$ for every function $0 < \rho \in C^\infty(M)$ satisfying $\rho|_{\partial M} = 1$.

In the present paper, we study Problem 1.1 for a compact and simply connected Riemannian surface (M, g) with a nonempty boundary. Such a surface is diffeomorphic to the disc

$$D = \{(x, y) \mid x^2 + y^2 \le 1\} \subset \mathbb{R}^2 = \mathbb{C}.$$

Without loss of generality, we can assume $M = D$. For a Riemannian metric g on D, we abbreviate the notation $\mathfrak{S}(D, \Lambda_g)$ to $\mathfrak{S}(\Lambda_g)$, where Λ_g is the pseudodifferential operator on the unit circle

$$\gamma = \partial D = \{e^{i\theta} \mid \theta \in \mathbb{R}\}. \tag{1.1}$$

Given a Riemannian metric g on D and diffeomorphism $\Phi : D \to D$, the metric $g' = \Phi^* g$ is defined by $g'_p(v, w) = g_{\Phi(p)}((d_p\Phi)v, (d_p\Phi)w)$ for a point $p \in D$ and vectors v and w belonging to the tangent space T_pD, where $d_p\Phi : T_pD \to T_{\Phi(p)}D$ is the differential of Φ. The statement “the metrics g and g' on D are isometric” is equivalent to the statement “there exists a diffeomorphism $\Phi : D \to D$ such that $g' = \Phi^* g$”.

We suggest that two mentioned ambiguities, the isometry invariance and conformal invariance, exhaust the non-uniqueness in Problem 1.1 for simply connected Riemannian surfaces. In other words, we have the following

Conjecture 1.2. Two Riemannian metrics g_j $(j = 1, 2)$ on D satisfy

$$\mathfrak{S}(\Lambda_{g_1}) = \mathfrak{S}(\Lambda_{g_2}) \tag{1.2}$$

if and only if there exist a diffeomorphism $\Phi : D \to D$ and function $0 < \rho \in C^\infty(D)$ such that $\rho|_\gamma = 1$ and

$$g_2 = \rho^2 \Phi^* g_1. \tag{1.3}$$

The question becomes much easier if the isospectral hypothesis (1.2) is replaced by the assumption on intertwining the operators Λ_{g_1} and Λ_{g_2}. Let us recall the definition.

For a manifold N, two linear operators $A_j : C^\infty(N) \to C^\infty(N)$ $(j = 1, 2)$ are *intertwined* if there exists a diffeomorphism $\varphi : N \to N$ such that $A_2 = \varphi^* \circ A_1 \circ \varphi^{*-1}$, where $\varphi^* : C^\infty(N) \to C^\infty(N)$ is defined by $\varphi^* f = f \circ \varphi$.

THEOREM 1.3. *For two Riemannian metrics g_j $(j = 1, 2)$ on D, the operators Λ_{g_1} and Λ_{g_2} are intertwined if and only if there exist a diffeomorphism $\Phi : D \to D$ and function $0 < \rho \in C^\infty(D)$ such that $\rho|_\gamma = 1$ and $g_2 = \rho^2 \Phi^* g_1$.*

If two linear operators are intertwined, they are isospectral. Is the converse statement true for some classes of operators? To authors' knowledge, the question is open even in the one-dimensional case. Nevertheless, by virtue of Theorem 1.3, Conjecture 1.2 is equivalent to the statement: For two Riemannian metrics g_j $(j = 1, 2)$ on D, the operators Λ_{g_1} and Λ_{g_2} are intertwined if $\mathfrak{S}(\Lambda_{g_1}) = \mathfrak{S}(\Lambda_{g_2})$.

The paper is organized as follows. In Section 2, after recalling some known facts on the Dirichlet-to-Neumann operator, we prove Theorem 1.3.

In Section 3, we demonstrate that Conjecture 1.2 has an equivalent form in terms of the problem of recovering a positive function a on the unit circle from the eigenvalue spectrum of the operator $a^{-1}\Lambda_e$, where e is the Euclidean metric. We prove an analogous of Theorem 1.3 for the latter problem.

In Section 4, we consider the problem of recovering an immersion $\mathfrak{I} : D \to \mathbb{R}^2$ from the Steklov spectrum $\mathfrak{S}(\mathfrak{I}^* e)$. In the particular case of an embedding $\mathfrak{I}$, this is equivalent to the classical problem [**3**] of recovering a planar domain $\Omega = \mathfrak{I}(D)$ from the Steklov spectrum $\mathfrak{S}(\Omega, \Lambda_e)$. We again give an equivalent version of Conjecture 1.2 in terms of immersions and prove an analogous of Theorem 1.3.

In Section 5, we recall qualitative properties of the Steklov spectrum and describe the eigenvalue spectrum of $a_m^{-1}\Lambda_e$ for functions a_m of the form $a_m(e^{i\theta}) = b_0 + 2b_1 \cos(m\theta)$. Our main results are Theorem 5.3 and Corollary 5.4 that describe the eigenvalue spectrum for the functions a_2.

In Section 6, we interpret the eigenvalue problem for the operator $a^{-1}\Lambda_e$ as a scalar Riemann – Hilbert problem in order to prove Theorem 5.3.

The appendix contains proofs of Theorems 5.1 and 5.2 and of Lemma 6.2.

2. The problem of determining a metric on the disc

A Riemannian metric g on the disc D induces the metric g_∂ on the circle $\gamma = \partial D$. The latter metric can be written in the form $g_\partial = ds_g^2$, where ds_g is a smooth one-form on γ which does not vanish at any point. The form is uniquely determined if it is assumed to be a *positive* form, i.e., $ds_g = c(\theta)\, d\theta$, where $0 < c \in C^\infty(\gamma)$ and the cyclic coordinate θ on γ is defined by (1.1). We call ds_g *the arc-length form* of the metric g. If, for a diffeomorphism $\Phi : D \to D$, we set $\varphi = \Phi|_\gamma : \gamma \to \gamma$, then $ds_{\Phi^* g} = \pm\varphi^*(ds_g)$, where the sign $(+)$ should be chosen if Φ preserves orientation, and $(-)$ otherwise.

As mentioned in the Introduction, $\Lambda_{\rho^2 g} = \Lambda_g$ for $0 < \rho \in C^\infty(D)$ if $\rho|_\gamma = 1$. We will need the following generalization.

Lemma 2.1. *Given a Riemannian metric g on D and function $0<\rho\in C^\infty(D)$, let $a=\rho|_\gamma\in C^\infty(\gamma)$. Then*

$$\Lambda_{\rho^2 g}=a^{-1}\Lambda_g \tag{2.1}$$

and

$$ds_{\rho^2 g}=a\,ds_g. \tag{2.2}$$

In particular,

$$\Lambda_{\rho^2 g}=\Lambda_g \quad \textit{and} \quad ds_{\rho^2 g}=ds_g \quad \textit{for} \quad 0<\rho\in C^\infty(D),\ \rho|_\gamma=1.$$

Proof. Given $f\in C^\infty(\gamma)$, let $u\in C^\infty(D)$ be the solution to the Dirichlet problem

$$\Delta_g u=0 \quad \text{in} \quad D, \quad u|_\gamma=f.$$

It also solves the problem

$$\Delta_{\rho^2 g}u=0 \quad \text{in} \quad D, \quad u|_\gamma=f.$$

Therefore

$$\Lambda_g f=\left.\frac{\partial u}{\partial\nu}\right|_\gamma, \quad \Lambda_{\rho^2 g}f=\left.\frac{\partial u}{\partial\nu'}\right|_\gamma,$$

where ν and ν' are unit outward normal vectors to γ in metrics g and $\rho^2 g$ respectively. The vectors are related by $\nu'=a^{-1}\nu$. Therefore

$$\Lambda_{\rho^2 g}f=\left.\frac{\partial u}{\partial\nu'}\right|_\gamma=\left.\frac{\partial u}{\partial(a^{-1}\nu)}\right|_\gamma=a^{-1}\left.\frac{\partial u}{\partial\nu}\right|_\gamma=a^{-1}\Lambda_g f.$$

Equality (2.2) is obvious. □

Remark. Since we will refer to [**14**] several times, we emphasize the following difference between our notations and that of [**14**]. We systematically use the equality $g'=\rho^2 g$ for conformally equivalent metrics while the formula $g'=\rho g$ is used in [**14**]. In particular, (2.1) and (2.2) take the form $\Lambda_{\rho g}=\frac{1}{\sqrt{a}}\Lambda_g$ and $ds_{\rho g}=\sqrt{a}\,ds_g$ in notations of [**14**]. Here, we use the notation $\rho^2 g$ just to avoid the appearance of the square root in (2.1) and (2.2).

The arc-length form ds_g is uniquely determined by the DN map Λ_g. This follows from the statement: the full symbol of the operator Λ_g is equal to $|\xi|$, where ξ is the Fourier-dual variable of the arc-length s_g. The statement is proved in [**10**] and [**3**]. Taking this fact into account, Theorem 1.2 of [**14**] can be stated as follows:

Proposition 2.2. If $\Lambda_{g_1}=\Lambda_{g_2}$ for two Riemannian metrics on D, then there exist a diffeomorphism $\Phi: D\to D$ and function $0<\rho\in C^\infty(D)$ such that $\Phi|_\gamma=I$ (= the identity), $\rho|_\gamma=1$, and $g_2=\rho^2\Phi^* g_1$.

Proposition 2.2 differs from [**14**, Theorem 2.1] by the absence of the hypothesis $ds_{g_1}=ds_{g_2}$. But the latter equality follows from $\Lambda_{g_1}=\Lambda_{g_2}$ as we have just mentioned.

We also refer the reader to [**9**] for the unique determination of a two-dimensional compact connected Riemannian manifold M with a nonempty boundary ∂M from its DN map up to a conformal factor and up to an isometry that leaves invariant ∂M.

Proof of Theorem 1.3. If $\rho|_\gamma = 1$, then (1.3) implies

$$\Lambda_{g_2} = \Lambda_{\Phi^* g_1} = \varphi^* \circ \Lambda_{g_1} \circ \varphi^{*-1}$$

for $\varphi = \Phi|_\gamma$.

Conversely, let two metrics g_j $(j = 1, 2)$ on D satisfy

$$\Lambda_{g_2} = \varphi^* \circ \Lambda_{g_1} \circ \varphi^{*-1} \tag{2.3}$$

for some diffeomorphism $\varphi : \gamma \to \gamma$. Extend φ to a diffeomorphism $\Phi : D \to D$ and set

$$\widetilde{g}_2 = \Phi^* g_1. \tag{2.4}$$

Then

$$\Lambda_{\widetilde{g}_2} = \varphi^* \circ \Lambda_{g_1} \circ \varphi^{*-1}. \tag{2.5}$$

Comparing (2.3) and (2.5), we have

$$\Lambda_{\widetilde{g}_2} = \Lambda_{g_2}.$$

By Proposition 2.2, there exist a diffeomorphism $\Psi : D \to D$ and function $0 < \rho \in C^\infty(D)$ such that $\rho|_\gamma = 1$ and

$$g_2 = \rho^2 \Psi^* \widetilde{g}_2. \tag{2.6}$$

From (2.4) and (2.6),

$$g_2 = \rho^2 \Psi^* \Phi^* g_1 = \rho^2 (\Phi \circ \Psi)^* g_1.$$

□

3. The problem of determining a positive function on the circle

For a smooth map $\varphi : \gamma \to \gamma$, the derivative $\frac{d\varphi}{d\theta} \in C^\infty(\gamma)$ is defined by $\varphi^*(d\theta) = \frac{d\varphi}{d\theta}\, d\theta$, where θ is the cyclic coordinate defined by (1.1).

Let e be the Euclidean metric on $\mathbb{R}^2$. The following proposition is an extended version of [**14**, Theorem 1.3].

Proposition 3.1. Given a Riemannian metric g on D, there exist an orientation preserving diffeomorphism $\Phi : D \to D$ and two positive functions $\rho, \mu \in C^\infty(D)$ satisfying

$$\rho|_\gamma = 1, \quad \Delta_e(\ln \mu) = 0$$

such that

(a) $\Phi^* g = \rho^2 \mu^2 e$.

(b) There exists a smooth immersion $\mathfrak{I} : D \to \mathbb{R}^2$ such that $\mathfrak{I}^* e = \widetilde{\rho}^2 g$, where $\widetilde{\rho} = (\rho \circ \Phi^{-1})^{-1}$. In particular, $h = \widetilde{\rho}^2 g$ is a flat metric, i.e., its Gaussian curvature is identically equal to zero.

(c) If $\varphi = \Phi|_\gamma$ and $a = \mu|_\gamma$, then

$$\varphi^* \circ \Lambda_g \circ \varphi^{*-1} = a^{-1} \Lambda_e \tag{3.1}$$

and

$$\varphi^*(ds_g) = a\, d\theta. \tag{3.2}$$

Only statement (c) is explicitly formulated in [**14**, Theorem 1.3] although statements (a) and (b) also participate in the proof. Let us briefly recall our arguments for the proof.

Sketch of the proof of Proposition 3.1. Given a Riemannian metric g on D, we can find a function $0 < \widetilde{\rho} \in C^\infty(D)$ such that $\widetilde{\rho}|_\gamma = 1$ and $h = \widetilde{\rho}^2 g$ is a flat metric, see [**14**, Lemma 2.1]. Extend h to a flat metric on the open disc $D_\varepsilon = \{(x,y) \mid x^2+y^2 < 1+\varepsilon\}$ for some $\varepsilon > 0$ and denote the extension by h again. The flat metric h is locally isometric to the Euclidean metric e, i.e., for every point $p \in D_\varepsilon$, there exist a neighborhood $U \subset D_\varepsilon$ and orientation preserving isometric embedding $(U,h) \to (\mathbb{R}^2, e)$ which is defined uniquely up to the composition of a rotation and parallel translation. Continuing such embeddings along curves and using the monodromy principle, we obtain an immersion $\mathfrak{I} : D \to \mathbb{R}^2$ such that $\mathfrak{I}^* e = h$. This proves statement (b).

The flat metric h determines the complex structure $\mathcal{C}_h$ on D_ε, see [**14**, Lemma 2.5]. We consider D as a closed domain in the complex manifold $(D_\varepsilon, \mathcal{C}_h)$. On the other hand, D is a closed domain in $(\mathbb{C}, \mathcal{C}_e)$, where $\mathcal{C}_e$ is the standard complex structure. By the Riemann theorem on the existence of a conformal map between two simply connected domains, there exists a biholomorphism of closed domains $\Phi : (D, \mathcal{C}_e|_{\mathrm{int}\,D}) \to (D, \mathcal{C}_h|_{\mathrm{int}\,D})$. By statement (iii) of [**14**, Lemma 2.5], $\Phi^* h = \mu^2 e$ for some function $0 < \mu \in C^\infty(D)$ such that $\ln\mu$ is an e-harmonic function.

From equalities $h = \widetilde{\rho}^2 g$ and $\Phi^* h = \mu^2 e$, we derive

$$\rho^2\mu^2 e = \rho^2 \Phi^* h = \rho^2 \Phi^*(\widetilde{\rho}^2 g) = \rho^2 (\widetilde{\rho}\circ\Phi)^2 \Phi^* g = \Phi^* g$$

if $\rho = (\widetilde{\rho}\circ\Phi)^{-1}$. This proves statement (a).

Set $\varphi = \Phi|_\gamma$. Repeating the arguments from the beginning of [**14**, Section 3], we prove that

$$\varphi^* \circ \Lambda_h \circ \varphi^{*-1} = a^{-1}\Lambda_e, \tag{3.3}$$

where the function $a \in C^\infty(\gamma)$ is defined by

$$\varphi^*(ds_h) = a\,d\theta. \tag{3.4}$$

Equalities $h = \widetilde{\rho}^2 g$ and $\widetilde{\rho}|_\gamma = 1$ imply that $\Lambda_g = \Lambda_h$ and $ds_g = ds_h$. Together with (3.3) and (3.4), this gives (3.1) and (3.2). Finally, statement (a) together with the boundary condition $\rho|_\gamma = 1$ implies $\varphi^*(ds_g) = \mu|_\gamma\,d\theta$. Comparing the last equality with (3.2), we obtain $a = \mu|_\gamma$. □

Statement (c) of Proposition 3.1 relates the inverse problem of recovering a metric on D to the problem of finding a positive function $a \in C^\infty(\gamma)$ from the eigenvalue spectrum $\mathfrak{S}(a^{-1}\Lambda_e)$ of the operator $a^{-1}\Lambda_e$.

Statement (b) of Proposition 3.1 has an important corollary. Choosing a biholomorphism of $(D, \mathcal{C}_h|_{\mathrm{int}\,D})$ onto the upper half-plane, we can repeat all arguments from the proof of [**3**, Theorem 1] to obtain the following:

Proposition 3.2. For a Riemannian metric g on D, let L be the length of γ in g and let

$$\mathfrak{S}(\Lambda_g) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_k \le \dots\}$$

be the Steklov spectrum. Then

$$\lambda_k = \frac{2\pi}{L}\Big[\frac{k+1}{2}\Big] + O(k^{-N}) \quad \text{as} \quad k \to \infty$$

for every $N > 0$, where $[x]$ is the integer part of x. In particular, L is uniquely determined by $\mathfrak{S}(\Lambda_g)$.

The same statement is valid for the asymptotics of the spectrum $\mathfrak{S}(a^{-1}\Lambda_e)$, where the constant L is defined by $L = \int_0^{2\pi} a(\theta)\,d\theta$.

Lemma 3.3. *Let $\Phi : D \to D$ be a smooth map such that $\Phi(\gamma) \subset \gamma$ and $\Phi|_{int\,D} : int\,D \to \mathbb{C}$ is either a holomorphic or antiholomorphic function. Set $\varphi = \Phi|_\gamma : \gamma \to \gamma$. Assume the derivative $d\varphi/d\theta$ does not vanish at any point. Then*

$$\Lambda_e \circ \varphi^* = \left|\frac{d\varphi}{d\theta}\right| \varphi^* \circ \Lambda_e. \tag{3.5}$$

If two functions $0 < a_j \in C^\infty(\gamma)$ $(j = 1, 2)$ are related by the equation

$$a_2 = \left|\frac{d\varphi}{d\theta}\right| a_1 \circ \varphi, \tag{3.6}$$

then

$$(a_2^{-1}\Lambda_e) \circ \varphi^* = \varphi^* \circ (a_1^{-1}\Lambda_e) \tag{3.7}$$

and

$$\mathfrak{S}(a_1^{-1}\Lambda_e) \subset \mathfrak{S}(a_2^{-1}\Lambda_e). \tag{3.8}$$

Proof. Let, for definiteness, $\Phi|_{\mathrm{int}\,D}$ be a holomorphic function. Given a function $f \in C^\infty(\gamma)$, let $u \in C^\infty(D)$ be the solution to the Dirichlet problem

$$\Delta_e u = 0 \quad \text{in} \quad D, \quad u|_\gamma = f.$$

The function $\widetilde{u} = u \circ \Phi$ solves the problem

$$\Delta_e \widetilde{u} = 0 \quad \text{in} \quad D, \quad \widetilde{u}|_\gamma = \varphi^* f.$$

Therefore

$$\Lambda_e f = \left.\frac{\partial u}{\partial \nu}\right|_\gamma, \quad \Lambda_e(\varphi^* f) = \left.\frac{\partial \widetilde{u}}{\partial \nu}\right|_\gamma. \tag{3.9}$$

We are going to relate the right-hand sides of (3.9). For $z \in \gamma$, let ν_z be the unit outward vector normal to γ at the point z. Then

$$\frac{\partial \widetilde{u}}{\partial \nu}(z) = (d_z(u \circ \Phi))\nu_z = (d_{\varphi(z)}u \circ d_z\Phi)\nu_z. \tag{3.10}$$

The differential

$$d_z\Phi : T_zD = \mathbb{C} \to \mathbb{C} = T_{\Phi(z)}D$$

is just the multiplication by $\Phi'(z)$. In particular, the differential commutes with the multiplication by the imaginary unit. Since $\Phi(\gamma) \subset \gamma$ and the vector $i\nu_z$ is tangent to γ, the vector $i(d_z\Phi)\nu_z$ is also tangent to γ. This implies

$$(d_z\Phi)\nu_z = |\Phi'(z)|\, \nu_{\Phi(z)} = \left|\frac{d\varphi}{d\theta}(z)\right|\, \nu_{\varphi(z)}.$$

With the help of this, formula (3.10) becomes

$$\frac{\partial \widetilde{u}}{\partial \nu}(z) = \left|\frac{d\varphi}{d\theta}(z)\right| (d_{\varphi(z)}u)\nu_{\varphi(z)} = \left|\frac{d\varphi}{d\theta}(z)\right| \frac{\partial u}{\partial \nu}(\varphi(z)).$$

Comparing this with (3.9), we obtain

$$(\Lambda_e \circ \varphi^*)f = \Big(\left|\frac{d\varphi}{d\theta}\right| \varphi^* \circ \Lambda_e\Big) f.$$

This proves (3.5).

Let two functions $0 < a_j \in C^\infty(\gamma)$ $(j = 1,2)$ satisfy (3.6). Multiply (3.5) by a_2^{-1} from the left

$$(a_2^{-1}\Lambda_e)\circ\varphi^* = a_2^{-1}\left|\frac{d\varphi}{d\theta}\right|\varphi^*\circ(a_1a_1^{-1}\Lambda_e) = a_2^{-1}\left|\frac{d\varphi}{d\theta}\right|(a_1\circ\varphi)\,\varphi^*\circ(a_1^{-1}\Lambda_e) = \varphi^*\circ(a_1^{-1}\Lambda_e).$$

This proves (3.7). Inclusion (3.8) obviously follows from (3.7). $\square$

Definition 3.4. Two functions $0 < a_j \in C^\infty(\gamma)$ $(j = 1,2)$ are said to be e-conformally equivalent if there exists an e-conformal (or e-anticonformal) transformation $\Phi : D \to D$ such that (3.6) holds for $\varphi = \Phi|_\gamma$.

Theorem 3.5. *Two functions $0 < a_j \in C^\infty(\gamma)$ $(j = 1,2)$ are e-conformally equivalent if and only if the operators $a_1^{-1}\Lambda_e$ and $a_2^{-1}\Lambda_e$ are intertwined.*

Proof. The "only if" statement follows from Lemma 3.3. Indeed, in our case φ^* is invertible and (3.7) can be written as

$$a_2^{-1}\Lambda_e = \varphi^* \circ a_1^{-1}\Lambda_e \circ \varphi^{*-1},$$

i.e., φ intertwines $a_1^{-1}\Lambda_e$ and $a_2^{-1}\Lambda_e$.

Conversely, let two functions $0 < a_j \in C^\infty(\gamma)$ $(j = 1,2)$ be such that the operators $a_1^{-1}\Lambda_e$ and $a_2^{-1}\Lambda_e$ are intertwined, i.e.,

$$a_2^{-1}\Lambda_e = \psi^* \circ a_1^{-1}\Lambda_e \circ \psi^{*-1} \tag{3.11}$$

for some diffeomorphism $\psi : \gamma \to \gamma$. For each $j = 1,2$, we extend a_j to a function $0 < \rho_j \in C^\infty(D)$,

$$\rho_j|_\gamma = a_j$$

and define the metric g_j on D by

$$g_j = \rho_j^2 e. \tag{3.12}$$

Hence,

$$ds_j = a_j\, d\theta. \tag{3.13}$$

By Lemma 2.1,

$$\Lambda_{g_j} = a_j^{-1}\Lambda_e. \tag{3.14}$$

From (3.11) and (3.14),

$$\Lambda_{g_2} = \psi^* \circ \Lambda_{g_1} \circ \psi^{*-1}. \tag{3.15}$$

By Theorem 1.3, (3.15) implies the existence of a diffeomorphism $\Phi : D \to D$ and of a function $0 < \rho \in C^\infty(D)$ such that $\rho|_\gamma = 1$ and

$$g_2 = \rho^2\Phi^* g_1. \tag{3.16}$$

Hence

$$ds_{g_2} = \pm\varphi^*(ds_{g_1}), \quad \text{where} \quad \varphi = \Phi|_\gamma. \tag{3.17}$$

From (3.12) and (3.16),

$$\rho_2^2 e = g_2 = \rho^2\Phi^* g_1 = \rho^2\Phi^*(\rho_1^2 e) = \rho^2(\rho_1\circ\Phi)^2\Phi^* e,$$

i.e.,

$$\Phi^* e = \left(\frac{\rho_2}{\rho(\rho_1\circ\Phi)}\right)^2 e.$$

This means that Φ is an e-conformal (or e-anticonformal) transformation of the disc D.

Using (3.13) and (3.17), we obtain

$$a_2\, d\theta = ds_{g_2} = \pm\varphi^*(ds_{g_1}) = \pm\varphi^*(a_1\, d\theta) = (a_1 \circ \varphi)\left|\frac{d\varphi}{d\theta}\right| d\theta,$$

i.e.,

$$a_2 = (a_1 \circ \varphi)\left|\frac{d\varphi}{d\theta}\right|. \tag{3.18}$$

Since Φ is an e-conformal (or e-anticonformal) transformation and $\varphi = \Phi|_\gamma$, (3.18) means the e-conformal equivalence of a_1 and a_2. □

Conjecture 1.2 has the following form in terms of e-conformally equivalent functions.

CONJECTURE 3.6. For two functions $0 < a_j \in C^\infty(\gamma)$ $(j = 1, 2)$, the equality

$$\mathfrak{S}(a_1^{-1}\Lambda_e) = \mathfrak{S}(a_2^{-1}\Lambda_e)$$

holds if and only if these functions are e-conformally equivalent.

A wrong version of the conjecture was supposed in [**14**, Problem 3.2].

PROOF OF THE EQUIVALENCE OF CONJECTURES 1.2 AND 3.6. First of all, the "if" statement of Conjecture 3.6 follows from Theorem 3.5 since intertwined operators have coincident spectra. The implication Conjecture 1.2 $\longrightarrow$ Conjecture 3.6 is proved by the same arguments we have used in the proof of Theorem 3.5. The reverse implication is a little bit more tricky.

Let Riemannian metrics g_j $(j = 1, 2)$ on D satisfy (1.2). Applying Proposition 3.1 to each of the metrics, we find an orientation preserving diffeomorphism $\Phi_j : D \to D$ and positive functions $\rho_j, \mu_j \in C^\infty(D)$ such that $\rho_j|_\gamma = 1$, $\Delta_e(\ln \mu_j) = 0$,

$$\Phi_j^* g_j = \rho_j^2 \mu_j^2 e, \tag{3.19}$$

and for $\varphi_j = \Phi_j|_\gamma,\ a_j = \mu_j|_\gamma$,

$$\varphi_j^* \circ \Lambda_{g_j} \circ \varphi_j^{*-1} = a_j^{-1}\Lambda_e. \tag{3.20}$$

From (1.2) and (3.20), $\mathfrak{S}(a_1^{-1}\Lambda_e) = \mathfrak{S}(a_2^{-1}\Lambda_e)$. Assuming Conjecture 3.6 to be true, we find an e-conformal (or e-anticonformal) transformation Φ such that

$$a_2 = (a_1 \circ \varphi)\left|\frac{d\varphi}{d\theta}\right|, \quad \text{where} \quad \varphi = \Phi|_\gamma. \tag{3.21}$$

We remember also that

$$\Phi^* e = |\Phi'|^2 e, \tag{3.22}$$

where $\Phi' = d\Phi/dz$ $(\Phi' = d\Phi/d\bar{z})$ for the holomorphic (antiholomorphic) function Φ.

From (3.19) and (3.22), we derive

$$g_2 = \Phi_2^{*-1}(\rho_2^2\mu_2^2 e) = \Phi_2^{*-1}(\rho_2^2\mu_2^2|\Phi'|^{-2}\Phi^* e).$$

Substitute the value $e = \rho_1^{-2}\mu_1^{-2}\Phi_1^* g_1$, that follows from (3.19), into the right-hand side of the latter formula

$$\begin{aligned} g_2 &= \Phi_2^{*-1}\Big(\rho_2^2\mu_2^2|\Phi'|^{-2}\Phi^*(\rho_1^{-2}\mu_1^{-2}\Phi_1^* g_1)\Big) \\ &= \Phi_2^{*-1}\Big(\rho_2^2\mu_2^2|\Phi'|^{-2}(\rho_1\circ\Phi)^{-2}(\mu_1\circ\Phi)^{-2}\Phi^*\Phi_1^* g_1\Big). \end{aligned}$$

Rewrite this in the form

$$g_2 = \Phi_2^{*-1}\left\{\left(\frac{\rho_2}{\rho_1\circ\Phi}\right)^2\left[\frac{\mu_2}{(\mu_1\circ\Phi)|\Phi'|}\right]^2\Phi^*\Phi_1^* g_1\right\}. \tag{3.23}$$

Let us demonstrate that the function in brackets on the right-hand side of (3.23) is identically unit, i.e.,

$$\mu_2 = (\mu_1\circ\Phi)|\Phi'|. \tag{3.24}$$

Indeed, both

$$\ln\mu_2 \quad \text{and} \quad \ln(\mu_1\circ\Phi) + \ln|\Phi'|$$

are e-harmonic functions in D. Therefore, to prove (3.24), it suffices to show that this equality holds on γ. But on γ, (3.24) coincides with (3.21) since $\mu_j|_\gamma = a_j$ and $|\Phi'|_\gamma = |d\varphi/d\theta|$.

By (3.24), formula (3.23) simplifies to the following one:

$$g_2 = \Phi_2^{*-1}(\widetilde{\rho}^2\Phi^*\Phi_1^* g_1) = (\widetilde{\rho}\circ\Phi_2^{-1})^2\Phi_2^{*-1}\Phi^*\Phi_1^* g_1,$$

where $\widetilde{\rho} = \rho_2/\rho_1\circ\Phi$. Setting $\rho = \widetilde{\rho}\circ\Phi_2^{-1}$ and $\Psi = \Phi_1\circ\Phi\circ\Phi_2^{-1}$, we obtain

$$g_2 = \rho^2\Psi^* g_1, \quad \rho|_\gamma = 1.$$

□

Theorem 3.5 can be generalized. We first generalize Definition 3.4 as follows.

Definition 3.7. Let g be a Riemannian metric on D. Two functions $0 < a_j \in C^\infty(\gamma)$ $(j = 1, 2)$ are said to be g-conformally equivalent if there exists a g-conformal (or g-anticonformal) transformation

$$\Phi : D \to D, \quad \Phi^* g = \lambda^2 g$$

such that

$$a_2 = c\,(a_1\circ\varphi)$$

for $\varphi = \Phi|_\gamma$ and $c = \lambda|_\gamma$.

Theorem 3.8. *Let g be a Riemannian metric on D. Two functions $0 < a_j \in C^\infty(\gamma)$ $(j = 1, 2)$ are g-conformally equivalent if and only if the operators $a_1^{-1}\Lambda_g$ and $a_2^{-1}\Lambda_g$ are intertwined.*

Proof. The proof of the “if” statement repeats, with obvious changes, the corresponding part of the proof of Theorem 3.5. So, we present the proof of the statement “only if”.

Assume functions $0 < a_j \in C^\infty(\gamma)$ to be such that the operators $a_1^{-1}\Lambda_g$ and $a_2^{-1}\Lambda_g$ are intertwined. On using Proposition 3.1, we find an orientation preserving diffeomorphism $\Phi : D \to D$ and two functions $0 < \rho, \mu \in C^\infty(D)$ such that

$$\Phi^* g = \rho^2\mu^2 e, \quad \rho|_\gamma = 1, \quad \Delta_e(\ln\mu) = 0 \tag{3.25}$$

and

$$\varphi^* \circ \Lambda_g \circ \varphi^{*-1} = a^{-1}\Lambda_e \tag{3.26}$$

for $a = \mu|_\gamma$ and $\varphi = \Phi|_\gamma$.

We set

$$b_j = a\,(a_j \circ \varphi) \quad (j = 1, 2). \tag{3.27}$$

From this, we derive with the help of (3.26)

$$b_j^{-1}\Lambda_e = (a_j \circ \varphi)^{-1} a^{-1}\Lambda_e = (a_j \circ \varphi)^{-1}\varphi^* \circ \Lambda_g \circ \varphi^{*-1} = \varphi^* \circ a_j^{-1}\Lambda_g \circ \varphi^{*-1}.$$

This means that the operators $b_j^{-1}\Lambda_e$ and $a_j^{-1}\Lambda_g$ are intertwined. Since $a_1^{-1}\Lambda_g$ and $a_2^{-1}\Lambda_g$ are assumed to be intertwined, we conclude: $b_1^{-1}\Lambda_e$ and $b_2^{-1}\Lambda_e$ are intertwined.

By Theorem 3.5, b_1 and b_2 are e-conformally equivalent, i.e., there exists an e-conformal (or e-anticonformal) $\Psi : D \to D$ such that

$$b_2 = \left|\frac{d\psi}{d\theta}\right| b_1 \circ \psi \quad \text{for} \quad \psi = \Psi|_\gamma. \tag{3.28}$$

From (3.27) and (3.28), we find the relationship between a_1 and a_2

$$a_2 \circ \varphi = a^{-1} b_2 = a^{-1}\left|\frac{d\psi}{d\theta}\right| b_1 \circ \psi = a^{-1}\left|\frac{d\psi}{d\theta}\right| (a\, a_1 \circ \varphi) \circ \psi = a^{-1}\left|\frac{d\psi}{d\theta}\right| (a \circ \psi)(a_1 \circ \varphi \circ \psi),$$

i.e.,

$$a_2 = \frac{a \circ \psi \circ \varphi^{-1}}{a \circ \varphi^{-1}}\left|\frac{d\psi}{d\theta} \circ \varphi^{-1}\right| (a_1 \circ \varphi \circ \psi \circ \varphi^{-1}).$$

Introducing the notations

$$\widetilde{\Psi} = \Phi \circ \Psi \circ \Phi^{-1}, \quad \widetilde{\psi} = \widetilde{\Psi}|_\gamma,$$

we write the result as

$$a_2 = \frac{a \circ \psi \circ \varphi^{-1}}{a \circ \varphi^{-1}}\left|\frac{d\psi}{d\theta} \circ \varphi^{-1}\right| (a_1 \circ \widetilde{\psi}). \tag{3.29}$$

Let us demonstrate that $\widetilde{\Psi}$ is a g-conformal (or g-anticonformal) transformation. Indeed, by twice usage of (3.25), we deduce

$$\begin{aligned}
\widetilde{\Psi}^* g &= \Phi^{*-1}\Psi^*\Phi^* g = \Phi^{*-1}\Psi^*(\rho^2\mu^2 e)\\
&= (\rho \circ \Psi \circ \Phi^{-1})^2(\mu \circ \Psi \circ \Phi^{-1})^2\Phi^{*-1}\Psi^* e\\
&= (\rho \circ \Psi \circ \Phi^{-1})^2(\mu \circ \Psi \circ \Phi^{-1})^2\Phi^{*-1}(|\Psi'|^2 e)\\
&= (\rho \circ \Psi \circ \Phi^{-1})^2(\mu \circ \Psi \circ \Phi^{-1})^2|\Psi' \circ \Phi^{-1}|^2\Phi^{*-1} e\\
&= (\rho \circ \Psi \circ \Phi^{-1})^2(\mu \circ \Psi \circ \Phi^{-1})^2|\Psi' \circ \Phi^{-1}|^2\Phi^{*-1}(\rho^{-2}\mu^{-2}\Phi^* g)\\
&= \Big(\frac{\rho \circ \Psi \circ \Phi^{-1}}{\rho \circ \Phi^{-1}}\Big)^2\Big(\frac{\mu \circ \Psi \circ \Phi^{-1}}{\mu \circ \Phi^{-1}}\Big)^2|\Psi' \circ \Phi^{-1}|^2 g.
\end{aligned}$$

We write the result in the form

$$\widetilde{\Psi}^* g = \widetilde{\lambda}^2 g, \tag{3.30}$$

where

$$\widetilde{\lambda} = \frac{\rho \circ \Psi \circ \Phi^{-1}}{\rho \circ \Phi^{-1}}\,\frac{\mu \circ \Psi \circ \Phi^{-1}}{\mu \circ \Phi^{-1}}|\Psi' \circ \Phi^{-1}|.$$

In particular, (3.30) means that $\widetilde{\Psi}$ is a g-conformal (or g-anticonformal) transformation. Restricting the last formula to γ and using $\rho|_\gamma = 1$, $\mu|_\gamma = a$, and $\Psi'|_\gamma = d\psi/d\theta$, we get

$$\widetilde{\lambda}|_\gamma = \frac{a \circ \psi \circ \varphi^{-1}}{a \circ \varphi^{-1}} \left| \frac{d\psi}{d\theta} \circ \varphi^{-1} \right|.$$

With the help of this, formula (3.29) takes the form

$$a_2 = \widetilde{\lambda}|_\gamma (a_1 \circ \widetilde{\psi}). \tag{3.31}$$

Formulas (3.30) and (3.31) mean that a_1 and a_2 are g-conformally equivalent. $\square$

Finally, Conjecture 3.6 is equivalent to the following more general statement. The equivalence is proved by the same arguments as we have used in the proof of Theorem 3.8.

Conjecture 3.9. Let g be a Riemannian metric on D. For two functions $0 < a_j \in C^\infty(\gamma)$, the equality

$$\mathfrak{S}(a_1^{-1}\Lambda_g) = \mathfrak{S}(a_2^{-1}\Lambda_g)$$

holds if and only if these functions are g-conformally equivalent.

4. The problem of recovering a planar domain

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain bounded by a smooth closed curve. How far is such a domain determined by the Steklov spectrum $\mathfrak{S}(\Omega, \Lambda_e)$, where e is the Euclidean metric? The problem was considered in [**3**]. We are going to generalize the problem a little bit by involving multisheet domains. A typical example of such a domain is shown on the picture.

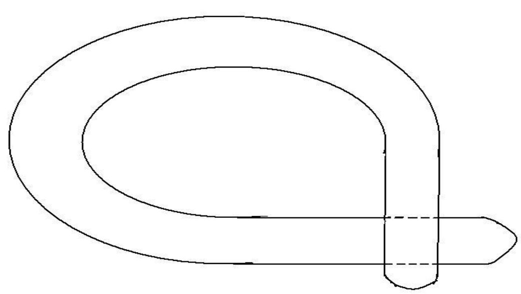

Figure 1

How can one define rigorously a (smooth simply connected) multisheet domain? One of possible definitions is as follows: this is a domain diffeomorphic to the disc D and situated on the Riemann surface of a multivalued analytic function. However, the following definition is more appropriate for our purposes.

Definition 4.1. Let $\mathcal{I}$ be the set of all smooth immersions $\mathfrak{I} : D \to \mathbb{R}^2$. Introduce the equivalence relation on $\mathcal{I}$ as follows: $\mathfrak{I}_1 \sim \mathfrak{I}_2$ if there exists a diffeomorphism $\Phi : D \to D$ such that $\mathfrak{I}_2 = \mathfrak{I}_1 \circ \Phi$. Elements of the factor-set $\mathcal{I}/\!\sim$ are called (smooth, closed, bounded, simply connected) multisheet domains in $\mathbb{R}^2$.

At first sight, the definition can seem too complicated. Nevertheless, in our opinion, every reasonable definition of a multisheet domain should be equivalent to this one. In particular, if an immersion $\mathfrak{I} : D \to \mathbb{R}^2$ is injective, then its equivalence class is uniquely determined by the image $\Omega = \mathfrak{I}(D)$.

The natural conjecture is that a smooth bounded simply connected domain $\Omega \subset \mathbb{R}^2$ is determined by the Steklov spectrum $\mathfrak{S}(\Omega, \Lambda_e)$ uniquely up to an isometry. For multisheet domains, the conjecture sounds as follows:

Conjecture 4.2. For two smooth immersions $\mathfrak{I}_j : D \to \mathbb{R}^2$ $(j = 1, 2)$, the equality

$$\mathfrak{S}(\Lambda_{\mathfrak{I}_1^* e}) = \mathfrak{S}(\Lambda_{\mathfrak{I}_2^* e}) \tag{4.1}$$

holds if and only if there exist a diffeomorphism $\Phi : D \to D$ and isometry $I : (\mathbb{R}^2, e) \to (\mathbb{R}^2, e)$ such that

$$\mathfrak{I}_2 = I \circ \mathfrak{I}_1 \circ \Phi. \tag{4.2}$$

We are going to demonstrate the conjecture is equivalent to our previous Conjectures 1.2 and 3.6. But first we will prove the following:

Theorem 4.3. *For two smooth immersions $\mathfrak{I}_j : D \to \mathbb{R}^2$ $(j = 1, 2)$, the operators $\Lambda_{\mathfrak{I}_1^* e}$ and $\Lambda_{\mathfrak{I}_2^* e}$ are intertwined if and only if there exist a diffeomorphism $\Phi : D \to D$ and isometry $I : (\mathbb{R}^2, e) \to (\mathbb{R}^2, e)$ such that $\mathfrak{I}_2 = I \circ \mathfrak{I}_1 \circ \Phi$.*

Proof. The "if" statement is obvious. Indeed, since $I^* e = e$, (4.2) implies $\mathfrak{I}_2^* e = \Phi^*(\mathfrak{I}_1^* e)$ and the operators $\Lambda_{\mathfrak{I}_1^* e}$ and $\Lambda_{\mathfrak{I}_2^* e}$ are intertwined by $\varphi = \Phi|_\gamma$.

Now, we prove the statement "only if". Let the operators $\Lambda_{\mathfrak{I}_1^* e}$ and $\Lambda_{\mathfrak{I}_2^* e}$ be intertwined. By Theorem 1.3, there exist a diffeomorphism $\Phi : D \to D$ and function $0 < \rho \in C^\infty(D)$ such that

$$\mathfrak{I}_2^* e = \rho^2 \Phi^* \mathfrak{I}_1^* e, \quad \rho|_\gamma = 1.$$

We rewrite this in the form

$$h_2 = \rho^2 h_1, \quad \text{where} \quad h_1 = \Phi^* \mathfrak{I}_1^* e,\ h_2 = \mathfrak{I}_2^* e.$$

Both h_1 and h_2 are flat metrics. For flat metrics, equality $h_2 = \rho^2 h_1$ holds if and only if $\Delta_{h_1}(\ln \rho) = 0$. Together with the boundary condition $\rho|_\gamma = 1$, this gives $\rho \equiv 1$. Thus,

$$\mathfrak{I}_2^* e = (\mathfrak{I}_1 \circ \Phi)^* e. \tag{4.3}$$

Two immersions $\widetilde{\mathfrak{I}}_1 = \mathfrak{I}_1 \circ \Phi$ and $\mathfrak{I}_2$ satisfy $\widetilde{\mathfrak{I}}_1^* e = \mathfrak{I}_2^* e$ if and only if $\mathfrak{I}_2 = I \circ \widetilde{\mathfrak{I}}_1$ for some isometry $I : (\mathbb{R}^2, e) \to (\mathbb{R}^2, e)$. Therefore (4.3) implies (4.2). $\square$

Proof of the equivalence of Conjectures 1.2 and 4.2. The implication Conjecture 1.2 $\longrightarrow$ Conjecture 4.2 is proved by the same arguments as we have used in the proof of Theorem 4.3. Let us prove the reverse implication.

Assume metrics g_j $(j = 1, 2)$ on D to satisfy (1.2). By Proposition 3.1, there exist a function $0 < \rho_j \in C^\infty(D)$, $\rho_j|_\gamma = 1$, and immersion $\mathfrak{I}_j : D \to \mathbb{R}^2$ such that

$$h_j = \rho_j^2 g_j = \mathfrak{I}_j^* e.$$

Hence,

$$\mathfrak{S}(\Lambda_{\mathfrak{I}_j^* e}) = \mathfrak{S}(\Lambda_{g_j}).$$

Together with (1.2), this gives (4.1). Assuming Conjecture 4.2 to be true, we obtain (4.2). From (4.2),

$$h_2 = \mathfrak{I}_2^* e = (I \circ \mathfrak{I}_1 \circ \Phi)^* e = \Phi^* \mathfrak{I}_1^* I^* e = \Phi^* \mathfrak{I}_1^* e = \Phi^* h_1,$$

i.e.,

$$h_2 = \Phi^* h_1.$$

Now,

$$g_2 = \rho_2^{-2} h_2 = \rho_2^{-2} \Phi^* h_1 = \rho_2^{-2} \Phi^* (\rho_1^2 g_1) = \Big(\frac{\rho_1 \circ \Phi}{\rho_2}\Big)^2 \Phi^* g_1.$$

Setting $\rho = \rho_1 \circ \Phi / \rho_2$, we have

$$g_2 = \rho^2 \Phi^* g_1, \quad \rho|_\gamma = 1.$$

This is the statement of Conjecture 1.2. □

The following statement was proved by Weinstock [**17**]:

If $\mathfrak{S}(\Omega, \Lambda_e) = \{0 = \lambda_0 < \lambda_1 \le \dots\}$ is the Steklov spectrum of a smooth bounded simply connected domain $\Omega \subset \mathbb{R}^2$ and L is the length of $\partial\Omega$, then

$$\lambda_1 \le \frac{2\pi}{L} \tag{4.4}$$

and the equality holds if and only if Ω is a disc.

Weinstock's proof works for multisheet domains as well and the statement can be presented in the form:

Proposition 4.4. For a smooth immersion $\mathfrak{I} : D \to \mathbb{R}^2$, let $\mathfrak{S}(\Lambda_{\mathfrak{I}^* e}) = \{0 = \lambda_0 < \lambda_1 \le \dots\}$ be the Steklov spectrum of the metric $\mathfrak{I}^* e$ and L be the length of γ in the metric. Then (4.4) is valid. The equality in (4.4) holds if and only if there exist a diffeomorphism $\Phi : D \to D$ and isometry $I : (\mathbb{R}^2, e) \to (\mathbb{R}^2, e)$ such that $\mathfrak{I} = I \circ \mathfrak{I}_L \circ \Phi$, where $\mathfrak{I}_L(z) = \frac{L}{2\pi} z$.

On using the equivalence of Conjecture 4.2 to Conjectures 1.2 and 3.6, the latter statement can be transformed to two other forms.

Proposition 4.5. For a function $0 < a \in C^\infty(\gamma)$, let $\mathfrak{S}(a^{-1}\Lambda_e) = \{0 = \lambda_0 < \lambda_1 \le \dots\}$ and $L = \int_0^{2\pi} a(\theta)\, d\theta$. Then (4.4) is valid. The equality in (4.4) holds if and only if the function a is e-conformally equivalent to the constant $L/2\pi$.

Proposition 4.6. For a Riemannian metric g on D, let $\mathfrak{S}(\Lambda_g) = \{0 = \lambda_0 < \lambda_1 \le \dots\}$ be the Steklov spectrum and L be the length of γ in the metric g. Then (4.4) is valid. The equality in (4.4) holds if and only if there exist a diffeomorphism $\Phi : D \to D$ and function $0 < \rho \in C^\infty(D)$ such that $\rho|_\gamma = 1$ and $g = (\frac{L}{2\pi}\rho)^2 \Phi^* e$.

5. Two classes of Riemannian metrics

In the rest of the text we use the following notations. $L^2(\gamma)$ is the space of square integrable complex functions on γ with the norm and scalar product defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \bar{g}(e^{i\theta})\, d\theta, \quad \|f\|^2 = \langle f, f \rangle.$$

For $f \in L^2(\gamma)$, we denote by f_n the n-th Fourier coefficient,

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta.$$

For $1 \le m \in \mathbb{N}$, we denote by $H^m(\gamma)$ the Hilbert space

$$H^m(\gamma) = \Big\{ f \in L^2(\gamma) \mid \sum_{n\in\mathbb{Z}} |n|^{2m} |f_n|^2 < \infty \Big\}$$

endowed with the scalar product defined by

$$\langle f, g\rangle_{H^m} = \sum_{n\in\mathbb{Z}} (1+n^2)^m f_n \bar{g}_n \quad \text{for} \quad f, g \in H^m(\gamma).$$

Then $\Lambda_e(e^{in\theta}) = |n|e^{in\theta}$ for $n \in \mathbb{Z}$, and $\mathfrak{S}(\Lambda_e) = \mathbb{N}$ where 0 is a simple eigenvalue while others eigenvalues are double. We will denote by e_n the function $e_n(e^{i\theta}) = e^{in\theta}$ for $n \in \mathbb{Z}$.

5.1. General statements on the eigenvalue spectrum. Given a function $0 < a \in C^\infty(\gamma)$, observe that $a^{-1}\Lambda_e = a^{-\frac{1}{2}}(a^{-\frac{1}{2}}\Lambda_e a^{-\frac{1}{2}})a^{\frac{1}{2}}$, where $a^{-\frac{1}{2}}\Lambda_e a^{-\frac{1}{2}} : H^1(\gamma) \to L^2(\gamma)$ is a positive self-adjoint operator, i.e., $\langle a^{-\frac{1}{2}}\Lambda_e a^{-\frac{1}{2}} u, u\rangle \ge 0$ for $u \in H^1(\gamma)$. Then $\mathfrak{S}(a^{-1}\Lambda_e) = \mathfrak{S}(a^{-\frac{1}{2}}\Lambda_e a^{-\frac{1}{2}})$ consists of an increasing sequence of eigenvalues λ_n $(n = 0, 1, 2, \dots)$ counted with multiplicities, where $\lambda_0 = 0$ is a simple eigenvalue and every λ_n is of a finite multiplicity. The multiplicity of λ_1 is not greater than 2, see [**6, 8**]. The fact that $\lambda_0 = 0$ is simple follows from the equality $\Lambda_e 1 = 0$ and $\|a^{-1}\Lambda_e u\| \ge \frac{\|\Lambda_e u\|}{\max a} \ge \frac{\|u-u_0\|}{\max a}$ for $u \in H^1(\gamma)$.

Every eigenspace $\ker(a^{-1}\Lambda_e - \lambda_n)$ consists of smooth functions and can be spanned by real-valued functions. Indeed, if $u \in C^\infty(\gamma)$ is an eigenvector for $a^{-1}\Lambda_e$, then the complex conjugate function $\bar{u}$ is also an eigenvector, and $a^{-1}\Lambda_e(\Re u) = \lambda \Re u$ and $a^{-1}\Lambda_e(\Im u) = \lambda \Im u$.

The spectrum of $a^{-1}\Lambda_e$ is generically simple (see [**1, 16**]) which means, in particular, that there exists a dense subset $\mathcal{S}$ of $\{0 < a \in C^\infty(\gamma)\}$ endowed with its usual topology such that all eigenvalues of $a^{-1}\Lambda_e$ are simple for any $a \in \mathcal{S}$.

We will now describe the eigenvalue spectrum of $a^{-1}\Lambda_e$ for two classes of functions a, first when $a(e^{i\theta}) = a_0 + 2a_1 \cos\theta$ for $(a_0, a_1) \in \mathbb{R}^2$, $0 < 2|a_1| < a_0$, and then when $a(e^{i\theta})^{-1} = b_0 + 2b_1 \cos(m\theta)$ for $(b_0, b_1, m) \in \mathbb{R}^2 \times \mathbb{N}$, $0 < 2|b_1| < b_0$ and $m \ge 2$.

When $a(e^{i\theta})^{-1} = b_0 + 2b_1 \cos\theta$ for $(b_0, b_1) \in \mathbb{R}^2$, $0 < 2|b_1| < b_0$, then a is e-conformally equivalent to the constant-valued function a_0:

$$a_0 \left| \frac{d\varphi}{d\theta} \right| = a, \tag{5.1}$$

where $\varphi(e^{i\theta}) = \exp\Big(i a_0^{-1} \int_0^\theta a(e^{is})ds\Big) = \frac{z-r}{1-rz}\Big|_{z=e^{i\theta}}$ and $r \in (-1, 1)$ is a root of the polynomial $b_1X^2 + b_0X + b_1$ (see also [**13**, Remark 3]). Hence in that case $\mathfrak{S}(a^{-1}\Lambda_e) = \mathfrak{S}(a_0^{-1}\mathbb{N}) = a_0^{-1}\mathbb{N}$, where 0 is a simple eigenvalue and all others are double eigenvalues.

5.2. Simple eigenvalues. First we exhibit an example of a class of functions a that belongs to $\mathcal{S}$ defined above. We have the following result.

THEOREM 5.1. *All eigenvalues of $a^{-1}\Lambda_e$ are simple when $a(e^{i\theta}) = a_0 + 2a_1 \cos\theta$ for $(a_0, a_1) \in \mathbb{R}^2$, $0 < 2|a_1| < a_0$.*

The proof of Theorem 5.1 is given in the appendix. It is based on the following reformulation of the eigenvalue problem for the operator $a^{-1}\Lambda_e$. For $\lambda \in (0, +\infty)$ and $u \in C^\infty(\gamma)$, the equation $\Lambda_e u = \lambda a u$ holds if and only if the Fourier coefficients

$(u_n)_{n\in\mathbb{Z}}$ of the functions u satisfy the following linear second order recurrence with non-constant coefficients

$$-a_1u_{n+1} + (\frac{|n|}{\lambda} - a_0)u_n - a_1u_{n-1} = 0 \quad \text{for} \quad n \in \mathbb{Z}.$$

We introduce some notations before studying the second class of functions a. Given $0 < a \in C^\infty(\gamma)$ and $2 \le m \in \mathbb{N}$, define the holomorphic function $\Phi(z) = z^m$ on $\operatorname{int} D$ and set $\varphi = \Phi|_\gamma$. Then $|\Phi'|\big|_\gamma = m$ and Lemma 3.3 implies

$$(a\circ\varphi)^{-1}\Lambda_e\varphi^* f = m\varphi^*\big(a^{-1}\Lambda_e f\big) \quad \text{for} \quad f \in C^\infty(\gamma). \tag{5.2}$$

Let $t_m(a)$ denote $a\circ\varphi$. For $j = 0,\dots,m-1$, we denote by $V_{j,m}$ (resp. $\tilde V_{j,m}$) the closure in $H^1(\gamma)$ (resp. $L^2(\gamma)$) of the vector space spanned by the vectors e_{j+km} ($k \in \mathbb{Z}$). For $j = 1,\dots,[\frac{m}{2}]$, we denote by $W_{j,m}$ (resp. $\tilde W_{j,m}$) the vector space $V_{j,m}\bigoplus V_{m-j,m}$ (resp. $\tilde V_{j,m}\bigoplus \tilde V_{m-j,m}$). Note that $W_{j,m}$ is invariant under the complex conjugation. If m is even, then $W_{\frac{m}{2},m} = V_{\frac{m}{2},m}$.

Given a bounded linear operator $B : H^1(\gamma) \to L^2(\gamma)$, we denote by $B|_{V_{j,m}}$ (resp. $B|_{W_{j,m}}$) the restriction of B to $V_{j,m}$ (resp. $W_{j,m}$) for every $0 \le j \le m-1$. The operator Λ_e maps $V_{j,m}$ to $\tilde V_{j,m}$ and the multiplication operators $t_m(a)^{-\frac12}$ and $t_m(a)^{\frac12}$ map $V_{j,m}$ to $V_{j,m}$. Therefore $\Lambda_e|_{V_{j,m}}$, $t_m(a)^{-\frac12}\Lambda_e t_m(a)^{-\frac12}|_{V_{j,m}}$, and $t_m(a)^{-1}\Lambda_e|_{V_{j,m}}$ define unbounded operators in $\tilde V_{j,m}$ where their domain is $V_{j,m}$. The same statement is valid when $V_{j,m}$ and $\tilde V_{j,m}$ are replaced by $W_{j,m}$ and $\tilde W_{j,m}$ respectively. We also have

$$\begin{aligned}\mathfrak{S}((t_m(a))^{-1}\Lambda_e) &= \bigcup_{j=0}^{m-1}\mathfrak{S}((t_m(a))^{-1}\Lambda_e|_{V_{j,m}})\\ &= \mathfrak{S}((t_m(a))^{-1}\Lambda_e|_{V_{0,m}})\bigcup_{j=1}^{[\frac{m}{2}]}\mathfrak{S}((t_m(a))^{-1}\Lambda_e|_{W_{j,m}}).\end{aligned}$$

The following statement is proved along the same lines as Theorem 5.1. The proof is presented in the appendix.

THEOREM 5.2. *Given integers $m \ge 2$ and $1 \le j \le m-1$, let $a(e^{i\theta}) := (b_0 + 2b_1\cos\theta)^{-1}$ for some $(b_0, b_1) \in \mathbb{R}^2$ satisfying $b_0 > 2|b_1| > 0$. All eigenvalues of $(t_m(a))^{-1}\Lambda_e|_{V_{j,m}}$ are simple.*

On using the invariance of $t_m(a)^{-1}\Lambda_e$ with respect to the complex conjugation, we obtain the corollary: in notations of Theorem 5.2, positive eigenvalues of the operator $(t_m(a))^{-1}\Lambda_e|_{W_{j,m}}$ ($1 \le j < m/2$) are double, and these eigenvalues of the operator $(t_m(a))^{-1}\Lambda_e$ are at least double.

5.3. Double eigenvalues.

THEOREM 5.3. *For $(b_0, b_1) \in \mathbb{R}^2$ satisfying $b_0 > 2|b_1| > 0$, let $a(e^{i\theta}) := (b_0 + 2b_1\cos\theta)^{-1}$. Then*

(1) *$\mathfrak{S}((t_m(a))^{-1}\Lambda_e|_{V_{0,m}}) = ma_0^{-1}\mathbb{N}$ for every integer $m \ge 2$.*

(2) *For integers $m \ge 2$ and $k > 0$, mka_0^{-1} is a double eigenvalue of the operator $(t_m(a))^{-1}\Lambda_e$.*

The first statement of the theorem follows immediately from (5.2) and (5.1). The second statement will be proved in Section 6.2. The proof is based on the reformulation of the eigenvalue problem for the operator $(t_m(a))^{-1}\Lambda_e$ as a scalar Riemann – Hilbert problem, see Section 6.

In the case of $m = 2$, we have the following more precise statement:

Corollary 5.4. In notations of Theorem 5.3,

$$\mathfrak{S}((t_2(a))^{-1}\Lambda_e) = \bigcup_{k=1}^{\infty}\{\lambda_{k,-},\lambda_{k,+}\}\bigcup(2a_0^{-1}\mathbb{N}).$$

For $k>0$, $2ka_0^{-1}$ is a double eigenvalue while $\lambda_{k,-}$ and $\lambda_{k,+}$ are simple eigenvalues of the operator $(t_2(a))^{-1}\Lambda_e$. The following inequalities hold

$$(2k-2)a_0^{-1} < \lambda_{k,-} < \lambda_{k,+} < 2ka_0^{-1} \quad \text{for} \quad k>0. \tag{5.3}$$

Corollary 5.4 follows from Theorems 5.2 and 5.3 taking into account the continuity of eigenvalues with respect to a continuous perturbation of $(t_2(a))^{-1}\Lambda_e$, see [**7**]. More precisely, let $a(\varepsilon, e^{i\theta}) = a_0\sqrt{b_0^2 - 4\varepsilon^2 b_1^2}\,(b_0 + 2\varepsilon b_1\cos(2\theta))^{-1}$ for $\varepsilon\in(0,1]$. Then $a_0(\varepsilon,.) = a_0$ and eigenvalues of $(a(\varepsilon,.)^{-1}\Lambda_e)|_{V_{1,2}}$ are simple by Theorem 5.2. Let $\{\lambda_{\pm,\varepsilon,k}\mid k\in\mathbb{N}\}$ be the set of positive eigenvalues of the latter operator ordered so that $\lambda_{-,\varepsilon,k} < \lambda_{+,\varepsilon,k} < \lambda_{-,\varepsilon,k+1}$. Then, by continuity of eigenvalues with respect to ε, we have $\lambda_{\pm,\varepsilon,k}\to(2k+1)a_0^{-1}$ as $\varepsilon\to 0^+$. With the help of Theorem 5.3, this implies (5.3).

Quite similarly, for an integer $m\geq 3$, we can prove: mka_0^{-1} is a double eigenvalue of the operator $(t_m(a))^{-1}\Lambda_e$ for every $0<k\in\mathbb{N}$ and

$$\mathfrak{S}((t_m(a))^{-1}\Lambda_e)\backslash(ma_0^{-1}\mathbb{N}) = \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{2m}\{\lambda_{k,j}\}, \quad \text{where} \quad ma_0^{-1}(k-1) < \lambda_{k,j} < ma_0^{-1}k.$$

We also mention a possible direction for a generalization of Corollary 5.4 at the end of Section 6.3.

To authors' knowledge, no bounded simply connected domain is known with a Steklov eigenvalue of multiplicity greater than 2.

5.4. The Hilbert transform and an integral equation. Observe that (see for example [**12**])

$$\Lambda_e = \sqrt{-\frac{d^2}{d\theta^2}} = H\circ\frac{d}{d\theta} = \frac{d}{d\theta}\circ H,$$

where H is the Hilbert transform on $L^2(\gamma)$ defined by

$$(Hu)(e^{i\theta}) = i^{-1}\sum_{n\in\mathbb{Z},\ n\neq 0}\operatorname{sgn}(n)u_n e^{in\theta} = (2\pi)^{-1}\text{p.v.}\int_0^{2\pi}\frac{1+\cos(\theta-t)}{\sin(\theta-t)}u(e^{it})\,dt. \tag{5.4}$$

In particular, $H(\chi_\theta)(e^{it}) = \frac{1}{\pi}(-\ln)\left(\left|\sin\left(\frac{\theta-t}{2}\right)\right|/\left|\sin\left(\frac{t}{2}\right)\right|\right)$ for $(t,\theta)\in\mathbb{R}\times(0,2\pi)$, where $\chi_\theta(e^{it}) = 1$ for $t\in(0,\theta)$ and $\chi_\theta(e^{it}) = 0$ for $t\in[\theta,2\pi]$. Hence the eigenvalue problem for $a^{-1}\Lambda_e$ can be transformed into an integral equation as follows:

$$\left(a^{-1}\Lambda_e u = \lambda u\right) \Leftrightarrow u(e^{i\theta}) = u(1) + \frac{\lambda}{\pi}\int_0^{2\pi}(-\ln)\left(\left|\frac{\sin\left(\frac{\theta-t}{2}\right)}{\sin\left(\frac{t}{2}\right)}\right|\right)a(e^{it})u(e^{it})\,dt$$

for $u\in C^\infty(\gamma)$ and $\lambda\in(0,+\infty)$. We do not use the integral equation in the rest of the text. However, the Fredholm determinant of the square of the self-adjoint compact operator in $L^2([0,2\pi])$ with the nonnegative integral kernel $a^{1/2}(e^{i\theta})\big(-$

$\ln\big)\left(\left|\sin\left(\frac{\theta-t}{2}\right)\right|\right)a^{1/2}(e^{it})$ may be of some interest for the Steklov inverse spectral problem.

It is also of some interest to mention that the Steklov eigenvalues asymptotics presented in Proposition 3.2 can be derived from Proposition 3.1 on using the following fact: the commutator $[H,b]=Hb-bH$ is a smoothing operator for any $b\in C^\infty(\gamma)$. Indeed, as follows from (5.4), the kernel of $[H,b]$ is the smooth function $\mathcal{K}$ on $\gamma\times\gamma$ defined by

$$\mathcal{K}(e^{i\theta},e^{it})=(2\pi)^{-1}\cot\big(\frac{\theta-t}{2}\big)(b(e^{it})-b(e^{i\theta}))\quad\text{for}\quad(\theta,t)\in[0,2\pi]\times[0,2\pi].$$

Hence

$$(a^{-1/2}\Lambda_e a^{-1/2})^2=-\Big(a^{-1/2}\frac{d}{d\theta}a^{-1}\frac{d}{d\theta}a^{-1/2}\Big)+a^{-1/2}\frac{d}{d\theta}[H,a^{-1}]\Lambda_e a^{-1/2}\tag{5.5}$$

and $a^{-1/2}\frac{d}{d\theta}[H,a^{-1}]\Lambda_e a^{-1/2}$ is a smoothing operator.

Using (5.5) and the min-max principle, one obtains the following asymptotics for eigenvalues and eigenvectors of $a^{-1/2}\Lambda_e a^{-1/2}$. Define

$$v_{\pm,n}(e^{i\theta})=\Big(\frac{a(e^{i\theta})}{a_0}\Big)^{1/2}\exp\left(\pm i\left[\frac{n+1}{2}\right]\int_0^\theta\frac{a(e^{is})}{a_0}\,ds\right)$$

and let $(u_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{S}^1)$ such that $a^{-1/2}\Lambda_e a^{-1/2}u_n=\lambda_n u_n$. Then

$$\lambda_n=a_0^{-1}\left[\frac{n+1}{2}\right]+O(n^{-N})\quad\text{as}\quad n\to+\infty,$$
$$u_n-\langle u_n,v_{+,n}\rangle v_{+,n}-\langle u_n,v_{-,n}\rangle v_{-,n}=O(n^{-N})\quad\text{as}\quad n\to+\infty$$

for every $N>0$. The multiplicity of the eigenvalue λ_n is at most 2 for n large enough, see [**8**, Proposition 1.4.2].

In the case of $a_0=1$, the operator $a^{-1/2}\Lambda_e a^{-1/2}-h^*\Lambda_e(h^{-1})^*$ is a smoothing operator, where the diffeomorphism $h:\gamma\to\gamma$ is defined by $h(e^{i\theta})=\exp\Big(i\int_0^\theta a(e^{is})\,ds\Big)$, see [**3**].

In the next section, we relate the eigenvalue problem for $a^{-1}\Lambda_e$ to a scalar Riemann – Hilbert problem. The relationship will be used in our proof of Theorem 5.3, but it is also of some independent interest. We also refer the reader to [**11**, pp. 182] for a connection between Fredholm operators and Riemann – Hilbert problems.

6. A scalar Riemann – Hilbert problem

6.1. Preliminaries. In this paragraph we obtain equivalent formulations of the eigenvalue problem for $a^{-1}\Lambda_e$ as a scalar Riemann-Hilbert problem.

Let $\mathcal{H}(O)$ be the space of holomorphic functions on an open set $O\subset\mathbb{C}$. Given $u\in C^\infty(\gamma)$, define the functions $u_+\in\mathcal{H}(\mathrm{int}\,D)\cap C^\infty(D)$ and $u_-\in\mathcal{H}(\mathbb{C}\backslash D)\cap C^\infty(\mathbb{C}\backslash\mathrm{int}\,D)$ by $u_+(z)=u_0/2+\sum_{n=1}^{+\infty}u_nz^n$ and $u_-(z)=u_0/2+\sum_{n=1}^{+\infty}u_{-n}z^{-n}$ respectively. Then, for $z=e^{i\theta}$, $u(z)=u_+(z)+u_-(z)$ and

$$\Lambda_e u(z)=z\frac{d(u_+-u_-)}{dz}(z)=i^{-1}\frac{du_+}{d\theta}(e^{i\theta})-i^{-1}\frac{du_-}{d\theta}(e^{i\theta}).\tag{6.1}$$

If u is a real function, then $u_n=\bar{u}_{-n}$ and $u_-(z)=\overline{u_+(\bar{z}^{-1})}$.

On using (6.1) and the complex conjugation invariance of $a^{-1}\Lambda_e$, we obtain for a real valued function $u \in C^\infty(\gamma)$ and $\lambda \in \mathbb{R}$

$$(6.2)\qquad \big(\Lambda_e u = \lambda a u\big) \Leftrightarrow \left(\Re\left(z\frac{du_+}{dz}(z) - \lambda a(z)u_+(z)\right) = 0 \quad \textit{for} \quad z\in\gamma\right).$$

Remark 6.1. Statement (6.2) can be written in two other equivalent forms. First, for a complex function $u \in C^\infty(\gamma)$,

$$\big(\Lambda_e u = \lambda a u\big) \Leftrightarrow \left(z\frac{du_+}{dz}(z) - \lambda a(z)u_+(z) = z\frac{du_-}{dz}(z) + \lambda a(z)u_-(z) \quad \textit{for} \quad z\in\gamma\right).$$

Second, given a complex function $u \in C^\infty(\gamma)$, let $v_\pm$ be the 2π-periodic and bounded function on the closed half-plane $\{x+iy \mid \pm y \geq 0\}$ defined by $v_\pm(z) = u_0/2 + \sum_{n=1}^{+\infty} u_{\pm n}e^{inz}$. Then

$$\big(\Lambda_e u = \lambda a u\big) \Leftrightarrow \left(\frac{dv_+}{dx}(x) - i\lambda a(x)v_+(x) = \frac{dv_-}{dx}(x) + i\lambda a(x)v_-(x) \quad \textit{for} \quad x\in\mathbb{R}\right).$$

6.2. The case of a rational function a. We are going to reformulate (6.2) when a is a rational function of $z \in \gamma$. We will need the following

Lemma 6.2. *Let $f \in \mathcal{H}(int D) \cap C(D)$ and $n > 0$. Then $\Re(f(z)/z^n) = 0$ for all $z \in \gamma$ if and only if there exists a sequence $(\alpha_0, \dots, \alpha_n) \in \mathbb{R}\times\mathbb{C}^n$ such that $f(z) = i\alpha_0 z^n + \sum_{j=1}^n(\alpha_j z^{n+j} - \bar\alpha_j z^{n-j})$ for $z \in D$.*

For the sake of consistency, the proof of Lemma 6.2 is postponed to the appendix.

Let $(\zeta_1, \dots, \zeta_M)$ and $(\eta_1, \dots, \eta_N)$ be two sequences of points from $\operatorname{int} D\backslash\{0\}$ such that $\zeta_j \neq \eta_l$ for all j and l. Define functions p and q on $\mathbb{C}\backslash\{0\}$ by

$$(6.3)\quad p(z) = c_1 z^{-M}\Pi_{j=1}^M(z-\zeta_j)(z-\bar\zeta_j^{\,-1}), \quad q(z) = c_2 z^{-N}\Pi_{j=1}^N(z-\eta_j)(z-\bar\eta_j^{\,-1}),$$

where the constants $c_k \neq 0$ $(k = 1, 2)$ are such that $\bar c_1 = c_1\Pi_{j=1}^M(\zeta_j/\bar\zeta_j)$ and $\bar c_2 = c_2\Pi_{j=1}^N(\eta_j/\bar\eta_j)$. Then $\overline{p(\bar z^{-1})} = p(z)$ and $\overline{q(\bar z^{-1})} = q(z)$ for $z \in \mathbb{C}\backslash\{0\}$. The functions p and q are real-valued on γ and do not vanish at any point of γ. Next, set

$$(6.4)\qquad a(z) = \frac{p(z)}{q(z)} \quad \text{for} \quad z \in \mathbb{C}\backslash(Z\cup\{0\}),$$

where Z is the set of zeros of q. For so chosen a, (6.2) takes the form

$$\big(\Lambda_e u = \lambda a u\big) \Leftrightarrow \left(\Re\left(zq(z)\frac{d}{dz}u_+(z) - \lambda p(z)u_+(z)\right) = 0 \quad \textit{for} \quad z\in\gamma\right).$$

With the help of Lemma 6.2, this implies the statement: λ is an eigenvalue of $a^{-1}\Lambda_e$ if and only if there exist a function $u_+ \in \mathcal{H}(\operatorname{int} D)\cap C^\infty(D)$ and sequence $(\alpha_0, \dots, \alpha_s) \in \mathbb{R}\times\mathbb{C}^s$ such that

$$(6.5)\quad zq(z)\frac{du_+}{dz}(z) - \lambda p(z)u_+(z) = i\alpha_0 + \sum_{j=1}^s(\alpha_j z^j - \bar\alpha_j z^{-j}) \quad \text{for} \quad z \in D\backslash\{0\},$$

where $s = \max(M, N-1)$. This is a linear first order differential equation with rational coefficients and rational right-hand side. Therefore the solution u_+ admits a holomorphic continuation to $\mathbb{C}\backslash Z_-$, where Z_- denotes the set of zeros of q which are outside D. If $u_+ \in \mathcal{H}(\mathbb{C}\backslash Z_-)$ solves equation (6.5) and c is a real constant, then $u_+ + ic$ solves a similar equation that differs from (6.5) by values of parameters $(\alpha_0, \dots, \alpha_s) \in \mathbb{R} \times \mathbb{C}^s$ only. Since the integral $p_0 = (2\pi^{-1}) \int_0^{2\pi} p(e^{i\theta})\, d\theta$ is real and nonzero, we can choose the constant c such that $\alpha_0 = 0$ in the corresponding equation. We have thus proved

LEMMA 6.3. *Let the function $a \in C^\infty(\gamma)$ be defined by (6.3) and (6.4). A real function $u \in C^\infty(\gamma)$ is an eigenvector of $a^{-1}\Lambda_e$ with eigenvalue λ if and only if there exists $(\alpha_1, \dots, \alpha_s) \in \mathbb{C}^s$ such that*

$$zq(z)\frac{du_+}{dz}(z) - \lambda p(z)u_+(z) = \sum_{j=1}^{s}(\alpha_j z^j - \bar{\alpha}_j z^{-j}) \quad \textit{for} \quad z \in \mathbb{C}\backslash\big(Z_- \cup \{0\}\big), \tag{6.6}$$

where $s = \max(M, N-1)$ and Z_- is the set of zeros of q which are outside D.

REMARK 6.4. If $N = 0$, then q is a nonzero constant function, $Z_- = \emptyset$, and u_+ in (6.6) is an entire function. If $s = 0$, then the right-hand side of (6.6) is zero.

6.3. Proof of Theorem 5.3. Let $a(e^{i\theta}) = (b_0 + 2b_1\cos(m\theta))^{-1}$ for some $(b_0, b_1) \in \mathbb{R}^2$ satisfying $b_0 > 2|b_1| > 0$ and $m \in \mathbb{N}$. We can set $b_0 = 1$ and replace b_1 by some $b \in (0, \frac{1}{2})$ without loss of generality. Then a admits a holomorphic continuation given by

$$a(z) = \frac{z^m}{b + z^m + bz^{2m}} \quad \text{for} \quad z \in \mathbb{C}\backslash Z_m, \tag{6.7}$$

where $Z_m := \{\omega e^{i\frac{2j\pi}{m}},\ \bar{\omega}^{-1}e^{i\frac{2j\pi}{m}} \mid 1 \le j \le m\}$ and ω is a root in D of the polynomial $b + X^m + bX^{2m}$.

Formula (6.7) is the partial case of (6.4) for $p(z) = 1$ and $q(z) = bz^{-m} + 1 + bz^m$. We denote by $Z_{m,-}$ the set $\{\bar{\omega}^{-1}e^{i\frac{2j\pi}{m}} \mid 1 \le j \le m\}$. Lemma 6.3 implies: a real $u \in C^\infty(\gamma)$ satisfies $a^{-1}\Lambda_e u = \lambda u$ if and only if there exists $(\alpha_1, \dots, \alpha_{m-1}) \in \mathbb{C}^{m-1}$ such that

$$(b+z^m+bz^{2m})\frac{du_+}{dz}(z) - \lambda z^{m-1}u_+(z) = \sum_{l=1}^{m-1}(\alpha_l z^{m-1+l} - \bar{\alpha}_l z^{m-1-l}) \quad \text{for} \quad z \in \mathbb{C}\backslash Z_{m,-}. \tag{6.8}$$

REMARK 6.5. Since $W_{j,m}$ $(1 \le j < m/2)$ is spanned by $e_{\pm j+nm}$ for $n \in \mathbb{Z}$, we obtain: if λ is an eigenvalue for $a^{-1}\Lambda_e|_{W_{j,m}}$, we can set $\alpha_l = 0$ for $l \notin \{j, m-j\}$ on the right-hand side of (6.8). If λ is an eigenvalue for $a^{-1}\Lambda_{e|W_{m/2,m}}$ and m is even, we can set $\alpha_l = 0$ for $l \ne m/2$ on the right-hand side of (6.8).

Let $r \in (-1, 0)$ be a root of the polynomial $b + X + bX^2$ (we recall that $a_0^{-1} = (1 - 4b^2)^{1/2}$). When $m = 1$ and $\lambda = ka_0^{-1}$ for some positive integer k, equation (6.8) becomes $du_+(z)/dz - k\big(a_0 b(z-r)(z-r^{-1})\big)^{-1}u_+(z) = 0$ or

$$\frac{1}{i}\frac{d}{d\theta}u_+(e^{i\theta}) - \frac{k}{a_0(1 + 2b\cos\theta)}u_+(e^{i\theta}) = 0.$$

Integrating the equation and using $\big(a_0 b(z-r)(z-r^{-1})\big)^{-1} = (z-r)^{-1} - (z-r^{-1})^{-1}$, we obtain

$$u_+(z) = C\left(\frac{z-r}{1-rz}\right)^k \quad \text{for} \quad z \ne r^{-1}, \quad u_+(e^{i\theta}) = C\exp\left(\frac{ik}{a_0}\int_0^\theta \frac{ds}{1+2b\cos s}\right)$$

with a complex constant C. This implies for u being a real eigenfunction of $a^{-1}\Lambda_e|_{V_{0,m}}$ related to mka_0^{-1}

$$u_+(z) = C\left(\frac{z^m - r}{z^m - r^{-1}}\right)^k \text{ for } z \in \mathbb{C}\backslash Z_{m,-}. \tag{6.9}$$

Given $(\alpha_1, \ldots, \alpha_{m-1}) \in \mathbb{C}^{m-1}$, let us consider the first order differential equation

(6.10)

$$(b+z^m+bz^{2m})\frac{dv_+}{dz}(z) - kma_0^{-1}z^{m-1}v_+(z) = \sum_{j=1}^{m-1}(\alpha_j z^{m-1+j} - \bar{\alpha}_j z^{m-1-j}),\ z \in \mathbb{C}\backslash Z_{m,-}.$$

We are going to prove that the existence of a holomorphic solution v_+ on $\mathbb{C}\backslash Z_{m,-}$ implies that $\alpha_l = 0$ $(1 \leq l \leq m-1)$. Together with (6.9), this will give us the statement of Theorem 5.3.

Assume the existence of a solution $v_+ \in \mathcal{H}(\mathbb{C}\backslash Z_{m,-})$ to equation (6.10). We use the variation of constants. Define the function C by

$$C(z) = u_+(z)^{-1}v_+(z) = \left(\frac{z^m - r^{-1}}{z^m - r}\right)^k v_+(z) \text{ for } z \in \mathbb{C}\backslash Z_m, \tag{6.11}$$

where $u_+(z) = \left(\frac{z^m - r}{z^m - r^{-1}}\right)^k$. The function C is holomorphic on $\mathbb{C}\backslash Z_m$. Hence using (6.10) and (6.11) we obtain

$$(b+z^m+bz^{2m})u_+(z)\frac{d}{dz}C(z) = \sum_{j=1}^{m-1}(\alpha_j z^{m-1+j} - \bar{\alpha}_j z^{m-1-j}) \text{ for } z \in \mathbb{C}\backslash Z_m. \tag{6.12}$$

We set

(6.13)

$$R(z) = \frac{d}{dz}C(z) = b^{-1}\frac{(z^m - r^{-1})^{k-1}}{(z^m - r)^{k+1}}\sum_{j=1}^{m-1}(\alpha_j z^{m-1+j} - \bar{\alpha}_j z^{m-1-j}) \text{ for } z \in \mathbb{C}\backslash Z_m.$$

As follows from (6.13), there exists a constant δ such that $|R(z)| \leq \delta|z|^{-2}$ for $z \in \mathbb{C}$, $|z| \geq 2(-r)^{1/m}$.

Since the rational function R is the complex derivative of a holomorphic function on $\mathbb{C}\backslash Z_m$, the residue of R at each point of Z_m is equal to zero. Therefore R integrates to zero over the curve $\gamma_l : \mathbb{R} \to \mathbb{C}$ $(0 \leq l \leq m-1)$ defined by $\gamma_l(t) = -te^{i\frac{2\pi l}{m}}$ for $t \leq 0$ and $\gamma_l(t) = te^{\frac{i2\pi(l+1)}{m}}$ for $t > 0$:

$$\int_0^{+\infty} R\big(te^{i\frac{2\pi l}{m}}\big)\,dt - e^{i\frac{2\pi}{m}}\int_0^{+\infty} R\big(te^{i\frac{2\pi(l+1)}{m}}\big)\,dt = 0. \tag{6.14}$$

We remind that $-r > 0$ and $z^m - r \neq 0$ for any point z of γ_l. As is seen from (6.13), R is integrable on γ_l and $\rho\int_0^{2\pi} R(\rho e^{i\theta})d\theta = O(\rho^{-1})$ as $\rho \to +\infty$.

Using (6.12) and (6.14), we obtain the following linear system of equation on $(\alpha_1, \ldots, \alpha_{m-1})$:

(6.15)

$$\sum_{j=1}^{m-1}\Big(\alpha_j e^{i\frac{2\pi l(j-1)}{m}}(1 - e^{i\frac{2\pi j}{m}})I_{m-1+j}(-r) - \bar{\alpha}_j e^{i\frac{2\pi l(-j-1)}{m}}(1 - e^{-i\frac{2\pi j}{m}})I_{m-1-j}(-r)\Big) = 0$$

for $0 \le l \le m-1$, where

$$I_{l'}(\sigma) = \int_0^{+\infty} \frac{(\sigma t^m + 1)^{k-1}}{(t^m+\sigma)^{k+1}} t^{l'}\, dt \text{ for } (\sigma, l') \in (0,1)\times\mathbb{N},\ 0 \le l' \le 2m-2. \tag{6.16}$$

From (6.15) we obtain

$$\sum_{j=1}^{m-1} (\beta^{l(j-1)} - \beta^{lj})\,(\alpha_j I_{m-1+j}(-r) - \bar{\alpha}_{m-j} I_{j-1}(-r)) = 0 \quad \text{for } 0 \le l \le m-1,$$

where $\beta = e^{i\frac{2\pi}{m}}$. Since the Van der Monde matrix $(\beta^{j(l-1)})_{1\le j\le m, 1\le l\le m}$ is invertible, we obtain $\alpha_j I_{m-1+j}(-r) - \bar{\alpha}_{m-j} I_{j-1}(-r) = 0$ for $j = 1 \dots m-1$. In other words,

$$\alpha_j\big(I_{m-1+j}(-r) I_{2m-j-1}(-r) - I_{j-1}(-r) I_{m-j-1}(-r)\big) = 0 \quad \text{for } 1 \le j \le m-1. \tag{6.17}$$

We use the following

Lemma 6.6. *For $(\sigma, l) \in (0,1)\times\mathbb{N}$ satisfying $1 \le l \le m-1$, the inequalities $0 < I_{2m-l-1}(\sigma) < I_{l-1}(\sigma)$ hold.*

By Lemma 6.6, $I_{m-1+j}(-r) I_{2m-j-1}(-r) - I_{j-1}(-r) I_{m-j-1}(-r) < 0$ for $1 \le j \le m-1$. Therefore (6.17) implies $\alpha_j = 0$ for $j = 1 \dots m-1$. This finishes the proof of Theorem 5.3. □

Proof of Lemma 6.6. Given $\sigma \in (0,1)$ and $l \in \mathbb{N}$ satisfying $1 \le l \le m-1$, we set $\alpha = l/m \in (0,1)$. We change the integration variable $s = t^m$ on (6.16) to obtain

$$I_{2m-l-1}(\sigma) = \frac{1}{m}\int_0^{+\infty} \frac{(\sigma s+1)^{k-1}}{(s+\sigma)^{k+1}} s^{1-\alpha}\, ds, \quad I_{l-1}(\sigma) = \frac{1}{m}\int_0^{+\infty} \frac{(\sigma s+1)^{k-1}}{(s+\sigma)^{k+1}} s^{\alpha-1}\, ds.$$

The integration variable change $s := s^{-1}$ gives

$$\int_1^{+\infty} \frac{(\sigma s+1)^{k-1}}{(s+\sigma)^{k+1}} s^{1-\alpha}\, ds = \int_0^1 \frac{(\sigma+s)^{k-1}}{(1+\sigma s)^{k+1}} s^{\alpha-1}\, ds.$$

Therefore

$$\begin{aligned} I_{2m-l-1}(\sigma) &= \frac{1}{m}\int_0^1 \left(\frac{(\sigma s+1)^{k-1}}{(s+\sigma)^{k+1}} s^{1-\alpha} + \frac{(\sigma+s)^{k-1}}{(1+\sigma s)^{k+1}} s^{\alpha-1} \right) ds,\\ I_{l-1}(\sigma) &= \frac{1}{m}\int_0^1 \left(\frac{(\sigma s+1)^{k-1}}{(s+\sigma)^{k+1}} s^{\alpha-1} + \frac{(\sigma+s)^{k-1}}{(1+\sigma s)^{k+1}} s^{1-\alpha} \right) ds. \end{aligned}$$

Two last formulas imply

$$m(I_{l-1}(\sigma) - I_{2m-l-1}(\sigma)) = \int_0^1 s^{\alpha-1}(1-s^{2(1-\alpha)}) \left(\frac{(\sigma s+1)^{k-1}}{(s+\sigma)^{k+1}} - \frac{(\sigma+s)^{k-1}}{(1+\sigma s)^{k+1}} \right) ds. \tag{6.18}$$

Then note that

$$\frac{(\sigma s+1)^{k-1}}{(s+\sigma)^{k+1}} > \frac{(\sigma+s)^{k-1}}{(1+\sigma s)^{k+1}} \text{ and } (1-s^{2(1-\alpha)}) > 0 \quad \text{for } s \in (0,1) \text{ since } \alpha \in (0,1).$$

Thus the integrand on the right-hand side of (6.18) is positive and the inequality $I_{l-1}(\sigma) - I_{2m-l-1}(\sigma) > 0$ holds. □

We indicate a possible direction for a generalization of Corollary 5.4. Let $(\lambda, u) \in (0,+\infty) \times C^\infty(\gamma)$ and $1 \le j \le [m/2]$ so that $u \in W_{j,m}$. Hence $u_0 = 0$ and $u_+(0) = 0$, where u_+ is defined at the beginning of Section 6. Assume $\lambda \notin m a_0^{-1}\mathbb{N}$ and $a^{-1}\Lambda_e u = \lambda u$ for $a(e^{i\theta}) = (1+2b\cos(m\theta))^{-1}$. Then, using Remark 6.5, we see that $u_+ \in \mathcal{H}(\mathbb{C}\backslash Z_{m,-})$ and

$$(b+z^m+bz^{2m})\frac{d}{dz}u_+(z) - \lambda z^{m-1}u_+(z) = \alpha_{m-j}z^{2m-1-j} + \alpha_j z^{m-1+j} - \bar{\alpha}_{m-j}z^{j-1} - \bar{\alpha}_j z^{m-1-j}$$

for $z \in \mathbb{C}\backslash Z_{m,-}$ with some $(\alpha_j, \alpha_{m-j}) \in \mathbb{C}^2$. Integrate the equation to obtain
(6.19)

$$\begin{aligned} u_+(z) = z a_0 \int_0^1 & \left(\frac{(tz)^m - r^{-1}}{(tz)^m - r}\right)^{\frac{\lambda a_0}{m}} \Big(\frac{1}{(tz)^m - r^{-1}} - \frac{1}{(tz)^m - r}\Big) \\ & \times \Big(\alpha_{m-j}(tz)^{2m-1-j} + \alpha_j (tz)^{m-1+j} - \bar{\alpha}_{m-j}(tz)^{j-1} - \bar{\alpha}_j (tz)^{m-1-j}\Big) dt \end{aligned}$$

for $z \in \mathbb{C}\backslash K$, where $K = \bigcup_{l=1}^{m} \Big(e^{i\pi/m + 2\pi(l-1)/m}\Big[(-r)^{1/m}, (-r)^{-1/m}\Big]\Big)$. In (6.19), we have actually used the following branch of the logarithm: $\ln(z) = \ln|z| + i\arg z$ for $z \in \mathbb{C}\backslash(-\infty, 0]$ and $\arg z \in (-\pi, \pi)$. As is seen from (6.19), $u_+ \in \mathcal{H}(\mathbb{C}\backslash K)$. The condition $u_+ \in \mathcal{H}(\mathbb{C}\backslash Z_{m,-})$ gives
(6.20)

$$\begin{aligned} 0 &= u_+(s^{\frac{1}{m}}\omega_l e^{i0^+}) - u_+(s^{\frac{1}{m}}\omega_l e^{i0^-}) \\ &= \frac{2}{\lambda}\sin\left(\frac{\lambda a_0 \pi}{m}\right)\left(\frac{s-1}{r^{-2}-s}\right)^{\frac{\lambda a_0}{m}} \Big[\alpha_{m-j}\omega_l^{m-j} c_k(1-\frac{j}{m}, \mu) \\ &\qquad + \alpha_j \omega_l^{j} c_k(\frac{j}{m}, \mu) - \bar{\alpha}_{m-j}\omega_l^{j-m} c_k(\frac{j}{m} - 1, \mu) - \bar{\alpha}_j \omega_l^{-j} c_k(-\frac{j}{m}, \mu)\Big] \end{aligned}$$

for $s \in (1, r^{-2})$ and for $0 \le l \le m-1$, where $\omega_l = e^{i\pi/m + i2\pi l/m}(-r)^{1/m}$, $k = [\lambda a_0/m]$, and $\mu = \lambda - m a_0^{-1} k$. The function c_k is smooth on $(-1,1)\times(0,1)$ and

$$c_0(\alpha, \mu) = r^{-2\mu} + \alpha\int_0^1 \left(\Big(\frac{r^{-2} - t}{1-t}\Big)^{\mu} - r^{-2\mu}\right) t^{\alpha-1} dt, \quad (\alpha, \mu) \in (-1,1)\times(0,1).$$

We also define for $(\alpha, \mu) \in (0, \frac{1}{2}] \times (0,1)$

$$F_k(\alpha, \mu) = c_k(\alpha - 1, \mu)c_k(-\alpha, \mu) - r^2 c_k(1-\alpha, \mu)c_k(\alpha, \mu).$$

For fixed $k \in \mathbb{N}$ and $\mu \in (0,1)$, the linear system of equations (6.20) has a nontrivial solution $(\alpha_j, \alpha_{m-j}) \in \mathbb{C}^2$ if and only if $F_k(\alpha, \mu) = 0$. We therefore obtain, for $\mu \in (0,1)$ and $(j,k) \in \mathbb{N}^2$ $(1 \le j \le [m/2])$,

$$m k a_0^{-1} + \mu \in \mathfrak{S}(a^{-1}\Lambda_{e|W_{j,m}}) \Leftrightarrow F_k(\frac{j}{m}, \mu) = 0.$$

Finally, studying growth properties of functions $F_k(., \mu)$ at fixed $\mu \in (0,1)$, one may get some knowledge of multiplicities of eigenvalues of $a^{-1}\Lambda_e$ which belong to the interval $(mka_0^{-1}, m(k+1)a_0^{-1})$. More precisely, if the inequality $\frac{\partial F_0}{\partial \alpha}(\alpha, \mu) > 0$ was proved, then one would obtain the statement: the eigenvalues of $a^{-1}\Lambda_e|_{W_{j,m}}$ $(1 \le j < m/2)$ belonging to $(0, m a_0^{-1})$ are double eigenvalues of the operator $a^{-1}\Lambda_e$; and for even m, the eigenvalues of $a^{-1}\Lambda_{e|W_{\frac{m}{2},m}}$ belonging $(0, m a_0^{-1})$ are simple eigenvalues of $a^{-1}\Lambda_e$.

Appendix A. Proofs of Theorems 5.1 and 5.2 and of Lemma 6.2

To prove Theorem 5.1, we need the following lemma that will be proved at the end of the section.

Lemma A.1. *Given $(\lambda, a_0, a_1) \in (0, +\infty)^2 \times \mathbb{C}$ satisfying $|a_1| > 0$, let G_λ denote the two-dimensional complex vector space of complex-valued sequences $(v_n)_{n\in\mathbb{N}}$ satisfying*

$$-\lambda a_1 v_{k+1} + (k - \lambda a_0) v_k - \lambda a_1 v_{k-1} = 0 \tag{A.1}$$

for $k \geq 1$. Let also $\tilde{G}_\lambda$ be the subspace of G_λ consisting of sequences satisfying $\sum_{k=0}^{+\infty} (1 + k^2)|v_k|^2 < \infty$. Then the dimension of the vector space $\tilde{G}_\lambda$ is at most one.

Proof of Theorem 5.1. We remind that 0 is a simple eigenvalue of $a^{-1}\Lambda_e$. If $v \in C^\infty(\gamma)$ solves the equation $\Lambda_e v = \lambda a v$ with $\lambda > 0$, Fourier coefficients of v satisfy (A.1) for all $k \in \mathbb{Z}$. Hence $v_{k-1} = \lambda^{-1} a_1^{-1}(-\lambda a_1 v_{k+1} + (k - \lambda a_0) v_k)$ $(k \leq 1)$; and $v \equiv 0$ if $v_n = 0$ for all $n \geq 1$. We have thus the embedding $f : \ker(a^{-1}\Lambda_e - \lambda) \to \tilde{G}_\lambda$ defined by $f(v) = (v_n)_{n\in\mathbb{N}}$. This implies with the help of Lemma A.1 $\dim_{\mathbb{C}}(\ker(a^{-1}\Lambda_e - \lambda)) \leq \dim_{\mathbb{C}}(\tilde{G}_\lambda) \leq 1$. □

To prove Theorem 5.2, we need the following lemma that will be proved at the end of the section.

Lemma A.2. *Let $(\lambda, b_0, b_1, m) \in [0, +\infty) \times (0, +\infty) \times \mathbb{C} \times \mathbb{N}$ satisfy $m \geq 2$ and $b_0 > 2|b_1| > 0$. We denote by $F_{\lambda,j}$ $(1 \leq j \leq m - 1)$ the two-dimensional complex vector space of complex-valued sequences $(u_n)_{n\in\mathbb{N}}$ satisfying*

$$(mk + m + j) b_1 u_{k+1} + ((mk + j) b_0 - \lambda) u_k + (mk - m + j) b_1 u_{k-1} = 0 \tag{A.2}$$

for $k \geq 1$. Let also $S_{\lambda,j}$ be the subspace of $F_{\lambda,j}$ consisting of sequences satisfying $\sum_{k=0}^{+\infty} (1 + k^2)|u_k|^2 < \infty$. Then the dimension of the complex vector space $S_{\lambda,j}$ is at most one.

Proof of Theorem 5.2. We mimic the proof of Theorem 5.1. If $u \in C^\infty(\gamma)$ solves the equation $\Lambda_e u = \lambda t_m(a) u$ and $u \in V_{j,m}$, then the Fourier coefficients $(u_{mk+j})_{k\in\mathbb{Z}}$ of u satisfy (A.2) for every $k \in \mathbb{Z}$. Hence $u_{mk-m+j} = -((mk - m + j) b_1)^{-1}((mk + m + j) b_1 u_{mk+m+j} + ((mk + j) b_0 - \lambda) u_{mk+j})$ $(k \leq 1)$; and $u \equiv 0$ if $u_{mn+j} = 0$ for all $n \geq 1$. We have thus the embedding $f : \ker((t_m(a))^{-1}\Lambda_e|_{V_{j,m}} - \lambda) \to S_{\lambda,j}$ defined by $f(u) = (u_{mn+j})_{n\in\mathbb{N}}$. This implies with the help of Lemma A.2 $\dim_{\mathbb{C}}(\ker((t_m(a))^{-1}\Lambda_e|_{V_{j,m}} - \lambda)) \leq 1$. □

Proof of Lemma A.1. For $(u_n)_{n\in\mathbb{N}} \in G_\lambda$,

$$\begin{pmatrix} u_k \\ u_{k+1} \end{pmatrix} = \tilde{M}_k \begin{pmatrix} u_{k-1} \\ u_k \end{pmatrix} \quad \text{for} \quad k \geq 1,$$

where $\tilde{M}_k$ is the invertible 2×2-matrix

$$\tilde{M}_k = \begin{pmatrix} 0 & 1 \\ -1 & \frac{k - \lambda a_0}{\lambda a_1} \end{pmatrix}. \tag{A.3}$$

Let $\tilde{k}_\lambda \geq 2$ be such that

$$k - \lambda a_0 \geq 2\lambda |a_1| \quad \text{for} \quad k \geq \tilde{k}_\lambda. \tag{A.4}$$

Then using (A.3) we have for any $(u_n)_{n\in\mathbb{Z}}$

$$\begin{pmatrix} u_{\tilde{k}_\lambda-1} \\ u_{\tilde{k}_\lambda} \end{pmatrix} = \Pi_{i=1}^{\tilde{k}_\lambda-1}\tilde{M}_i \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Since $\Pi_{i=1}^{\tilde{k}_\lambda-1}\tilde{M}_i$ is invertible, we can consider the sequence $(v_n)_{n\in\mathbb{N}} \in G_\lambda$ such that $(v_{\tilde{k}_\lambda-1}, v_{\tilde{k}_\lambda}) = (0,1)$. We will prove that

$$|v_k| \geq |v_{k-1}| \geq 1 \quad \text{for} \quad k \geq \tilde{k}_\lambda. \tag{A.5}$$

This will prove that $\sum_{k=0}^{\infty}|v_k|^2 = \infty$, and hence $(v_n)_{n\in\mathbb{N}} \notin \tilde{G}_\lambda$. Therefore we will obtain $\tilde{G}_\lambda \neq G_\lambda$. Since $\tilde{G}_\lambda \subset G_\lambda$ and G_λ is two-dimensional, we will obtain that $\tilde{G}_\lambda$ is at most one-dimensional. This will prove Lemma A.1.

We prove (A.5) by induction. The inequalities (A.5) are trivially satisfied for $k = \tilde{k}_\lambda$. Assume (A.5) to hold for some $k \geq \tilde{k}_\lambda$. Then from (A.1)

$$|v_{k+1}| \geq \frac{|k - a_0\lambda|}{\lambda|a_1|}|v_k| - |v_{k-1}|.$$

Using the induction hypothesis (A.5) and (A.4), we obtain

$$|v_{k+1}| \geq \frac{|k - \lambda a_0| - \lambda|a_1|}{\lambda|a_1|}|v_k| \geq |v_k| \geq 1$$

and the induction step is done. □

Remark A.3. The solution $(v_k)_{k\in\mathbb{N}}$ of (A.1) used in the proof of Lemma A.1 grows at a rate faster than the exponential one. Indeed, taking (A.1) and (A.5) into account,

$$\frac{k - \lambda a_0 - \lambda|a_1|}{\lambda|a_1|} \leq \frac{|v_{k+1}|}{|v_k|} \leq \frac{k - \lambda a_0 + \lambda|a_1|}{\lambda|a_1|} \quad \text{for} \quad k \geq \tilde{k}_\lambda$$

and

$$\frac{k - \lambda a_0 - \lambda|a_1|}{\lambda|a_1|} \to +\infty \quad \text{as} \quad k \to +\infty,$$

where $\tilde{k}_\lambda$ is defined by (A.4).

Proof of Lemma 6.2. First we reduce the question to the case of $n = 0$. Indeed, given $f \in \mathcal{H}(\mathrm{int}D) \cap C(D)$, let $\alpha_{n-j} = \frac{1}{j!}\frac{d^j}{dz^j}f(0)$ $(0 \leq j \leq n-1)$ for some $n > 0$. Since $\Re(\frac{\sum_{j=1}^{n}(\alpha_j z^{n-j} - \bar{\alpha}_j z^{n+j})}{z^n}) = 0$ for $z \in \gamma$, we have $(\Re\tilde{f})|_\gamma = 0$, where the function $\tilde{f}(z) = \frac{f(z) - \sum_{j=1}^{n}(\alpha_j z^{n-j} - \bar{\alpha}_j z^{n+j})}{z^n}$ is continuous in D and holomorphic in $\mathrm{int}D$. The condition $(\Re\tilde{f})|_\gamma = 0$ is equivalent to $\tilde{f}|_D = i\alpha_0$ for some $\alpha_0 \in \mathbb{R}$, which proves the Lemma modulo the case of $n = 0$.

Now, assume f to satisfy $f|_D = i\alpha_0$ for some $\alpha_0 \in \mathbb{R}$. Then obviously $\Re(f(z)) = 0$ for $z \in \gamma$. Conversely assume that $\Re(f)|_\gamma = 0$. This implies, by maximum principle, $\Re f|_D = 0$ since $\Re f$ is a harmonic function in $\mathrm{int}D$. Since $\tilde{f}$ is holomorphic in D, it must be identically equal to some pure imaginary constant. □

Proof of Lemma A.2. For $(u_n)_{n\in\mathbb{N}} \in F_{\lambda,j}$,

$$\begin{pmatrix} u_k \\ u_{k+1} \end{pmatrix} = M_k \begin{pmatrix} u_{k-1} \\ u_k \end{pmatrix} \quad \text{for} \quad k \geq 1,$$

where M_k is the invertible 2×2-matrix

$$M_k = \begin{pmatrix} 0 & 1 \\ -\frac{mk-m+j}{mk+m+j} & -\frac{(mk+j)b_0-\lambda}{(mk+m+j)b_1} \end{pmatrix}. \tag{A.6}$$

Let $k_\lambda \geq 2$ be such that

$$b_0(mk+j) - \lambda \geq 2(mk+m+j)|b_1| \quad \text{for} \quad k \geq k_\lambda \tag{A.7}$$

(such a k_λ exists since $b_0 > 2|b_1| > 0$). Then using (A.6) we have for any $(u_n)_{n\in\mathbb{Z}}$

$$\begin{pmatrix} u_{k_\lambda-1} \\ u_{k_\lambda} \end{pmatrix} = \Pi_{i=1}^{k_\lambda-1} M_i \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Since $\Pi_{i=1}^{k_\lambda-1} M_i$ is invertible, we can consider the sequence $(v_n)_{n\in\mathbb{N}} \in F_{\lambda,j}$ such that $(v_{k_\lambda-1}, v_{k_\lambda}) = (0,1)$. We are going to demonstrate that

$$|v_k| \geq |v_{k-1}| \geq 1, \text{ for } k \geq k_\lambda. \tag{A.8}$$

This will prove $\sum_{k=0}^{\infty} |v_k|^2 = \infty$, and hence $(v_n)_{n\in\mathbb{N}} \notin S_{\lambda,j}$. Therefore we will obtain $S_{\lambda,j} \neq F_{\lambda,j}$. Since $S_{\lambda,j} \subset F_{\lambda,j}$ and $F_{\lambda,j}$ is two-dimensional, we will obtain that $S_{\lambda,j}$ is at most one-dimensional. This will prove Lemma A.2.

We prove (A.8) by induction. The inequalities (A.8) are trivially satisfied for $k = k_\lambda$. Assume (A.8) to hold for some $k \geq k_\lambda$. Then from (A.2),

$$|v_{k+1}| \geq \frac{|(mk+j)b_0 - \lambda|}{(mk+m+j)|b_1|}|v_k| - \frac{mk-m+j}{mk+m+j}|v_{k-1}|.$$

Using the induction hypothesis (A.8) and (A.7), we obtain

$$|v_{k+1}| \geq \frac{|(mk+j)b_0 - \lambda| - (mk-m+j)|b_1|}{(mk+m+j)|b_1|}|v_k| \geq |v_k| \geq 1$$

and the induction step is done. □

Note that similarly to Remark A.3 the solution $(v_k)_{k\in\mathbb{N}}$ of (A.2) has an exponential increase.

References

1. J. H. Albert, *Genericity of simple eigenvalues for elliptic PDE's*, Proc. Amer. Math. Soc. 48 (1975), pp. 413–418.
2. B. Colbois, A. El Soufi and A. Girouard, *Isoperimetric control of the Steklov spectrum*, J. Funct. Anal. 261(5) (2011), pp. 1384–1399.
3. J. Edward, *An inverse spectral result for the Neumann operator on planar domains*, J. Funct. Anal. 111(2) (1993), pp. 312–322.
4. A. Fraser and R. Schoen, *The first Steklov eigenvalue, conformal geometry, and minimal surfaces*, Adv. Math. 226(5) (2011), pp. 4011–4030.
5. C. Gordon, D. L. Webb and S. Wolpert, *Isospectral plane domains and surfaces via Riemannian orbifolds*, Invent. Math. 110(1) (1992), pp. 1–22.
6. P. Jammes, *Prescription du spectre de Steklov dans une classe conforme*, preprint 2012, arXiv:1209.4571.
7. T. Kato, *Perturbation theory for linear operators.* Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.
8. M. Karpukhin, G. Kokarev and I. Polterovich, *Multiplicity bounds for Steklov eigenvalues on Riemannian surfaces*, preprint 2012, arXiv:1209.4869.
9. M. Lassas and G. Ulhmann, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map*, Ann. Sci. École Norm. Sup., Sér. 4, 34(5) (2001), pp. 771–787.
10. J. M. Lee and G. Uhlmann, *Determining anisotropic real-analytic conductivities by boundary measurements*, Comm. Pure Appl. Math. 42(8) (1989), pp. 1097–1112.
11. S. G. Mikhlin and S. Prössdorf, *Singular Integral operators.* Springer-Verlag, Berlin, 1986.

12. L. Pestov and G. Uhlmann, *Two dimensional compact simple manifolds are boundary rigid*, Ann. of Math. 161(2) (2005), pp. 1093–1110.
13. I. Kh. Sabitov, *Locally Euclidean metrics with a given geodesic curvature of the boundary*, (Russian) Tr. Mat. Inst. Steklova 266 (2009), Geometriya, Topologiya i Matematicheskaya Fizika. II, pp. 218–226; translation in Proc. Steklov Inst. Math. 266(1) (2009), pp. 210–218.
14. V. Sharafutdinov, *Geometric problem of electrical impedance tomography on the disk*, Sib. Math. J. 52(1) (2011), pp. 178–190.
15. W. Stekloff, *Sur les problémes fondamentaux de la physique mathématique (suite et fin)*, Ann. Sci. École Norm. Sup. 19(3) (1902), pp. 455–490.
16. K. Uhlenbeck, *Generic properties of eigenfunctions*, Amer. J. Math. 98(4) (1976), pp. 1059–1078.
17. R. Weinstock, *Inequalities for a classical eigenvalue problem*, J. Rational Mech. Anal. 3 (1954), pp. 745–753.

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