Spectral gap for kinetically constrained spin models

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- The process in infinite volume is described by the following generator

$$\mathbf{L}f(\sigma) = \sum_{x \in \mathbb{Z}^d} c_x(\sigma)\,[\mu_x(f) - f(\sigma)] \qquad f \text{ local.}$$

- c_x is the constraint that depends on the model

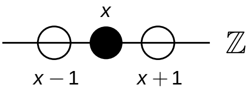
$$c_x(\sigma) = \begin{cases} 1 & \text{if the constraint around site } x \text{ is satisfied by } \sigma \\ 0 & \text{otherwise.} \end{cases}$$

- μ_x are independent Bernoulli-p probability measures, $p \in [0,1]$.

Two examples

- The one dimentional East Model (Eisinger-Jackle 91):

$$c_x(\sigma) = \begin{cases} 1 & \text{if } \sigma(x+1) = 0 \\ 0 & \text{otherwise.} \end{cases}$$



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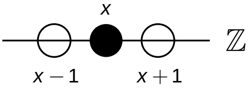
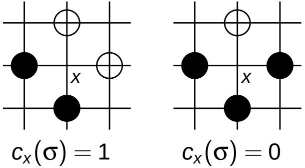


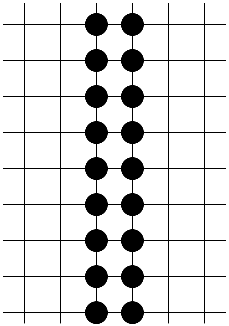
- The two dimentional FA2f model (Fredrickson-Andersen (84)):

$c_x(\sigma) = 1$ if at least 2 neighboors of x are empty;
$c_x(\sigma) = 0$ otherwise.



A blocking structure

There might exist blocking structures, as *e.g.* for the FA2f model :



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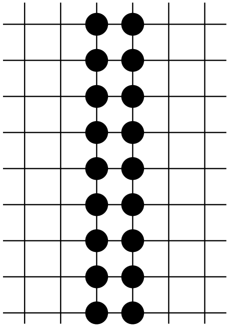

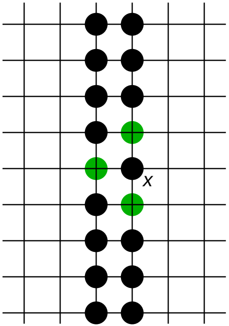

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The site x have 3 occupied neighbors (*i.e.* more than 2).

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(a) $\lim_{t\to\infty} P_t f = \mu(f)$ in $\mathbb{L}^2(\mu)$ for all $f \in \mathbb{L}^2(\mu)$ (ergodicity).

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Due to the definition of the constraints c_x,

$$q > q_c \qquad \Rightarrow \qquad 0 \text{ is a simple eigenvalue for } \mathbf{L}.$$

Bootstrap percolation

Define the (deterministic) Bootstrap percolation map :

$$\begin{aligned} T : \{0,1\}^{\mathbb{Z}^d} &\rightarrow \{0,1\}^{\mathbb{Z}^d} \\ \sigma &\mapsto T(\sigma)(x) = \begin{cases} 0 & \text{if either } \sigma(x) = 0 \text{ or } c_x(\sigma) = 1 \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

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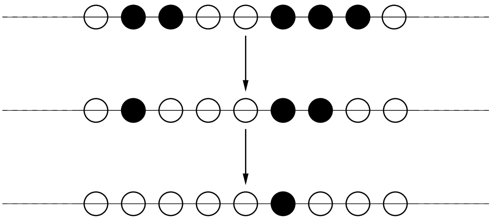
For the east model, the map T applied twice:



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Let

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q_{bp} = \inf\{q \in [0,1] : \mu(\{\sigma : T^{\infty}(\sigma) \equiv 0\}) = 1\}
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i.e. the infimum of the values q such that, with probability one, the lattice can be entirely emptied.

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$$q_c = q_{bp}.$$

Thus, for the east model and the FA2f model (Schonmann), we have

$$q_c = 0.$$

The process in finite volume

In a finite volume $\Lambda \subset \mathbb{Z}^d$, the process is defined by the generator

$$\mathbf{L}_\Lambda f(\sigma) = \sum_{x \in \Lambda} c_{x,\Lambda}(\sigma) \left[\mu_x(f) - f(\sigma)\right] \qquad \forall f$$

with

$$c_{x,\Lambda}(\sigma) = c_x \left(\sigma_\Lambda \cdot \tau_{\mathbb{Z}^d \setminus \Lambda}\right)$$

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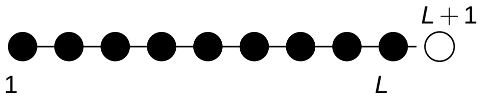
For the east model on $\Lambda = \{1, \ldots, L\}$, $\tau(L+1) = 0$ is a boundary condition that makes the system irreducible.



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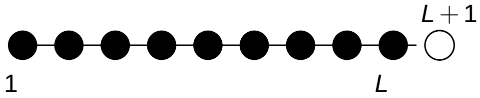

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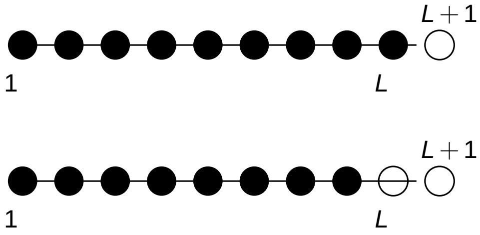

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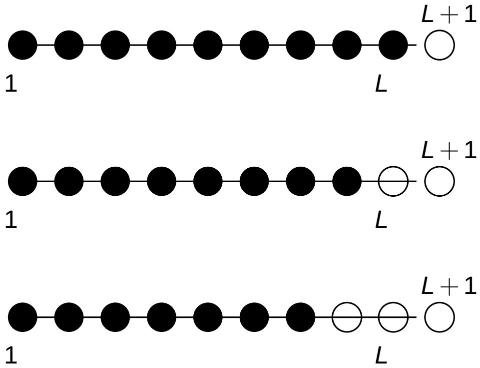

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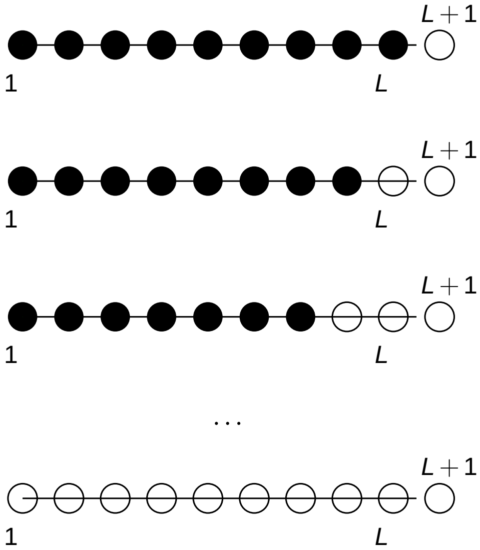
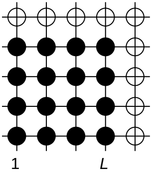


Good boundary condition for the FA2f model

In a volume $\Lambda = \{1, \ldots, L\} \times \{1, \ldots, L\}$ an example of nice boundary condition is :



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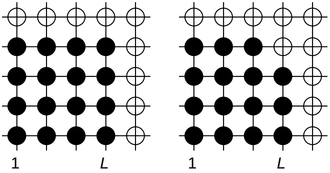

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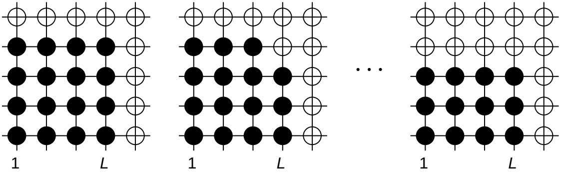

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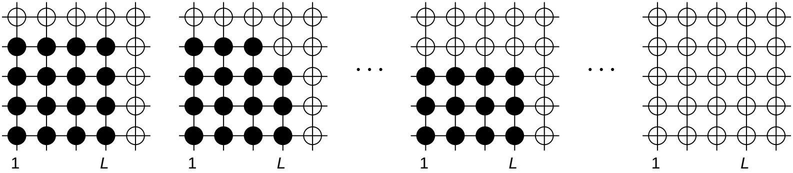

Results

Define $\mathcal{D}(f) = \sum_{x \in \mathbb{Z}^d} \mu(c_x \mathrm{Var}_x(f))$ and

$$\mathrm{gap}(\mathbf{L}) = \inf_{\substack{f \in \mathrm{Dom} \\ f \neq \mathrm{const}}} \frac{\mathcal{D}(f)}{\mathrm{Var}_\mu(f)}.$$

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Theorem (Cancrini-Martinelli-Roberto-Toninelli/Aldous-Diaconis)

For any $q \in (0,1)$, the spectral gap of the East model is positive and

$$\lim_{q\to 0} \frac{\log\left(\frac{1}{\operatorname{gap}(\mathbf{L})}\right)}{(\log\frac{1}{q})^2} = \frac{1}{2\log 2}.$$

Results

Theorem

The spectral gap of the FA2f model is positive and there exists C such that

$$\exp\left(-\frac{1}{Cq^5}\right) \leq \mathrm{gap}(\mathbf{L}) \leq \exp\left(-\frac{C}{q}\right) \qquad \forall q \in (0,1).$$

A word on the proof

By approximation, we have

$$\mathrm{gap}(\mathbf{L}) \geq \inf_{\Lambda \subset \mathbb{Z}^d} \mathrm{gap}(\mathbf{L}_\Lambda).$$

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The bisection-constrained technique on the east model

The aim is to get a bound of the type

$$\mathrm{gap}(\{1,\ldots,L\})^{-1} \leq (1+\varepsilon(L))\mathrm{gap}(\{1,\ldots,\frac{L}{2}\})^{-1}.$$

If $\varepsilon(L)$ is sufficiently small,

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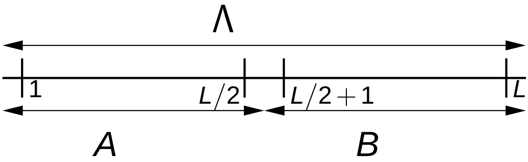
Consider $\Lambda = \{1,\dots,L\}$, $A = \{1,\dots,\frac{L}{2}\}$ and $B = \{\frac{L}{2}+1,\dots,L\}$



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Since $\mu = \mu_A \otimes \mu_B$ is product,
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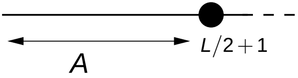

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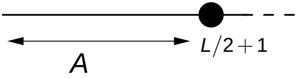

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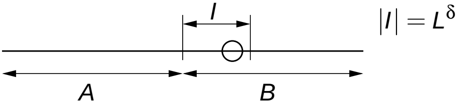
One has to force a good boundary condition. This is achieved by means of an auxiliary constrained two block dynamics.



$$\mathrm{Var}_{\mu_\Lambda}(f) \leq (1+\varepsilon(L))\,\mu_\Lambda\left(c_A \mathrm{Var}_{\mu_A}(f) + \mathrm{Var}_{\mu_B}(f)\right) \qquad \text{with}$$

$$c_A(\sigma) = \begin{cases} 1 \text{ if } \sigma(x) = 0 \text{ for some } x \in I \\ 0 \text{ otherwise.} \end{cases}, \quad \varepsilon(L)^2 \approx \mathbb{P}(c_A = 0) \leq e^{-qL^\delta} \ll 1$$

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From there the expected result follows

$$\mathrm{gap}(\{1,\ldots,L\})^{-1} \leq (1+\varepsilon(L))\mathrm{gap}(\{1,\ldots,\frac{L}{2}\})^{-1}.$$

Extensions

- Non-product measures.
- For some models, on general graphs.
- Link with information storage (Aldous).
- Conservative dynamics (Kawasaki type) with boundary sources.

Thanks for your attention!