Complete monotone coupling for Markov processes

Pierre-Yves Louis
Universität Potsdam

(joint work with P. Dai Pra and I. Minelli,
Università di Padova)

Journées de probabilités, Lille

5 septembre 2008

Outline

Motivation

Monotonicity vs. complete Monotonicity - Discrete Time case

- Monotonicity vs. complete Monotonicity
- When does the Equivalence hold?

Monotonicity vs. complete Monotonicity - Continuous Time case

- Monotonicity
- What is the complete monotonicity?
- When does the Equivalence hold?
  - Low cardinality
  - General cardinality

Motivation

The use of Markov chains for simulation purposes (like MCMC methods, perfect sampling...) motivates the study of the representation as random dynamical system:

Let (f, m) defines a (discrete time-) random dynamical system

- $(\theta_i)_{i \geq 1}$ r.v. on $(\Theta, \mathcal{A})$, i.i.d. with law m
- f measurable (updating function):

$$\begin{array}{rccl} f: & \Theta \times E & \longrightarrow & E \\ & (\theta, x) & \longmapsto & f(\theta, x) = f_\theta(x) \end{array}$$

We associate a Markov chain $(X_n)_{n \geq 0}$ to (f, m):

- X_0 a r.v. such that $\big((\theta_i)_{i \geq 1}, X_0\big)$ are independent
- $X_{n+1} = f(\theta_{n+1}, X_n)$

conversely:
for any Markov chain one can construct a r.d.s. (f, m)

Motivation

- Monotonicity of the representation is an important feature
- for generality, we require the state space to be a partially ordered set (poset) $\preccurlyeq$
- examples: spins systems (for instance with block updating), Markov chains on structures like lozenge-tiling of the hexagon

Definition

The Markov kernel P is said to have a monotone representation (f, m) if

$$x \preccurlyeq y \;\Rightarrow\; f_\theta(x) \preccurlyeq f_\theta(y) \qquad \theta - m\text{-a.s.}$$

Example of a monotone representation

P: symmetrical random walk on $E = \{1, 2, 3, 4, 5\}$ with reflection

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$(f, m := \mathcal{U}_{[0,1]})$ is a monotone representation as r.d.s of the Markov kernel P

$$f(x, \theta) = \begin{cases} x - 1 \text{ if } \theta \in [0, \frac{1}{2}[\\ x + 1 \text{ if } \theta \in [\frac{1}{2}, 1] \end{cases}$$

$$f(1, \theta) = \begin{cases} 1 \text{ if } \theta \in [0, \frac{1}{2}[\\ 2 \text{ if } \theta \in [\frac{1}{2}, 1] \end{cases}$$

$$f(5, \theta) = \begin{cases} 4 \text{ if } \theta \in [0, \frac{1}{2}[\\ 5 \text{ if } \theta \in [\frac{1}{2}, 1] \end{cases}$$

When can we find a monotone representation for a given Markov kernel?

Coupling preserving the ordering

Let P be a transition kernel s.t. it admits a monotone representation (f, m). Then:

- there exists, on the same prob. space, a family $(X_n^x)_{n\geq 0}$ of Markov chains with transition kernel P, parametrised with $x \in E$ s.t. $\begin{cases} X_{n+1}^x = f_\theta(X_n^x) \\ X_0^x = x \end{cases}$ and $x \preccurlyeq y \ \Rightarrow \ \forall n \ X_n^x \preccurlyeq X_n^y \ \mathbb{P}\text{-a.s.}$
- Moreover: monotone coupling of several copies of the same Markov dynamics

$$x_1 \preccurlyeq x_2 \preccurlyeq x_3$$
$$\Downarrow$$
$$\forall n \ X_n^{x_1} \preccurlyeq X_n^{x_2} \preccurlyeq X_n^{x_3} \ \mathbb{P}\text{-a.s.}$$

Outline

Definitions

Let $(E, \preccurlyeq)$ be a partially ordered finite set.

Definition

If there exists (f, m) a monotone representation of the transition kernel P then P is said to be complete monotone (or realisable monotone).

Definition

P is said to be (simply) monotone (attractivity) if

$$\forall f : E \to \mathbb{R} \text{ increasing, } P(f) \text{ is increasing,}$$

where $P(f) : u \mapsto \int f(v)P(u, dv)$.

Characterisation of the (simple) monotonicity

Proposition (see Strassen, Ann. Math. Statist, 1965)

P is monotone

$$\begin{aligned} &\iff \quad \forall \mu_1,\ \mu_2 \ \textit{prob. meas. on } E,\ \mu_1 \preccurlyeq \mu_2 \ \Rightarrow\ \mu_1 P \preccurlyeq \mu_2 P \\ &\iff \quad \forall x \preccurlyeq y,\ P(x,\ .\ ) \preccurlyeq P(y,\ .\ ) \\ &\iff \quad E \ni x \mapsto F(x,\ .\ ) \textit{ is decreasing}, \end{aligned}$$

where:

⋆ $\mu_1 \preccurlyeq \mu_2$ means for every $f : E \to \mathbb{R}$ increasing,
$\mu_1(f) := \int f \, d\mu_1 \leq_{\mathbb{R}} \mu_2(f)$ holds;

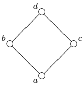
⋆ $\mu_1 P\ (A) = \int P(u, A)\mu_1(du)$;
⋆ $F(x,\ .\ ) : u \mapsto F(u, x) = \sum_{s \preccurlyeq u} P(x, s)$ (distribution function of $P(x,\ .\ )$).

Hasse Diagram (Order Diagram)

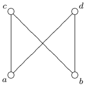
Definition

Hasse diagram of $(E, \preccurlyeq)$:= oriented graph, vertices: $x \in E$,
edges: (x, y) if $(x \preccurlyeq y$ and $\forall s,\ x \preccurlyeq s \preccurlyeq y \Rightarrow s = x \text{ or } s = y)$

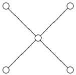
Examples



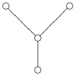
(*Diamond*)



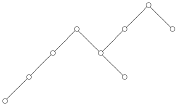
(*Bowtie*)

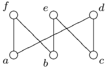


(*Star*)



(*Y*)





An other characterisation

Definition

$\Gamma \subset E$ is said to be an up-set if

$$x \in \Gamma, x \preccurlyeq y \Rightarrow y \in \Gamma.$$

Proposition

$$P \text{ is monotone} \iff \begin{array}{c} \forall x \preccurlyeq y,\ x \neq y,\ \forall \Gamma \ \textit{up-set} \\ \sum_{z\in\Gamma} P(x,z) \leq \sum_{z\in\Gamma} P(y,z) =: P(y,\Gamma) \end{array}$$

First results

Theorem (Kamae, Krengel, O'Brien, Ann. of Prob., 1977)

P monotone $\iff$ *$\exists$ a monotone coupling of two copies of the dynamics P*

Remark

P completely monotone $\Rightarrow$ P monotone

Remark

If the poset is totally ordered then
P completely monotone $\iff$ P monotone

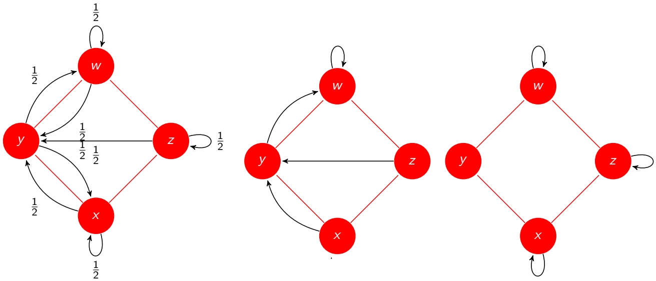
Outline

Question

How should be the partial order $\preccurlyeq$ on E in order to have, for any P transition kernel,

$$P \text{ monotone} \Rightarrow P \text{ completely monotone ?}$$

Equivalence does not always hold: Example



The kernel P is monotone but not completely monotone!
since: if $\exists$ (X^x, X^y, X^z, X^w) r.v. on E then

$$\frac{1}{2} = \mathbb{P}(X^x = y) = \mathbb{P}(X^x = y, X^y = w, X^z = y, X^w = w)$$

$$\frac{1}{2} = \mathbb{P}(X^z = z) = \mathbb{P}(X^x = x \qquad , X^z = z, X^w = w)$$

$$\Rightarrow \qquad \mathbb{P}(X^w = w) \geq \mathbb{P}(X^x = y) + \mathbb{P}(X^z = z) = 1 \text{ Contradiction}$$

Remark: it may be perturbed in order to have an irreducible aperiodic counter-example!

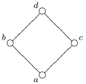
Answer

Theorem (Fill and Machida, Ann. of Prob. 2001)

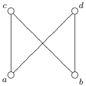
Let P be a monotone kernel. It is completely monotone iff there is *no loop* $x_0, x_1, \ldots, x_n, x_{n+1} = x_0$ *(pairwise unequal) s.t.*

$$\begin{cases} x_i \preccurlyeq x_{i+1} \ \mathit{or}\ x_{i+1} \preccurlyeq x_i \\ x_i \preccurlyeq y \preccurlyeq x_{i+1} \ \mathit{or}\ x_{i+1} \preccurlyeq y \preccurlyeq x_i \Rightarrow y = x_i \ \mathit{or}\ y = x_{i+1} \end{cases}$$

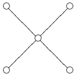
Examples



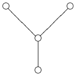
(Diamond)



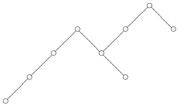
(Bowtie)

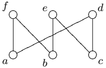


(Star)



(Y)

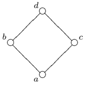




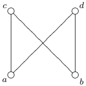
Idea of the proof

The nontrivial proof of the above statement consists of three steps.

1. For each *minimal* cyclic poset an example is found of a monotone Markov chain which is not completely monotone.



(*Diamond*)



(*Bowtie*)

2. Given a general cyclic poset, a monotone but not completely monotone Markov chain is constructed by "lifting" one of the examples in step 1.
3. A proof by induction on the cardinality of the poset shows that, in an acyclic poset, monotone Markov chains are completely monotone.

Outline

Frame

Let $(P_t)_{t\geq 0}$ the semi-group of a regular Markov process $(X_t)_{t\geq 0}$ with values in a partially ordered set $(E, \preccurlyeq)$ (poset).

- Meaning of $(P_t)_{t\geq 0}$ completely monotone?
- Assumptions on the ordering $\preccurlyeq$ s.t. the equivalence between monotonicity and complete monotonicity holds ?

Definition of the monotonicity

Definition

$(P_t)_{t\geq 0}$ is said to be monotone if

$$\forall f : E \to \mathbb{R} \text{ increasing}, \ \forall t \geq 0, P_t(f) \text{ is increasing},$$

where $P_t(f) : u \mapsto \int f(v)P_t(u, dv)$.

Let $L = (L_{x,y})_{x,y\in E^2}$ be the infinitesimal generator of $(P_t)_{t\geq 0}$.
$(L_{x,x} = -\sum_{y\neq x} L_{x,y})$

Characterisation of the monotonicity

Proposition

$(P_t)_{t\geq 0}$ is monotone

$$\underset{\text{(Kamae et al., '77)}}{\iff} \quad \forall y \preccurlyeq z\ \exists \text{ a M. process } (X_t(y,z))_{t\geq 0} \text{ on } E\times E \text{ s.t.} \begin{cases} X_0(y,z) = (y,z) \\ \forall i\in\{1,2\},\ (X_t^i(y,z))_{t\geq 0} \text{ is a M. process with } (P_t) \\ \forall t\geq 0 \quad X_t^1(y,z) \preccurlyeq X_t^2(y,z) \quad a.s. \end{cases}$$

$$\underset{\text{(Massey, '87)}}{\iff} \quad \forall \Gamma \text{ up-set } \forall x \preccurlyeq y,\ x,y\in\Gamma \text{ or } x,y\notin\Gamma \quad \sum_{z\in\Gamma} L_{x,z} \leq \sum_{z\in\Gamma} L_{y,z}$$

Outline

Definition

Definition

$(P_t)_{t\geq 0}$ is completely monotone if there exists a M. process $(\xi_t(\ .\ ))_{t\geq 0}$ on E^E s.t.

- $\xi_0(\ .\ ) = \text{ Identity}$
- $\forall z \in E$, $(\xi_t(z))_{t\geq 0}$ is a M. process with semi-group $(P_t)_{t\geq 0}$ (z is the starting condition)
- if $y \preccurlyeq z$ then $\forall t \geq 0\ \xi_t(y) \preccurlyeq \xi_t(z)$ a.s.

It means $\exists$ a representation as monotone random dynamical system in continuous time.

Characterisation

Proposition

A generator L is the generator of a complete monotone Markov chain if and only if there exists $\Lambda : \mathcal{M} \to \mathbb{R}^+$ such that

$$L_{x,y} = \sum_{f \in \mathcal{M} : f(x) = y} \Lambda(f)$$

holds.

Formalisation

$E_2 := E \times E \setminus \{(x,x) : x \in E\}$

L may be identified with an element of $(\mathbb{R}^+)^{E_2}$

$\mathbb{I}_f \in (\mathbb{R}^+)^{E_2}$ be defined by

$$(\mathbb{I}_f)_{x,y} = \begin{cases} 1 & \text{if } f(x) = y \\ 0 & \text{otherwise.} \end{cases}$$

- $\mathcal{G}_{c.mon}$ cone given by $L = \sum_{f \in \mathcal{M}} \Lambda_f \mathbb{I}_f, \text{ with } \Lambda_f \geq 0$
- $\mathcal{G}_{mon}$ cone given by

$$\langle L, W^{\Gamma,x,y} \rangle \geq 0 \text{ for every } \Gamma, x, y.$$

where

$$W^{\Gamma,x,y}_{v,z} = \begin{cases} 1 & \text{for} \quad \begin{array}{l} x \leq y \notin \Gamma,\ v = y,\ z \in \Gamma \text{ or} \\ x \geq y \in \Gamma,\ v = y,\ z \notin \Gamma \end{array} \\ -1 & \text{for} \quad \begin{array}{l} x \leq y \notin \Gamma,\ v = x,\ z \in \Gamma \text{ or} \\ x \geq y \in \Gamma,\ v = x,\ z \notin \Gamma \end{array} \\ 0 & \text{in all other cases.} \end{cases}$$

Outline

Continuous time vs. discrete time

Proposition (Dai Pra, L., Minelli 2006)

Let $(E, \preccurlyeq)$ *be a poset.*

Equivalence (monotonicity-complete monotonicity)
for any (discrete-time) Markov chain

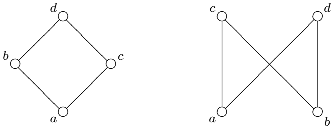
$$\Downarrow$$

Equivalence (monotonicity-complete monotonicity)
for any (continuous time) Markov process

Corollary

For continuous-time, on any poset whose Hasse diagram is without loop, the equivalence holds

The reciprocal is false: Example

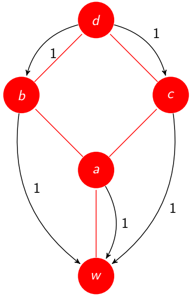


(*Diamond*) (*Bowtie*)

For these examples there is:

- no equivalence in discrete time
- but equivalence in continuous time

Example of posets where the equivalence fails



Monotone Markov process but not completely monotone

Problem for the complete monotonicity, it should be: $L_{aw} \geq 2$

Main reasons why the monotone representation fails

Problem due to synchronous movements:

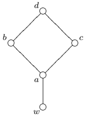
- forbidden group (> 2) - movements (like in the discrete time case)
- jump-intensity incompatibility

Results

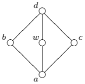
there are 63 unlabelled posets whose cardinality is 5.

Theorem (Dai Pra, L., Minelli 2006)

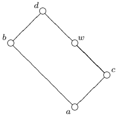
The only posets E with $|E| \leq 5$ where the equivalence in continuous time does not hold are those whose Hasse-Diagram are:



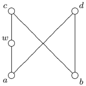
$(\mathcal{S}_1)$



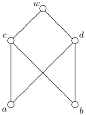
$(\mathcal{S}_2)$



$(\mathcal{S}_3)$



$(\mathcal{S}_4)$



$(\mathcal{S}_5)$

More results

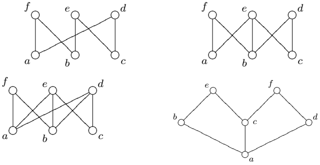
there are 318 unlabelled posets whose cardinality is 6.

Definition

A subset E' of E is said to be an (induced) sub-poset if for all $x, y \in E'$, $x \preccurlyeq y$ in E' is equivalent to $x \preccurlyeq y$ in E.

Theorem (Dai Pra, L., Minelli 2006)

The posets E, for $|E| = 6$, where the equivalence fails are either the ones which admits as sub-poset a poset of the previous slide or the following:



Outline

Motivation

Monotonicity vs. complete Monotonicity - Discrete Time case

Monotonicity vs. complete Monotonicity

When does the Equivalence hold?

Monotonicity vs. complete Monotonicity - Continuous Time case

Monotonicity

What is the complete monotonicity?

When does the Equivalence hold?

Low cardinality

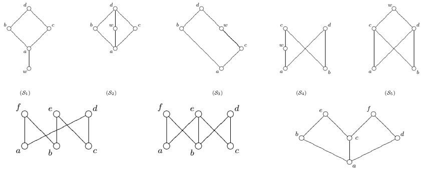
General cardinality

What if $|E| > 6$?

for $|E| = 16$ there are $4,48.10^{15}$ posets;
number unknown for $|E| = 17$...

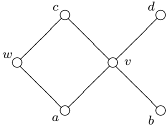
Theorem (Dai Pra, L., Minelli 2008)

If a poset E admits as induced sub-poset a poset E', whose Hasse-Diagram is one of the followings then the monotonicity equivalence fails in E as well.

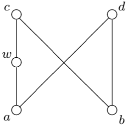


For instance

The equivalence does not hold for the following poset E:



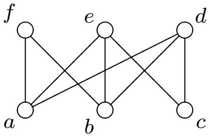
since
it admits as sub-poset $E' = E \setminus \{c\}$:

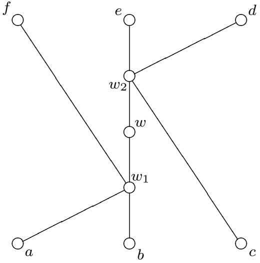


$(\mathcal{S}_4)$

Method used for proving the non-equivalence

- Let S be an induced sub-poset of a given poset S' and let L be a monotone generator on S which has a monotone extension L' to S'. Then $L' \in \mathcal{G}_{c.mon}(S') \Rightarrow L \in \mathcal{G}_{c.mon}(S)$.
- Nevertheless: non-equivalence is in general not preserved by extending the poset, for instance there are no monotone extension for counter-examples in the following case:

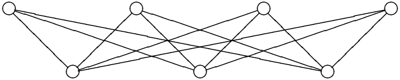




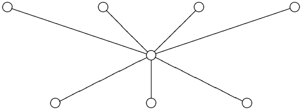
Complete classification ?

Proposition (Dai Pra, L., Minelli 2008)

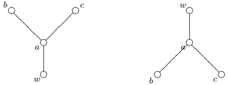
For the family of the complete crowns, equivalence holds!



since it is a sub-poset of



Conjecture

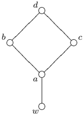


Let S be a connected poset having no acyclic extension. Then monotonicity equivalence holds if and only if the following conditions hold:

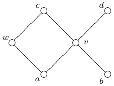
i) the Hasse diagram of S has a unique cycle, which is a diamond;

ii) S has no Y-shaped subposet having at most one point in common with the cycle in point i) and there is no induced subposet of the following types (up to symmetries).

Forbidden subposets:



(S_1)



Literature

- Dai Pra, P.; Louis, P.-Y. and Minelli, I., *Monotonicity and complete monotonicity for continuous-time Markov chains*, C. R. Math. Acad. Sci. Paris **342** (2006), no. 12, 965–970
- Dai Pra, P.; Louis, P.-Y. and Minelli, I., *Complete monotone coupling for Markov Processes*, Preprint Universität Potsdam (2008)
- J. A. Fill and M. Machida, *Stochastic monotonicity and realizable monotonicity*, Ann. of Prob. **29** (2001), no. 2, 938–978.
- T.E. Harris, *A correlation inequality for markov processes in partially ordered state spaces*, Ann. of Prob. **5** (1977), no. 3, 451–454.
- T. Kamae; U. Krengel, and G.L. O'Brien, *Stochastic inequalities on partially ordered spaces*, Ann. of Prob. **5** (1977), no. 6, 899–912.
- T. Kamae and U. Krengel, *Stochastic partial ordering*, Ann. of Prob. **6** (1978), no. 6, 1044–1049.
- W. A. Massey, *Stochastic orderings for Markov processes on partially ordered spaces*, Math. Oper. Res. **12** (1987), no. 2, 350–367.