Spectral measure of Brownian field on hyperbolic plane

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Summary

Euclidean case

Let $(B_x)_{x\in\mathbb{R}^2}$ be a real valued centered Gaussian field such that $B_0 = 0$ a.s. and

$$\mathbb{E}\left(B_x - B_y\right)^2 = Cst\|x - y\|. \tag{1}$$

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$$B_x = \int_{\mathbb{R}^2} \frac{e^{i\langle x,\xi\rangle}-1}{\|\xi\|^{3/2}} W(d\xi),$$

where $W(d\xi)$ is a Gaussian white noise.

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where $u \in S_1$.

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$$\phi_s(x) = \int_{S_1} cos(s\langle x,\theta\rangle) \frac{d\theta}{2\pi}$$

is the spherical function in the Euclidean case.

Hyperbolic plane

The *disk model* is a unit disk $\mathcal{D} = \{z \in \mathbb{C}, |z| < 1\}$ on the complex plane.

$$d(z_1,z_2) = \log \frac{|1 - \bar{z}_1 z_2| + |z_2 - z_1|}{|1 - \bar{z}_1 z_2| - |z_2 - z_1|} \ . \tag{3}$$

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The group of isometries of $\mathcal{D}$ is the set

$$SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

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If $g \in SU(1,1)$ then $g(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$. $\mathcal{D}$ is a Riemannian manifold and we have a Laplace Beltrami operator expressed in polar coordinate.

$$\Delta f = \frac{\partial^2}{\partial r^2} f + \coth(r) \frac{\partial}{\partial r} f + \frac{4}{\sinh^2(r)} \frac{\partial^2}{\partial \theta^2} f \tag{4}$$

where r denotes hyperbolic distance from 0.

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and its inverse

$$f(z) = \frac{1}{4\pi} \int_{\mathbb{R}} \int_{S_1} \tilde{f}(\lambda,\theta) \left(\cosh r - \sinh r \cdot \cos\theta\right)^{\frac{i\lambda+1}{2}} \lambda \; \tanh(\pi\lambda/2) \; d\theta \; d\lambda. \qquad (6)$$

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We denote by $\mathcal{S}$ the spectrum of the Laplace Beltrami operator

$$\mathcal{S} = \left(0, \frac{1}{2}\right) \cup \left\{ \frac{1+i\lambda}{2}, \lambda \geq 0 \right\}. \tag{8}$$

Field with stationary increments on hyperbolic planes

$\{X_z, t \in \mathcal{D}\}$ is a Gaussian field with stationary increments if $\forall z_1, z_2 \in \mathcal{D}, g \in SU(1,1)$

$$\mathbb{E}|X_{g(z_1)} - X_{g(z_2)}|^2 = \mathbb{E}|X_{z_1} - X_{z_2}|^2. \tag{9}$$

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Theorem (Lévy Khintchine)

Let $V(z) = \mathbb{E}|X_z|^2$ be a structure function of a field X with stationary increments. Then there exists a unique measure ν on the spectral set $\mathcal{S}$ and $c \geq 0$ such that

$$V(z) = c\,Q(z) + \int_S [1 - \omega_s(\eta)]\nu(ds), \qquad z \in \mathcal{D}, \tag{10}$$

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$$\int_{\mathcal{S}} (|s| \wedge 1)\ \nu(ds) < \infty. \tag{11}$$

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The corresponding Gaussian field with stationary increments is

$$X(x,y) = x\xi_1 + y\xi_2,$$

where ξ_i are independent standard Gaussian random variables.

Spectral measure of Lévy Brownian field

For Lévy Brownian field one can write

$$r = d(0,z) = c\,Q(z) + \int_{\mathcal{S}} [1 - \omega_s(z)]\nu(ds), \qquad z \in \mathcal{D}. \tag{12}$$

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For $0 < s < 1/2$
$\omega_s(z) = C(s)e^{-sr}\left(1 + O(e^{-2r}) + O(e^{-2r(\frac{1}{2}-s)})\right).$

Spectral measure of Lévy Brownian field: Higher spectrum

Theorem

The spectral decomposition of the Brownian field on the hyperbolic plane is

$$r = Q(r) + \int_0^\infty [1 - \omega_{\frac{1}{2}+i\frac{\lambda}{2}}(r)]p(\lambda)d\lambda, \tag{13}$$

where the spectral density p is given by

$$p(\lambda) = \frac{\hat{\varphi}(\lambda)\ \lambda\ \tanh(\pi\lambda/2)}{\sqrt{2\pi}(\lambda^2+1)}$$

and

$$\varphi(u) = \int_0^{\pi/2} \frac{2\sin^2\theta - 1}{\sqrt{\cosh^2 u - \sin^2\theta}}\ d\theta.$$

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Hint of the proof

Let apply Δ on both hands of the following equations

$$r - Q(r) = \int_0^\infty [1 - \omega_{\frac{1}{2}+i\frac{\lambda}{2}}(z)]p(\lambda)d\lambda$$

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If $f(z) = \coth(r) - 1$, the Fourier transform $\tilde{f}$ does not depend on θ and the inversion formula (6)

$$f(z) = \frac{1}{2\pi}\int_0^\infty \tilde{f}(\lambda)\ \lambda\ \tanh(\pi\lambda/2)\ \omega_{\frac{1}{2}+i\frac{\lambda}{2}}(z)d\lambda$$

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Hence we have to compute the Fourier transform on hyperbolic space of $\coth(r) - 1$. Actually the Fourier transform on hyperbolic space can be computed with the help of the Abel transform and a classical Fourier transform on $\mathbb{R}$.

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References

- Clerc J.L., Faraut J, Rais M., Eymard P., Takahashi R. (1982) Analyse harmonique. Ser. Les Cours du CIMPA, 1980.
- Faraut, J., Harzallah, K. (1974) Distances hilbertiennes invariantes sur un espace homogène. Ann. Inst. Fourier 24 , no. 3, 171–217.
- Helgason, S. (1978) Differential Geometry Lie Groups and Symmetric Spaces Academic Press, second edition, vol.80.