Approximation de complexité pour des champs additifs

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Introduction

- Calcul de complexité pour des problèmes multivariés
- Idée: Etudier la tractabilité

$$f(t)\,, t \in T \subset \mathbb{R}^d$$
$$\|f - A(f)\|$$

  A algorithme, $\|\cdot\|$ norme
- H. Wozniakowski

Cadre général

- Champ aléatoire

$$X(t) = \sum_{k=1}^{\infty} \xi_k \phi_k(t) , \quad t \in T \subset \mathbb{R}^d ,$$

ξ_k v.a. et $\phi_k : T \to \mathbb{R}$

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- Norme $\|\cdot\|$
- Approximation en moyenne

$$n^{avg}(\varepsilon) = \inf\{n\,;\; E\|X - X_n\|^2 \le \varepsilon^2\}\,, \quad \varepsilon \to 0$$

Champs additifs

- Exemple

$$X(t) = \sum_{l=1}^{d} X_l(t_l)\,,\, t \in \mathbb{R}^d$$

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- Notre cadre: champ à d paramètres, somme de champs dépendant chacun de b paramètres

- $d, b \in \mathbb{N}^*$
  $D = \{1, \cdots d\}$
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- $A = \{a_1, \cdots, a_b\} \in D_b$

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- $$X(t) = \sum_{A \in D_b} X_A(\Pi_A(t)) \, ,$$

  X_A copies i.i.d. d'un champ b-paramétrique

Produit de tenseur

- $\{Y(u)\,, u \in [0,1]\}$ processus de dim 1, ordre 2 t.q.

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$$Y^{\otimes b}(t) = \sum_{k \in \mathbb{N}^b} \prod_{l=1}^{b} \lambda(k_l)\phi_{k_l}(t_l)\xi_k\,, \qquad t \in T_b,$$

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d – Champ additif

$$X_{d,b}(t) = \sum_{A\in D_b} \sum_{k\in\mathbb{N}^b} \left(\prod_{l=1}^{b} \lambda(k_l)\phi_{k_l}\left([\Pi_A(t)]_l\right) \right) \xi_k^A$$

$$= \sum_{A\in D_b} \sum_{k\in\mathbb{N}^A} \left(\prod_{a\in A} \lambda(k_a) \prod_{a\in A} \phi_{k_a}(t_a) \right) \xi_k^A .$$

$\mathcal{K}_Y$ covariance de Y: $\mathrm{Cov}(Y(u), Y(u')) = \mathcal{K}_Y(u, u')$

$$\mathrm{Cov}(X_{d,b}(t), X_{d,b}(t')) = \sum_{A \in D_b} \prod_{a \in A} \mathcal{K}_Y(t_a, t'_a)\,.$$

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- Lemma
  $k, k' \in \mathbb{N}^d$ et $A, A' \subset D$, alors $\psi(t) = \prod_{a \in A} \phi_{k_a}(t_a)$ et $\psi'(t) = \prod_{a \in A'} \phi_{k'_a}(t_a)$ sont identiques ou orthogonales dans $L_2(T_d)$.

On peut alors réécrire

- $$X_{d,b}(t) = \sum_{h=0}^{b} \sum_{\substack{C \subset D \\ |C|=h}} \left[\sum_{k \in (\mathbb{N}^*)^C} \prod_{a \in C} \phi_{k_a}(t_a) \prod_{a \in C} \lambda(k_a) \right] \eta^C ,$$

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- $(\eta^C)_{C\in D}$ centrées, non corrélées

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- $$\mathrm{Var}(\eta^C) = C_{d-h}^{b-h} \lambda(0)^{2(b-h)} .$$

Approximation en moyenne

Approximation de $X_{d,b}$ par X_n (n v.p. maximales)

$$n^{avg}(\varepsilon, d, b) = \inf\{n\,;\ E\|X_{d,b} - X_n\|^2_{L_2(T_d)} \leq \varepsilon^2\}\,.$$

- $\Lambda := \sum_{i=0}^{\infty} \lambda(i)^2 < \infty$,
  $\lambda(i) \sim \mu i^{-r} (\log i)^q$
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- $\alpha = q/r$

Valeurs propres

- $h \in \{1, \cdots, b\}$ et $k \in (\mathbb{N}^*)^h$

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- $(\overline{\lambda}^2_{n,h}\,, n \in \mathbb{N})$ réarrangement décroissant

$$\overline{\lambda}^2_{n,h} \sim B_h^2 n^{-2r} (\log n)^{2r\beta}\,, \quad n \to \infty, \tag{1}$$

where

- $\alpha > -1 : \begin{cases} B_h = \mu^h \left(\frac{\Gamma(\alpha+1)^h}{\Gamma(h(\alpha+1))} \right)^r \\ \beta = (h-1) + h\alpha \end{cases}$
- $\alpha < -1 : \begin{cases} B_h = \mu h^r \left[\sum_{i \geq 1} \lambda(i)^{1/r} \right]^{(h-1)r} \\ \beta = \alpha \end{cases}$

(Csaki ; Li ; Karol, Nikitin & Nazarov ; Lifshits & Tulyakova)

Proposition

- $\alpha > -1$

$$n^{avg}(\varepsilon, d, b) \sim [C_d^b]^{\frac{2r}{2r-1}} \left(\frac{B_b}{\sqrt{2}(r-1/2)^{r\beta+1/2}} \frac{|\log \varepsilon|^{r\beta}}{\varepsilon} \right)^{(r-1/2)^{-1}}, \varepsilon \to 0$$

- $\alpha < -1$

$$n^{avg}(\varepsilon, d, b) \sim \left(\frac{\sqrt{Q}}{\sqrt{2}(r-1/2)^{r\alpha+1/2}} \frac{|\log \varepsilon|^{r\alpha}}{\varepsilon} \right)^{(r-1/2)^{-1}}, \varepsilon \to 0$$

$$Q = \left(\sum_{h=1}^{b} C(h)^{\frac{1}{2r}} \right)^{2r} \quad and \quad C(h) = C_{d-h}^{b-h} [C_d^h]^{2r} \lambda(0)^{2(b-h)} B_h^2.$$

Dimension croissante

$d \to \infty$

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- b fixé et $d \to \infty$
- $b, d \to \infty$ $b/d \to l$

b fixé

Proposition

$$\tilde{n}^{avg}(\varepsilon, b, d) \sim \frac{d^b}{b!} \Lambda^{-b/(2r-1)} \left(\frac{B_b}{\sqrt{2}(r-1/2)^{r\beta+1/2}} \frac{|\log \varepsilon|^{r\beta}}{\varepsilon} \right)^{(r-1/2)^{-1}} .$$

$b \to \infty$ et $b/d \to l$

Proposition

$b/d \to l \in [0,1]$

$$\lim_{d\to\infty} \frac{\log \tilde{n}^{avg}(\varepsilon, b, d)}{d} = \log V,$$

où

$$V = (1-lp)^{-1+lp} l^{-lp} (1-p)^{(1-p)l} A^l \,,$$

$$\tilde{\Lambda} = \sum_{i=1}^{\infty} \lambda(i)^2 \,, \quad p = \frac{\tilde{\Lambda}}{\Lambda} \,,$$

$$M = \sum_{i=0}^{\infty} (-\log \lambda(i)) \frac{\lambda(i)^2}{\tilde{\Lambda}} \,, \quad A = e^{2M} \Lambda \,.$$

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