Geometry of multivariate stable laws

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Symmetric stable random vectors

A random vector ξ in $\mathbb{R}^d$ is symmetric stable with characteristic exponent $\alpha \neq 0$ (notation $S\alpha S$) if $\xi \stackrel{\mathcal{D}}{=} (-\xi)$ and for all $a, b > 0$,

$$a^{1/\alpha}\xi_1 + b^{1/\alpha}\xi_2 \stackrel{\mathcal{D}}{=} (a+b)^{1/\alpha}\xi\,,$$

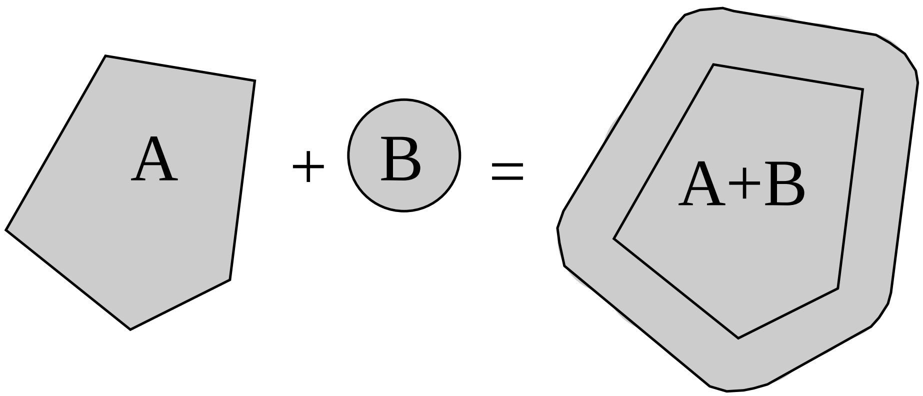
where ξ_1, ξ_2 are independent realisations of ξ.

Relationships to convex geometry

Minkowski sums

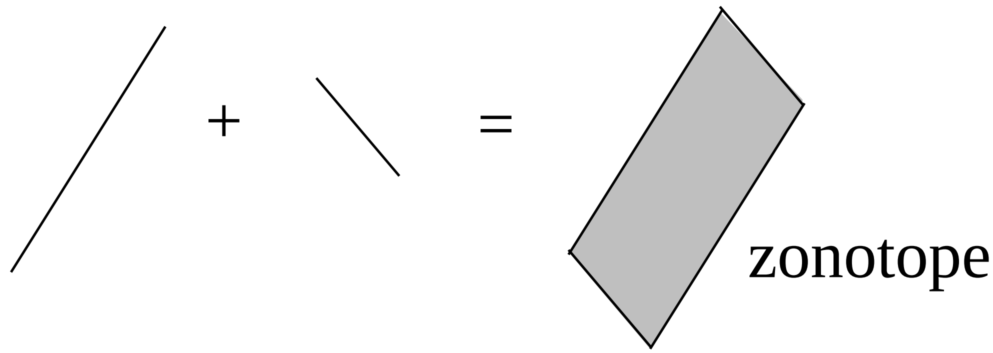
Sums of convex bodies in $\mathbb{R}^d$

$$A + B = \{x + y : \ x \in A, \ y \in B\}$$



Zonotopes

- Zonotopes are sums of segments, e.g. parallelograms.



- Translate all segments, so that their centres are at the origin. Then

$$\sum_{i=1}^{n} [-a_i, a_i] = \mathbf{E}\, X$$

is the expectation of the random segment X that equally likely takes values $[-na_i, na_i]$. More general, $\sum_{i=1}^{n} p_i [-a_i, a_i] = \mathbf{E}\, X$

Central symmetry

- Zonotopes are centrally symmetric — very much centrally symmetric!

- A polytope is a zonotope if and only if each its face of any dimension is centrally symmetric (e.g. icosaedron is not a zonotope).

- Each centrally symmetric planar polygone is a zononotope.

Zonoids

- Zonotopes are expected random segments with a discrete distribution.

- Zonoids are limits of zonotopes in the Hausdorff metric, i.e. expectations of general random segments.

- Zonoids are convex and centrally symmetric.

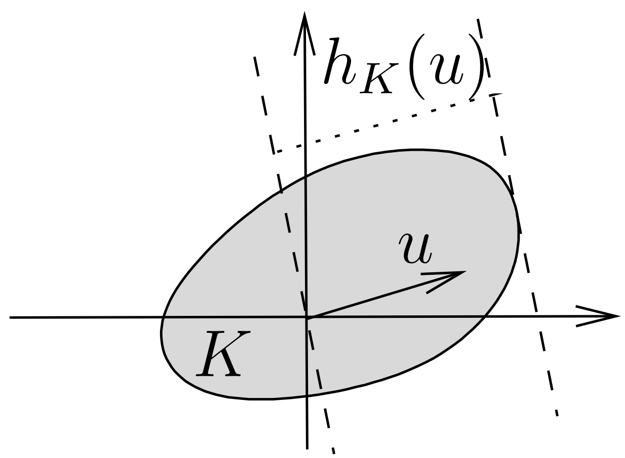
- In the plane each centrally symmetric convex compact set is a zonoid. This is wrong for dimensions $d \geq 3$ (e.g. icosaedron).

- Translations are not important; assume that all segments are origin symmetric, so the sums are also origin symmetric (centred).

Support function

- For compact $K \subset \mathbb{R}^d$ the support function is

$$h_K(u) = \sup\{\langle u, x\rangle : \ x \in K\}\,, \quad u \in \mathbb{R}^d$$

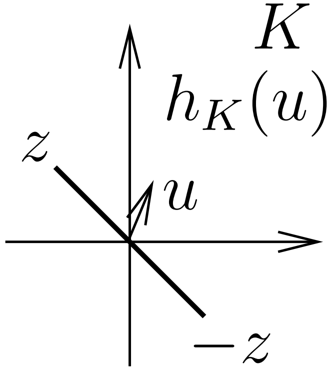


- Minkowski sum of convex compact sets translates into the arithmetic sum of their support functions, i.e.

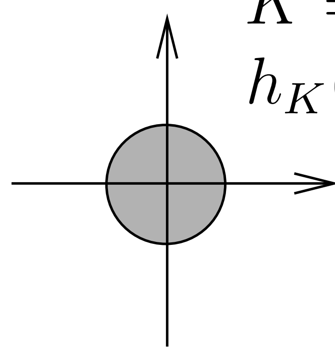
$$h_{K+L}(u) = h_K(u) + h_L(u)$$

Examples of support functions

$$K = [-z, z]$$
$$h_K(u) = |\langle z, u \rangle|$$



$$K = B_r(0)$$
$$h_K(u) = r\|u\|$$



Cosine transform

- Zonotope Z is a sum of segments, so its support function is a sum of support functions of the summands:

$$h_Z(u) = \sum_{i=1}^{n} h_{[-a_i,a_i]}(u) = \mathbf{E}\, h_X(u)$$

- Thus, zonoid becomes the expectation of a random segment X, i.e.

$$h_Z(u) = \mathbf{E}\, h_X(u)$$

- If $X = [-\eta, \eta]$ is centred, then $h_X(u) = |\langle u, \eta\rangle|$, so that Z is a centred zonoid if and only if

$$h_Z(u) = \mathbf{E}\,|\langle u, \eta\rangle| = \int_{\mathbb{R}^d} |\langle u, z\rangle| \mathbf{P}_\eta(dz) = \int_{\mathbb{S}^{d-1}} |\langle u, x\rangle| \sigma(dx)\,,$$

where the spectral measure σ is a finite measure on the unit sphere $\mathbb{S}^{d-1}$.

Equivalent characterisations of zonoids

Z is zonoid is equivalent to

- $h_Z(u) = \int_{\mathbb{S}^{d-1}} |\langle u, x\rangle| \sigma(dx)$ for a finite measure σ.

- Z is the range of an $\mathbb{R}^d$-valued measure.

- $\mathbb{R}^d$ with the norm $\|u\|_F = h_Z(u)$ is embeddable in $L_1([0,1])$, where $F = \{u : h_Z(u) \le 1\}$ is the unit ball in this norm (polar set to Z).

- $\varphi(u) = e^{-h_Z(u)}$, $u \in \mathbb{R}^d$, is a positive definite function.

- etc. etc.

Positive definiteness and zoonoids

- Z is zonoid if and only if $\varphi(u) = e^{-h_Z(u)}$ is positive definite.

- Note that $h_Z(tu) = th_Z(u)$, function $\varphi(u)$ is continuous and $\varphi(0) = 1$.

- Thus φ is the characteristic functions of a stable law with characteristic exponent $\alpha = 1$, i.e. the Cauchy distribution.

- T.Ferguson (1962) noticed the difference between Cauchy laws in dimensions 2 and 3, but did not explain it in terms of zonoids.

Lévy representation

- A random vector ξ is $S\alpha S$ with $0 < \alpha < 2$ if and only if there exists a unique symmetric finite (spectral) measure σ on the unit sphere $\mathbb{S}^{d-1}$ such that

$$\varphi_\xi(u) = \mathbf{E}\, e^{i\langle \xi, u\rangle} = \exp\left\{-\int_{\mathbb{S}^{d-1}} |\langle u, z\rangle|^\alpha \sigma(dz)\right\} .$$

(if $\alpha = 2$, then $|\langle u, z\rangle|^2$ is a quadratic form and σ is not unique).

- Density is not known analytically apart from the cases $\alpha = 2$ (normal law) and $\alpha = 1$ (Cauchy distribution).

Star bodies

- F is star body if $[0, u) \subset \mathrm{Int F}$ for every $u \in F$ and the Minkowski functional

$$\|u\|_F = \inf\{s \geq 0 : \ u \in sF\} = \frac{1}{\text{radial function of } F \text{ in direction } u}$$

is continuous.

- F is called centred if it is origin-symmetric.

If F is also convex, then $\|u\|_F$ is a (convex) norm on $\mathbb{R}^d$.

Stable laws, zonoids and star bodies

- For $\alpha \in (0, 2]$,

$$\int_{\mathbb{S}^{d-1}} |\langle u, z\rangle|^\alpha \sigma(dz) = \|u\|_F^\alpha \quad \Rightarrow \quad \varphi_\xi(u) = e^{-\|u\|_F^\alpha},$$

for an origin symmetric star body F.

- If $\alpha \in [1, 2]$, then

$$\int_{\mathbb{S}^{d-1}} |\langle u, z\rangle|^\alpha \sigma(dz) = h_Z(u)^\alpha \quad \text{and} \quad \varphi_\xi(u) = e^{-h_Z(u)^\alpha}$$

for a convex body Z being an L_α-zonoid called the associated zonoid of ξ.

Then $F = \{u : \ h_Z(u) \le 1\}$ is convex and is the polar set to Z.

First examples

$$\varphi_\xi(u) = e^{-\|u\|_F^\alpha}$$

F is called the associated star body of ξ

- If ξ has independent components, $F = \{x : x_1^\alpha + \cdots + x_d^\alpha \le r^\alpha\}$ is ℓ_α-ball in $\mathbb{R}^d$ (not convex if $\alpha < 1$).

- If $\xi = (\xi_1, \ldots, \xi_1)$ (completely dependent), then $\|u\|_F = |\sum u_i|$; its spectral measure σ is not full-dimensional; F is an infinite strip.

L_p-balls

F is associated star body of $S\alpha S$ law

$$\Updownarrow$$

$e^{-\|u\|_F^\alpha}$ is positive definite

$$\Updownarrow$$

F is an L_p-ball, i.e. $(\mathbb{R}^d, \|\cdot\|_F)$ is embeddable in $L_p([0,1])$ with $p = \alpha$

1966: J. Bretagnolle, D. Dacunha Castelle, J.L. Krivine. Lois stables et espaces L^p.

L_1-balls are polar sets to zonoids

One known result

Theorem 1. *If F is an L_p-ball for $p \in (0,2]$, then F is an L_r-ball for all $r \in (0,p]$.*

Proof. ξ is $S\alpha S$ with $\alpha = p$ and star body F. Let $\zeta > 0$ be strictly stable with exponent $\beta \in (0,1)$ and independent of ξ. Define $\xi' = \zeta^{1/\alpha}\xi$ (sub-stable law).

$$\mathbf{E}\, e^{i\langle \xi', u\rangle} = \mathbf{E}(\mathbf{E}(e^{i\zeta^{1/\alpha}\langle \xi, u\rangle} | \zeta)) = \mathbf{E}\, e^{-\zeta \|u\|_F^\alpha} = e^{-\|u\|_F^{\alpha\beta}}$$

Thus, F is the associated star body of ξ' with the characteristic exponent $r = \alpha\beta \in (0,p)$. □

Summary so far

- Each $S\alpha S$ law is determined by $\alpha \in (0, 2]$ and star body F.
- The associated star body F is an L_α-ball.
- If $\alpha \in [1, 2]$, then F is convex, its polar is L_α-zonoid K, and

$$\varphi_\xi(u) = e^{-\|u\|_F^\alpha} = e^{-h(K,u)^\alpha} .$$

- Independent coordinates if and only if F is an ℓ_α-ball.

Sub-Gaussian laws

- Sub-Gaussian laws appear as products $\sqrt{\zeta}\xi$, where ξ is Gaussian and $\zeta > 0$ is positive (one-sided) strictly stable.

- Ellipsoids are associated zonoids (and star bodies) of Gaussian laws, so all ellipsoids are L_p-zonoids for $p \in [1, 2]$ and L_p-balls for $p \in (0, 2]$.

- Sub-Gaussian laws can be characterised as those having ellipsoids as associated star bodies (and zonoids).

Approximation by sub-Gaussian laws

Theorem 2. *A law is $S\alpha S$ with $\alpha \in [1, 2]$ if and only if it can be obtained as a weak limit for sums of independent sub-Gaussian laws with the same characteristic exponent α.*

Proof. Grinberg–Zhang (1999) result on approximation of L_p-balls by sums of ellipsoids. □

Dvoretzky's theorem (1960): Each $S\alpha S$ law of sufficiently high dimension can be projected on a lower dimensional subspace, such that the projection lies arbitrarily close to a sub-Gaussian law.

Stable density

ξ is $S\alpha S$ with associated star body F and density f. Then

$$\varphi_\xi(u) = e^{-\|u\|_F^\alpha} = \mathbf{P}\{\zeta \geq \|u\|_F\} = \mathbf{E}\,\mathbf{1}_{\zeta \geq \|u\|_F} = \mathbf{E}\,\mathbf{1}_{u \in \zeta F}\,,$$

where

$$\mathbf{P}\{\zeta \geq x\} = e^{-x^\alpha}\,, \quad x > 0$$

Fourier inversion

$$(2\pi)^d f(x) = \mathbf{E} \int_{\mathbb{R}^d} e^{-i\langle u,x\rangle} \mathbf{1}_{u \in \zeta F} du = \mathbf{E} \int_{\zeta F} e^{-i\langle u,x\rangle} du$$

E.g.

$$f(0) = \frac{1}{(2\pi)^d}\,\Gamma(1 + \frac{d}{\alpha})\mathrm{Vol}_d(F)$$

Generalised functions

f is the stable density, g is a generalised function

$$(g, f) = \frac{1}{(2\pi)^d} (\hat{g}, \mathbf{E}\, \mathbf{1}_{u \in \zeta F})$$

- Need to find the action of the Fourier transform $\hat{g}$ on $\mathbf{1}_{u \in \zeta F}$.

- Important example $g(x) = \|x\|^\lambda$.

Moments of the Euclidean norm

Theorem 3. *For* $\lambda \in (-d, \alpha)$

$$\mathbf{E}\,\|\xi\|^\lambda = \frac{2^{\lambda-1}}{\pi^{d/2}}\Gamma(\frac{d+\lambda}{2})\frac{\Gamma(1-\frac{\lambda}{\alpha})}{\Gamma(1-\frac{\lambda}{2})}\int_{\mathbb{S}^{d-1}} \|u\|_F^\lambda du\,.$$

Proof. Fourier transform for the generalised function $\|u\|^\lambda$ or plain-wave expansion of the norm. □

Known: in the isotropic case $F = B_{\sigma^{-1}}$ (σ is the scale parameter) and

$$\mathbf{E}\,\|\xi\|^\lambda = 2^\lambda \frac{\Gamma(\frac{2+\lambda}{2})}{\Gamma(\frac{d}{2})}\,\frac{\Gamma(1-\frac{\lambda}{\alpha})}{\Gamma(1-\frac{\lambda}{2})}\sigma^\lambda\,.$$

Wanted $\int_{\mathbb{S}^{d-1}} \|u\|_F^\lambda du$!

- Not easy even if $\|u\|_F^2$ is a bilinear form.

- This quantity is a dual mixed volume of F. The dual mixed volume inequality implies

$$\mathbf{E}\,\|\xi\|^\lambda \geq 2^\lambda \, \frac{\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{d}{2})} \, \frac{\Gamma(1-\frac{\lambda}{\alpha})}{\Gamma(1-\frac{\lambda}{2})} \left(\frac{\kappa_d}{\mathrm{Vol}_d(F)}\right)^{\lambda/d} .$$

with the equality if and only if F is a Euclidean ball (κ_d is the volume of the unit Euclidean ball).

- Sophisticated bounds (Litvak–Milman–Schechtman, 1998: Averages of norms and quasi-norms) imply inequalities between moments of different orders.

Other moments

- Expressions for mixed moments $\mathbf{E}(|\xi_1|^{\lambda_1}\cdots|\xi_d|^{\lambda_d})$ and signed powers $\mathbf{E}(\xi_1^{\langle\lambda_1\rangle}\cdots\xi_d^{\langle\lambda_d\rangle})$.

- For instance, if d is even,

$$\mathbf{E}\,\mathrm{sign}(\xi_1\cdots\xi_d)=\frac{i^d}{\pi^d}\int_F\frac{du}{u_1\cdots u_d}\,.$$

The integral is scale-invariant and does not exceed π^d in absolute value.

Integrals of the density

$$\int_{\mathbb{R}} f(tu)dt = \frac{1}{(2\pi)^{d-1}}\Gamma\Big(1+\frac{d-1}{\alpha}\Big)\mathrm{Vol}_{d-1}(F\cap u^{\perp})$$

- Busemann problem: Does $\mathrm{Vol}_{d-1}(F_1\cap u^{\perp}) \le \mathrm{Vol}_{d-1}(F_2\cap u^{\perp})$ for all u and centred F_1, F_2 imply $\mathrm{Vol}_d(F_1) \le \mathrm{Vol}_d(F_2)$?
Gardner et al., 1999: yes if $d \le 4$, otherwise not.

- F is an L_p-ball, and so is an intersection body. Thus, in all dimensions

$$\int_{\mathbb{R}} f_1(tu)dt \le \int_{\mathbb{R}} f_2(tu)dt\,, \quad u\in\mathbb{S}^{d-1} \quad \Longrightarrow f_1(0)\le f_2(0)$$

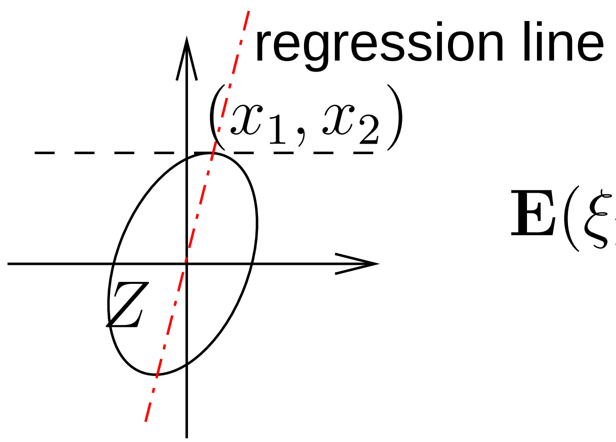
Covariation

ξ is $S\alpha S$ in $\mathbb{R}^2$ with $\alpha > 1$ and associated zonoid Z

The covariation of ξ is defined as $[\xi_1, \xi_2]_\alpha = \int_{\mathbb{S}^1} s_1 s_2^{\langle \alpha - 1 \rangle} \sigma(ds)$

Theorem 4. *If* $\{(x_1, x_2)\}$ *is the support point (necessarily unique!) of* Z *in direction* $(0, 1)$*, then*

$$[\xi_1, \xi_2]_\alpha = x_1 x_2^{\alpha - 1}$$



$$\mathbf{E}(\xi_1 | \xi_2) = \frac{x_1}{x_2} \xi_2 \text{ a.s.}$$

Extension for multiple regression.

One-sided stable laws

- How to describe geometrically a strictly stable (not symmetric!) ξ with values in $\mathbb{R}^d_+$?

- How to describe geometrically distributions stable with respect to other operations, e.g. max-stable laws?

$\mathbb{R}_+^d$ with maximum operation

- ξ is max-stable random vector in $\mathbb{R}_+^d$, i.e.

$$a^{1/\alpha}\xi_1 \vee b^{1/\alpha}\xi_2 \stackrel{\mathcal{D}}{=} (a+b)^{1/\alpha}\xi$$

- Assume $\alpha = 1$; then ξ is said to have a semi-simple max-stable distribution. Up to a scale of coordinates, all marginals are unit Fréchet

$$\Phi_1(x) = \begin{cases} 0, & x < 0\,, \\ e^{-x^{-1}}, & x \geq 0\,, \end{cases}$$

and ξ has a simple max-stable distribution.

Representation of semi-simple max-stable distributions

Theorem 5. *A random vector ξ is semi-simple max-stable if and only if*

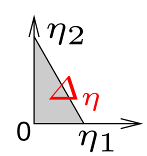
$$\mathbf{P}\{\xi \le x\} = \exp\{-h_Z(x^{-1})\}\,, \quad x \in \mathbb{R}^d_+\,,$$

where $x^{-1} = (x_1^{-1}, \dots, x_d^{-1})$ and

$$Z = c\,\mathbf{E}\,\Delta_\eta$$

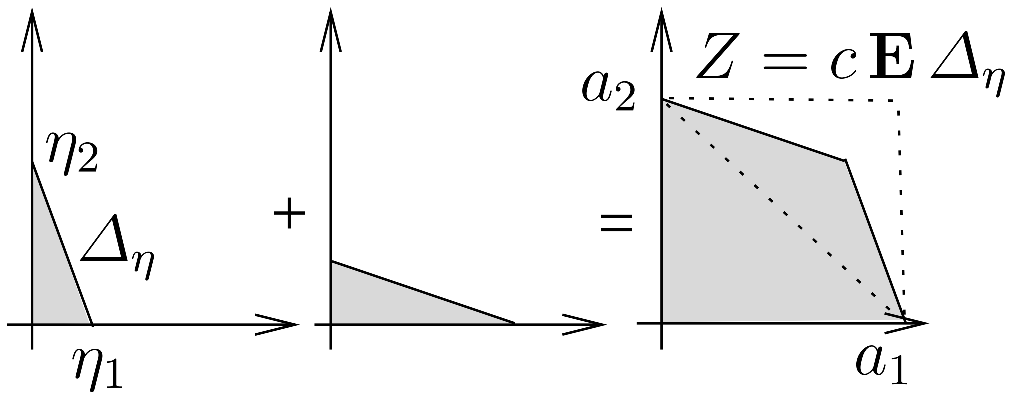
is the expectation of the random crosspolytope

$$\Delta_\eta = \mathrm{conv}(0, \eta_1 e_1, \dots, \eta_d e_d)$$



for $c > 0$ and a random vector η in $\mathbb{S}^{d-1}_+$ (unit sphere in $\mathbb{R}^d_+$).

Max-zonoids



- The set $Z = c\,\mathbf{E}\,\Delta_\eta$ is said to be a max-zonoid.

- If $d = 2$, then each convex set Z satisfying $\Delta_a \subset Z \subset [0, a_1] \times [0, a_2]$ for $a = (a_1, a_2) \in (0, \infty)^2$ is a max-zonoid. This is not the case if $d \geq 3$.

Summary so far

- Stable ξ has the chacteristic function

$$\varphi(u) = e^{-h_Z(u)^\alpha},$$

for an L_α-zonoid Z (L_p-expectation of a segment).

- Max-stable ξ with $\alpha = 1$ has the cumulative distribution function

$$F(x) = e^{-h_Z(x^{-1})},$$

where Z is a max-zonoid (expectation of a random triangle).

Between the sum and the maximum

- p-addition (assume $p > 0$)

$$s +_p t = (s^p + t^p)^{1/p}, \quad s, t \geq 0$$

Special cases: $p = 1$ (arithmetic sum); $p = \infty$ (maximum).

- The p-sum $x +_p y$ for $x, y \in \mathbb{R}^d_+$ is defined coordinatewisely.

Stability for p-sums: definition

- Random vector ξ in $\mathbb{R}_+^d$ is $S\alpha S$ for p-sums if

$$a^{1/\alpha}\xi_1 +_p b^{1/\alpha}\xi_2 \stackrel{\mathcal{D}}{=} (a+b)^{1/\alpha}\xi$$

for all $a, b > 0$, some $\alpha \neq 0$, and ξ_1, ξ_2 being independent copies of ξ. Assume $p \in (0, \infty)$.

- Stability on semigroups $\Rightarrow \alpha \in (0, p]$.

Stability for p-sums: characterisation

Theorem 6. *ξ is $S\alpha S$ for p-sums with $\alpha = 1$ if and only if*

$$\mathbf{E}\, e^{-\sum (u_i \xi_i)^p} = e^{-h_Z(u)}\,, \quad u \in \mathbb{R}^d_+\,,$$

where $Z = \mathbf{E}\, X$ is the expectation of

$$X = \{(\eta_1 v_1, \ldots, \eta_d v_d) : \ \|v\|_q \leq 1,\ v \in \mathbb{R}^d_+\}$$

being randomly rescaled ℓ_q-ball.

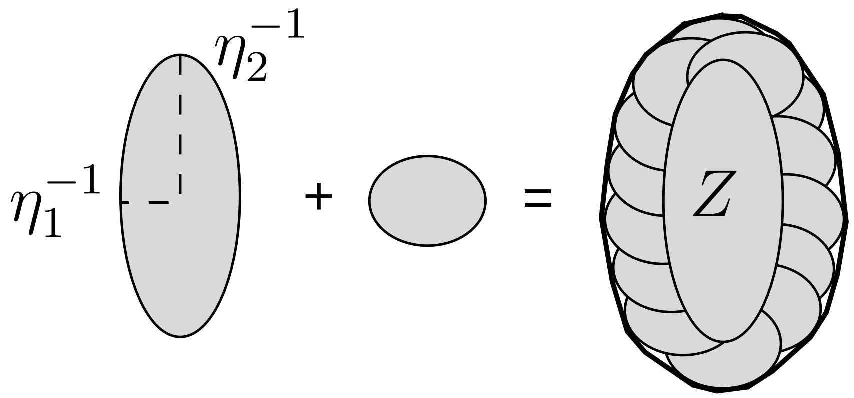
- The set Z is said to be an $L_1(p)$-zonoid.

Case $p = 2$, i.e. $s +_2 t = \sqrt{s^2 + t^2}$

- $L_1(2)$-zonoid Z is the expectation of the ellipsoid

$$X = \{(\eta_1 v_1, \dots, \eta_d v_d) : \ \|v\|_2 \le 1, \ v \in \mathbb{R}^d_+\}$$

with random semi-axes $\eta_1^{-1}, \dots, \eta_d^{-1}$.



- The Laplace transform of ξ is given by

$$\mathbf{E} \exp\left\{-\sum(\xi_i u_i)^2\right\} = e^{-h_Z(u)}$$

Arithmetic sums and $\alpha \in (0,1)$

- ξ is $S\alpha S$ in $\mathbb{R}^d_+$ with $\alpha \in (0,1)$.

- Then ξ^α is $S1S$ for p-sums with $p = \frac{1}{\alpha}$.

- Finally

$$\mathbf{E}\, e^{-\sum u_i \xi_i} = e^{-h_Z(u^\alpha)},$$

where Z is the expectation of randomly rescaled ℓ_q-ball with $q = 1/(1-\alpha)$, e.g. expectation of an ellipsoid if $\alpha = \frac{1}{2}$.

References

- Multivariate $S\alpha S$ laws:

I. Molchanov, ***Convex and star-shaped sets associated with stable distributions***
http://www.arxiv.org/abs/math.PR/0707.0221

- Max-zonoids:

I. Molchanov, ***Convex geometry of max-stable distributions***
Extremes, 2008. http://www.arxiv.org/abs/math.PR/0603423

- General stability for semigroups:

Y. Davydov, I. Molchanov and S. Zuyev, ***Strictly stable distributions on convex cones*** Electronic J. Probab. 2008