Épuisement de ressources partagées en environnement Markovien : l'approche des grandes déviations

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Joint work with
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François Delarue et René Schott

Algorithm

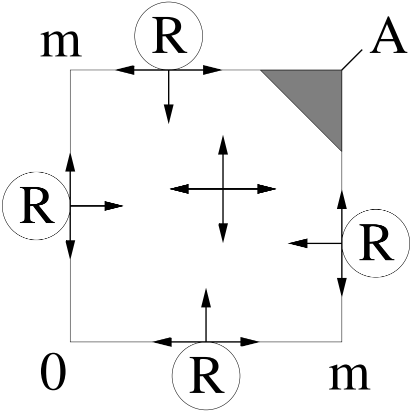
1. Resource **sharing** problem between d customers
   - Resource $\equiv$ money, memory
   - Maximum need for a customer $\equiv m$
   - Total amount of available resource $\equiv L \times m < d \times m$, $L > 0$
2. Modelization by a process

$$R_n = (R_n^1, \dots, R_n^d)$$

3. R_n^i = resource allocated
   - to the customer $i \in \{1, \dots, d\}$
   - at time n
4. Questions of interest
   - **Dynamics** for $(R_n)_{n \geq 0}$?
   - **Behavior** for m large ?
   - **Deadlock** = breakdown of the system

Modelization

1. **Markovian** dynamics for $(R_n)_{n \geq 0}$



Modelization

1. **Markovian** dynamics for $(R_n)_{n\geq 0}$
   - **Constant** transition probabilities (Ellis'77, Flajolet'86, Louchard-Schott' 91)
   - Influence of the **spatial** position (Maier'91)
   - Probabilities governed by a **dynamical system** (Guillotin-Schott'02)

$$q(T^n x, \pm e_1),\ \ldots,\ q(T^n x, \pm e_d)$$

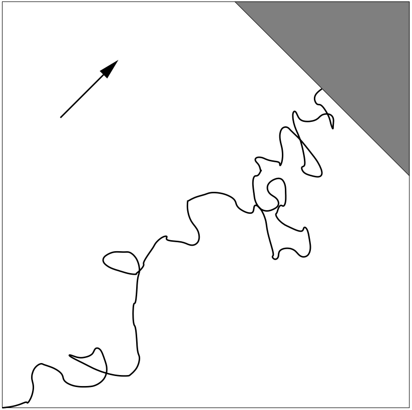
2. CDS: Transitions governed by a **Markovian environment**

$$q(\xi_n, \pm e_1),\ \ldots,\ q(\xi_n, \pm e_d)$$

3. Properties of the **deadlock time** ?
   - Behavior for m large: **scaling** properties ?
   - Expectation ? Law ? (After scaling)
4. Influence of the environment ?

Typical Behavior (Constant Transition Probabilities)

1. If the drift is directed **to the absorption area**



2. Absorption time $\sim m$ due to **typical paths**

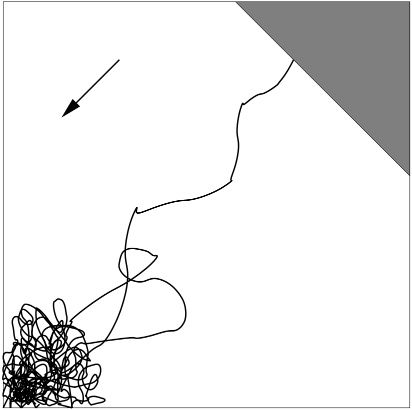
Typical Behavior (Constant Transition Probabilities)

1. If the drift is **null** ("diffusive case")

2. Absorption time $\sim m^2$ due to **normal fluctuations**

Typical Behavior (Constant Transition Probabilities)

1. If the drift is directed **to the origin** ("stable case")



2. Absorption time $\sim \exp(Cm)$ due **to large deviations**

Part I- Framework and assumptions

- **Environment**: Markov chain $(\xi_n)_{n\geq 0}$
  - ☛ **Finite state** space: transition matrix P (H1)
  - ☛ **Irreducible** chain: invariant probab. measure μ
- Transition probabilities of the **non-reflected** walk $(S_n)_{n\geq 0}$

$$p(S_n/m, \xi_n, \pm e_j),\ j \in \{1, \dots, d\}$$

  - ☞ $p(y, i, \pm e_j)$ **Lipschitz** in y (H2)
  - ☞ **Ellipticity**: $p(y, i, \pm e_j) > 0$
- Transition probabilities of the **reflected** walk $(R_n)_{n\geq 1}$

$$q(R_n/m, \xi_n, \pm e_j),\ j \in \{1, \dots, d\}$$

$$\text{Reflection} \begin{cases} q(y, i, -e_j) & = & 0 & \text{for } y_j = 0 \\ q(y, i, e_j) & = & 0 & \text{for } y_j = 1 \end{cases}$$

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$$\begin{cases} q(y,i,\pm e_j) = p(y,i,\pm e_j) & \text{for } 0 < y_j < 1 \\ q(y,i,e_j) = p(y,i,e_j) + p(y,i,-e_j) & \text{for } y_j = 0 \\ q(y,i,-e_j) = p(y,i,e_j) + p(y,i,-e_j) & \text{for } y_j = 1 \end{cases}$$

Averaging

1. **Local** drift

$$f(y,i) = \sum_{u=\pm e_j} [p(y,i,u)u],$$

y = location of the walk, i = state of the environment

2. **Averaged** drift

$$\bar{f}(y) = \sum_{i \in E} \mu(i) f(y,i)$$

3. Decomposition of the **non-reflected** walk

$$S_n = S_0 + \sum_{k=0}^{n-1} f(m^{-1} S_k, \xi_k) + \text{Mart.}$$

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$$\bar{S}_t^{(m)} = m^{-1} S_{\lfloor mt \rfloor} \approx \bar{S}_0^{(m)} + \int_0^t \bar{f}(\bar{S}_s^{(m)}) ds + m^{-1}\text{Mart.}$$

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Hyperbolic Scaling

- $(\bar{S}_t^{(m)} = m^{-1} S_{\lfloor mt \rfloor})_{t \geq 0} \underset{\text{proba}}{\longrightarrow} (s_t)_{t \geq 0}$

ODE $$\frac{ds_t}{dt} = \bar{f}(s_t)$$

Hyperbolic Scaling

1. **Non-reflected** $(\bar{S}_t^{(m)} = m^{-1} S_{\lfloor mt \rfloor})_{t \geq 0} \underset{\text{proba}}{\longrightarrow} (s_t)_{t \geq 0}$

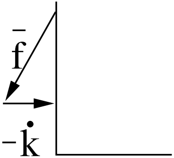
   ODE $$\frac{ds_t}{dt} = \bar{f}(s_t)$$

2. **Reflected** $(\bar{R}_t^{(m)} = m^{-1} R_{\lfloor mt \rfloor})_{t \geq 0} \underset{\text{proba}}{\longrightarrow} (r_t)_{t \geq 0}$

   Reflected DE $$\frac{dr_t}{dt} = \bar{f}(r_t) - \frac{dk_t}{dt}$$

3. $(k_t)_{t \geq 0}$ pushing process

$$\frac{dk_t^i}{dt} = \begin{cases} 0 & \text{if } 0 < r_t^i < 1 \\ \leq 0 & \text{if } r_t^i = 0 \\ \geq 0 & \text{if } r_t^i = 1 \end{cases}$$



Reflection: formal definition

1. **Skorohod problem**:
$\forall w : t \in [0, +\infty) \mapsto w_t \in \mathbb{R}^d (w_0 \in [0,1]^d)$ continuous,
$\exists ! (x_\cdot, k_\cdot) : t \in [0, +\infty) \mapsto (x_t, k_t) \in [0,1]^d \times \mathbb{R}^d$ with $k_\cdot$ locally bounded variation, such that

$$w_t = x_t + k_t,$$

$$k_t = \int_0^t n_s d|k|_s \ , \quad |k|_t = \int_0^t \mathbf{1}_{\{x_s \in \partial [0,1]^d\}} d|k|_s \ ,$$

with n_s unit outward normals to $\partial[0,1]^d$ at x_s.
Let Ψ_{Sk} the Skorohod map $w_\cdot \mapsto x_\cdot$.

Reflection: formal definition

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   with n_s unit outward normals to $\partial[0,1]^d$ at x_s.
   Let Ψ_{Sk} the Skorohod map $w_. \mapsto x_.$.

2. **Reflected DE** $\dfrac{dr_t}{dt} = \bar{f}(r_t) - \dfrac{dk_t}{dt}$

$$= \text{unique solution of } r_. = \Psi_{Sk}\Big(r_0 + \int_0^{\cdot} \bar{f}(r_t)dt\Big)$$

Part II- Diffusive case ($\bar{f} = 0$) CDS'07

1. Law of Large Numbers for **Reflected** walk

$$(\bar{R}_t^{(m)} = m^{-1} R_{\lfloor mt \rfloor})_{t \geq 0} \underset{\text{proba}}{\longrightarrow} (r_t)_{t \geq 0}$$

Ref.DE $$\frac{dr_t}{dt} = \bar{f}(r_t) - \frac{dk_t}{dt}$$

2. Assume $\boxed{\bar{f} = 0}$. Then, the limit is **trivial** !

Ref. DE $$\frac{dr_t}{dt} = 0.$$

Theorem [CDS, Rand.Struct.Algo.'07]

Diffusive scaling: $(\hat{R}_t^{(m)} = m^{-1} R_{\lfloor m^2 t \rfloor})_{t \geq 0} \underset{\text{law}}{\longrightarrow} (r_t)_{t \geq 0}$

Stoch. Ref. DE $$dr_t = \bar{b}(r_t)dt + \bar{\sigma}(r_t)dB_t - dk_t$$

$(k_t)_{t \geq 0}$ of bounded variation

Deadlock Time under Diffusive case ($\bar{f} = 0$)

1. Prove convergence to $dr_t = \bar{b}(r_t)dt + \bar{\sigma}(r_t)dB_t - dk_t$ by the corrector method:
2. v solution of the **Poisson** equation

$$\underbrace{(P - I)}_{Generator} v(y, \cdot) = -f(y, \cdot)$$

3. **Itô's** like formula $\Rightarrow$ **kill** f

$$m^{-1}S_{\lfloor m^2 t \rfloor} + m^{-1}v(\xi_{\lfloor m^2 t \rfloor}, S_{\lfloor m^2 t \rfloor}/m)$$
$$= \text{Init.Cond.} + m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} b(\xi_k, S_k/m) + m^{-1}\text{Mart.}_{\lfloor m^2 t \rfloor}$$

Corollary (Diffusive case) ($\bar{f} = 0$)

1. $T_L^{(m)} = \inf\{n \geq 0,\ |R_n|_1 \geq Lm\},\ T_L = \inf\{t \geq 0,\ |r_t|_1 \geq L\}$

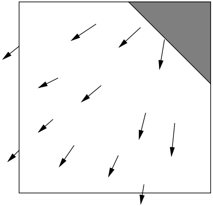
$$m^{-2} T_L^{(m)} \underset{\text{law}}{\rightarrow} T_L$$

1. Hyperbolic scaling: $(\bar{R}_t^{(m)} = m^{-1} R_{\lfloor mt \rfloor})_{t \geq 0} \underset{\text{proba}}{\longrightarrow} (r_t)_{t \geq 0}$

   Ref. DE
$$\frac{dr_t}{dt} = \bar{f}(r_t) - \frac{dk_t}{dt}.$$

2. **Stable case**
$$\bar{f}_i(x) < 0, \;\; x_i \in (0, 1] \qquad (H3)$$

3. Dimension 2



4. **Behaviour** of RDE
$$r_t \longrightarrow 0 \text{ as } t \longrightarrow +\infty$$

Part III- Deadlock Time in Stable Case

1. Hyperbolic scaling: $(\bar{R}_t^{(m)} = m^{-1} R_{\lfloor mt \rfloor})_{t \geq 0} \underset{\text{proba}}{\longrightarrow} (r_t)_{t \geq 0}$

Ref. DE
$$\frac{dr_t}{dt} = \bar{f}(r_t) - \frac{dk_t}{dt}.$$

2. **Stable case**

$$\bar{f}_i(x) < 0, \ \ x_i \in (0, 1] \qquad (H3)$$

$$T_L = \inf\{t \geq 0, \ \sum_{i=1}^{d} (r_t)_i \geq L\} = +\infty$$

3. **Conclusion**

$\{T_L^{(m)} < C\}$ is a rare event for m large

Freidlin and Wentzell Theory

1. **Non-reflected** walk $(S_n)_{n\geq 0}$

$$p(S_n/m, \xi_n, \pm e_j),\ j \in \{1, \dots, d\}$$

2. **Large Deviation Principle** for $(\bar{S}_t^{(m)} = m^{-1} S_{\lfloor mt \rfloor})_{0\leq t\leq T}$

$$\liminf_{m\to\infty} m^{-1} \ln \mathbb{P}\big\{(\bar{S}_t^{(m)})_{0\leq t\leq T} \in A\big\} \geq - \inf_{\phi\in \mathring{A}, \phi_0 = x_0} I(\phi)$$

$$\limsup_{m\to\infty} m^{-1} \ln \mathbb{P}\big\{(\bar{S}_t^{(m)})_{0\leq t\leq T} \in A\big\} \leq - \inf_{\phi\in \bar{A}, \phi_0 = x_0} I(\phi)$$

with action functional

$$I(\phi) = \int_0^T L(\phi_s, \dot{\phi}_s) ds \qquad \text{if } \phi \text{ abs. cont.}$$

- LDP for averaging: Freidlin-Wenzell'84, Gulinsky-Veretennikov'93, Feng-Kurtz'05
- LDP for reflected processes: Dupuis'88

Freidlin and Wentzell Theory

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$$I(\phi) = \int_0^T L(\phi_s, \dot{\phi}_s) ds \qquad \text{if } \phi \text{ abs. cont.}$$

3. **Hamiltonian**

$$H(x, \alpha) = \lim_{n\to 0} n^{-1} \ln \mathbb{E}\Big[\prod_{k=1}^{n} \underbrace{\Big[\sum_{v=\pm e_j} \exp(\langle \alpha, v\rangle) p(x, \xi_k, v) \Big]}_{Laplace\ trans.\ of\ jumps\ for\ freezed\ position} \Big]$$

4. **Lagrangian** $L(x, v) = \sup_{\alpha} \big[\langle \alpha, v \rangle - H(x, \alpha) \big]$

1. **Non-reflected** walk $(S_n)_{n\geq 0}$

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3. **Hamiltonian**

$$H(x, \alpha) = \text{Perron}-\text{Froebenius}\Big[P_{i,j} \sum_{v=\pm e_k} \exp(\langle \alpha, v \rangle) p(x, i, v)\Big]$$

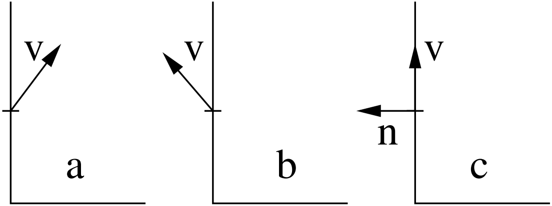
4. **Lagrangian** $L(x, v) = \sup_{\alpha} \big[\langle \alpha, v \rangle - H(x, \alpha)\big]$

Lagrangian in the Reflected Setting

1. Lagrangian is a **local** cost

$$x \in (0,1)^d \Rightarrow L^{ref}(x,v) = L(x,v)$$

2. **Boundary**



a $L^{ref}(x,v) = L(x,v)$
b $L^{ref}(x,v) = +\infty$
c $L^{ref}(x,v) = \inf_{\beta>0} L(x, v+\beta n)$ Continuity, convexity lost!

LDP in the Reflected Setting

Theorem [CDS preprint'07]

$\bar{R}^{(m)}_{\cdot}$ obeys Large Deviation Principle with this Lagrangian.

☐ Sketch: represent the walk as random recurrent sequence,

$$F(x, i, U) \sim p(x, i, \cdot)$$

U_k i.i.d. uniform, and projection $\Pi : \mathbb{R}^d \to [0,1]^d$,

$$\bar{R}^{(m)}_{\frac{k+1}{m}} = (2\Pi - Id)\Big(\bar{R}^{(m)}_{\frac{k}{m}} + \frac{1}{m} F(\bar{R}^{(m)}_{\frac{k}{m}}, \xi_k, U_k)\Big)$$

(reflected walk). Introduce Y, Z (unprojected, projected)

$$\begin{cases} Y^{(m)}_{\frac{k+1}{m}} = Y^{(m)}_{\frac{k}{m}} + \frac{1}{m} F(\bar{R}^{(m)}_{\frac{k}{m}}, \xi_k, U_k) \\ Z^{(m)}_{\frac{k+1}{m}} = \Pi\big(Z^{(m)}_{\frac{k}{m}} + \frac{1}{m} F(\bar{R}^{(m)}_{\frac{k}{m}}, \xi_k, U_k)\big) \end{cases}$$

Then,

$$Z^{(m)} = \Psi_{Sk}(Y^{(m)}) , \quad |Z^{(m)}_t - \bar{R}^{(m)}_t|_\infty \le \frac{1}{m}$$

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LDP Reflected: Sketch of Proof

$$Y^{(m)}_{\frac{k+1}{m}} = Y^{(m)}_{\frac{k}{m}} + \frac{1}{m} F(\bar{R}^{(m)}_{\frac{k}{m}}, \xi_k, U_k)$$

and

$$Z^{(m)} = \Psi_{Sk}(Y^{(m)}) \;, \quad |Z^{(m)}_t - \bar{R}^{(m)}_t|_\infty \leq \frac{1}{m}$$

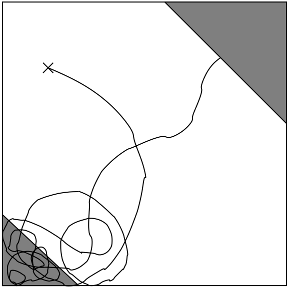
1. $Y^{(m)}$ has LDP with $J_Y(\theta) = \int_0^T L(\Psi_{Sk}(\theta_t), \dot{\theta}_t) dt$
2. $Z^{(m)}$ has LDP with $J_Z(\phi) = \inf\{\int_0^T L(\phi_t, \dot{\theta}_t) dt; \Psi_{Sk}(\theta) = \phi\}$. [Hence, $\bar{R}^{(m)}$ too]
3. identify $J_Z = I$ with our Lagrangian. □

Deadlock Phenomenon

1. **Quasi-potential** : "global cost"

$$V(x) = \inf\Big[\int_0^T L^{ref}(\phi_s, \dot{\phi}_s)ds;\ \phi_0 = 0,\ \phi_T = x,\ T > 0\Big]$$

2. Deadlock phenomenon $\simeq$ first success in a **Bernoulli** scheme



3. **Probability** of success

$$\sim \exp\big[-m\bar{V}\big],\ \ \bar{V} = \inf(V(x); |x|_1 = L)$$

4. Length of an **excursion** = O(m)

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Theorem [CDS preprint'07]

- $\mathbb{E}(T_L^{(m)}) = \exp\big[m(\bar V + o(1))\big]$
- $\mathbb{P}\{\exp\big[m(\bar V - \delta)\big] \le T_L^{(m)} \le \exp\big[m\big(\bar V + \delta\big)\big]\} \to 1$
- $R_{T_L^{(m)}}$ converges in probability towards the set of minimizers of $\bar V$
- $T_L^{(m)}/\sigma_m \underset{\text{law}}{\to} Exp.(1)$ for some $\sigma_m = e^{m(\bar V + o(1))}$ (+ assumpt.)

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- $R_{T_L^{(m)}}$ converges in probability towards the set of minimizers of $\bar{V}$
- $T_L^{(m)}/\sigma_m \underset{\text{law}}{\to} Exp.(1)$ for some $\sigma_m = e^{m(\bar{V}+o(1))}$ (+ assumpt.)

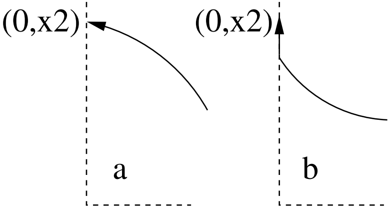
Quasi-potential

1. If V C^1 inside the domain

$$\mathbf{H(x, \nabla V(x)) = 0} \qquad \text{Hamilton} - \text{Jacobi}$$

2. **Boundary condition**

   Reflection $\Rightarrow$ (Weak) Neumann boundary condition

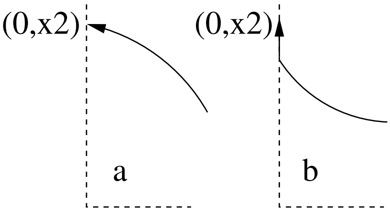


3. Quasi-potential increases along **optimal** paths
   a) $\partial V / \partial x_1(0, x_2) < 0$
   b) $\partial V / \partial x_1(0, x_2) > 0$

1. If V C^1 inside the domain

$$\mathbf{H}(\mathbf{x}, \nabla\mathbf{V}(\mathbf{x})) = \mathbf{0} \qquad \text{Hamilton} - \text{Jacobi}$$

2. **Boundary condition**



3. Quasi-potential increases along **optimal** paths
   a) $\partial V/\partial x_1(0, x_2) < 0$, $\quad H\big((0, x_2), \nabla\mathbf{V}(\mathbf{0}, \mathbf{x_2})\big) = 0$
   b) $\partial V/\partial x_1(0, x_2) > 0$, $\quad H\big((0, x_2), (\mathbf{0}, [\partial\mathbf{V}/\partial\mathbf{x_2}](\mathbf{0}, \mathbf{x_2}))\big) = 0$

Quasi-potential

1. If V C^1 inside the domain

$$\mathbf{H}(\mathbf{x}, \nabla \mathbf{V}(\mathbf{x})) = \mathbf{0} \qquad \text{Hamilton} - \text{Jacobi}$$

2. **Boundary condition**

$$H(x, \nabla_+ \mathbf{V}(\mathbf{x})) = 0$$

$$(\nabla_+ V(x))_i = \begin{cases} \min(\frac{\partial V}{\partial x_i}(x), 0) \text{ if } x_i = 0 \\ \max(\frac{\partial V}{\partial x_i}(x), 0) \text{ if } x_i = 1 \\ \frac{\partial V}{\partial x_i}(x) \text{ if } x_i \in (0, 1) \end{cases}$$

Identification of the Quasi-potential

Proposition

Let W be C^1 on $[0,1]^d \setminus \{0\}$ with $W(0)=0$ and

1. $$H(x, \nabla W(x)) = 0$$
2. Boundary condition
   - $H(x, \nabla_+ W(x)) = 0$
   - $\langle \nabla_\alpha H(x, \nabla_+ W(x)), n \rangle \geq 0$ for n outward normal at x

Then, W is the quasi-potential, and optimal path is the backward RDE

$$\dot{\phi} = \nabla_\alpha H(\phi_t, \nabla_+ W(\phi_t)) - \dot{k}_t$$

with $\phi_{-\infty} = 0, \phi_0 = x$ ("typical deadlock path")

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Two-Stacks Model

Example. Generalization of Maier'91

1. Environment with two states $= \{1, 2\}$, $d = 2$

$$p(x,i,v) = \begin{cases} \frac{1}{2}\ell_i[1 - g_1(x_1)] & v = e_1 \\ \frac{1}{2}\ell_i[1 + g_1(x_1)] & v = -e_1 \\ \frac{1}{2}(1-\ell_i)[1 - g_2(x_2)] & v = e_2 \\ \frac{1}{2}(1-\ell_i)[1 + g_2(x_2)] & v = -e_2 \end{cases} \qquad \ell_i \in (0,1)$$

2. $g_1, g_2 : [0,1] \to [0,1)$ Lipschitz, $g_j(z) > 0$ for $z > 0$
3. Mean trend

$$\bar{f}(x) = -\begin{pmatrix} \ell g_1(x_1) \\ (1-\ell)g_2(x_2) \end{pmatrix} \qquad \ell = \sum_{i \in \{1,2\}} \ell_i \mu(i)$$

4. $W(x) = 2\int_0^{x_1} \tanh^{-1}(g_1(y))dy + 2\int_0^{x_2} \tanh^{-1}(g_2(y))dy$

Merci !