Angular average of time-harmonic transport solutions

Guillaume Bal Alexandre Jollivet Ian Langmore François Monard

July 22, 2010

Abstract

We consider the angular averaging of solutions to time-harmonic transport equations. Such quantities model measurements obtained for instance in optical tomography, a medical imaging technique, with frequency-modulated sources. Frequency modulated sources are useful to separate ballistic photons from photons that undergo scattering with the underlying medium. This paper presents a precise asymptotic description of the angularly averaged transport solutions as the modulation frequency ω tends to ∞. Provided that scattering vanishes in the vicinity of measurements, we show that the ballistic contribution is asymptotically larger than the contribution corresponding to single scattering. Similarly, we show that singly scattered photons also have a much larger contribution to the measurements than multiply scattered photons. This decomposition is a necessary step toward the reconstruction of the optical coefficients from available measurements.

1 Introduction

Transport (linear Boltzmann) equations offer accurate descriptions for the propagation of particles in scattering media such as e.g. photons in human tissues in the application of optical tomography [1]. In the latter medical imaging modality, optical parameters are reconstructed from available photon density readings at detectors. Such detectors are typically optical fibers, which collect photons coming from different directions. The optical sources are also typically optical fibers and photons are thus emitted over a larger range of directions. When steady state sources are used, the reconstruction of the optical parameters from available measurements is severely ill-posed; see e.g. [7] as well as [4] for a review of inverse transport theory in several regimes of interest in optical tomography.

When time dependent sources are being used instead, some optical coefficients can be reconstructed stably from angularly averaged measurements as described in [6]. However, because light speed is very large, measurements in the time domain are typically not available. An intermediate regime consists of using frequency modulated sources, for instance sources of the form $(1 + \cos(\omega t + \phi))S(x, v)$ where ϕ is a constant phase shift, x is position and v direction of propagation. Heuristically, large values of ω correspond to good temporal sampling (inversely proportional to ω) in the time domain. As ω increases, we thus expect to be able to achieve accurate reconstructions of (some) optical parameters as in the time domain. That the reconstruction of the optical parameters is greatly improved when ω increases was demonstrated in the numerical simulations performed in e.g. [12, 13]; see also [2] for a similar behavior in the diffusive regime.

The main objective of this paper is to give a precise description of the asymptotic behavior of the angular averaging of transport solutions in the limit of large ω. The reason why reconstructions are more stable for large values of ω is that scattering is damped, in a sense that will be made precise, as ω increases. We decompose the transport solution into three components: the ballistic component, the single scattering component, and the multiple scattering component. These contributions exhibit different behaviors for large values of ω. By using stationary phase techniques, we present a precise asymptotic description of these three components as $\omega \to \infty$. Although we shall not do so here, such a decomposition can then be used to obtain stable reconstructions of (some) optical coefficients; see also [3].

That stationary phases appear in the analysis may be understood as follows. As "time harmonic" photons propagate from the source (located at a point x_0) to the detector (located at a point x_c), they accumulate a phase described by $e^{i\omega d}$, where d is the distance traveled by the photon along its (possibly complicated) path. For ballistic photons, the distance is fixed and given by the distance between the source and the detector. The amplitude of ballistic photons is not affected by the modulation frequency ω. Scattered photons, however, may scatter at different locations and thus arrive at the detectors' location with interfering phases. The precise averaging of phases is then described by stationary phase. It turns out that the points where the phase is stationary are precisely the points on the line segment joining the source to the detector. There is therefore a continuum of stationary points, which is the main mathematical difficulty we overcome in this paper by a careful estimate of the remainders that appear in standard stationary phase expansions. Multiple scattering is then handled similarly and shown to have a contribution that is asymptotically negligible compared to that of the ballistic and single scattering components.

The rest of the paper is structured as follows. Our main results are stated in section 2. The decomposition of the angularly averaged transport solution (defined in (2.4) below) into ballistic, single scattering, and multiple scattering contributions is presented in (2.6) below. The asymptotic behavior of these terms for large values of ω is described in Theorems 2.2, 2.5 and 2.6. Section 3 introduces notation on transport theory and decomposes the transport solution into terms involving increasing orders of scattering. Section 4 gives a proof of lemma 3.1, which states how the so-called subcriticality conditions in the steady-state regime can be relaxed in the time-harmonic regime. Finally, sections 5 and 6-7 give proofs for the single and multiple scattering estimates, respectively, using a careful stationary phase expansion.

2 Statement of the main results

Let $X \subset \mathbb{R}^n, n \geq 2$ be an open convex bounded domain with $\mathcal{C}^1$ boundary ∂X and diameter $\Delta > 0$. Denote the incoming and outgoing boundaries

$$\Gamma_\pm = \left\{(x,v) \in \partial X \times \mathbb{S}^{n-1} \mid v \in \mathbb{S}^{n-1}_{x,\pm}\right\}, \quad \text{where} \quad \mathbb{S}^{n-1}_{x,\pm} := \{v \in \mathbb{S}^{n-1} : \pm\nu_x \cdot v > 0\}, \tag{2.1}$$

where ν_x is the outer normal to ∂X at $x \in \partial X$. We consider the transport equation in the time-harmonic regime for the density $u(x,v)$, with isotropic ingoing boundary conditions

$$\begin{aligned} v \cdot \nabla u(x,v) + (\sigma(x,v) + i\omega)u(x,v) &= \int_{\mathbb{S}^{n-1}} k(x,v',v)u(x,v')\, dv', \quad (x,v) \in X \times \mathbb{S}^{n-1}, \\ u(x,v) &= g(x), \quad (x,v) \in \Gamma_-, \end{aligned} \tag{2.2}$$

where $\omega \geq 0$ and the input function g takes the form $g(x) = \delta(x - x_0), (x_0, x) \in (\partial X)^2$ (call it g_{x_0}). By $\delta(x - x_0)$ we mean the delta distribution that satisfies for each smooth function ψ defined at the boundary:

$$\int_{\partial X} \delta(x - x_0)\psi(x)\, d\mu(x) = \psi(x_0), \tag{2.3}$$

where $d\mu(x)$ is the standard measure on the boundary. The coefficient $\sigma(x,v) \geq 0$ accounts for particles that were absorbed by the medium, and $k(x,v',v) \geq 0$ accounts for particles that scattered at point x from direction v' to direction v. It is customary to write $\sigma(x,v) = \sigma_a(x,v) + \sigma_p(x,v)$, where σ_a is the intrinsic absorption of the medium, and

$$\sigma_p(x,v) = \int_{\mathbb{S}^{n-1}} k(x,v,v')\, dv',$$

represents the loss of particles that have scattered at x from direction v to other directions. An existence theory for (2.2) is given in the next section.

From the solution u of (2.2), we consider the angularly averaged outgoing measurements:

$$T^\omega(x_0, x_c) = \int_{\mathbb{S}^{n-1}_{x_c,+}} u_{|\Gamma_+}(x_c, v)|v \cdot \nu_{x_c}|\, dv, \tag{2.4}$$

where u solves (2.2). Here and below, $x_0 \in \partial X$ will denote the emitter's position and $x_c \in \partial X$, the captor's position. The measurement function (2.4) can be seen as the distributional kernel of the operator

$$\mathcal{M}^\omega(f,g) := \int_{\partial X}\int_{\partial X} T^\omega(x_0,x_c) f(x_0) g(x_c)\, d\mu(x_0)\, d\mu(x_c), \quad (f,g) \in (L^1(\partial X))^2, \tag{2.5}$$

defined and studied in [7] in the case $\omega = 0$.

Now, as will be justified in the next section, assuming that ω is large enough, the measurement operator defined in (2.4) admits the following decomposition

$$T^\omega(x_0, x_c) = T_0^\omega(x_0, x_c) + T_1^\omega(x_0, x_c) + T_{2+}^\omega(x_0, x_c), \tag{2.6}$$

where T_0^ω accounts for ballistic particles emitted at x_0 and captured at x_c, T_1^ω/T_{2+}^ω accounts for particles that scattered once/multiple-times inside the domain, respectively.

For the work that follows, and in order to avoid effects of scattering at the boundary, we formulate the following crucial hypothesis,

Hypothesis 2.1. *Define the spatial support of the scattering function*

$$supp_X\ k = \overline{\left\{ x \in X,\ \sup_{v,v' \in \mathbb{S}^{n-1}} k(x, v, v') > 0 \right\}},$$

then we make the following assumption:

$$D := dist\ (\partial X, supp_X\ k) > 0.$$

Let us now give the main results. The ballistic term is given by

$$T_0^\omega(x_0, x_c) = \frac{e^{-i\omega|x_c - x_0|} E(x_0, x_c)}{|x_c - x_0|^{n-1}} |\nu_{x_0} \cdot e_0||\nu_{x_c} \cdot e_0|, \tag{2.7}$$

where we have defined $e_0 := \widehat{x_c - x_0}$ and where $E(x_0, x_c)$ is defined by (3.4) given below. One can readily see from expression (2.7) that the magnitude of the ballistic part is unaffected by the frequency ω. This is because the ballistic particles are the first ones to reach the detector, and there is only one ballistic path from x_0 to x_c, hence no interference due to a difference of paths.

The single scattering term admits the following expression

$$\begin{aligned} T_1^\omega(x_0, x_c) &= \int_X e^{i\omega\varphi(x, x_0, x_c)} E(x_0, x, x_c) k(x, \widehat{x - x_0}, \widehat{x_c - x}) c(x, x_0, x_c)\ dx, \qquad \text{where} \\ c(x, x_0, x_c) &:= (|x - x_0||x - x_c|)^{-n+1} |\nu_{x_0} \cdot \widehat{x - x_0}||\nu_{x_c} \cdot \widehat{x_c - x}|, \\ \varphi(x, x_0, x_c) &:= -|x_0 - x| - |x - x_c|. \end{aligned} \tag{2.8}$$

where $E(x_0, x, x_c)$ is defined below by (3.5). We define the following function

$$f(x_0, x_c, x) := E(x_0, x, x_c) k(x, \widehat{x - x_0}, \widehat{x_c - x}) c(x, x_0, x_c). \tag{2.9}$$

Due to the fact that there is a continuum of single scattering paths from x_0 to x_c, particles having taken different paths will interfere. As a result, the single scattering becomes an oscillatory integral, the leading behavior of which, as $\omega \to \infty$, is ruled by the stationary points of the phase function $\varphi(x, x_0, x_c) := -|x - x_0| - |x - x_c|$. The following theorem gives the leading-order term in the asymptotic expansion of (2.8) for large ω (here and below, we denote by $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$ the floor and ceiling functions, respectively):

Theorem 2.2. *Assume that the integrand defined in* (2.9) *satisfies* $f \in L^\infty((\partial X)^2; \mathcal{C}^{\lceil \frac{n+3}{2} \rceil}(X))$ *and assume hypothesis 2.1. Then there exists a constant* C *such that the following decomposition holds for every* $(x_0, x_c) \in \partial X^2$, $x_0 \neq x_c$ *(denote* $d_0 = |x_c - x_0|$ *and* $e_0 = \widehat{x_c - x_0}$*) and for every* $\omega > 0$

$$\begin{aligned} T_1^\omega(x_0, x_c) = e^{-i\omega d_0} \left(\frac{2\pi}{d_0 \omega} \right)^{\frac{n-1}{2}} e^{-i(n-1)\frac{\pi}{4}} E(x_0, x_c) |\nu_{x_0} \cdot e_0| |\nu_{x_c} \cdot e_0| \\ \times \int_0^{d_0} \frac{k(x_0 + u e_0, e_0, e_0)}{(u(d_0 - u))^{\frac{n-1}{2}}} \, du + R^\omega(x_0, x_c), \end{aligned} \tag{2.10}$$

where the remainder R^ω *belongs to* $L^\infty(\partial X \times \partial X)$ *and satisfies the estimate*

$$\|R^\omega\|_\infty \leq \frac{C}{\omega^{\frac{n+1}{2}}} \|f\|_{\mathcal{C}^{\lceil \frac{n+3}{2} \rceil}}. \tag{2.11}$$

Remark 2.3. *A sufficient condition for the integrand* f *to satisfy the regularity prescribed in theorem 2.2 is to assume that the optical coefficients* (σ, k) *are of class* $\mathcal{C}^{\lceil \frac{n+3}{2} \rceil}$.

Remark 2.4. *Besides increasing the ballistic-scattering separation in the measurements, the leading-order of* (2.10) *contains a weighted integral transform of* k *which will motivate a mildly ill-posed reconstruction formula for* k*. This is to be compared with [7], where the same inverse problem was investigated in the case* $\omega = 0$ *and was shown to be severely ill-posed. This weighted integral transform also appears in the time-dependent setting [6].*

Finally, the multiple scattering part of the measurements is a bounded function and satisfies the following estimate:

Theorem 2.5. *Let* $(\sigma, k) \in \mathcal{C}^{\lceil \frac{n+1}{2} \rceil}(\bar{X} \times \mathbb{S}^{n-1}) \times \mathcal{C}^{\lceil \frac{n+1}{2} \rceil}(X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$*, and assume hypothesis 2.1. Then there exists a frequency* $\omega_0 \geq 2$ *and a constant* C *such that for every* $\omega \geq \omega_0$ *the multiple scattering* $T_{2+}^\omega \in L^\infty(\partial X \times \partial X)$ *and satisfies the estimate*

$$\|T_{2+}^\omega\|_\infty \leq \begin{cases} C\ \omega^{-1}, & n = 2, \\ C\ \omega^{-2} \ln(\omega), & n = 3, \\ C\ \omega^{-\frac{n+1}{2}}, & n \geq 4. \end{cases} \tag{2.12}$$

These results are summarized in the following theorem, using the most regularity of the optical coefficients that is required for theorems 2.2 and 2.5 to hold:

Theorem 2.6. *Assume that* $(\sigma, k) \in \mathcal{C}^{\lceil \frac{n+3}{2} \rceil}(\bar{X} \times \mathbb{S}^{n-1}) \times \mathcal{C}^{\lceil \frac{n+3}{2} \rceil}(X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$*, and assume hypothesis 2.1. Then there exists a frequency* $\omega_0 \geq 2$ *and a constant* C *such that for every* $\omega \geq \omega_0$ *the measurement function* T^ω *admits the following singular decomposition*

$$T^\omega(x_0, x_c) = T_0^\omega(x_0, x_c) + T_1^\omega(x_0, x_c) + T_{2+}^\omega(x_0, x_c), \tag{2.13}$$

for a.e. $(x_0, x_c) \in \partial X \times \partial X$*, where* T_1^ω *is given by* (2.8) *and has the asymptotic behavior described in* (2.10) *and*

$$T_0^\omega(x_0, x_c) := \frac{e^{-i\omega|x_c - x_0|}E(x_0, x_c)}{|x_c - x_0|^{n-1}}|\nu_{x_0} \cdot e_0||\nu_{x_c} \cdot e_0|, \tag{2.14}$$

$$T_{2+}^\omega \in L^\infty(\partial X \times \partial X) \quad \textit{and} \ \|T_{2+}^\omega\|_\infty \le \begin{cases} C\ \omega^{-1}, & n = 2, \\ C\ \omega^{-2}\ln(\omega), & n = 3, \\ C\ \omega^{-\frac{n+1}{2}}, & n \ge 4. \end{cases} \tag{2.15}$$

3 Forward theory

Let us now return to the forward model (2.2) and present the necessary results that will be useful for the subsequent sections.

Integral equation: Let us recall some notation. For $(x, v) \in (X \times \mathbb{S}^{n-1}) \bigcup \Gamma_+ \bigcup \Gamma_-$, let $\tau_\pm(x, v)$ be the distance from x to ∂X traveling in the direction of $\pm v$, and $x_\pm(x, v) = x \pm \tau_\pm(x, v)v$ be the boundary point encountered when we travel from x in the direction of $\pm v$. We also define $\tau = \tau_+ + \tau_-$.

As it is done in many settings, we integrate the PDE in (2.2) along the direction v. We obtain that u is a solution of the following integro-differential equation

$$(I - \mathcal{K}_\omega)u = J_\omega g, \tag{3.1}$$

where we have defined, for $\phi \in L^1(X \times \mathbb{S}^{n-1})$ and $\psi \in L^1(\partial X)$

$$\mathcal{K}_\omega\phi(x, v) := \int_0^{\tau_-(x,v)} e^{-i\omega t}E(x - tv, x)\int_{\mathbb{S}^{n-1}} k(x - tv, v', v)\phi(x - tv, v')dv'dt, \tag{3.2}$$

$$J_\omega\psi(x, v) := e^{-i\omega\tau_-(x,v)}E(x_-(x, v), x)\psi(x_-(x, v)), \quad \text{and} \tag{3.3}$$

$$E(x', x) := \exp\left(-\int_0^{|x-x'|} \sigma(x' + s\,\widehat{x - x'}, \widehat{x - x'})\ ds\right). \tag{3.4}$$

For future reference, we also define iteratively

$$E(x_1, \dots, x_{i+1}) := E(x_1, \dots, x_i)E(x_i, x_{i+1}). \tag{3.5}$$

The operators $\mathcal{K}_\omega$ and J_ω are well-defined and continuous operators in $\mathcal{L}(L^1(X \times \mathbb{S}^{n-1}))$ and $\mathcal{L}(L^1(\partial X), L^1(X \times \mathbb{S}^{n-1}))$, respectively [9, 11]. Now we must also make sense of J_ω and $\mathcal{K}_\omega$ when the inputs are the singular distributions $g_{x_0} = \delta(x - x_0)$ defined in (2.3). One can show that, in the sense of distributions, we have

$$J_\omega g_{x_0}(x, v) = E(x_0, x)e^{-i\omega|x-x_0|}|\nu_{x_0} \cdot v|\frac{\delta_S(v - \widehat{x - x_0})}{|x - x_0|^{n-1}}, \quad (x, v) \in X \times \mathbb{S}^{n-1},$$

where δ_S stands for delta distribution on $\mathbb{S}^{n-1}$. Extending naturally the definition (3.2) to distributions in the angular variable and applying it to the previous equality, we obtain that $\mathcal{K}_\omega J_\omega g_{x_0}$ is well-defined in $L^1(X\times\mathbb{S}^{n-1})$ and its expression is given by, for a.e. $(x,v)\in X\times\mathbb{S}^{n-1}$,

$$\mathcal{K}_\omega J_\omega g_{x_0}(x,v) = \int_0^{\tau_-(x,v)} e^{-i\omega(t+|x-tv-x_0|)} k(x-tv, \widehat{x-x_0}, v) E(x_0, x-tv, x) \frac{|\nu_{x_0}\cdot \widehat{x-tv-x_0}|}{|x-tv-x_0|^{n-1}}\, dt. \tag{3.6}$$

Moreover, taking the L^1 norm of (3.6), we obtain that

$$\begin{aligned}\int_{X\times\mathbb{S}^{n-1}} |\mathcal{K}_\omega J_\omega g_{x_0}(x,v)|\, dx\, dv &\le \|k\|_\infty \int_{X\times\mathbb{S}^{n-1}}\int_0^\Delta \frac{dt}{|x-tv-x_0|^{n-1}}\, dx\, dv \\ &\le \|k\|_\infty |\mathbb{S}^{n-1}|^2\Delta^2,\end{aligned} \tag{3.7}$$

and the bound is uniform in $x_0\in\partial X$, where we recall that Δ denotes the diameter of X. We will use this result later on to prove theorem 2.5.

Subcriticality conditions: Like in the stationary setting for $\omega = 0$ (see e.g. [8, 7, 5, 14]), equation (3.1) is solvable for u if the operator $(I-\mathcal{K}_\omega)$ is invertible in $\mathcal{L}^1 := \mathcal{L}(L^1(X\times\mathbb{S}^{n-1}))$. Such a condition is met when there exists an integer $r\ge 1$ such that the operator $(I-\mathcal{K}_\omega^r)$ is invertible in $\mathcal{L}^1$. Then the solution of (3.1) admits either one of the following equivalent expressions:

$$u = (I-\mathcal{K}_\omega^r)^{-1}\sum_{m=0}^{r-1}\mathcal{K}_\omega^m J_\omega g = \sum_{m=0}^{r-1}\mathcal{K}_\omega^m (I-\mathcal{K}_\omega^r)^{-1} J_\omega g = \sum_{m=0}^{N}\mathcal{K}_\omega^m J_\omega g + (I-K_\omega)^{-1}K_\omega^{N+1}J_\omega g, \tag{3.8}$$

for any integer N.

In the stationary setting (i.e. $\omega=0$), concrete examples of invertibility of $I-\mathcal{K}_0^r$, $r=1$, are given by either one of the following so-called subcriticality conditions [4, 9, 11]

$$\sigma-\sigma_p \ge 0, \tag{3.9}$$
$$\|\tau\sigma_p\|_\infty < 1. \tag{3.10}$$

In the present time-harmonic regime, invertibility is still achieved with $r=1$ under the above conditions. However, it can also be obtained with $r=2$ for large enough ω, independently of any positivity assumption, though we require that the optical coefficients have bounded C^1-norms and k be supported away from the spatial boundary ∂X (lemma 3.1 and corollary 3.3).

Lemma 3.1. *Let $(\sigma, k) \in \mathcal{C}^1(\overline{X} \times \mathbb{S}^{n-1}) \times \mathcal{C}^1(X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ and assume hypothesis 2.1. Then there exists a constant C such that for every $\omega \geq 2$ the following estimates hold*

$$\|\mathcal{K}_\omega^2\|_{\mathcal{L}^1} \leq \begin{cases} C\ (\|\sigma\|_{\mathcal{C}^1}, \|k\|_{\mathcal{C}^1})\ \omega^{-\frac{1}{2}} \ln(\omega), & n = 2, \\ C\ (\|\sigma\|_{\mathcal{C}^1}, \|k\|_{\mathcal{C}^1})\ \omega^{-1}(\ln(\omega))^2, & n = 3, \\ C\ (\|\sigma\|_{\mathcal{C}^1}, \|k\|_{\mathcal{C}^1})\ \omega^{-1} \ln(\omega), & n \geq 4. \end{cases} \tag{3.11}$$

Remark 3.2. *The same result holds even if k is supported up to the boundary ∂X.*

Lemma 3.1 will allow us to assess the decay of multiple scattering terms in further sections. From this lemma we obtain the following corollary.

Corollary 3.3. *Let $(\sigma, k) \in C^1(\overline{X} \times \mathbb{S}^{n-1}) \times C^1_0(X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$. Then there exists ω_0 which depends on $\|\sigma\|_{C^1}$ and $\|k\|_{C^1}$, X and n such that $I - \mathcal{K}_\omega$ is invertible and $\|(I - \mathcal{K}_\omega)^{-1}\|_{\mathcal{L}^1} \leq C_0$ uniformly for $\omega \geq \omega_0$. C_0 can be chosen to be $2(1 + \Delta|\mathbb{S}^{n-1}|\|k\|_\infty)$.*

The uniform estimate of the norm of $(I - \mathcal{K}_\omega)^{-1}$ at large ω given in corollary 3.3 will also be useful to estimate the decay of multiple scattering terms (see section 6).

Proof of Corollary 3.3. From lemma 3.1, we have that $\lim_{\omega \to \infty} \|\mathcal{K}_\omega^2\|_{\mathcal{L}^1} = 0$, so there exists ω_0 such that $\|\mathcal{K}_\omega^2\|_{\mathcal{L}^1} \leq \frac{1}{2}$ for $\omega \geq \omega_0$. For such an ω, $I - \mathcal{K}_\omega^2$ is invertible with inverse $(I - \mathcal{K}_\omega^2)^{-1} = \sum_{m=0}^\infty \mathcal{K}_\omega^{2m}$ and in turn, $I - \mathcal{K}_\omega$ is invertible with inverse $(I - \mathcal{K}_\omega)^{-1} = (I + \mathcal{K}_\omega)(I - \mathcal{K}_\omega^2)^{-1} = \sum_{m=0}^\infty \mathcal{K}_\omega^m$. In this case, and noting that $\|\mathcal{K}_\omega\|_{\mathcal{L}^1} \leq \Delta|\mathbb{S}^{n-1}|\|k\|_\infty$, we get

$$\|(I - \mathcal{K}_\omega)^{-1}\|_{\mathcal{L}^1} \leq \frac{1 + \|\mathcal{K}_\omega\|_{\mathcal{L}^1}}{1 - \|\mathcal{K}_\omega^2\|_{\mathcal{L}^1}} \leq 2(1 + \Delta|\mathbb{S}^{n-1}|\|k\|_\infty).$$

This completes the proof. □

Derivation of decomposition (2.6)**:** Whenever subcriticality conditions are satisfied and the solution u of (3.1) admits the expressions (3.8), one obtains the following decomposition of the measurement function

$$T^\omega(x_0, x_c) = \sum_{m=0}^N T_m^\omega(x_0, x_c) + R_{N+1}(x_0, x_c), \tag{3.12}$$

for any integer $N \geq 2$ where

$$T_m^\omega(x_0, x_c) = \int_{\mathbb{S}^{n-1}_{x_c,+}} [\mathcal{K}_\omega^m J_\omega g_{x_0}](x_c, v)|v \cdot \nu_{x_c}|\ dv, \text{ for } m \geq 0, \tag{3.13}$$

and

$$R_m^\omega(x_0, x_c) = \int_{\mathbb{S}^{n-1}_{x_c,+}} [(I - \mathcal{K})^{-1}\mathcal{K}_\omega^m J_\omega g_{x_0}](x_c, v)|v \cdot \nu_{x_c}|\ dv, \text{ for } m \geq 2.$$

From this latter expression (3.13) of the T^ω_m's, one can deduce the expressions (2.7), (2.8) and (6.1) of the ballistic, single and multiple scattering components of the measurement function T^ω. This can be done by following the calculations in [7, Section 3.1] with minor modifications. We do not repeat these calculations here.

Then the decomposition (2.6) follows from (3.12) and the equality

$$T^\omega_{2+}(x_0,x_c)=\sum_{m=2}^{N}T^\omega_m(x_0,x_c)+R_{N+1}(x_0,x_c),\ \text{for any integer } N\geq 2. \tag{3.14}$$

4 Proof of Lemma 3.1

From (3.2) it follows that

$$\mathcal{K}^2_\omega\phi(x,v)=\int_{X\times\mathbb{S}^{n-1}}\beta_\omega(x,v,x',v')\phi(x',v')dx'dv', \tag{4.1}$$

for $(x,v)\in X\times\mathbb{S}^{n-1}$ and for $\phi\in L^1(X\times\mathbb{S}^{n-1})$, where the distributional kernel β_ω of $\mathcal{K}^2_\omega$ is given by

$$\begin{aligned}\beta_\omega(x,v,x',v')&=\int_0^{\tau_-(x,v)}e^{-i\omega(t+|x-x'-tv|)}\frac{f(t,x,v,x',v')}{|x-tv-x'|^{n-1}}\,dt,\qquad\text{where}\\ f(t,x,v,x',v')&:=E(x',x-tv,x)k(x-tv,v_1,v)k(x',v',v_1),\quad v_1:=\widehat{x-tv-x'},\end{aligned} \tag{4.2}$$

for a.e. $(x,v,x',v')\in X\times\mathbb{S}^{n-1}\times X\times\mathbb{S}^{n-1}$. Note that f is uniformly bounded by $\|k\|^2_\infty$.

The goal here is to bound the quantity

$$\sup_{x',v'}\int_{X\times\mathbb{S}^{n-1}}|\beta_\omega(x,v,x',v')|\ dx\ dv.$$

Let $\omega\geq 2$. We extend σ (resp. k) by 0 to $\mathbb{R}^2\times\mathbb{S}^{n-1}$ (resp. $\mathbb{R}^2\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$). Then since k is compactly supported inside $X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$, f is thus extended smoothly to $\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^{n-1}\times\mathbb{R}^n\times\mathbb{S}^{n-1}$. We now fix $(x',v',v)\in X\times(\mathbb{S}^{n-1})^2$ and study the quantity

$$I(x',v',v,\omega)=\int_X|\beta_\omega(x,v,x',v')|\ dx. \tag{4.3}$$

It is now customary to study the phase function $\varphi(t)=-t-|x-tv-x'|$ (the dependency of φ on (x,x',v) is made implicit here), especially, for what x's the function is stationary. We have

$$\frac{\partial\varphi}{\partial t}=-\frac{|x-tv-x'|-v\cdot(x-tv-x')}{|x-tv-x'|}=-\frac{|x-x'|^2-((x-x')\cdot v)^2}{|x-tv-x'|(|x-tv-x'|+v\cdot(x-tv-x'))}. \tag{4.4}$$

Hence $\partial_t\varphi = 0$ whenever $x = x' + uv, u \in [-\Delta, \Delta]$. In order to adapt x to the geometry of the problem, we make the change of variable

$$x = x' + uv + \rho\theta, \quad u \in [-\Delta, \Delta],\ \rho \in [0, \Delta],\ \theta \in \mathbb{S}^{n-2}(\{v\}^\perp). \tag{4.5}$$

In these new variables, we have the following facts

$$\begin{aligned} dx &= \rho^{n-2}\ d\rho\ d\theta\ du, & |x - tv - x'| &= ((u-t)^2 + \rho^2)^{\frac{1}{2}} \\ |x - x'|^2 - ((x - x')\cdot v)^2 &= \rho^2, & \frac{1}{\partial_t\varphi} &= -\frac{(t-u)^2 + \rho^2 + (u-t)\sqrt{(t-u)^2+\rho^2}}{\rho^2}. \end{aligned}$$

The integral (4.3) becomes

$$I(x', v', v, \omega) = \int_{-\Delta}^{\Delta}\int_{\mathbb{S}^{n-2}}\int_0^{\Delta} |\beta_\omega(x(u,\rho,\theta), v, x', v')|\rho^{n-2}\ d\rho\ d\theta\ du.$$

In this set of variables, the phase is non-stationary for $\rho > 0$, so we split the above integral into two parts $I := I_1 + I_2$ by chopping the integral in ρ: $\int_0^\Delta\ d\rho = \int_0^\varepsilon\ d\rho + \int_\varepsilon^\Delta\ d\rho$, respectively. The small parameter $\varepsilon \le 1/2$ will be chosen at the end in terms of ω, in order to get the optimal estimate.

Let us estimate I_1 first. Note that using (4.2) we obtain

$$|\beta_\omega(x(u,\rho,\theta), v, x', v')| \le \|k\|_\infty^2 \int_0^{\tau_-(x'+uv+\rho\theta)} \frac{dt}{((t-u)^2+\rho^2)^{\frac{n-1}{2}}} \le 2\|k\|_\infty^2 \int_0^{\Delta}\frac{dt}{(t^2+\rho^2)^{\frac{n-1}{2}}},$$

thus

$$\begin{aligned} I_1(x', v', v, \omega) &\le 2\|k\|_\infty^2 \int_{-\Delta}^{\Delta} du \int_{\mathbb{S}^{n-2}} d\theta \int_0^\varepsilon d\rho \int_0^\Delta dt \frac{\rho^{n-2}}{(t^2+\rho^2)^{\frac{n-1}{2}}} \\ &\le 4\|k\|_\infty^2 \Delta |\mathbb{S}^{n-2}| \int_0^\varepsilon\int_0^\Delta \frac{\rho^{n-2}}{(t^2+\rho^2)^{\frac{n-1}{2}}}\ dt\ d\rho. \end{aligned}$$

Estimating the integral on the above right-hand side, we write $(t,\rho) = (r\cos\alpha, r\sin\alpha)$, with $\alpha \in [0, \frac{\pi}{2}]$ and $r \in [0, r(\alpha)]$, where $r(\alpha) = \frac{\Delta}{\cos\alpha}$ if $\alpha \le \alpha_0 := \tan^{-1}\frac{\varepsilon}{\Delta}$ and $r(\alpha) = \frac{\varepsilon}{\sin\alpha}$ if $\alpha \ge \alpha_0$. The above double integral becomes, after simplification:

$$\int_0^\varepsilon\int_0^\Delta \frac{\rho^{n-2}}{(t^2+\rho^2)^{\frac{n-1}{2}}}\ dt\ d\rho = \Delta\int_0^{\alpha_0}\frac{\sin^{n-2}\alpha}{\cos\alpha}\ d\alpha + \varepsilon\int_{\alpha_0}^{\frac{\pi}{2}} \sin^{n-3}\alpha\ d\alpha, \le C\begin{cases} \varepsilon|\ln\varepsilon|, & n = 2, \\ \varepsilon, & n \ge 3, \end{cases}$$

whence the bound on $I_1(x', v', v, \omega)$.

Now, let us estimate I_2. For $\rho \geq \varepsilon$, $\partial_t \varphi$ is never zero, so we can write $e^{i\omega\varphi} = \frac{1}{i\omega\partial_t\varphi}\dfrac{\partial}{\partial t}e^{i\omega\varphi}$ and integrate by parts the right-hand-side of (4.2). Doing so, we write

$$\beta_\omega = \frac{1}{i\omega}\left(\beta_{1,\omega} + \beta_{2,\omega} + \beta_{3,\omega}\right),$$

where we have defined

$$\beta_{1,\omega}(x,v,x',v') = \frac{e^{-i\omega(\tau_-(x,v)+|x-\tau_-(x,v)v-x'|)} f(\tau_-(x,v),x,v,x',v')}{|x-\tau_-(x,v)v-x'|^{n-2}(|x-\tau_-(x,v)v-x'|+\tau_-(x,v)-(x-x')\cdot v)} \tag{4.6}$$

$$\beta_{2,\omega}(x,v,x',v') = -\frac{e^{-i\omega|x-x'|} f(0,x,v,x',v')}{|x-x'|^{n-2}(|x-x'|-(x-x')\cdot v)}, \tag{4.7}$$

$$\beta_{3,\omega}(x,v,x',v') = \int_0^{\tau_-(x,v)} e^{-i\omega(t+|x-tv-x'|)} \frac{d}{dt}\left(\frac{f(t,x,v,x',v')}{|x-tv-x'|^{n-2}(|x-tv-x'|+t-(x-x')\cdot v)}\right). \tag{4.8}$$

The term $\beta_{1,\omega}$ is identically zero provided that k is supported away from the boundary (hypothesis 2.1).

We now estimate $\beta_{2,\omega}$. From (4.7) it follows that

$$\begin{aligned}
|\beta_{2,\omega}(x,v,x',v')| &\leq \frac{\|k\|_\infty^2}{|x-x'|^{n-2}(|x-x'|-(x-x')\cdot v)} \\
&\leq \|k\|_\infty^2 \frac{|x-x'|+(x-x')\cdot v}{|x-x'|^{n-2}(|x-x'|^2-((x-x')\cdot v)^2)} \\
&\leq \frac{2\|k\|_\infty^2}{|x-x'|^{n-3}(|x-x'|^2-((x-x')\cdot v)^2)} \\
&\leq \frac{2\|k\|_\infty^2}{(u^2+\rho^2)^{\frac{n-3}{2}}\rho^2},
\end{aligned} \tag{4.9}$$

where we expressed the change of variable (4.5) in the last line. We get

$$\int_{-\Delta}^{\Delta}\int_{\mathbb{S}^{n-2}}\int_{\varepsilon}^{\Delta} |\beta_{2,\omega}(x(u,\rho,\theta),v,x',v')|\ \rho^{n-2}\ d\rho\ d\theta\ du$$

$$\leq 4\|k\|_\infty^2|\mathbb{S}^{n-2}|\int_0^{\Delta}\int_{\varepsilon}^{\Delta} \frac{\rho^{n-4}}{(u^2+\rho^2)^{\frac{n-3}{2}}}\ d\rho\ du \tag{4.10}$$

$$\leq 4\|k\|_\infty^2|\mathbb{S}^{n-2}|\begin{cases} \sqrt{2}\Delta^2\ \varepsilon^{-1}, & n=2, \\ \Delta\ |\ln(\varepsilon)|, & n=3, \\ \frac{\pi}{2}\Delta, & n\geq 4. \end{cases} \tag{4.11}$$

Finally we estimate $\beta_{3,\omega}$. Plugging the change of variable (4.5) into (4.8), we get

$$\begin{aligned}\beta_{3,\omega}(x(u,\rho,\theta),v,x',v') &= \frac{1}{\rho^2}\int_0^{\tau_-(x'+uv+\rho\theta)} e^{i\omega\varphi} \\ &\times\frac{d}{dt}\left(\frac{t-u-\sqrt{(t-u)^2+\rho^2}}{((t-u)^2+\rho^2)^{\frac{n-2}{2}}} f(t,x(u,\rho,\theta),v,x',v')\right)\,dt.\end{aligned}$$

Hence, we have that

$$\begin{aligned}|\beta_{3,\omega}(x(u,\rho,\theta),v,x',v')| &\leq \frac{C\|f\|_{\mathcal{C}^1}}{\rho^2}\int_0^{\tau_-(x'+uv+\rho\theta)} \frac{dt}{((t-u)^2+\rho^2)^{\frac{n-2}{2}}} \\ &\leq \frac{2C\|f\|_{\mathcal{C}^1}}{\rho^2}\int_0^{\Delta} \frac{dt}{(t^2+\rho^2)^{\frac{n-2}{2}}}.\end{aligned}$$

Finally ,

$$\begin{aligned}&\int_{-\Delta}^{\Delta}\int_{\mathbb{S}^{n-2}}\int_{\varepsilon}^{\Delta} |\beta_{3,\omega}(x(u,\rho,\theta),v,x',v')|\ \rho^{n-2}\ d\rho\ d\theta\ du \\ &\leq 4C\|f\|_{\mathcal{C}^1}\Delta|\mathbb{S}^{n-2}|\int_0^{\Delta}\int_{\varepsilon}^{\Delta} \frac{\rho^{n-4}}{(t^2+\rho^2)^{\frac{n-2}{2}}}\ d\rho\ dt \leq 2C\|f\|_{\mathcal{C}^1}\Delta|\mathbb{S}^{n-2}| \left\{\begin{array}{ll} \Delta\varepsilon^{-1}, & n=2, \\ \frac{\pi}{2}(\ln\varepsilon)^2, & n=3, \\ \frac{\pi}{2}|\ln\varepsilon|, & n\geq 4.\end{array}\right.\end{aligned}$$

To sum up, we have the following estimates:

$$I_1 \leq C\left\{\begin{array}{ll} \varepsilon|\ln\varepsilon|, & n=2, \\ \varepsilon, & n\geq 3,\end{array}\right. , \qquad I_2 \leq C\left\{\begin{array}{ll} (\omega\varepsilon)^{-1}, & n=2, \\ \omega^{-1}(\ln\varepsilon)^2, & n=3, \\ \omega^{-1}|\ln\varepsilon|, & n\geq 4.\end{array}\right.$$

Choosing $\varepsilon=\omega^{-\frac{1}{2}}$ when $n=2$ and $\varepsilon=\omega^{-1}$ when $n\geq 3$ finally gives the estimate (3.11). This completes the proof.

5 Proof of single scattering asymptotics (theorem 2.2)

In order to prove theorem 2.2, we will use the following

Lemma 5.1. *Let $X\subset\mathbb{R}^d$ be an open, bounded set containing 0, $m\geq -d+1$ and $f\in\mathcal{C}_0^{\lceil\frac{m+d}{2}\rceil}(X)$ such that $f(0)\neq 0$. Consider the following oscillatory integral*

$$I=\int_{\mathbb{R}^d}|x|^m f(x)e^{-i\omega\varphi(|x|)}\ dx, \tag{5.1}$$

where $\varphi(r) = r^2 g(r^2)$ with g a smooth nonnegative function such that $g(0) \neq 0$, and $r = 0$ is the only point satisfying $\varphi'(r) = 0$. Then the following estimate holds

$$|I| \leq \frac{C}{\omega^{\frac{m+d}{2}}} \|f\|_{\mathcal{C}^{\lceil \frac{m+d}{2} \rceil}}.$$

Proof of lemma 5.1. Define $f_1(r) := \int_{\mathbb{S}^{d-1}} f(r\hat{\theta})\ dS(\theta)$. Note that f_1 is an even function of r and has compact support in $[0, \Delta)$, where Δ denotes the diameter of X. It is also $\lceil \frac{m+d}{2} \rceil$ times continuously differentiable and for every $0 \leq \alpha \leq \lceil \frac{m+d}{2} \rceil$ there exist a constant C_α such that

$$\|f_1\|_{\mathcal{C}^\alpha[0,\Delta)} \leq C_\alpha \|f\|_{\mathcal{C}^\alpha(X)}.$$

Applying a polar change of variable to (5.1), we get

$$I = \int_0^\infty r^{m+d-1} f_1(r) e^{-i\omega\varphi(r)}\ dr.$$

Now notice that the ratio $h(r) = \frac{r}{\varphi'(r)}$ is a smooth, bounded function and that by means of $\lfloor \frac{m+d-1}{2} \rfloor$ integrations by parts with zero boundary terms, one can write I as

$$I = \left(\frac{1}{i\omega}\right)^{\lfloor \frac{m+d-1}{2} \rfloor} \int_0^\infty \left(\frac{d}{dr} \circ \frac{1}{\varphi'} \cdot\right)^{\lfloor \frac{m+d-1}{2} \rfloor} \left(r^{m+d-1} f_1(r)\right) e^{-i\omega\varphi(r)}\ dr.$$

We now split cases: if $m + d$ is even, then $\lfloor \frac{m+d-1}{2} \rfloor = \frac{m+d}{2} - 1$ and we have that

$$\left(\frac{d}{dr} \circ \frac{1}{\varphi'} \cdot\right)^{\frac{m+d}{2}-1} \left(r^{m+d-1} f_1(r)\right) = r f_2(r),$$

thus we can integrate by parts one more time and get that

$$I = \left(\frac{1}{i\omega}\right)^{\frac{m+d}{2}-1} \frac{1}{i\omega} \left\{ f_2(0)h(0) - \int_0^\infty \frac{d}{dr} \left(f_2 h\right) e^{-i\omega\varphi}\ dr \right\},$$

in which case we get the bound

$$|I| \leq \frac{C}{\omega^{\frac{m+d}{2}}} \|f\|_{\mathcal{C}^{\frac{m+d}{2}}}.$$

Now if $m + d$ is odd, then $\lfloor \frac{m+d-1}{2} \rfloor = \frac{m+d-1}{2}$. We call $f_3(r) := \left(\frac{d}{dr} \circ \frac{1}{\varphi'} \cdot\right)^{\frac{m+d-1}{2}} \left(r^{m+d-1} f_1(r)\right)$, and I has the expression

$$I = \left(\frac{1}{i\omega}\right)^{\frac{m+d-1}{2}} \int_0^\infty f_3(r) e^{-i\omega\varphi(r)} dr.$$

Using Morse lemma, there is an interval $[0,\eta]$ where the change of variable $v(r)=\sqrt{\varphi(r)}$ is a diffeomorphism. Let $\chi \in \mathcal{C}^\infty([0,+\infty))$ be identically one on a neighborhood of 0 and $\chi(s)=0$ for $s\geq \eta$. Then I can be split into two integrals:

$$I=\left(\frac{1}{i\omega}\right)^{\frac{m+d-1}{2}}\left(\int_0^{v(\eta)}\chi(r(v))f_3(r(v))\frac{dr}{dv}e^{-i\omega v^2}\,dv+\int_0^\infty(1-\chi(r))f_3(r)e^{-i\omega\varphi(r)}dr\right), \quad (5.2)$$

where we applied the change of variable $v=\sqrt{\varphi(r)}$ in the first integral. In the first integral, we integrate by parts and obtain that

$$\int_0^{v(\eta)}\chi(r(v))f_3(r(v))\frac{dr}{dv}e^{-i\omega v^2}\,dv=-\frac{1}{\omega^{\frac{1}{2}}}\int_0^\infty\frac{d}{dv}\left(\chi(r(v))f_3(r(v))\frac{dr}{dv}\right)H(\sqrt{\omega}v)dv,$$

where the function $H(t):=\int_0^t e^{-iv^2}dv$ is related to the Fresnel integral and is known to be uniformly bounded on $[0,\infty)$. For the second integral in (5.2) the integrand is supported away from zero, so we can write $e^{-i\omega\varphi(r)}=\frac{-1}{i\omega\varphi'(r)}\frac{d}{dr}e^{-i\omega\varphi(r)}$ and integrate by parts. We obtain that

$$\int_0^\infty(1-\chi(r))f_3(r)e^{-i\omega\varphi(r)}dr=\frac{1}{i\omega}\int_0^\infty\frac{d}{dr}\left(\frac{(1-\chi(r))f_3(r)}{\varphi'(r)}\right)e^{-i\omega\varphi(r)}dr.$$

This finally shows that

$$|I|\leq\frac{C}{\omega^{\frac{m+d}{2}}}\|f_3\|_{\mathcal{C}^1}\leq\frac{C}{\omega^{\frac{m+d}{2}}}\|f\|_{\mathcal{C}^{\frac{m+d+1}{2}}}.$$

The proof is complete. □

Note also the following two lemmas, useful to get uniform bounds in theorem 2.2

Lemma 5.2. *Under hypothesis 2.1, there exists $\delta_1>0$ such that for all $(x_0,x_c)\in(\partial X)^2$*

$$|x_0-x_c|\leq\delta_1 \Rightarrow \textit{dist}\,([x_0,x_c],\textit{supp}_X k)>\delta_1. \quad (5.3)$$

Lemma 5.3. *Assume hypothesis 2.1 and let $\delta_1>0$ be given by lemma 5.2. Then there exists $\delta_2>0$ such that for all $(x,x_0,x_c)\in \textit{supp}_X k\times(\partial X)^2$ with $|x_0-x_c|\geq\delta_1$, we have*

$$|x-x_0-(x-x_0)\cdot\widehat{x_c-x_0}|\leq\delta_2\Rightarrow\begin{cases}(x-x_0)\cdot\widehat{x_c-x_0}>\delta_2\\|x_c-x_0|-(x-x_0)\cdot\widehat{x_c-x_0}>\delta_2.\end{cases} \quad (5.4)$$

Proof of theorem 2.2. Pick $(x_0,x_c)\in(\partial X)^2$ with $x_0\neq x_c$, and define $e_0:=\widehat{x_c-x_0}$. The stationary points of φ with respect to x solve the equation

$$\widehat{x-x_c}+\widehat{x-x_0}=0,$$

which characterizes exactly the points located on the segment $[x_0, x_c]$. Hence it is convenient to introduce the notation $x(x_0, x_c, u, v) = x_0 + ue_0 + v$, where $u \in \mathbb{R}$ and $v \in {e_0}^\perp$. Then $\{v = 0, u \in [0, d_0]\}$ is a parametrization of the stationary segment. In this new set of variables the phase function is given by

$$\varphi(u, v) = -(u^2 + |v|^2)^{\frac{1}{2}} - ((d_0 - u)^2 + |v|^2)^{\frac{1}{2}}, \tag{5.5}$$

and the function $f(x_0, x_c, x)$ (defined in (2.9)) with $x = x(x_0, x_c, u, v)$ is now supported in

$$\left\{u \in [-\Delta, \Delta], v \in {e_0}^\perp, |v| \leq \Delta, \text{ s.t. } x(x_0, x_c, u, v) \in \operatorname{supp}_X k\right\}.$$

In the sequel, we make the dependency on x_0 and x_c implicit for readability. Let δ_1, δ_2 be given by lemmas 5.2, 5.3 respectively, denote $\delta := \min(\delta_1, \delta_2) > 0$, and let $\chi \in \mathcal{C}_0^\infty([0, \delta))$, $\chi \geq 0$, $\chi \equiv 1$ over $[0, \frac{\delta}{2}]$. We then write

$$\begin{aligned}
T_1^\omega(x_0, x_c) &= I_1(x_0, x_c) + I_2(x_0, x_c), \qquad \text{where} \\
I_1(x_0, x_c) &:= \int_{-\Delta}^{\Delta} \int_{|v| \leq \Delta} f(x_0, x_c, u, v) \chi(v) e^{i\omega\varphi(u,v)}\, dv\, du \\
I_2(x_0, x_c) &:= \int_{-\Delta}^{\Delta} \int_{|v| \leq \Delta} f(x_0, x_c, u, v)(1 - \chi(v)) e^{i\omega\varphi(u,v)}\, dv\, du.
\end{aligned}$$

The integral I_1 contains stationary points whereas I_2 does not.

Let us take care of I_1 first. We start by saying that if $|x_0 - x_c| \leq \delta_1$, then lemma 5.2 ensures that $\chi(v) f(x_0, x_c, u, v) = 0$ everywhere, hence $I_1(u) = 0$ in this case. Now assume $|x_0 - x_c| \geq \delta_1$. Then lemma 5.3 ensures that for any (u, v) in the support of $\chi(\cdot) f(x_0, x_c, \cdot, \cdot)$, we have that $|v| \leq \delta \leq \delta_2 < \min(u, d_0 - u)$, therefore both squareroots in (5.5) are analytic in v, and φ can be identically written as

$$\varphi(u, v) = -d_0 + \frac{1}{2} q(u)|v|^2 + |v|^4 g(u, |v|),$$

where g is a smooth function, $-d_0 = \varphi(u, 0)$ and the quadratic term in v comes from the Hessian matrix with respect to v at $v = 0$

$$H_\varphi(u, 0) = \frac{-d_0}{u(d_0 - u)} I_{n-1} := q(u) I_{n-1}.$$

Since u and $d_0 - u$ are bounded away from zero uniformly over the support of $\chi(\cdot) f(x_0, x_c, \cdot, \cdot)$, the Hessian matrix is never degenerate, i.e. the function $q(u)$ is bounded away from zero. Now for $s \in [0, 1]$, let us define the phase

$$\varphi_s(u, v) := -d_0 + \frac{1}{2} q(u)|v|^2 + s|v|^4 g(u, |v|),$$

so that $\varphi_0(u,v) = -d_0 + \frac{1}{2}q(u)|v|^2$ and $\varphi_s = \varphi$ for $s = 1$. We now write a Taylor formula with integral remainder with respect to the parameter s:

$$\begin{aligned}\int_{|v|\le\Delta} \chi(v)f(u,v)e^{i\omega\varphi_1(u,v)}\,dv = &\int_{|v|\le\Delta} \chi(v)f(u,v)e^{i\omega\varphi_0(u,v)}\,dv\\ &+ i\omega\int_0^1\int_{|v|\le\Delta} |v|^4 g(u,|v|)\chi(v)f(u,v)e^{i\omega\varphi_s(u,v)}\,dv\,ds\\ &:= I_{11}(u) + I_{12}(u).\end{aligned}$$

The term $I_{12}(u)$ is dealt with by applying lemma 5.1 with $(m,d) = (4, n-1)$, so $\frac{m+d}{2} = \frac{n+3}{2}$. Therefore we can bound by the following, everything being continuous with respect to the parameter s:

$$|I_{12}(u)| \le \frac{C}{\omega^{\frac{n+1}{2}}}\|f(x_0,x_c,\cdot,\cdot)\|_{\mathcal{C}^{\lceil\frac{n+3}{2}\rceil}}. \tag{5.6}$$

For the first integral $I_{11}(u)$, we write for $u \in [\delta_2, d_0 - \delta_2]$ and $v \in \mathbb{R}^{n-1}$

$$\chi(v)f(u,v) = f(u,0)\chi(v) + v\cdot f_1(u,v)\chi(v), \quad f_1(u,v) := \int_0^1 \nabla_v f(u,tv)\,dt,$$

and I_{11} becomes

$$\begin{aligned}I_{11}(u) = f(u,0)&\int_{\mathbb{R}^{n-1}} e^{i\omega\varphi_0(u,v)}\,dv - f(u,0)\int_{\mathbb{R}^{n-1}}(1-\chi(v))e^{i\omega\varphi_0(u,v)}\,dv\\ &+ \int_{|v|\le\Delta}\chi(v)v\cdot f_1(u,v)e^{i\omega\varphi_0(u,v)}\,dv\\ &:= I_{111}(u) + I_{112}(u) + I_{113}(u).\end{aligned} \tag{5.7}$$

The first term in (5.7) will give the leading-order term of the asymptotic expansion:

$$I_{111}(u) = \left(\frac{2\pi}{\omega}\right)^{\frac{n-1}{2}} e^{-i\omega d_0}\frac{e^{i\frac{\pi}{4}\sigma(H_\varphi(u,0))}}{|\det H_\varphi(u,0)|^{\frac{1}{2}}}f(u,0),$$

with $\sigma(H_\varphi(u,0)) = -(n-1)$ the signature of $H_\varphi(u,0)$, $|\det H_\varphi(u,0)| = d_0^{n-1}[u(d_0-u)]^{1-n}$, and we used the following formula (see [10, Sections 3.4, 7.6])

$$\int_{\mathbb{R}^d} e^{i\frac{1}{2}\langle Bx,x\rangle}\,dx = e^{i\frac{\pi}{4}\sigma(B)}\frac{(2\pi)^{\frac{d}{2}}}{|\det B|^{\frac{1}{2}}},$$

where B is a non-degenerate quadratic form.

The second term in (5.7) has rapid decay in ω because $1-\chi$ is supported away from $v=0$. We see that by noting the identity

$$\frac{1}{i\omega q(u)|v|^2} v\cdot\nabla_v\left(e^{i\omega\varphi_0}\right) = e^{i\omega\varphi_0},$$

so by means of $\lceil\frac{n+1}{2}\rceil$ integrations by parts, $I_{112}(u)$ becomes

$$I_{112}(u) = f(u,0)\left(\frac{-1}{i\omega q(u)}\right)^{\lceil\frac{n+1}{2}\rceil}\int_{\mathbb{R}^{n-1}} e^{i\omega\varphi_0}\left(\nabla_v\cdot\left(\frac{v}{|v|^2}\cdot\right)\right)^{\lceil\frac{n+1}{2}\rceil}[1-\chi(v)]\ dv,$$

which we can bound by

$$|I_{112}(u)| \leq \frac{C}{\omega^{\lceil\frac{n+1}{2}\rceil}}\|f\|_{\mathcal{C}^0}\|\chi\|_{\mathcal{C}^{\lceil\frac{n+1}{2}\rceil}}.$$

The third term is dealt with by integration by parts with zero boundary term:

$$I_{113}(u) = \frac{-e^{-i\omega d_0}}{i\omega q(u)}\int_{|v|\leq\Delta}(\nabla_v\cdot(\chi f_1))e^{i\omega\frac{1}{2}q(u)|v|^2}\ dv,$$

and we are now in the setting of lemma 5.1 with $(m,d)=(0,n-1)$, so we can bound

$$|I_{113}(u)| \leq \frac{1}{\omega}\frac{C}{\omega^{\frac{n-1}{2}}}\|\nabla_v\cdot(\chi f_1)\|_{\mathcal{C}^{\lceil\frac{n-1}{2}\rceil}} \leq \frac{C}{\omega^{\frac{n+1}{2}}}\|f(x_0,x_c,\cdot,\cdot)\|_{\mathcal{C}^{\lceil\frac{n+3}{2}\rceil}}.$$

Noting that we have

$$f(x_0,x_c,u,0) = E(x_0,x_c)k(x_0+ue_0,e_0,e_0)|\nu(x_0)\cdot e_0||\nu(x)\cdot e_0|[u(d_0-u)]^{-n+1},$$

we integrate relation (5.7) together with the estimate (5.6) in u and obtain the estimate

$$\begin{aligned} I_1(x_0,x_c) = {} & e^{i\omega d_0}E(x_0,x_c)|\nu(x_0)\cdot e_0||\nu(x)\cdot e_0|\left(\frac{2\pi}{\omega}\right)^{\frac{n-1}{2}} e^{-i(n-1)\frac{\pi}{4}} \\ & \times\int_0^{d_0}\frac{k(x_0+ue_0,e_0,e_0)}{[d_0u(d_0-u)]^{\frac{n-1}{2}}}\ du + R_1(x_0,x_c), \end{aligned}$$

$$\text{with}\quad |R_1(x_0,x_c)| \leq \frac{C}{\omega^{\frac{n+1}{2}}}\|f(x_0,x_c,\cdot)\|_{\mathcal{C}^{\lceil\frac{n+3}{2}\rceil}}.$$

For the integral I_2, let us introduce the differential operator $L^\star := \frac{1}{|\nabla_v\varphi|^2}\nabla_v\varphi\cdot\nabla_v$ such that $\frac{1}{i\omega}L^\star\left(e^{i\omega\varphi}\right) = e^{i\omega\varphi}$ (the $\star$ exponent denotes the formal adjoint). The function $(1-$

$\chi(v))f(x_0,x_c,u,v)$ is supported where $|v| \geq \frac{\delta}{2}$ and $\frac{1}{|\nabla_v\varphi|^2} \leq \frac{4\Delta^2}{\delta^2} < \infty$ there, so we can integrate by parts $\lceil\frac{n+1}{2}\rceil$ times and get that

$$I_2(x_0,x_c) = \left(\frac{-1}{i\omega}\right)^{\lceil\frac{n+1}{2}\rceil} \int_{-\Delta}^{\Delta}\int_{|v|\leq\Delta} e^{i\omega\varphi(u,v)} L^{\lceil\frac{n+1}{2}\rceil}\left[(1-\chi(v))f(x_0,x_c,u,v)\right]\, dv\, du,$$

where $Lg := \nabla_v \cdot (g\frac{\nabla_v\varphi}{|\nabla_v\varphi|^2})$, from which we deduce the estimate

$$|I_2(x_0,x_c)| \leq \frac{C}{\omega^{\lceil\frac{n+1}{2}\rceil}} \|f(x_0,x_c,\cdot)\|_{\mathcal{C}^{\lceil\frac{n+1}{2}\rceil}}.$$

This completes the proof. $\square$

6 The multiple scattering estimate

For $m \geq 2$, the multiple scattering of order m is given by

$$\begin{aligned} T_m^\omega(x_0,x_c) = &\int_{X^m} e^{-i\omega(\sum_{i=0}^{m-1}|x_i-x_{i+1}|+|x_m-x_c|)} \\ &\times \frac{E(x_0,x_1,\ldots,x_m,x_c)\Pi_{i=1}^m k(x_i,v_{i-1},v_i)}{\Pi_{i=0}^{m-1}|x_i-x_{i+1}|^{n-1}|x_m-x_c|^{n-1}} |v_0\cdot\nu_{x_0}||v_m\cdot\nu_{x_c}|\, dx_1\ldots\, dx_m, \end{aligned} \tag{6.1}$$

where $v_i = \widehat{x_{i+1}-x_i}, i = 0\ldots m-1$ and $v_m = \widehat{x_c - x_m}$. The following theorem bounds each T_m^ω separately.

Theorem 6.1. *Let $(\sigma,k) \in \mathcal{C}^{\lceil\frac{n+1}{2}\rceil}(\bar{X}\times\mathbb{S}^{n-1})\times\mathcal{C}^{\lceil\frac{n+1}{2}\rceil}(X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$, and assume hypothesis 2.1. Then for any integer $m \geq 2$ there exists a constant C_m such that for any $\omega \geq 2$, $T_m^\omega(x_0,x_c)$ is uniformly bounded in $\partial X \times \partial X$ by*

$$|T_m^\omega(x_0,x_c)| \leq \begin{cases} C_m\ \omega^{-1}, & n=2, \\ C_m\ \omega^{-2}\ln(\omega), & n=3, \\ C_m\ \omega^{-\frac{n+1}{2}}, & n\geq 4. \end{cases} \tag{6.2}$$

We relegate the proof of this theorem to the next section. We must now prove that one can control the remainder R_{N+1} of the decomposition (3.12) for N large enough. In order to do that, let us define the operator $\overline{\mathcal{K}_\omega}$ for $\phi \in L^1(X\times\mathbb{S}^{n-1})$ as

$$\overline{\mathcal{K}_\omega}\phi(x) = \int_{\mathbb{S}^{n-1}_{x,+}} (\mathcal{K}_\omega\phi)_{|\Gamma_+}(x,v)|\nu_x\cdot v|\, dv, \quad x\in\partial X. \tag{6.3}$$

Then the following lemma shows that $\overline{\mathcal{K}_\omega}$ is well-defined in $\mathcal{L}^{1,\infty} := \mathcal{L}(L^1(X\times\mathbb{S}^{n-1}), L^\infty(\partial X))$.

Lemma 6.2. *Under hypothesis 2.1 and assuming that k is bounded, the operator $\overline{\mathcal{K}_\omega} \in \mathcal{L}^{1,\infty}$ and satisfies the estimate (independent of ω)*

$$\|\overline{\mathcal{K}_\omega}\|_{\mathcal{L}^{1,\infty}} \leq D^{1-n}\|k\|_\infty. \tag{6.4}$$

Proof. Let $\phi \in L^1(X \times \mathbb{S}^{n-1})$ and $x \in \partial X$. Starting from (6.3), we have

$$\begin{aligned}
\overline{\mathcal{K}_\omega}\phi(x) &= \int_{\mathbb{S}^{n-1}_{x,+}} \int_0^{\tau_-(x,v)} e^{-i\omega t} E(x-tv,x) \int_{\mathbb{S}^{n-1}} k(x-tv,v',v)\phi(x-tv,v')|\nu_x \cdot v| \, dv' \, dt \, dv \\
&= \int_{X\times\mathbb{S}^{n-1}} \frac{e^{-i\omega|x-x'|}}{|x-x'|^{n-1}} k(x',v',\widehat{x-x'})E(x',x)\phi(x',v')|\nu_x \cdot \widehat{x-x'}| \, dx' \, dv',
\end{aligned}$$

where we changed variables $x' = x - tv, v \in \mathbb{S}^{n-1}_{x,+}, t \in [0, \tau_-(x,v)]$. The result is immediate since $|E(x',x)||\nu_x \cdot \widehat{x-x'}| \leq 1$ and $|x'-x| \geq D$ for all $x \in \partial X$ and $x' \in \mathrm{supp}_X k$. □

Proof of Theorem 2.5. Let $\omega \geq \omega_0$, $\omega \geq 2$, where ω_0 is given in Corollary 3.3. We write the following decomposition:

$$\begin{aligned}
T^\omega_{2+}(x_0,x_c) &= \sum_{m=2}^{2\alpha+1} T^\omega_m(x_0,x_c) + R_{2\alpha+2}(x_0,x_c), \qquad \text{where} \\
R_{2\alpha+2}(x_0,x_c) &= \overline{\mathcal{K}_\omega}\mathcal{K}^{2\alpha}_\omega(I-\mathcal{K}_\omega)^{-1}\mathcal{K}_\omega J_\omega g_{x_0}(x_c),
\end{aligned} \tag{6.5}$$

and where the integer α will be chosen at the end. For the first finite number of terms in (6.5), we bound each of them separately using Theorem 6.1 and obtain

$$\left\| \sum_{m=2}^{2\alpha+1} T^\omega_m(x_0,x_c) \right\|_\infty \leq \begin{cases} C\ \omega^{-1}, & n=2, \\ C\ \omega^{-2}\ln(\omega), & n=3, \\ C\ \omega^{-\frac{n+1}{2}}, & n \geq 4. \end{cases} \tag{6.6}$$

For the remainder term $R_{2\alpha+2}(x_0,x_c)$,

$$\begin{aligned}
\sup_{x_c\in\partial X} |R_{2\alpha+2}(x_0,x_c)| &\leq \|\overline{\mathcal{K}_\omega}\|_{\mathcal{L}^{1,\infty}} \|\mathcal{K}^2_\omega\|^\alpha_{\mathcal{L}^1} \|(I-\mathcal{K}_\omega)^{-1}\|_{\mathcal{L}^1} \|\mathcal{K}_\omega J_\omega g_{x_0}\|_{L^1(X\times\mathbb{S}^{n-1})}. \\
&\leq C_0 D^{1-n}\|k\|_\infty \|\mathcal{K}_\omega J_\omega g_{x_0}\|_{L^1(X\times\mathbb{S}^{n-1})} \begin{cases} \omega^{-\frac{\alpha}{2}}(\ln\omega)^\alpha, & n=2, \\ \omega^{-\alpha}(\ln\omega)^{2\alpha}, & n=3, \\ \omega^{-\alpha}(\ln\omega)^\alpha, & n\geq 4, \end{cases}
\end{aligned}$$

where we used lemma 6.2 and corollary 3.3. Choosing $\alpha = 3$ for $n = 2, 3$ and $\alpha = \lceil \frac{n+3}{2} \rceil$ if $n \geq 4$ ensures that the latter decays are faster than those in (6.6). Taking the supremum over $x_0 \in \partial X$ satisfies the same estimate because $\|K_\omega J_\omega g_{x_0}\|_{L^1(X\times\mathbb{S}^{n-1})}$ is uniformly bounded in x_0 as stated in (3.7), and the proof is complete. □

7 Proof of theorem 6.1

Changes of variable: Let us put ourselves in the appropriate geometry: denote $e_1 := \widehat{x_c - x_1}$ (always well-defined and real analytic for $x_1 \in \mathrm{supp}_X k$, since $|x_c - x_1| \geq D$), and write the following changes of variables:

$$x_1(\sigma, \Omega_1) = x_0 + \frac{\sigma^2(\Omega_1 \cdot e_0) + d_0((\Omega_1 \cdot e_0) + 1)}{2} e_0 + \frac{\sigma\sqrt{\sigma^2 + 2d_0}}{2}(\Omega_1 - (\Omega_1 \cdot e_0)e_0), \tag{7.1}$$

with $\sigma \in (0, \infty), \Omega_1 \in \mathbb{S}^{n-1}$. Formula (7.1) is an elliptic change of variable, where for given $\sigma \geq 0$, x_1 describes the ellipsoid of equation $|x_1 - x_0| + |x_1 - x_c| = \sigma^2 + d_0$. Moreover we have the following equalities

$$|x_1 - x_0| = \frac{1}{2}\left(\sigma^2 + d_0(1 + (\Omega_1 \cdot e_0))\right), \quad d_1 := |x_1 - x_c| = \frac{1}{2}\left(\sigma^2 + d_0(1 - (\Omega_1 \cdot e_0))\right), \tag{7.2}$$

$$dx_1 = 2^{1-n}\sigma^{n-2}(\sigma^2 + 2d_0)^{\frac{n-3}{2}}((\sigma^2 + d_0)^2 - d_0^2(\Omega_1 \cdot e_0)^2)\, d\sigma\, d\Omega_1, \tag{7.3}$$

where $d\Omega_1$ denotes the standard measure on the sphere. Now for $j = 2, \ldots, m$, we write

$$x_j = x_{j-1} + r_j\big(\Omega_{j,1}e_1 + \sum_{l=2}^{n}\Omega_{j,l}V_l(e_1)\big), \quad r_j \in [0, \infty), \quad \Omega_j := (\Omega_{j,1}, \ldots, \Omega_{j,n}) \in \mathbb{S}^{n-1}, \tag{7.4}$$

where $V_l \in \mathcal{C}^\infty(\overline{\mathbb{S}^{n-1}_{x_c,+}})$, $l = 2\ldots, n$ such that $(v, V_2(v), \ldots, V_n(v))$ is an orthonormal basis of $\mathbb{R}^n$ for any $v \in \mathbb{S}^{n-1}_{x,+}$. (When $n = 2$ one can choose $V_2(v) = (-v_2, v_1)$ for $v = (v_1, v_2) \in \mathbb{S}^1$.) Note that the vectors $V_l(e_1)$ vary when e_1 varies.

Then at fixed $x_1(\sigma, \Omega_1) \in \mathrm{supp}_X k$ we have $dx_j = r_j^{n-1}\, dr_j\, d\Omega_j$. Here and below, we will use the following notations $\mathbf{r} := (r_2, \ldots, r_m)$, $\boldsymbol{\Omega} := (\Omega_1, \ldots, \Omega_m)$, and for a given integer $2 \leq \alpha \leq m$, $\mathbf{r}^\alpha := (r_2, \ldots, r_\alpha)$ and $\boldsymbol{\Omega}^\alpha := (\Omega_1, \ldots, \Omega_\alpha)$.

In this new set of variables, we have that

$$\begin{aligned} |x_m - x_c| &= \left| x_1 - x_c + \sum_{j=2}^{m} r_j\big(\Omega_{j,1}e_1 + \sum_{l=2}^{n}\Omega_{j,l}V_l(e_1)\big)\right| \\ &= \left(\left(d_1 - \sum_{j=2}^{m} r_j\Omega_{j,1}\right)^2 + \sum_{l=2}^{n}\Big(\sum_{j=2}^{m} r_j\Omega_{j,l}\Big)^2\right)^{\frac{1}{2}}. \end{aligned} \tag{7.5}$$

Then we define the phase function φ on $[0,+\infty)^m \times (\mathbb{S}^{n-1})^m$ by

$$\begin{aligned}\varphi(\sigma, \mathbf{r}, \mathbf{\Omega}) &= \frac{1}{2}\left(\sigma^2 + d_0(1 + (\Omega_1 \cdot e_0))\right) + \sum_{j=2}^{m} r_j \\ &\quad + \left(\left(\frac{1}{2}\left(\sigma^2 + d_0(1 - (\Omega_1 \cdot e_0))\right) - \sum_{j=2}^{m} r_j \Omega_{j,1}\right)^2 + \sum_{l=2}^{n}\Big(\sum_{j=2}^{m} r_j \Omega_{j,l}\Big)^2\right)^{\frac{1}{2}}.\end{aligned} \tag{7.6}$$

The phase function φ is a continuous function on $[0,+\infty)^m \times (\mathbb{S}^{n-1})^m$. In addition from (7.2), (7.5) and (7.6) it follows that $\varphi \in \mathcal{C}^\infty(\mathcal{D})$ where

$$\mathcal{D} := \{(\sigma, \mathbf{r}, \mathbf{\Omega}) \in [0,+\infty)^m \times (\mathbb{S}^{n-1})^m) \mid x_1(\sigma, \Omega_1) \neq x_c \text{ and } x_m(\sigma, \mathbf{r}, \mathbf{\Omega}) \neq x_c\}.$$

And the first derivatives of φ are uniformly bounded on $\mathcal{D}$.

The multiple scattering integral becomes

$$T_m^\omega(x_0, x_c) = \int_{(0,\infty)^m \times (\mathbb{S}^{n-1})^m} e^{-i\omega\varphi(\sigma,\mathbf{r},\mathbf{\Omega})} \sigma^{n-2} g(\sigma, \mathbf{r}, \mathbf{\Omega})\, d\sigma\, d\mathbf{r}\, d\mathbf{\Omega}, \tag{7.7}$$

where the integrand g is given by

$$\begin{aligned}g(\sigma, \mathbf{r}, \mathbf{\Omega}) &= \frac{E(x_0, x_1, \ldots, x_m, x_c)}{|x_c - x_m|^{n-1}} |\nu_{x_0} \cdot \widehat{x_1 - x_0}||\nu_{x_c} \cdot \widehat{x_c - x_m}| \prod_{j=1}^{m} k(x_j, \widehat{x_j - x_{j-1}}, \widehat{x_{j+1} - x_j}) \\ &\quad \times 2^{1-n}(\sigma^2 + 2d_0)^{\frac{n-3}{2}}((\sigma^2 + d_0)^2 - d_0^2(\Omega_1 \cdot e_0)^2)\end{aligned} \tag{7.8}$$

with $x_1 = x_1(\sigma, \Omega_1)$ and $x_j = x_j(\sigma, \mathbf{r}^j, \mathbf{\Omega}^j)$ for $j = 2, \ldots, m$ defined by (7.1) and (7.4) and $x_{m+1} = x_c$. Moreover $g \in \mathcal{C}^{\lceil \frac{n+1}{2} \rceil}([0,+\infty)^m \times (\mathbb{S}^{n-1})^m)$ and g is compactly supported in the set

$$\left\{(\sigma, \mathbf{r}, \mathbf{\Omega}) \in (0, \sqrt{2\Delta}) \times (0, \Delta)^{m-1} \times (\mathbb{S}^{n-1})^m\right\}.$$

More precisely

$$\begin{aligned}\operatorname{supp} g = \Big\{&(\sigma, \mathbf{r}, \mathbf{\Omega}) \in [0, \sqrt{2\Delta}) \times [0, \Delta)^{m-1} \times (\mathbb{S}^{n-1})^m) \mid x_1(\sigma, \Omega_1) \in \operatorname{supp}_X k, \\ &x_j(\sigma, \mathbf{r}^j, \mathbf{\Omega}^j) \in \operatorname{supp}_X k,\ j = 2, \ldots, m\Big\}.\end{aligned}$$

The $\mathcal{C}^{\lceil \frac{n+1}{2} \rceil}$ norm of g is bounded by a constant depending on the $\mathcal{C}^{\lceil \frac{n+1}{2} \rceil}$ norm of σ and k.

We now give properties of the phase function in terms of σ and r_m. First note that we have

$$\frac{\partial \varphi}{\partial \sigma} = \sigma \Phi, \quad \text{where } \Phi := 1 + \widehat{x_c - x_1} \cdot \widehat{x_c - x_m}, \text{ for } (\sigma, \mathbf{r}, \mathbf{\Omega}) \in \mathcal{D}. \tag{7.9}$$

The next lemma shows that the function Φ is uniformly bounded away from zero on the set $(x_1, x_m, x_c) \in (\operatorname{supp}_X k)^2 \times \partial X$.

Lemma 7.1. *Assuming hypothesis 2.1, there exists a constant $C_D > 0$ such that*

$$1 + \widehat{x_c - x_1} \cdot \widehat{x_c - x_m} \geq C_D \tag{7.10}$$

for all $(x_1, x_m, x_c) \in (supp_X k)^2 \times \partial X$.

Proof. By contradiction, assume that (7.10) does not hold. Then using compactness of $(\mathrm{supp}_X k)^2$ and ∂X, and the continuity of inner products, one can construct limit points $(x_1^\star, x_m^\star, x_c^\star) \in (\mathrm{supp}_X k)^2 \times \partial X$ satisfying $1 + \widehat{x_c - x_1} \cdot \widehat{x_c - x_m} = 0$. This means that $x_c^\star \in [x_1^\star, x_m^\star]$, which by convexity of X implies $x_1^\star \notin X$ or $x_m^\star \notin X$, and this is a contradiction. Hence the result. $\square$

Let us introduce the differential operator L_σ defined by

$$L_\sigma f = \frac{1}{i} \frac{\partial}{\partial \sigma} \left(\frac{f}{\sigma \Phi} \right). \tag{7.11}$$

In addition straightforward computations show that the phase function φ satisfies the following equality

$$\varphi(\sigma, \mathbf{r}, \mathbf{\Omega}) = \varphi(0, \mathbf{r}, \mathbf{\Omega}) + \frac{1}{2} \sigma^2 \Psi(\sigma, \mathbf{r}, \mathbf{\Omega}), \ (\sigma, \mathbf{r}, \mathbf{\Omega}) \in \mathcal{D}, \tag{7.12}$$

where

$$\Psi(\sigma, \mathbf{r}, \mathbf{\Omega}) = \frac{|x_m(\sigma, \mathbf{r}, \mathbf{\Omega}) - x_c| \Phi(\sigma, \mathbf{r}, \mathbf{\Omega}) + |\tilde{x}_m(\mathbf{r}, \mathbf{\Omega}) - x_c| - (\tilde{x}_m(\mathbf{r}, \mathbf{\Omega}) - x_c) \cdot e_0}{|x_m(\sigma, \mathbf{r}, \mathbf{\Omega}) - x_c| + |\tilde{x}_m(\mathbf{r}, \mathbf{\Omega}) - x_c|} \tag{7.13}$$

where $\tilde{x}_m(\mathbf{r}, \mathbf{\Omega}) = x_0 + \big(2^{-1} d_0 (1 + \Omega_1 \cdot e_0) + \sum_{j=2}^m r_j \Omega_{j,1}\big) e_0 + \sum_{\substack{j=2\ldots m \\ l=2\ldots n}} r_j \Omega_{j,l} V_l(e_0)$. We have $\tilde{x}_m(\mathbf{r}, \mathbf{\Omega}) = x_m(0, \mathbf{r}, \mathbf{\Omega})$ when $(0, \mathbf{r}, \mathbf{\Omega}) \in \mathcal{D}$.

Note that $\frac{\partial^\alpha \Psi}{\partial \sigma^\alpha} \in \mathcal{C}(\mathcal{D})$, $\frac{\partial^{\alpha+1} \Psi}{\partial \sigma^\alpha \partial r_m} \in L^\infty_{\mathrm{loc}}(\mathcal{D})$ for any $\alpha \in \mathbb{N}$. In addition the function Ψ satisfies the following estimates.

Lemma 7.2. *Assuming hypothesis 2.1, we have*

$$0 < \frac{D C_D}{4 \Delta} \leq \Psi(\sigma, \mathbf{r}, \mathbf{\Omega}) \leq 2, \ \textit{for } (\sigma, \mathbf{r}, \mathbf{\Omega}) \in \textit{supp } g, \tag{7.14}$$

where C_D is the constant on the right-hand side of (7.10).

Proof of lemma 7.2. The estimate $\Psi(\sigma, \mathbf{r}, \mathbf{\Omega}) \leq 2$, for $(\sigma, \mathbf{r}, \mathbf{\Omega}) \in \mathrm{supp}\, g$, follows from (7.13) and the fact that $\Phi \leq 2$ uniformly in $(\sigma, \mathbf{r}, \mathbf{\Omega})$. From (7.13) and from Lemma 7.1 it also follows that

$$\Psi(\sigma, \mathbf{r}, \mathbf{\Omega}) \geq \frac{C_D |x_m(\sigma, \mathbf{r}, \mathbf{\Omega}) - x_c|}{|x_m(\sigma, \mathbf{r}, \mathbf{\Omega}) - x_c| + |\tilde{x}_m(\mathbf{r}, \mathbf{\Omega}) - x_c|}, \tag{7.15}$$

for $(\sigma, \mathbf{r}, \mathbf{\Omega}) \in \operatorname{supp} g$. Then note that

$$
\begin{aligned}
|\tilde{x}_m(\mathbf{r}, \mathbf{\Omega}) - x_c| &= \left(\left(\frac{1}{2} \left(d_0(1 - (\Omega_1 \cdot e_0)) \right) - \sum_{j=2}^{m} r_j \Omega_{j,1} \right)^2 + \sum_{l=2}^{n} \Big(\sum_{j=2}^{m} r_j \Omega_{j,l} \Big)^2 \right)^{\frac{1}{2}} \\
&\leq \frac{\sigma^2}{2} + \left| \frac{1}{2}(\sigma^2 + d_0(1 - (\Omega_1 \cdot e_0))) - \sum_{j=2}^{m} r_j \Omega_{j,1} \right| + \left(\sum_{l=2}^{n} \left(\sum_{j=2}^{m} r_j \Omega_{j,l} \right)^2 \right)^{\frac{1}{2}} \\
&\leq \frac{\sigma^2}{2} + \sqrt{2} |x_m(\sigma, \mathbf{r}, \mathbf{\Omega}) - x_c| \leq 3\Delta,
\end{aligned} \tag{7.16}
$$

for $(\sigma, \mathbf{r}, \mathbf{\Omega}) \in \operatorname{supp} g$ (we use the estimate $\frac{\sigma^2}{2} \leq \Delta$). The estimate (7.13) follows from (7.15),(7.16) and from the estimate $|x_m - x_c| \geq D$ for $x_m \in \operatorname{supp}_X k$. $\square$

For further use, we also rewrite

$$
\begin{aligned}
T_m^\omega(x_0, x_c) &= \int_{(0,\Delta)^{m-2} \times (\mathbb{S}^{n-1})^{m-1}} I(\mathbf{r}^{m-1}, \mathbf{\Omega}^{m-1}) \; d\mathbf{r}^{m-1} \; d\mathbf{\Omega}^{m-1}, \qquad \text{where} \\
I(\mathbf{r}^{m-1}, \mathbf{\Omega}^{m-1}) &= \int_0^\infty \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{-i\omega\varphi(\sigma, \mathbf{r}, \mathbf{\Omega})} \sigma^{n-2} g(\sigma, \mathbf{r}, \mathbf{\Omega}) \; d\Omega_m \; dr_m \; d\sigma.
\end{aligned} \tag{7.17}
$$

Control of the integral in (r_m, Ω_m) Consider the following integral

$$
I_m(\sigma, \mathbf{r}^{m-1}, \mathbf{\Omega}^{m-1}) = \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{-i\omega\varphi} h(\sigma, \mathbf{r}, \mathbf{\Omega}) \; d\Omega_m \; dr_m, \tag{7.18}
$$

for given $(\sigma, \mathbf{r}^{m-1}, \mathbf{\Omega}^{m-1})$, which implies that $x_1, \ldots, x_{m-1}$ are fixed. The function h is an arbitrary bounded integrand whose dependency on r_m is compactly supported in $[0, \Delta)$, and h satisfies the following bounds, uniform in all variables $(\sigma, \mathbf{r}, \mathbf{\Omega})$:

$$
\max \left(\|h\|_\infty, \left\| \frac{\partial h}{\partial r_m} \right\|_\infty \right) < \infty.
$$

For $(\sigma, \mathbf{r}, \mathbf{\Omega}) \in \mathcal{D}$, the phase function satisfies

$$
\frac{\partial \varphi}{\partial r_m} = 1 + \Omega_m \cdot \widehat{x_m - x_c}, \quad x_m = x_m(\sigma, \mathbf{r}, \mathbf{\Omega}), \quad \Omega_m = \big(\Omega_{m,1} e_1 + \sum_{j=2}^{n} \Omega_{m,j} V_j(e_1) \big), \tag{7.19}
$$

so $\dfrac{\partial \varphi}{\partial r_m} = 0$ whenever $\Omega_m = e_{m-1} := \widehat{x_c - x_{m-1}}$. This illustrates the fact that $|x_m - x_{m-1}| + |x_m - x_c|$ does not depend on $r_m = |x_m - x_{m-1}|$ when $x_m \in [x_{m-1}, x_c]$. This fact motivates the change of variables

$$
\Omega_m = \cos\theta \; e_{m-1} + \sin\theta \; w_m, \quad \theta \in (0, \pi), \; w_m \in \mathbb{S}^{n-2}(e_{m-1}^\perp), \tag{7.20}
$$

with change of volume $d\Omega_m = \sin^{n-2}\theta \; d\theta \; dw_m$. In these new variables, we have that

$$\frac{1}{\partial_{r_m}\varphi} = \frac{d_m(d_m - r_m + d_{m-1}\cos\theta)}{d_{m-1}^2 \sin^2\theta} = \frac{h_m(r_m,\theta)}{\sin^2\theta}, \tag{7.21}$$

with h_m a smooth function on the support of h. The integral (7.18) becomes

$$I_m(\sigma, \mathbf{r}^{m-1}, \mathbf{\Omega}^{m-1}) = \int_{\mathbb{S}^{n-2}} \int_0^\pi \sin^{n-2}\theta \int_0^\Delta e^{-i\omega\varphi} h(\sigma, \mathbf{r}, \mathbf{\Omega}^{m-1}, \Omega_m(\theta, w_m)) \; dr_m \; d\theta \; dw_m \tag{7.22}$$

(where $\int_{\mathbb{S}^0} f(w)dw := f(-1) + f(1)$).

We now split cases according to the dimension n.

Case $n = 2$**:** we split the integral in θ as follows: $\int_0^\pi = \int_0^\varepsilon + \int_\varepsilon^{\pi-\varepsilon} + \int_{\pi-\varepsilon}^\pi$, $0 < \varepsilon \le 2^{-1}$. In the second term, we write $e^{-i\omega\varphi} = \frac{-1}{i\omega\partial_{r_m}\varphi}\dfrac{\partial}{\partial r_m}e^{-i\omega\varphi}$ and integrate by parts in r_m. We get

$$\begin{aligned} I_m &= \int_{\mathbb{S}^0}\int_0^\varepsilon\int_0^\Delta e^{-i\omega\varphi}h \; dr_m \; d\theta \; dw_m + \int_{\mathbb{S}^0}\int_{\pi-\varepsilon}^\pi\int_0^\Delta e^{-i\omega\varphi}h \; dr_m \; d\theta \; dw_m \\ &+ \frac{1}{i\omega}\int_{\mathbb{S}^0}\int_\varepsilon^{\pi-\varepsilon}\frac{1}{\sin^2\theta}\left\{\left[h_m h e^{-i\omega\varphi}\right]_{r_m=0} - \int_0^\Delta \frac{\partial}{\partial r_m}(h_m h)\, e^{-i\omega\varphi} \; dr_m\right\} \; d\theta \; dw_m. \end{aligned}$$

Thus

$$|I_m| \le C\max\left(\|h\|_\infty, \left\|\frac{\partial h}{\partial r_m}\right\|_\infty\right)\left(\varepsilon + \frac{1}{\varepsilon\omega}\right) \le C\max\left(\|h\|_\infty, \left\|\frac{\partial h}{\partial r_m}\right\|_\infty\right)\omega^{-\frac{1}{2}}$$

by choosing $\varepsilon = \omega^{-\frac{1}{2}}$.

Case $n = 3$**:** splitting the integral in a similar way to $n = 2$, we get

$$\begin{aligned} I_m &= \int_{\mathbb{S}^1}\int_0^\varepsilon \sin\theta\int_0^\Delta e^{-i\omega\varphi}h \; d\rho \; d\theta \; dw_m + \int_{\mathbb{S}^1}\int_{\pi-\varepsilon}^\pi \sin\theta\int_0^\Delta e^{-i\omega\varphi}h \; d\rho \; d\theta \; dw_m \\ &+ \frac{1}{i\omega}\int_{\mathbb{S}^1}\int_\varepsilon^{\pi-\varepsilon}\frac{1}{\sin\theta}\left\{\left[h_m h e^{-i\omega\varphi}\right]_{r_m=0} - \int_0^\Delta \frac{\partial}{\partial r_m}(h_m h)\, e^{-i\omega\varphi} \; dr_m\right\} \; d\theta \; dw_m. \end{aligned}$$

Thus

$$|I_m| \le C\max\left(\|h\|_\infty, \left\|\frac{\partial h}{\partial r_m}\right\|_\infty\right)\left(\varepsilon^2 + \frac{|\ln\varepsilon|}{\omega}\right) \le C\max\left(\|h\|_\infty, \left\|\frac{\partial h}{\partial r_m}\right\|_\infty\right)\frac{\ln\omega}{\omega}$$

by choosing $\varepsilon = \omega^{-1}$.

Case $n \geq 4$: we integrate by part without splitting the integral, and obtain

$$I_m = \frac{1}{i\omega}\int_{\mathbb{S}^{n-2}}\int_0^{\pi}\sin^{n-4}\theta\left\{\left[h_m h e^{-i\omega\varphi}\right]_{r_m=0} - \int_0^{\Delta}\frac{\partial}{\partial r_m}\left(h_m h\right)e^{-i\omega\varphi}\,dr_m\right\}\,d\theta\,dw_m, \quad \text{so}$$

$$|I_m| \leq C\max\left(\|h\|_\infty, \left\|\frac{\partial h}{\partial r_m}\right\|_\infty\right)\frac{1}{\omega}.$$

Summary: To summarize, we have that

$$|I_m| \leq \max\left(\|h\|_\infty, \left\|\frac{\partial h}{\partial r_m}\right\|_\infty\right)\begin{cases} C\ \omega^{-\frac{1}{2}} & n=2, \\ C\ \omega^{-1}\ln\omega & n=3, \\ C\ \omega^{-1} & n\geq 4. \end{cases} \tag{7.23}$$

Completion of the proof: For fixed $(\mathbf{r}^{m-1}, \mathbf{\Omega}^{m-1})$, let us study the integral $I(\mathbf{r}^{m-1}, \mathbf{\Omega}^{m-1})$ from expression (7.17) and study cases according to the dimension n.

Case n even: Using the differential operator L_σ defined in (7.11), we integrate by parts w.r.t. σ with zero boundary terms $\frac{n-2}{2}$ times. One can show that we can write

$$L_\sigma^{\frac{n-2}{2}}\left(\sigma^{n-2}g\right) = g_1(\sigma, \mathbf{r}, \mathbf{\Omega}),$$

and the integral I becomes

$$I(\mathbf{r}^{m-1}, \mathbf{\Omega}^{m-1}) = \frac{1}{\omega^{\frac{n-2}{2}}}\int_0^{\infty}\int_{\mathbb{S}^{n-1}}\int_0^{\infty} e^{-i\omega\varphi(\sigma,\mathbf{r},\mathbf{\Omega})} g_1(\sigma, \mathbf{r}, \mathbf{\Omega})\,d\sigma\,d\Omega_m\,dr_m. \tag{7.24}$$

Then we perform the change of variables

$$\eta = \sigma\sqrt{\Psi(\sigma, \mathbf{r}, \mathbf{\Omega})} = \sqrt{2(\varphi(\sigma, \mathbf{r}, \mathbf{\Omega}) - \varphi(0, \mathbf{r}, \mathbf{\Omega}))}, \tag{7.25}$$

for the integral over the σ variable where Ψ is defined by (7.12) and satisfies (7.14). We have

$$\frac{\Phi(\sigma, \mathbf{r}, \mathbf{\Omega})}{\sqrt{2\Psi(\sigma, \mathbf{r}, \mathbf{\Omega})}}\,d\sigma = d\eta,$$

where the function Φ is defined by (7.9) and satisfies (7.10) on supp g. And we obtain

$$I(\mathbf{r}^{m-1}, \mathbf{\Omega}^{m-1}) = \frac{1}{\omega^{\frac{n-2}{2}}}\int_0^{\infty}\int_{\mathbb{S}^{n-1}} e^{-i\omega\varphi(0,\mathbf{r},\mathbf{\Omega})}\int_0^{\infty} e^{-i\omega\frac{\eta^2}{2}} g_2(\eta, \mathbf{r}, \mathbf{\Omega})\,d\eta\,d\Omega_m\,dr_m, \tag{7.26}$$

where $g_2 \in C([0,\infty)^m \times (\mathbb{S}^{n-1})^m)$, $\max_{\alpha=0,1}\left(\left\|\frac{\partial^\alpha g_2}{\partial \eta^\alpha}\right\|_\infty, \left\|\frac{\partial^{\alpha+1} g_2}{\partial \eta^\alpha \partial r_m}\right\|_\infty\right) \leq C\|g\|_{\mathcal{C}^{\frac{n}{2}+1}}$ for some constant C. The function g_2 is also compactly supported in $[0, 2\sqrt{2\Delta}) \times [0,\Delta)^{m-1} \times (\mathbb{S}^{n-1})^m$. Then integrating by parts in the η variable on the right hand side of (7.26) we obtain

$$\begin{aligned} I(\mathbf{r}^{m-1}, \mathbf{\Omega}^{m-1}) &= \sqrt{\frac{\pi}{2}} e^{-i\frac{\pi}{4}} \frac{1}{\omega^{\frac{n-1}{2}}} \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{-i\omega\varphi(0,\mathbf{r},\mathbf{\Omega})} g_2(0,\mathbf{r},\mathbf{\Omega})\, d\eta\, d\Omega_m\, dr_m \\ &+ \frac{1}{\omega^{\frac{n-1}{2}}} \int_0^\infty \int_{\mathbb{S}^{n-1}} \int_0^\infty \tilde{H}(\sqrt{\omega}\eta) e^{-i\omega\varphi(\sigma(\eta,\mathbf{r},\mathbf{\Omega}),\mathbf{r},\mathbf{\Omega})} \frac{\partial g_2}{\partial \eta}(\eta,\mathbf{r},\mathbf{\Omega})\, d\eta\, d\Omega_m\, dr_m, \end{aligned} \tag{7.27}$$

where $\tilde{H}(s) = \int_s^{+\infty} e^{-i\frac{t^2}{2}} dt$, $s \in [0,+\infty)$, is uniformly bounded on $[0,+\infty)$, and where $\sigma(\eta,\mathbf{r},\mathbf{\Omega})$ is the unique solution of (7.25) at fixed $(\mathbf{r},\mathbf{\Omega})$ (we also used the equality $\int_0^\infty e^{-i\omega\frac{\eta^2}{2}} d\eta = \sqrt{\frac{\pi}{2}} e^{-i\frac{\pi}{4}} \omega^{-\frac{1}{2}}$).

Hence the integral in σ in (7.24) is the sum of two terms. To each of these terms, we apply the estimate (7.23) to the integral in the (r_m, Ω_m) variables, and obtain

$$|I| \leq \max_{0\leq\alpha\leq\frac{n}{2}} \left(\left\|\frac{\partial^\alpha g}{\partial\sigma^\alpha}\right\|_\infty, \left\|\frac{\partial^{1+\alpha} g}{\partial\sigma^\alpha \partial r_m}\right\|_\infty\right) \begin{cases} C\ \omega^{-1} & n=2, \\ C\ \omega^{-\frac{n+1}{2}} & n\geq 4, n \text{ even}. \end{cases} \tag{7.28}$$

Case n odd: Using again the operator L_σ defined in (7.11), we integrate by parts w.r.t. σ with zero boundary terms $\frac{n-3}{2}$ times. One can show that we can write

$$L_\sigma^{\frac{n-3}{2}}\left(\sigma^{n-2} g\right) = \sigma g_3(\sigma,\mathbf{r},\mathbf{\Omega}),$$

and the integral I becomes

$$I(\mathbf{r}^{m-1}, \mathbf{\Omega}^{m-1}) = \frac{1}{\omega^{\frac{n-3}{2}}} \int_0^\infty \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{-i\omega\varphi(\sigma,\mathbf{r},\mathbf{\Omega})} \sigma g_3(\sigma,\mathbf{r},\mathbf{\Omega})\, d\Omega_m\, dr_m\, d\sigma.$$

Integrating by parts once again, we arrive at

$$I(\mathbf{r}_2^{m-1}, \mathbf{\Omega}_1^{m-1}) = \frac{1}{\omega^{\frac{n-1}{2}}} \int_0^\Delta \int_{\mathbb{S}^{n-1}} \left\{ -\left[e^{-i\omega\varphi} \frac{g_3}{i\Phi}\right]_{\sigma=0} + \int_0^{\sqrt{2\Delta}} e^{-i\omega\varphi} \frac{\partial}{\partial\sigma}\left(\frac{g_3}{i\Phi}\right) d\sigma \right\} d\Omega_m\, dr_m.$$

We then use the estimate (7.23) to treat the integral in the (r_m, Ω_m) variables, and obtain

$$|I| \leq \max_{0\leq\alpha_1\leq\frac{n-3}{2}, 0\leq\alpha_2\leq\frac{n-1}{2}} \left(\left\|\frac{\partial^{\alpha_1} g}{\partial\sigma^{\alpha_1}}\right\|_\infty, \left\|\frac{\partial^{1+\alpha_2} g}{\partial\sigma^{\alpha_2}\partial r_m}\right\|_\infty\right) \begin{cases} C\ \omega^{-2}\ln\omega & n=3, \\ C\ \omega^{-\frac{n+1}{2}} & n\geq 4, n \text{ odd}. \end{cases} \tag{7.29}$$

Finally, integrating (7.29) or (7.28) over $(\mathbf{r}^{m-1}, \mathbf{\Omega}^{m-1}) \in [0,\Delta)^{m-2} \times (\mathbb{S}^{n-1})^{m-1}$ does not change the estimates. The last precision to make is that the operators $\dfrac{\partial}{\partial \sigma}$ and $\dfrac{\partial}{\partial r_m}$ satisfiy the following equalities:

$$\begin{aligned}
\frac{\partial}{\partial r_m} &= \frac{\partial x_m}{\partial r_m} \cdot \nabla_{x_m} = \Omega_m \cdot \nabla_{x_m}, \quad \text{and}\\
\frac{\partial}{\partial \sigma} &= \sum_{j=1}^{m} \frac{\partial x_j}{\partial \sigma} \cdot \nabla_{x_j} = \sum_{j=1}^{m} v_j(\sigma) \cdot \nabla_{x_j}, \qquad \text{where}\\
v_1(\sigma) &= \sigma(\Omega_1 \cdot e_0)e_0 + \frac{\sigma^2 + d_0}{\sqrt{\sigma^2 + 2d_0}}(\Omega_1 - (\Omega_1 \cdot e_0)e_0),\\
v_j(\sigma) &= v_1(\sigma) + \sum_{m=2}^{j} \Big(\Omega_{m,1} \frac{\partial}{\partial \sigma} \widehat{x_c - x_1} + \sum_{l=2}^{n} \Omega_{m,l} \frac{\partial}{\partial \sigma} V_l(\widehat{x_c - x_1})\Big), \ j \geq 2,
\end{aligned}$$

are infinitely smooth on the support of g. This in turn allows us to bound estimates (7.29) and (7.28) by the suitable $\mathcal{C}^\alpha$ norms in the initial variables $x_1, \dots, x_m$. This concludes the proof of Theorem 6.1.

Acknowledgment

This work was supported in part by NSF grant DMS-0804696 and DOE grant DE-FG52-08NA28779.

References

[1] S. R. Arridge, *Optical tomography in medical imaging*, Inverse Problems, 15 (1999), pp. R41–R93.

[2] S. R. Arridge and W. R. B. Lionheart, *Nonuniqueness in diffusion-based optical tomography*, Optics Letters, 23 (1998), pp. 882–884.

[3] G. Bal, *Inverse transport from angularly averaged measurements and time harmonic isotropic sources*, in Mathematical Methods in Biomedical Imaging and Intensity-Modulated Radiation Therapy, Y. Censor, M. Jiang, and A. Louis, eds., CRM Series, Scuola Normale Superiore Pisa, Italy, 2008, pp. 19–35.

[4] ——, *Inverse transport theory and applications*, Inverse Problems, 25 (2009), p. 053001.

[5] G. Bal and A. Jollivet, *Stability estimates in stationary inverse transport*, Inverse Probl. Imaging, 2(4) (2008), pp. 427–454.

[6] ——, *Time-dependent angularly averaged inverse transport*, Inverse Problems, 25 (2009), p. 075010.

[7] G. Bal, I. Langmore, and F. Monard, *Inverse transport with isotropic sources and angularly averaged measurements*, Inverse Probl. Imaging, 2(1) (2008), pp. 23–42.

[8] M. Choulli and P. Stefanov, *Reconstruction of the coefficients of the stationary transport equation from boundary measurements*, Inverse Problems, 12 (1996), pp. L19–L23.

[9] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology. Vol.6*, Springer Verlag, Berlin, 1993.

[10] L. V. Hörmander, *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, Springer Verlag, 1983.

[11] M. Mokhtar-Kharroubi, *Mathematical Topics in Neutron Transport Theory*, World Scientific, Singapore, 1997.

[12] K. Ren, G. S. Abdoulaev, G. Bal, and A. H. Hielscher, *Algorithm for solving the equation of radiative transfer in the frequency domain*, Optics Letter, 29(6) (2004), pp. 578–580.

[13] K. Ren, G. Bal, and A. H. Hielscher, *Frequency domain optical tomography based on the equation of radiative transfer*, SIAM J. Sci. Comput., 28 (2006), pp. 1463–1489.

[14] P. Stefanov and A. Tamasan, *Uniqueness and non-uniqueness in inverse radiative transfer*, Proc. Amer. Math. Soc., 137 (2009), pp. 2335–2344.