Paper published in: Adv. Math. 223 (2010), pp. 2049-2096.

The Bar Complex of an E-infinity Algebra

Benoit Fresse[1]

UMR 8524 du CNRS et de l'Université de Lille 1, Sciences et Technologies
Cité Scientifique – Bâtiment M2
F-59655 Villeneuve d'Ascq Cédex (France)

Abstract

The standard reduced bar complex B(A) of a differential graded algebra A inherits a natural commutative algebra structure if A is a commutative algebra. We address an extension of this construction in the context of E-infinity algebras. We prove that the bar complex of any E-infinity algebra can be equipped with the structure of an E-infinity algebra so that the bar construction defines a functor from E-infinity algebras to E-infinity algebras. We prove the homotopy uniqueness of such natural E-infinity structures on the bar construction.

We apply our construction to cochain complexes of topological spaces, which are instances of E-infinity algebras. We prove that the n-th iterated bar complexes of the cochain algebra of a space X is equivalent to the cochain complex of the n-fold iterated loop space of X, under reasonable connectedness, completeness and finiteness assumptions on X.

Key words: Bar construction, Operads, E-infinity algebras, iterated loop spaces
2000 MSC: 57T30, 55P48, 18G55, 55P35

Introduction

This paper is concerned with the standard reduced bar complex $B(A)$ defined basically for an associative differential graded algebra A equipped with an augmentation over the ground ring $\Bbbk$. We also consider the natural extension of the bar construction to A_∞-algebras, differential graded algebras equipped with a set of coherent homotopies that make the structure associative in the strongest homotopical sense.

By a classical construction, the bar complex of an associative and commutative algebra inherits a multiplicative structure, unlike the bar complex of a non-commutative algebra, and still forms a differential graded associative and commutative algebra. In this paper, we address a generalization of this construction to E_∞-algebras (E-infinity algebras in plain words), the notion, par-

[1]Research supported in part by ANR grant JCJC06 OBTH

allel to the notion of an A_∞-algebra, which models a differential graded algebra equipped with a set of coherent homotopies that make the structure associative and commutative in the strongest homotopical sense. Our main theorems, Theorems 2.1.B-2.2.B, give the existence and the homotopy uniqueness of an E_∞-algebra structure on the bar construction so that:

(1) The bar construction $B(A)$ defines a functor from E_∞-algebras to E_∞-algebras.
(2) The E_∞-algebra structure of $B(A)$ reduces to the standard commutative algebra structure of the bar construction whenever A is a commutative algebra.

To make these assertions more precise, a model of the category of E_∞-algebras has to be fixed. For this purpose, we use that the algebra structures which occur in our problem are modeled by operads: an A_∞-algebra is equivalent to an algebra over an A_∞-operad, in our context a differential graded operad weakly-equivalent to the operad of associative algebras; an E_∞-algebra is an algebra over an E_∞-operad, a differential graded operad weakly-equivalent to the operad of associative and commutative algebras (see [26] for the original definition in the topological framework). Our existence and uniqueness theorems give a functorial E_∞-algebra structure on the bar construction, for every category of algebras over an E_∞-operad E, for any E_∞-operad E. To define the action on the target, we just have to take a cofibrant replacement of E with respect to the model structure of differential graded operads [7, 17].

The overall idea of our construction is to use modules over operads to represent functors on categories of algebras over operads. The bar construction itself is determined by a right module over a particular A_∞-operad, the chain operad of Stasheff's associahedra (*Stasheff's operad* for short). The existence and uniqueness of E_∞-algebra structures on the bar construction is proved at the module level by techniques of homotopical algebra. The arguments rely on the existence of a model structure for right modules over operads.

The existence of a dual E_∞-coalgebra structure on the cobar construction has already been obtained by a different method in [34]. But: the modelling of functors by modules over operads makes our construction more conceptual; our uniqueness theorem makes the definition of an E_∞-structure easier since a simple characterization ensures us to obtain the right result.

Since the bar construction defines a functor from E_∞-algebras to E_∞-algebras, we have a well-defined iterated bar complex $B^n(A)$ associated to any E_∞-algebra. Our motivation, explained next, is to have an iterated bar complex $B^n(C^*(X))$, for any cochain algebra $C^*(X)$, for every pointed topological space X, so that $B^n(C^*(X))$ is equivalent, under reasonable finiteness and connectedness assumptions on the space X, to $C^*(\Omega^n X)$, the cochain algebra of the iterated loop space $\Omega^n X$.

The usual cochain complexes $C^*(X)$ associated to topological spaces are examples of objects equipped with an E_∞-algebra structure (see [18] and the

more combinatorial constructions of [6, 27]). In positive characteristic, the existence of Steenrod operations represents a primary obstruction to the existence of a genuine commutative algebra equivalent to $C^*(X)$ and one has to use E_∞-algebras (or equivalent notions) to model faithfully the homotopy of a space X by a cochain complex (see [25]).

According to classical results of Adams [1] and Adams-Hilton [2], the bar complex $B(C^*(X))$, where $C^*(X)$ is the cochain algebra of a topological space X, is equivalent as a chain complex to $C^*(\Omega X)$, the cochain complex of the loop space ΩX. Since the cochain complex $C^*(X)$ forms an E_∞-algebra, we obtain by our structure theorem that the bar complex $B(C^*(X))$ comes equipped with a well-defined E_∞-algebra structure. To obtain the topological interpretation of the iterated bar complex $B^n(C^*(X))$, we prove that $B(C^*(X))$ is equivalent to $C^*(\Omega X)$ as an E_∞-algebra.

For this aim, we prove that, for a cofibrant E_∞-algebra, the usual bar construction is equivalent as an E_∞-algebra to a categorical version of the bar construction in which tensor products are replaced by algebra coproducts. Then we apply a theorem of Mandell [25] which asserts that the categorical bar construction of a cofibrant replacement of $C^*(X)$ defines an E_∞-algebra equivalent to $C^*(\Omega X)$.

The categorical bar construction preserves weak-equivalences between cofibrant E_∞-algebras only. Therefore we have to form a cofibrant replacement of $C^*(X)$ in E_∞-algebras in order to apply the categorical bar construction reasonably. In contrast, the usual bar construction preserves weak-equivalences between all E_∞-algebras which are cofibrant in the underlying category of dg-modules (all E_∞-algebras if the ground ring is a field). For this reason, we can apply the usual bar construction to the cochain algebra itself $C^*(X)$, and not only to a cofibrant replacement of $C^*(X)$, to still have an E_∞-algebra equivalent to $C^*(\Omega X)$.

The article [25] gives an attractive theoretical setting to model the homotopy of spaces in positive characteristic, but in practice one has to face deep difficulties to build cofibrant replacements in categories of E_∞-algebras. In this sense, our construction gives an effective substitute for the categorical bar construction used in [25].

According to [29], the bar complex of simplicial commutative algebras models the suspension in the homotopy category of simplicial commutative algebras. In passing, we prove that, in the differential graded setting, the bar complex of E_∞-algebras yields a model of the suspension in categories of E_∞-algebras.

Other attempts to define an iterated bar construction occur in the literature outside J.R.Smith's memoirs [33, 34]. Usually, authors deal with the dual cobar construction and chain complexes rather than cochain complexes. If we assume reasonable finiteness assumptions on spaces, then this dual construction is equivalent to the bar construction and nothing changes. To simplify we examine the previous results of the literature in the context of the bar construction.

(1) The original geometrical approach of Adams [1] and Adams-Hilton [2] is continued by Milgram in [28] and Baues in [3, 4, 5] to define a double bar

construction $B^2(C^*(X))$, for any cochain algebra $C^*(X)$, where X is a simplicial set (see also the survey article [10]).

(2) In [22], Kadeishvili-Saneblidze use perturbation lemmas and the classical chain equivalence $B(C^*(X)) \sim C^*(\Omega X)$ to obtain an inductive construction of an iterated bar complex $B^n(C^*(X))$ together with a chain equivalence $B^n(C^*(X)) \sim C^*(\Omega^n X)$, for every cochain algebra $C^*(X)$; this approach is used by Rubio-Sergeraert in the Kenzo program [31] to perform computer calculations.

(3) In [24], Karoubi uses ideas of non-commutative differential geometry and non-commutative analogues of difference calculus to introduce new cochain complexes $D^*(X)$ for which a modified iterated bar complex $B^n(D^*(X))$ can be defined so that $B^n(D^*(X)) \sim D^*(\Omega^n X)$.

The difficulty in (1) is to understand the geometry of certain cell complexes in order to define higher iterated bar complex $B^n(C^*(X))$ for $n > 2$ (see [4, 8]). In the approach of (1), and similarly in (3), the bar construction is only defined for complexes of a particular type. In the approach of (2), one has to keep track of a simplicial model of $\Omega^n X$, the iterated Kan construction $G^n(X)$, to define the differential of $B^n(C^*(X))$.

In contrast, our theorems imply the existence of a well-characterized iterated bar complex $B^n(A)$, for every E_∞-algebra A, and such that $B^n(A)$ incorporates minimal information in itself. Besides, we have to use multiplicative structures to relate the iterated bar complex $B^n(C^*(X))$ to the cochain complex of an iterated loop space $C^*(\Omega^n X)$, but the iterated bar complex $B^n(A)$ can be determined directly by using that a composite of functors associated to modules over operads, like the iterated bar complex, forms itself a functor determined by a module over an operad (see [14]). This observation, beyond the scope of this article, is the starting point of [16].

In this introduction, we adopt the usual convention to apply the bar construction to augmented unital algebras. In the context of the cochain complex of a space X, the augmentation is determined by the choice of a base point $* \in X$. But in the definition of the bar complex we have to replace an algebra A by its augmentation ideal $\bar{A}$, which forms a non-augmented non-unital algebra, and the cochain complex of a space $C^*(X)$ by the associated reduced complex $\bar{C}^*(X)$. Therefore it is more natural to use non-augmented non-unital algebras for our purpose and we take this convention in the core sections of the article (for details, see §1.1.2 and §1.1.4).

Contents

In the first part of the paper, *"Background"*, we survey new ideas introduced in [14] to model functors on algebras over operads by modules over operads. These preliminaries are necessary to make the conceptual setting of our constructions accessible to readers which are only familiar with standard definitions of the theory of operads.

The object of our study, the bar construction, appears in the second part, *"The bar construction and its multiplicative structure"*, where we prove the main results of the article. In the core sections, §§1-4, we define the bar module, the module over Stasheff's operad which represents the bar construction, we prove the existence and uniqueness of a multiplicative structure on the bar construction, and we give a homotopy interpretation of the bar construction in the model category of E_∞-algebras. For a more detailed outline, we refer to the introduction of this part.

In the concluding part, *"The iterated bar construction and iterated loop spaces"*, we address topological applications of our results. As explained in this introduction, we use the multiplicative structure of the bar construction to define an iterated bar construction $B^n(C^*(X))$, for any cochain algebra $C^*(X)$, so that $B^n(C^*(X)) \sim C^*(\Omega^n X)$. One aim of this part is to make explicit reasonable finiteness, completeness and connectedness assumptions on X which ensure this equivalence.

Background

Before studying the structure of the bar construction, we survey ideas introduced in the book [14] to make the overall setting of our constructions accessible to readers.

First, our use of functors and modules over operads motivates a review of the categorical background of operad theory, to which §0.1 are devoted. Then, in §0.2, we review the definition of an operad, of an algebra over an operad, and the definition of categories of modules associated to operads. The correspondence between modules over operads and functors is addressed in §0.3.

Throughout the paper, we use extensively extension and restriction functors in the context of algebras and modules over operads. The last subsection of this part, §0.4, is devoted to recollections on these topics.

0.1. Symmetric monoidal categories over dg-modules

As usual in the literature, we assume that operads consist of objects in a fixed base symmetric monoidal category – for our purpose, the category of unbounded differential graded modules (*dg-modules* for short) over a fixed ground ring $\Bbbk$ (see §0.1.1).

In contrast, we can assume that the underlying category of algebras over an operad is not the base category itself, to which the operad belongs, but some symmetric monoidal category over the category of dg-modules. Though we only use specific examples of such categories in applications, the category of dg-modules itself, the category of Σ_*-modules, and categories of right modules over an operad, for which alternative point of views are available (see §0.2), we prefer to review the definition of this general setting which gives the right conceptual background to understand our arguments.

0.1.1. Symmetric monoidal categories over dg-modules

Let $\Bbbk$ be a ground ring, fixed once and for all. Throughout the paper, the notation $\mathcal{C}$ refers to the category of dg-modules, where a dg-module consists of a lower $\mathbb{Z}$-graded $\Bbbk$-module $C = \oplus_{*\in\mathbb{Z}} C_*$ equipped with an internal differential, usually denoted by $\delta : C \to C$, that decreases degrees by 1. The usual convention $C^* = C_{-*}$ makes any upper graded module equivalent to an object of $\mathcal{C}$.

The category of dg-modules is equipped with the standard tensor product of dg-modules $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which provides $\mathcal{C}$ with the structure of a symmetric monoidal category. The unit object of dg-modules is formed by the ground ring itself $\Bbbk$, viewed as a dg-module concentrated in degree 0.

For us, a symmetric monoidal category over $\mathcal{C}$ is a symmetric monoidal category $\mathcal{E}$ equipped with an external tensor product $\otimes : \mathcal{C} \times \mathcal{E} \to \mathcal{E}$ so that an obvious generalization of relations of symmetric monoidal categories holds in $\mathcal{E}$, for any composite of the tensor products $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ and $\otimes : \mathcal{C} \times \mathcal{E} \to \mathcal{E}$. For details on this background we refer to [14, §1.1].

In principle, we assume that the internal tensor product of $\mathcal{E}$, as well as the external tensor product over dg-modules $\otimes : \mathcal{C} \times \mathcal{E} \to \mathcal{E}$, preserves colimits. Under mild set-theoretic assumptions, these conditions are equivalent to the existence of right adjoints for the internal tensor product and the external tensor product of $\mathcal{E}$. In the paper, we only use the existence of the external-hom

$$\mathrm{Hom}_{\mathcal{E}}(-,-) : \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{C},$$

which satisfies

$$\mathrm{Mor}_{\mathcal{E}}(C \otimes E, F) = \mathrm{Mor}_{\mathcal{C}}(C, \mathrm{Hom}_{\mathcal{E}}(E, F)),$$

for $C \in \mathcal{C}$, $E, F \in \mathcal{E}$.

0.1.2. Symmetric monoidal model categories over dg-modules

The category of dg-modules $\mathcal{C}$ is equipped with a cofibrantly generated model structure such that a morphism $f : C \to D$ is a weak-equivalence if f induces an isomorphism in homology, a fibration if f is degreewise surjective, and a cofibration if f has the left lifting properties with respect to acyclic fibrations.

This model structure is symmetric monoidal (see [20, §4]) in the sense that:

MM0. *The unit of the tensor product forms a cofibrant object in $\mathcal{C}$.*

MM1. *The tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfies the pushout-product axiom – explicitly: the natural morphism*

$$(i_*, j_*) : A \otimes D \bigoplus_{A \otimes C} B \otimes C \to B \otimes D$$

induced by cofibrations $i : A \rightarrowtail B$ and $j : C \rightarrowtail D$ forms a cofibration in $\mathcal{C}$, an acyclic cofibration if i or j is also acyclic.

In the paper, we use cofibrantly generated model categories $\mathcal{E}$ which are symmetric monoidal over the base category of dg-modules $\mathcal{C}$ and such that the

analogues of axioms MM0-MM1 are satisfied at the level of $\mathcal{E}$: the unit object $\mathbf{1} \in \mathcal{E}$ forms a cofibrant object in $\mathcal{E}$ and the internal tensor product of $\mathcal{E}$, as well as the external tensor product of $\mathcal{E}$ over the category of dg-modules $\mathcal{C}$, satisfies the pushout product axiom. In this context, we say that $\mathcal{E}$ forms a cofibrantly generated symmetric monoidal model category over dg-modules.

The books [19, 20] are our references on the background of model categories. For the definition of a symmetric monoidal model category, we refer more particularly to [20, §4]. For the generalization of this notion to our relative setting, we refer to [14, §11.3].

0.1.3. Enriched model category structures

The axioms of symmetric monoidal model categories are used implicitly when we define the model category of operads and the model category of algebras over an operad. In the article, we also use a dual version of the pushout-product axiom which holds for the external hom functor of a symmetric monoidal category over $\mathcal{C}$:

MM1'. *The natural morphism*

$$\mathrm{Hom}_{\mathcal{E}}(B, C) \xrightarrow{(i^*, p_*)} \mathrm{Hom}_{\mathcal{E}}(A, C) \times_{\mathrm{Hom}_{\mathcal{E}}(A,D)} \mathrm{Hom}_{\mathcal{E}}(B, D)$$

induced by a cofibration $i : A \rightarrowtail B$ and a fibration $p : C \twoheadrightarrow D$ forms a fibration in $\mathcal{C}$, an acyclic fibration if i or p is also acyclic.

The characterization of (acyclic) fibrations in a model category by the left lifting property with respect to (acyclic) cofibrations and the definition of the external hom imply readily that axiom MM1' is formally equivalent to the pushout product axiom MM1 for the external tensor product $\otimes : \mathcal{C} \times \mathcal{E} \to \mathcal{E}$.

0.2. Operads, algebras and modules over operads

In this subsection, we review basic definitions of the theory of operads in the context of symmetric monoidal categories over dg-modules. To begin with, we recall briefly the definition of a Σ_*-module, of an operad, and of module structures associated to operads. For details, we refer to relevant sections of [14].

0.2.1. Operads and modules over operads

Throughout the paper, we use the notation $\mathcal{M}$ to refer to the category of Σ_*-objects in dg-modules (Σ_*-*modules* for short), whose objects are collections $M = \{M(n)\}_{n \in \mathbb{N}}$, where $M(n)$ is a dg-module equipped with an action of the symmetric group in n letters Σ_n, for $n \in \mathbb{N}$.

In the classical theory, a module of symmetric tensors

$$S(M, E) = \bigoplus_{n=0}^{\infty} (M(n) \otimes E^{\otimes n})_{\Sigma_n}$$

is associated to any Σ_*-module $M \in \mathcal{M}$. The coinvariants $(M(n) \otimes E^{\otimes n})_{\Sigma_n}$ identify the natural action of permutations on $E^{\otimes n}$ with their action on $M(n)$.

For our purpose, we note that this construction makes sense in any symmetric monoidal category $\mathcal{E}$ over the category of dg-modules $\mathcal{C}$, so that the map $S(M) : E \mapsto S(M, E)$ defines a functor $S(M) : \mathcal{E} \to \mathcal{E}$.

The category of Σ_*-modules comes equipped with a composition product $\circ : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ such that $S(M \circ N, E) = S(M, S(N, E))$, for all $M, N \in \mathcal{M}$, $E \in \mathcal{E}$, and for every symmetric monoidal category over dg-modules $\mathcal{E}$. The composition product of Σ_*-modules is associative and unital. The composition unit is defined by the Σ_*-module

$$\mathsf{I}(n) = \begin{cases} \Bbbk, & \text{if } n = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and we have $S(\mathsf{I}) = \mathrm{Id}$, the identity functor on $\mathcal{E}$.

There are several equivalent definitions for the notion of an operad. According to one of them, an operad consists of a Σ_*-module P equipped with an associative product $\mu : \mathsf{P} \circ \mathsf{P} \to \mathsf{P}$, the composition product of P, together with a unit represented by a morphism $\eta : \mathsf{I} \to \mathsf{P}$.

The structure of a right module over an operad R is defined by a Σ_*-module M equipped with a right R-action determined by a morphism $\rho : M \circ \mathsf{R} \to M$ which is associative with respect to the operad composition product and unital with respect to the operad unit. The category of right R-modules is denoted by $\mathcal{M}_{\mathsf{R}}$.

There is a symmetrically defined notion of left module over an operad P consisting of a Σ_*-module N equipped with a left P-action determined by a morphism $\lambda : \mathsf{P} \circ N \to N$. One can also define the notion of a bimodule as a Σ_*-object N equipped with both a right R-action $\rho : N \circ \mathsf{R} \to N$ and a left P-action $\lambda : \mathsf{P} \circ N \to N$ that commute to each other. The notation ${}_{\mathsf{P}}\mathcal{M}$ refers to the category of left P-modules and the notation ${}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}}$ to the category of P-R-bimodules.

Note that an operad R forms obviously a right module (respectively, left module, bimodule) over itself.

The composition product of Σ_*-modules is not symmetric since this operation is supposed to represent the composition of functors. For this reason, left and right operad actions on Σ_*-modules have different nature though definitions are symmetrical. In §0.2.5, we observe that left modules (respectively, bimodules) over operads are equivalent to algebras over operads and this equivalent definition reflects the structure of left modules and bimodules more properly.

0.2.2. The symmetric monoidal category of Σ_*-modules

The category of Σ_*-modules, which defines an underlying category for operads and modules over operads, gives our primary example of a symmetric monoidal model category over dg-modules (outside the category of dg-modules itself).

The unit of the tensor structure of Σ_*-modules is the Σ_*-module $\mathbf{1}$ such that $\mathbf{1}(0) = \Bbbk$ and $\mathbf{1}(n) = 0$ for $n > 0$. The tensor product $C \otimes N \in \mathcal{M}$ of a

Σ_*-module $M \in \mathcal{M}$ with a dg-module $C \in \mathcal{C}$ is given by the obvious formula

$$(C \otimes M)(r) = C \otimes M(r),$$

for $r \in \mathbb{N}$. The tensor product $M \otimes N \in \mathcal{M}$ of Σ_*-modules $M, N \in \mathcal{M}$ is defined by a formula of the form:

$$(M \otimes N)(r) = \bigoplus_{s+t=r} \Sigma_r \otimes_{\Sigma_s \times \Sigma_t} M(s) \otimes N(t),$$

for $r \in \mathbb{N}$. At the functor level, the tensor operations of Σ_*-modules represent the pointwise tensor products

$$S(M \otimes N, E) = S(M, E) \otimes S(N, E) \quad \text{and} \quad S(C \otimes M, E) = C \otimes S(M, E),$$

for any symmetric monoidal category over dg-modules $\mathcal{E}$, where $E \in \mathcal{E}$ (we refer to [14, §2.1] for details on these recollections).

Since $\mathcal{M}$ forms a symmetric monoidal category over dg-modules, a Σ_*-module M gives rise to a functor $S(M) : \mathcal{M} \to \mathcal{M}$ on the category of Σ_*-modules itself. In fact, we have an identity $S(M, N) = M \circ N$, for all $M, N \in \mathcal{M}$ (see [14, §2.2]).

0.2.3. The symmetric monoidal category of right R*-modules*

According to [14, §6.1], the tensor product $M \otimes N$ of right modules over an operad R inherits the structure of a right R-module and similarly for the external tensor product $C \otimes M$ of a dg-module $C \in \mathcal{C}$ with a right R-module $M \in \mathcal{M}_{\mathsf{R}}$. Hence the category of right R-modules forms a symmetric monoidal category over dg-modules so that the forgetful functor $U : \mathcal{M}_{\mathsf{R}} \to \mathcal{M}$ preserves symmetric monoidal structures. In the context of right R-modules, the functor $S(M) : \mathcal{M}_{\mathsf{R}} \to \mathcal{M}_{\mathsf{R}}$ is still given by the formula $S(M, N) = M \circ N$, for all $M \in \mathcal{M}$, $N \in \mathcal{M}_{\mathsf{R}}$, where $M \circ N$ has an obvious right R-action induced by the right R-action on N.

0.2.4. Symmetric monoidal model structures

The category of Σ_*-modules $\mathcal{M}$ inherits a natural model structure such that a morphism $f : M \to N$ is a weak-equivalence (respectively, a fibration) if the underlying collection of dg-module morphisms $f : M(n) \to N(n)$ consists of weak-equivalences (respectively, fibrations) in the category of dg-modules. Cofibrations are determined by the right lifting property with respect to acyclic fibrations. The model category $\mathcal{M}$ is also cofibrantly generated and symmetric monoidal over dg-modules in the sense of §0.1.2 (see [14, §11.4]).

In [14, §14] we check that the category of right modules over an operad R forms a cofibrantly generated symmetric monoidal model category over dg-modules, like the category of Σ_*-modules, provided that the underlying collection of the operad $\{\mathsf{R}(n)\}_{n \in \mathbb{N}}$ consists of cofibrant dg-modules. Throughout the paper, we assume tacitely that an operad R satisfies this condition if we deal with model structures of the category of right R-modules. As usual, we assume

that a morphism of right R-modules $f : M \to N$ is a weak-equivalence (respectively, a fibration) if the underlying collection consists of weak-equivalences (respectively, fibrations) of dg-modules $f : M(n) \to N(n)$ and we characterize cofibrations by the left lifting property with respect to acyclic fibrations.

0.2.5. On algebras over operads

In standard definitions, one uses that the functor $S(\mathsf{P})$ associated to an operad P forms a monad in order to define the category of algebras associated to P. The usual definition can readily be extended in the context of symmetric monoidal categories over dg-modules since according to the construction of §0.2.1 we have a functor $S(\mathsf{P}) : \mathcal{E} \to \mathcal{E}$ for every symmetric monoidal category $\mathcal{E}$ over the category of dg-modules $\mathcal{C}$.

The structure of a P-algebra in $\mathcal{E}$ consists of an object $A \in \mathcal{E}$ equipped with an evaluation morphism $\lambda : S(\mathsf{P}, A) \to A$ that satisfies natural associativity and unit relations. The definition of $S(\mathsf{P}, A)$ implies that the evaluation morphism is also equivalent to a collection of equivariant morphisms

$$\lambda : \mathsf{P}(n) \otimes A^{\otimes n} \to A$$

formed in the category $\mathcal{E}$. Throughout the paper, we use the notation ${}_{\mathsf{P}}\mathcal{E}$ to refer to the category of P-algebras in $\mathcal{E}$.

For $E \in \mathcal{E}$, the object $S(\mathsf{P}, E) \in \mathcal{E}$ is equipped with a natural P-algebra structure and represents the free object associated to E in the category of P-algebras. In the paper, we use the notation $\mathsf{P}(E) = S(\mathsf{P}, E)$ to refer to the object $S(\mathsf{P}, E)$ equipped with the free P-algebra structure and we keep the notation $S(\mathsf{P}, E)$ to refer to the underlying object in $\mathcal{E}$.

In the case $\mathcal{E} = \mathcal{M}$, we have an identity $S(\mathsf{P}, M) = \mathsf{P} \circ M$ from which we deduce that a P-algebra in Σ_*-objects is equivalent to a left P-module. In the case $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$, we obtain that a P-algebra in right R-modules is equivalent to a P-R-bimodule. Our conventions for categories of algebras over operads is coherent with the notation of §0.2.5 for the category of left P-modules ${}_{\mathsf{P}}\mathcal{M}$ and for the category of P-R-bimodules ${}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}}$.

In the paper, we use repeatedly the observation, made in §0.2.1, that an operad forms a bimodule over itself, and hence an algebra over itself in the category of right modules over itself.

The categories of right modules over an operad carry the same structures as usual categories of modules over algebras. The categories of left modules over an operad, as well as the categories of bimodules, have structures of different nature that the notion of an algebra in a symmetric monoidal category over dg-modules reflects. The idea of an algebra in a symmetric monoidal category over dg-modules is also more natural in constructions of this article. Therefore, in this paper, we prefer to use the language of algebras in symmetric monoidal categories for left modules and bimodules over operads.

0.2.6. Model categories of algebras over operads

Let P be a Σ_*-cofibrant operad, an operad which forms a cofibrant object in the underlying category of Σ_*-modules.

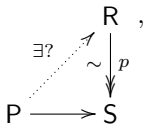
Let $\mathcal{E}$ be a cofibrantly generated symmetric monoidal model category over dg-modules. The category of P-algebras in $\mathcal{E}$ inherits a semi-model structure such that a morphism $f : A \to B$ defines a weak-equivalence (respectively, a fibration) in ${}_{\mathsf{P}}\mathcal{E}$ if f forms a weak-equivalences (respectively, fibrations) in the underlying category $\mathcal{E}$ (see [35], we also refer to [21] for the notion of a semi-model category). Roughly, all axioms of a model category are satisfied in ${}_{\mathsf{P}}\mathcal{E}$, including M4 and M5, as long as the source of the morphism $f : A \to B$ that occurs in these properties is assumed to be cofibrant.

This assertion can be applied to the category of Σ_*-modules $\mathcal{E} = \mathcal{M}$ (respectively, to the category of right modules over an operad $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$) to obtain that the left P-modules ${}_{\mathsf{P}}\mathcal{M}$ (respectively, the P-R-bimodules ${}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}}$) form a semi-model category.

0.2.7. Model categories of operads

The category of operads $\mathcal{O}$ carries a semi-model structure such that the forgetful functor $U : \mathcal{O} \to \mathcal{M}$ creates fibrations and weak-equivalences (see [35]). Thus, according to definitions for $\mathcal{M}$, a morphism $f : \mathsf{P} \to \mathsf{Q}$ is a weak-equivalence (respectively, a fibration) in $\mathcal{O}$ if the underlying morphisms of dg-modules $f : \mathsf{P}(n) \to \mathsf{Q}(n)$, $n \in \mathbb{N}$, are all weak-equivalences (respectively, fibration) in the category of dg-modules. In the core sections of the paper, we use operads P such that $\mathsf{P}(0) = 0$. According to [7, 17], the subcategory $\mathcal{O}_0 \subset \mathcal{O}$ formed by these operads inherits a full model category structure.

As usual, we characterize cofibrations by the right lifting property with respect to acyclic fibrations in $\mathcal{O}$. In particular, an operad $\mathsf{P} \in \mathcal{O}$ is cofibrant as an operad if the lifting exists in all diagrams of the form



where $p : \mathsf{R} \to \mathsf{S}$ is an acyclic fibration of operads.

Recall that an operad P is said to be Σ_*-cofibrant if P forms a cofibrant object in the underlying category of Σ_*-modules. One can check that cofibrant operads are Σ_*-cofibrant (see [7, Proposition 4.3]), but the converse assertion does not hold.

0.3. Modules over operads and functors

In this subsection, we recall the definition and categorical properties of functors associated to right modules over operads.

0.3.1. The functor associated to a right module over an operad

Let M be a right module over an operad R.

Let $\mathcal{E}$ be any symmetric monoidal category over dg-modules. For an R-algebra $A \in {}_{\mathsf{R}}\mathcal{E}$, we form the coequalizer:

$$S(M \circ \mathsf{R}, A) \overset{d_0}{\underset{d_1}{\rightrightarrows}} S(M, A) \longrightarrow S_{\mathsf{R}}(M, A) ,$$

where d_0 is the morphism

$$S(M \circ \mathsf{R}, A) \xrightarrow{S(\rho, A)} S(M, A)$$

induced by the right R-action on M and d_1 is the morphism

$$S(M \circ \mathsf{R}, A) = S(M, S(\mathsf{R}, A)) \xrightarrow{S(M, \lambda)} S(M, A)$$

induced by the left R-action on A.

The map $S_{\mathsf{R}}(M) : A \mapsto S_{\mathsf{R}}(M, A)$ defines the functor $S_{\mathsf{R}}(M) : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ associated to M. Let $\mathcal{F}_{\mathsf{R}}$ denote the category of functors $F : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$. The definition of $S_{\mathsf{R}}(M)$ is obviously natural in M so that $S_{\mathsf{R}} : M \mapsto S_{\mathsf{R}}(M)$ defines a functor $S_{\mathsf{R}} : \mathcal{M}_{\mathsf{R}} \to \mathcal{F}_{\mathsf{R}}$.

The definition of the functor $S_{\mathsf{R}}(M) : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ can be applied to the category of Σ_*-modules $\mathcal{E} = \mathcal{M}$, or to another category of right modules $\mathcal{E} = \mathcal{M}_{\mathsf{S}}$, for any operad S. In this context, the object $S_{\mathsf{R}}(M, N)$ is identified with the classical relative composition product $M \circ_{\mathsf{R}} N$ of the operad literature. Indeed, the relative composition product $M \circ_{\mathsf{R}} N$ is defined by a coequalizer of the same form where the objects $S(M, N)$ are replaced by the equivalent composites $S(M, N) = M \circ N$ in the category of Σ_*-modules (see for instance [13, §2.1.7] for this definition).

0.3.2. Categorical operations on functors associated to right-modules over operads

To determine the functor $S_{\mathsf{R}}(M) : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ associated to a right R-module M, we essentially use:

(1) For the unit object $M = \mathbf{1}$, the functor $S_{\mathsf{R}}(\mathbf{1}) : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ is the constant functor $S_{\mathsf{R}}(\mathbf{1}, A) \equiv \Bbbk$.

(2) We have a natural isomorphism $S_{\mathsf{R}}(M \otimes N, A) = S_{\mathsf{R}}(M, A) \otimes S_{\mathsf{R}}(N, A)$, for all $M, N \in \mathcal{M}_{\mathsf{R}}$, $A \in {}_{\mathsf{R}}\mathcal{E}$, and a natural isomorphism $S_{\mathsf{R}}(C \otimes M, A) = C \otimes S_{\mathsf{R}}(M, A)$, for all $C \in \mathcal{C}$, $M \in \mathcal{M}_{\mathsf{R}}$, $A \in {}_{\mathsf{R}}\mathcal{E}$, so that the map $S_{\mathsf{R}} : M \mapsto S_{\mathsf{R}}(M)$ defines a functor of symmetric monoidal categories over dg-modules $S_{\mathsf{R}} : (\mathcal{M}_{\mathsf{R}}, \otimes, \mathbf{1}) \to (\mathcal{F}_{\mathsf{R}}, \otimes, \Bbbk)$, the tensor structure of functors being defined pointwise.

(3) The functor $S_{\mathsf{R}} : \mathcal{M}_{\mathsf{R}} \to \mathcal{F}_{\mathsf{R}}$ preserves colimits.

We refer to [14, §§5-6] for the proof of these assertions.

The functor $S_{\mathsf{R}} : M \mapsto S_{\mathsf{R}}(M)$ is uniquely characterized by (1-3) and assertion (1) of §0.3.3 (use the form of generating objects in $\mathcal{M}_{\mathsf{R}}$, see [14, §7.1]). If we forget algebra structures on the target, then this latter assertion implies:

(4) The functor $S_{\mathsf{R}}(\mathsf{R}) : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ associated to the operad R, viewed as a right module over itself, represents the forgetful functor $U : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$.

0.3.3. On algebras in right-modules over operads and functors

The assertions of §0.3.2 imply that the evaluation morphism of a P-algebra in right R-modules

$$\lambda : \mathsf{P}(n) \otimes N^{\otimes n} \to N$$

give rise to natural evaluation morphisms at the functor level:

$$\mathsf{P}(n) \otimes S_{\mathsf{R}}(N, A)^{\otimes n} = S_{\mathsf{R}}(\mathsf{P}(n) \otimes N^{\otimes n}, A) \to S_{\mathsf{R}}(N, A),$$

where $A \in {}_{\mathsf{R}}\mathcal{E}$. Thus we obtain that the map $S_{\mathsf{R}}(N) : A \mapsto S_{\mathsf{R}}(N, A)$ defines a functor $S_{\mathsf{R}}(N) : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{P}}\mathcal{E}$.

According to [14, Observation 9.2.2]:

(1) The identity functor $\mathrm{Id} : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{R}}\mathcal{E}$ is realized by the functor $S_{\mathsf{R}}(\mathsf{R}) : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{R}}\mathcal{E}$ associated to the operad R considered as an algebra over itself in right modules over itself.

The definition of the functor $S_{\mathsf{R}}(N) : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{P}}\mathcal{E}$ is obviously natural in $N \in {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}}$ so that the map $N \mapsto S_{\mathsf{R}}(N)$ defines a functor $S_{\mathsf{R}} : {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}} \to {}_{\mathsf{P}}\mathcal{F}_{\mathsf{R}}$, where ${}_{\mathsf{P}}\mathcal{F}_{\mathsf{R}}$ denotes the category of functors $F : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{P}}\mathcal{E}$ from the category of R-algebras in $\mathcal{E}$ to the category of P-algebras in $\mathcal{E}$. According to [14, Proposition 9.2.1]:

(2) For a free P-algebra in right R-modules we have the identity $S_{\mathsf{R}}(\mathsf{P}(M), A) = \mathsf{P}(S_{\mathsf{R}}(M, A))$, where on the right hand side we consider the free P-algebra generated by the object $S_{\mathsf{R}}(M, A) \in \mathcal{E}$ associated to $A \in {}_{\mathsf{R}}\mathcal{E}$ by the functor $S_{\mathsf{R}}(M) : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ determined by $M \in \mathcal{M}_{\mathsf{R}}$.

(3) The functor $S_{\mathsf{R}} : {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}} \to {}_{\mathsf{P}}\mathcal{F}_{\mathsf{R}}$ preserves colimits.

0.4. On extension and restriction of structure

Any operad morphism gives rise to adjoint extension and restriction functors on module categories, as well as on algebra categories. The purpose of this subsection is to recall the definition of these functors.

0.4.1. On extension and restriction of structure for right modules over operads

On module categories, the adjoint extension and restriction functors

$$\psi_! : \mathcal{M}_{\mathsf{R}} \rightleftarrows \mathcal{M}_{\mathsf{S}} : \psi^*$$

associated to an operad morphism $\psi : \mathsf{R} \to \mathsf{S}$ are very analogous to the classical extension and restriction functors of linear algebra.

The right R-module $\psi^* N$ obtained by restriction of structure from an S-module N is defined by the object underlying N on which the operad R acts through S by way of the morphism $\psi : \mathsf{R} \to \mathsf{S}$. Usually, we omit marking the restriction of structure in notation, unless this abuse of notation creates confusion.

Recall that an operad S forms a bimodule over itself. By restriction, we obtain that S is acted on by the operad R on the left so that S forms an R-S-bimodule as well. The extension functor is defined by the relative composition product $\psi_! M = M \circ_{\mathsf{R}} \mathsf{S}$.

Usually, we use the expression of the relative composition product $M \circ_{\mathsf{R}} \mathsf{S}$ to refer to the object $\psi_! M$. This convention has the advantage of distinguishing extensions of structure on the right from extensions of structure on the left (whose definition is recalled next, in §0.4.2) and to stress the analogy with extension of scalars in linear algebra. Nevertheless we keep using the notation $\psi_!$ to refer to the extension of structure as a functor $\psi_! : \mathcal{M}_{\mathsf{R}} \to \mathcal{M}_{\mathsf{S}}$.

The right R-module $\psi^* N$ associated to a P-algebra in right S-module $N \in {}_{\mathsf{P}}\mathcal{M}_{\mathsf{S}}$, where P is another operad, inherits an obvious P-algebra structure and forms a P-algebra in right R-modules. In the converse direction, one checks that the relative composition product $M \circ_{\mathsf{R}} \mathsf{S}$ preserve tensor products, from which we obtain that the right S-module $\psi_! M = M \circ_{\mathsf{R}} \mathsf{S}$ associated to a P-algebra in right R-modules $M \in {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}}$ inherits a P-algebra structure and forms a P-algebra in right S-modules. Finally, we have induced extension and restriction functors

$$\psi_! : {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}} \rightleftarrows {}_{\mathsf{P}}\mathcal{M}_{\mathsf{S}} : \psi^*$$

which are obviously adjoint to each other.

0.4.2. On extension and restriction of structure for algebras over operads

An operad morphism $\phi : \mathsf{P} \to \mathsf{Q}$ yields adjoint extension and restriction functors on algebra categories

$$\phi_! : {}_{\mathsf{P}}\mathcal{E} \rightleftarrows {}_{\mathsf{Q}}\mathcal{E} : \phi^*,$$

for any symmetric monoidal category $\mathcal{E}$ over the base category of dg-modules $\mathcal{C}$.

Again, the P-algebra $\phi^* B$ obtained by restriction of structure from a Q-algebra B is defined by the object underlying B on which the operad P acts through Q by the morphism $\phi : \mathsf{P} \to \mathsf{Q}$. In the other direction, the P-algebra $\phi_! A$ obtained by extension of structure from a P-algebra A is just characterized by the adjunction relation

$$\mathrm{Mor}_{{}_{\mathsf{P}}\mathcal{E}}(\phi_! A, B) = \mathrm{Mor}_{{}_{\mathsf{Q}}\mathcal{E}}(A, \phi^* B).$$

In fact, the P-algebra $\phi_! A$ can be identified with the object $S_{\mathsf{P}}(\mathsf{Q}, A) \in \mathsf{Q}$ associated to A by the functor $S_{\mathsf{P}}(\mathsf{Q}) : {}_{\mathsf{P}}\mathcal{E} \to {}_{\mathsf{Q}}\mathcal{E}$ where the operad Q is considered as an algebra over itself in right modules over P (use the restriction of structure on the right of §0.4.1).

In the case $\mathcal{E} = \mathcal{M}$ and $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$, we obtain extension and restriction functors for left modules over operads

$$\phi_! : {}_{\mathsf{P}}\mathcal{M} \rightleftarrows {}_{\mathsf{Q}}\mathcal{M} : \phi^*$$

and extension and restriction functors on the left for bimodules over operads

$$\phi_! : {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}} \rightleftarrows {}_{\mathsf{Q}}\mathcal{M}_{\mathsf{R}} : \phi^*.$$

In the context of bimodules, the extension and restriction of structure on the left commute with the extension and restriction of structure on the right.

From the module point of view, the extension and the restriction of structure on the left is defined in a symmetric fashion to the extension and the restriction of structure on the right. In particular, for extension of structure, we have an identity $\phi_! M = \mathsf{Q} \circ_{\mathsf{P}} M$. Nevertheless we prefer to use the notation $\phi_! M$ to refer to an extension of structure on the left, rather than the notation of a relative composition product, because we view the functor $\phi_! : M \mapsto \phi_! M$ as an instance of an extension of structure of R-algebras and this point of view reflects the nature of extensions of structure on the left more properly.

0.4.3. On extension and restriction of functors

According to [14, §7.2], extension and restriction of structure of modules over operads reflect extension and restriction operations at the functor level. For extensions on the right, we have natural isomorphisms

$$S_{\mathsf{S}}(M \circ_{\mathsf{R}} \mathsf{S}, B) \simeq S_{\mathsf{R}}(M, \psi^* B),$$

for every $M \in \mathcal{M}_{\mathsf{R}}$ and all $B \in {}_{\mathsf{S}}\mathcal{E}$, as well as natural isomorphisms

$$S_{\mathsf{R}}(N, A) \simeq S_{\mathsf{S}}(N, \psi_! A),$$

for every $N \in \mathcal{M}_{\mathsf{S}}$ and all $A \in {}_{\mathsf{R}}\mathcal{E}$, and similarly in the context of bimodules over operads $M \in {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}}$, $N \in {}_{\mathsf{P}}\mathcal{M}_{\mathsf{S}}$ (in this context, the identities hold in the category of P-algebras). Symmetrically, for extensions on the left, we have identities of P-algebras

$$S_{\mathsf{R}}(\phi_! M, A) \simeq \phi_! \, S_{\mathsf{R}}(M, A),$$

for every $M \in {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}}$, and

$$S_{\mathsf{R}}(\phi^* N, A) \simeq \phi^* \, S_{\mathsf{R}}(N, A),$$

for every $N \in {}_{\mathsf{Q}}\mathcal{M}_{\mathsf{R}}$, where in both cases $A \in {}_{\mathsf{R}}\mathcal{E}$.

The bar construction and its multiplicative structure

In this part, we apply the general theory recalled in §§0.1-0.4 to prove our main results on the bar construction.

In §1, we recall the definition of the bar construction of differential graded algebras and we check that this construction is an instance of a functor determined by a module over an operad, the *bar module*. For this purpose, we observe that a generalized bar construction is defined in the setting of modules over operads. In fact, the bar module is an instance of a bar construction in that category, where an operad is considered as an algebra over itself in right modules over itself. This idea is also used to check homotopical properties of the bar module associated to an operad.

The multiplicative structure of the bar complex is examined in §2, where we use constructions of §1 to prove the existence and uniqueness of an E_∞-structure on the bar complex of E_∞-algebras.

In §3, we recall the definition of a categorical analogue of the bar construction, where tensor products are replaced by categorical coproducts, and we check that this categorical bar construction forms an instance of a functor determined by a module over an operad. By [25, Theorems 3.2 and 3.5], the categorical bar construction defines a model of the suspension in the homotopy category of algebras over an operad. In §4, we use an equivalence of modules over operads to prove that the usual bar construction is equivalent to the categorical bar construction as an E_∞-algebra, from which we conclude that the usual bar construction defines a model of the suspension in the homotopy category of E_∞-algebras. This relationship is used in the next part to deduce from results of [25] that the bar complex of a cochain algebra $C^*(X)$ is equivalent as an E_∞-algebra to $C^*(\Omega X)$, the cochain algebra of the loop space of X.

Conventions

In the remainder of the article, the notation $\mathcal{E}$ refers either to the category of dg-modules $\mathcal{E} = \mathcal{C}$ or to a category of right modules over an operad $\mathcal{E} = \mathcal{M}_{\mathsf{S}}$ and we do not consider other examples of symmetric monoidal categories over dg-modules. The concept of a symmetric monoidal category over dg-modules is essential to understand our arguments, but in applications we are only interested in these examples.

From now on, we use the subcategory $\mathcal{O}_0 \subset \mathcal{O}$ formed by operads P such that $\mathsf{P}(0) = 0$, and we assume tacitely that any given operad satisfies this condition. The assumption $\mathsf{P}(0) = 0$ amounts to considering algebras without 0-ary operations $\lambda : \mathsf{P}(0) \to A$. In the sequel, we say that an operad P which has $\mathsf{P}(0) = 0$ is *non-unitary* and that the associated algebras are *non-unital*. This setting simplifies the definition of the bar complex (see §1.1.2 and §1.1.4).

1. The bar construction and the bar module

Introduction

In this section, we check that the bar construction $A \mapsto B(A)$ is identified with the functor associated to a right module over Stasheff's operad and we check properties of this module.

For our needs, we study restrictions of the bar construction to categories of algebras over operads R, where R is any operad under Stasheff's operad K. In this context, we prove:

Proposition 1.A. *Let R be any operad under Stasheff's operad K. There is a right R-module naturally associated to R, the bar module B_{R}, such that $B(A) = S_{\mathsf{R}}(B_{\mathsf{R}}, A)$, for all $A \in {}_{\mathsf{R}}\mathcal{E}$.*

In §1.1, we recall the definition of Stasheff's operad K and the definition of the bar construction for algebras over this operad. In §1.2, we study the bar construction of a K-algebra in a category of right modules over an operad R.

In §1.3, we note that all E_∞-operads form operads under K. As a consequence we obtain that all E_∞-algebras have an associated bar complex. In §1.4, we use the generalized bar complex of K-algebras in right modules over an operad R to define the bar module B_{R} associated to an operad R under K. For this aim, we just observe that an operad under K forms a K-algebra in right modules over itself. We study the structure of this right R-module B_{R} and the functoriality of the construction $\mathsf{R} \mapsto B_{\mathsf{R}}$.

1.1. On Stasheff's operad and the bar complex

In this section, we use that the structure of an algebra over an operad P, defined by a collection of evaluation morphisms

$$\lambda : \mathsf{P}(n) \otimes A^{\otimes n} \to A,$$

amounts to associating an actual operation $p : A^{\otimes n} \to A$ to any homogeneous element $p \in \mathsf{P}(n)$, at least in the case $\mathcal{E} = \mathcal{C}$, the category of dg-modules, and $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$, the category of right modules over an operad R.

For this purpose, we use the adjunction relation

$$\mathrm{Mor}_{\mathcal{E}}(\mathsf{P}(n) \otimes A^{\otimes n}, A) = \mathrm{Mor}_{\mathcal{C}}(\mathsf{P}(n), \mathrm{Hom}_{\mathcal{E}}(A^{\otimes n}, A))$$

and an explicit representation of the dg-hom

$$\mathrm{Hom}_{\mathcal{E}}(-,-) : \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{C}$$

on these categories.

In the context of dg-modules, an element $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ is simply a homogeneous map $f : C \to D$. In the context of right modules over an operad, an element $f \in \mathrm{Hom}_{\mathcal{M}_{\mathsf{R}}}(M, N)$ consists of a collection of homogeneous maps of dg-modules $f : M(n) \to N(n)$, $n \in \mathbb{N}$, which commute with the action of symmetric groups and so that the action of the operad R is preserved by $f : M \to N$. In general, the evaluation morphism of a P-algebra associates an element $p \in \mathrm{Hom}_{\mathcal{E}}(A^{\otimes n}, A)$ to any operation $p \in \mathsf{P}(n)$.

The standard bar complex is an instance of a construction where the internal differential of a dg-module C is twisted by a cochain $\partial \in \mathrm{Hom}_{\mathcal{C}}(C, C)$ to produce a new dg-module, which has the same underlying graded module as C, but whose differential is given by the sum $\delta + \partial : C \to C$. One has simply to assume that a twisting cochain ∂ satisfies the equation $\delta(\partial) + \partial^2 = 0$ in $\mathrm{Hom}_{\mathcal{C}}(C, C)$ to obtain that the map $\delta + \partial$ verifies the equation of differentials $(\delta + \partial)^2 = 0$. This construction makes sense in the context of right modules over an operad. In this case, the twisting cochain $\partial : M \to M$ is supposed to represents an element of $\mathrm{Hom}_{\mathcal{M}_{\mathsf{R}}}(M, M)$ and this condition ensures that the sum $\delta + \partial : M \to M$ defines a differential of right R-modules (for details, compare with definitions of [13, §2.1.11]).

From these observations, a bar complex in the category of right modules over an operad can be defined in parallel to the standard bar complex in dg-modules. Before doing this construction, we recall the definition of Stasheff's operad, at least for the sake of completeness.

1.1.1. On the chain operad of Stasheff's associahedra

The structure of Stasheff's operad K is specified by a pair $\mathsf{K} = (\mathsf{F}(M), \partial)$, where $\mathsf{F}(M)$ is a free operad and $\partial : \mathsf{F}(M) \to \mathsf{F}(M)$ is an operad derivation that defines the differential of K. The generating Σ_*-module M is given by

$$M(r) = \begin{cases} 0, & \text{if } r = 0, 1, \\ \Sigma_r \otimes \Bbbk\ \mu_r, & \text{otherwise,} \end{cases}$$

where μ_r is a generating operation of degree $r - 2$. The derivation $\partial : \mathsf{F}(M) \to \mathsf{F}(M)$ is determined on generating operations by the formula

$$\partial(\mu_r) = \sum_{s+t-1=r} \Big\{ \sum_{i=1}^{s} \pm \mu_s \circ_i \mu_t \Big\}.$$

Let A be operad of associative algebras. The Stasheff operad is endowed with an operad equivalence $\epsilon : \mathsf{K} \xrightarrow{\sim} \mathsf{A}$ defined by $\epsilon(\mu_r) = 0$ for $r > 2$ and $\epsilon(\mu_2) = \mu$, where $\mu \in \mathsf{A}(2)$ is the operation which represents the product of associative algebras.

1.1.2. The bar complex

Let A be a K-algebra in $\mathcal{E}$, where $\mathcal{E} = \mathcal{C}$, the category of dg-modules, or $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$, the category right modules over an operad.

The (reduced) bar complex of A is defined by the pair $B(A) = (T^c(\Sigma A), \partial)$ formed by the (non-augmented) tensor coalgebra

$$T^c(\Sigma A) = \bigoplus_{n=1}^{\infty} (\Sigma A)^{\otimes n}$$

where ΣA is the suspension of A in $\mathcal{E}$, together with a twisting cochain $\partial \in \mathrm{Hom}_{\mathcal{E}}(T^c(\Sigma A), T^c(\Sigma A))$, called the bar coderivation, defined pointwise by the formula

$$\partial(a_1 \otimes \cdots \otimes a_n) = \sum_{r=2}^{n} \Big\{ \sum_{i=1}^{n-r+1} \pm a_1 \otimes \cdots \otimes \mu_r(a_i, \ldots, a_{i+r-1}) \otimes \cdots \otimes a_n \Big\}.$$

The internal differential of the bar complex $B(A)$ is the sum $\delta + \partial$ of the natural differential of the tensor coalgebra $\delta : T^c(\Sigma A) \to T^c(\Sigma A)$, induced by the internal differential of A, with the bar coderivation $\partial : T^c(\Sigma A) \to T^c(\Sigma A)$, determined by the K-operad action.

In the case of an associative algebra, the bar coderivation reduces to terms

$$\partial = \sum_{i=1}^{n-1} \pm a_1 \otimes \cdots \otimes \mu(a_i, a_{i+1}) \otimes \cdots \otimes a_n$$

since the operations $\mu_r \in \mathsf{K}(r)$ vanish in $\mathsf{A}(r)$ for $r > 2$. Hence, in this case, we recover the standard definition of the bar complex of associative algebras.

According to the definition, the bar complex forms naturally a coalgebra in $\mathcal{E}$, but we do not use coalgebra structures further in this article.

1.1.3. Remark

In the context of right modules over an operad $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$, we have essentially to form the tensor coalgebra $T^c(\Sigma A)$ in $\mathcal{M}_{\mathsf{R}}$. The suspension of an object $M \in \mathcal{M}_{\mathsf{R}}$ can be defined by a tensor product $\Sigma M = \bar{N}_*(S^1) \otimes M$, as in the context of dg-modules, where $\bar{N}_*(S^1)$ is the reduced normalized chain complex of the circle. The pointwise definition of the bar coderivation $\partial : T^c(\Sigma A) \to T^c(\Sigma A)$, makes sense if we recall that $(\Sigma A)^{\otimes n}$ is generated by tensors $a_1 \otimes \cdots \otimes a_n \in \Sigma A(r_1) \otimes \cdots \otimes \Sigma A(r_n)$ (we apply the principle of generalized point-tensors of [14, §0.5]). In both cases $\mathcal{E} = \mathcal{C}$ and $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$, the bar coderivation can be defined as a sum of homomorphism tensor products

$$\mathrm{id} \otimes \cdots \otimes \mu_r \otimes \cdots \otimes \mathrm{id} \in \mathrm{Hom}_{\mathcal{E}}((\Sigma A)^{\otimes n}, (\Sigma A)^{\otimes n-r+1})$$

as well.

1.1.4. Remark

The definition of §1.1.2 is the right one for a non-unital algebra. Similarly, we consider a non-augmented tensor coalgebra in the definition of $B(A)$, or equivalently the augmentation ideal of the standard tensor coalgebra, so that our bar complex forms a non-unital object. In general it is simpler for us to deal with non-unital algebras and therefore we take this convention. In the unital context we have to assume that A is augmented and, in the definition of $B(A)$, we have to replace the algebra A by its augmentation ideal $\bar{A}$.

1.2. The generalized bar complex

The bar construction gives by definition a functor $B : {}_{\mathsf{K}}\mathcal{E} \to \mathcal{E}$, for $\mathcal{E} = \mathcal{C}$ and $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$. In this subsection we check that standard properties of the usual bar construction of K-algebras in dg-modules hold in the context of right modules over an operad R.

First, we have the easy propositions:

Proposition 1.2.1. *Let $\psi : \mathsf{R} \to \mathsf{S}$ be any operad morphism. For any K-algebra in right R-modules M, we have a natural isomorphism $B(M) \circ_{\mathsf{R}} \mathsf{S} \simeq B(M \circ_{\mathsf{R}} \mathsf{S})$ in the category of right S-modules.*

Proof. Use simply that extension functors $\psi_!(M) = M \circ_{\mathsf{R}} \mathsf{S}$ commute with tensor products to obtain this isomorphism (see [14, §7.2] and recollections in §0.4.1). □

Proposition 1.2.2. *If $\phi : M \to N$ is a fibration of K-algebras in right R-modules, then the induced morphism $B(\phi) : B(M) \to B(N)$ defines a fibration in the category of right R-modules*

Proof. Recall that fibrations in the category of right R-modules are created in the category of dg-modules and, as such, are just degreewise epimorphisms. Therefore the assertion is an immediate consequence of the definition of the bar complex as a twisted module $B(N) = (T^c(\Sigma N), \partial)$. Note simply that the tensor coalgebra $T^c(\Sigma N)$ preserves epimorphisms because the tensor product of right R-modules, inherited from Σ_*-modules, has this property. □

Our main task is to check that the bar construction, preserves cofibrations, acyclic cofibrations, and all weak-equivalences between K-algebras which are cofibrant as a right R-module. For this aim we prove that the bar complex has a natural cell decomposition.

Let D^n be the dg-module spanned by an element e_n in degree n and an element b_{n-1} in degree $n-1$ so that $\delta(e_n) = b_{n-1}$. Consider the submodule $C^{n-1} \subset D^n$ spanned by b_{n-1}. To define the cells, we use the dg-module embeddings $i_n : C^{n-1} \to D^n$, which are generating cofibrations of the category of dg-modules.

Lemma 1.2.3. *For any K-algebra in right R-modules N, the bar complex $B(N)$ decomposes into a sequential colimit*

$$0 = B_{\leq 0}(N) \xrightarrow{j_1} B_{\leq 1}(N) \to \cdots \to B_{\leq n-1}(N) \xrightarrow{j_n} B_{\leq n}(N) \to \cdots \\ \cdots \to \operatorname{colim}_n B_{\leq n}(N) = B(N)$$

so that $B_{\leq n}(N)$ is obtained from $B_{\leq n-1}(N)$ by a pushout of the form

$$\begin{array}{ccc} C^{n-1} \otimes N^{\otimes n} & \xrightarrow{f_n} & B_{\leq n-1}(N) \, . \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle j_n} \\ D^n \otimes N^{\otimes n} & \xrightarrow[g_n]{} & B_{\leq n}(N) \end{array}$$

This decomposition is also functorial with respect to N.

Proof. Indeed, the object $B(N)$ has a canonical filtration

$$0 = B_{\leq 0}(N) \hookrightarrow B_{\leq 1}(N) \hookrightarrow \cdots \hookrightarrow B_{\leq n}(N) \hookrightarrow \cdots \hookrightarrow \operatorname{colim}_n B_{\leq n}(N) = B(N)$$

defined by

$$B_{\leq n}(N) = T^c_{\leq n}(\Sigma N) = \bigoplus_{m=1}^{n} (\Sigma N)^{\otimes m}.$$

The summand $(\Sigma N)^{\otimes m}$ is preserved by the natural differential of the tensor coalgebra $T^c(\Sigma N)$. Moreover the bar coderivation satisfies

$$\partial((\Sigma N)^{\otimes n}) \subset \bigoplus_{r \geq 2} (\Sigma N)^{\otimes n-r+1} = T^c_{\leq n-1}(\Sigma N).$$

Accordingly, we obtain that $B_{\leq n}(N)$ forms a subobject of $B(N)$ in the category of (differential graded) right R-modules.

Besides, our observation implies that $B_{\leq n}(N)$ splits into a twisted direct sum

$$B_{\leq n}(N) = (B_{\leq n-1}(N) \oplus (\Sigma N)^{\otimes n}, \partial),$$

where $\partial : (\Sigma N)^{\otimes n} \to B_{\leq n-1}(N)$ represents the restriction of the bar coderivation to the summand $(\Sigma N)^{\otimes n}$ (compare with [14, §11.2.2]). By definition, the

differential of such a twisted object is the sum of the internal differential of $B_{\leq n-1}(N) \oplus (\Sigma N)^{\otimes n}$ with the twisting map $\partial : (\Sigma N)^{\otimes n} \to B_{\leq n-1}(N)$ on the summand $(\Sigma N)^{\otimes n}$. Hence the identity $B_{\leq n}(N) = (B_{\leq n-1}(N) \oplus (\Sigma N)^{\otimes n}, \partial)$ is obvious.

One checks readily that a twisted direct sum of this form is equivalent to a pushout of the form of the lemma, where the attaching map $f_n : C^{n-1} \otimes N^{\otimes n} \to B_{\leq n-1}(N)$ is yielded by the twisting map $\partial : (\Sigma N)^{\otimes n} \to B_{\leq n-1}(N)$. Observe simply that

$$\Sigma(C^{n-1} \otimes N^{\otimes n}) = \Sigma^n(N^{\otimes n}) = (\Sigma N)^{\otimes n}$$

to obtain that any twisting map $\partial : (\Sigma N)^{\otimes n} \to B_{\leq n-1}(N)$, homogeneous of degree -1, is equivalent to a morphism $f_n : C^{n-1} \otimes N^{\otimes n} \to B_{\leq n-1}(N)$, homogeneous of degree 0. □

Proposition 1.2.4. *The bar complex $B(N)$ associated to a K-algebra in right R-modules N is cofibrant if the K-algebra N defines itself a cofibrant object in the underlying category of right R-modules $\mathcal{M}_{\mathsf{R}}$.*

Proof. The axioms of monoidal model categories imply that the morphism $i_n \otimes N^{\otimes n} : C^{n-1} \otimes N^{\otimes n} \to C^n \otimes N^{\otimes n}$ forms a cofibration in the category of right R-modules $\mathcal{M}_{\mathsf{R}}$ if M is cofibrant as a right R-module. As a consequence, we obtain that the morphism $j_n : B_{\leq n-1}(N) \to B_{\leq n}(N)$ defines a cofibration, for each $n \geq 1$, since this morphism is obtained by a pushout of i_n. The proposition follows. □

A morphism of Σ_*-modules $i : M \to N$ is called a $\mathcal{C}$-cofibration (respectively, an acyclic $\mathcal{C}$-cofibration) if the morphisms $i : M(n) \to N(n)$, $n \in \mathbb{N}$, are cofibrations in the category of dg-modules $\mathcal{C}$. Similarly, a Σ_*-module M is $\mathcal{C}$-cofibrant if its underlying collection consists of cofibrant dg-modules.

Lemma 1.2.5. *Let $i : M \to N$ be a morphism of K-algebras in right R-modules such that the K-algebra M is $\mathcal{C}$-cofibrant.*

The morphism $B(i) : B(M) \to B(N)$ induced by i forms a $\mathcal{C}$-cofibration (respectively, an acyclic $\mathcal{C}$-cofibration) if i forms itself a $\mathcal{C}$-cofibration (respectively, an acyclic $\mathcal{C}$-cofibration).

Proof. The morphism $B(i) : B(M) \to B(N)$ can be decomposed naturally into a sequential colimit of morphisms $j_n : B_{\leq n-1}(N/M) \to B_{\leq n}(N/M)$, where

$$B_{\leq n}(N/M) = B(M) \bigoplus_{B_{\leq n}(M)} B_{\leq n}(N)$$

and j_n is induced componentwise by the morphisms

$$\begin{array}{ccccc} B(M) & \longleftarrow & B_{\leq n-1}(M) & \longrightarrow & B_{\leq n-1}(N) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle B_{\leq n-1}(i)} & & \downarrow{\scriptstyle B_{\leq n-1}(i)} \\ B(M) & \longleftarrow & B_{\leq n}(M) & \longrightarrow & B_{\leq n}(N) \end{array} .$$

One checks readily that j_n fits a pushout of the form

$$\begin{array}{ccc} C^{n-1} \otimes N^{\otimes n} \underset{C^{n-1}\otimes M^{\otimes n}}{\bigoplus} D^n \otimes M^{\otimes n} & \longrightarrow & B_{\leq n-1}(N/M) \\ \downarrow & & \downarrow{\scriptstyle j_n} \\ D^n \otimes N^{\otimes n} & \longrightarrow & B_{\leq n}(N/M) \end{array} .$$

The underlying dg-modules of the tensor power $M^{\otimes r}$, where M is any right R-module, have an expansion of the form:

$$\begin{aligned} M^{\otimes r}(m) &= \bigoplus_{m_1+\cdots+m_r=m} \Sigma_m \otimes_{\Sigma_{m_1}\times\cdots\times\Sigma_{m_r}} M(m_1)\otimes\cdots\otimes M(m_r) \\ &= \bigoplus_{m_1+\cdots+m_r=m} (\Sigma_m/\Sigma_{m_1}\times\cdots\times\Sigma_{m_r})\otimes M(m_1)\otimes\cdots\otimes M(m_r), \end{aligned}$$

where the tensor product of the dg-module $T = M(m_1)\otimes\cdots\otimes M(m_r)$ with the coset $K = \Sigma_m/\Sigma_{m_1}\times\cdots\times\Sigma_{m_r}$ is defined by a sum of copies of T indexed by K, as usual. By the monoidal model structure of dg-modules, we obtain that the morphism $i^{\otimes n} : M^{\otimes n} \to N^{\otimes n}$ forms a $\mathcal{C}$-cofibration (respectively, an acyclic $\mathcal{C}$-cofibration) if i is so, as long as M is $\mathcal{C}$-cofibrant. Under this assumption, the pushout product-axiom in dg-modules implies that the left-hand side morphism of the pushout above is a $\mathcal{C}$-cofibration (respectively, an acyclic $\mathcal{C}$-cofibration), from which we deduce that our morphism $j_n : B_{\leq n-1}(N/M) \to B_{\leq n}(N/M)$ forms a cofibration (respectively, an acyclic cofibration) as well. The conclusion follows. (Recall that the forgetful functor which maps a right R-module M to its underlying collection of dg-modules $\{M(n)\}_{n\in\mathbb{N}}$ creates all colimits in $\mathcal{M}_{\mathsf{R}}$. Hence we obtain that $\mathcal{C}$-cofibrations and acyclic $\mathcal{C}$-cofibrations are preserved by pushouts in the category of right R-modules $\mathcal{M}_{\mathsf{R}}$.) □

Proposition 1.2.6. *The morphism $B(i) : B(M) \to B(N)$ induced by a weak-equivalence of K-algebras in right R-modules $i : M \xrightarrow{\sim} N$ forms itself a weak-equivalence if the underlying collection of the K-algebras M and N consist of cofibrant dg-modules.*

Proof. Cofibrant algebras over operads form cofibrant objects in the underlying category by [7, Corollary 5.5] (see also [14, Proposition 12.3.2]). This assertion enables us to use the standard Brown's lemma (see for instance [20, Lemma 1.1.12]) to obtain the proposition as an immediate consequence of Lemma 1.2.5. □

1.3. Operads under Stasheff's operad and the bar complex

In this subsection, we examine restrictions of the bar complex to categories of algebras associated to operads P equipped with a morphism $\eta : \mathsf{K} \to \mathsf{P}$. For our purpose, we record that any E_∞-operad E can be equipped with such a morphism $\eta : \mathsf{K} \to \mathsf{E}$, so that any algebra over an E_∞-operad has a bar complex. By the way, we recall the definition of an E_∞-operad, at least to fix conventions.

1.3.1. Operads under Stasheff's operad and the bar complex

The category of (non-unitary) operads under K, for which we use the notation $\mathcal{O}_0 \setminus \mathsf{K}$, is the comma category of operad morphisms $\eta : \mathsf{K} \to \mathsf{P}$, where $\mathsf{P} \in \mathcal{O}_0$. According to this definition, an operad under Stasheff's operad K is defined by a pair (P, η) formed by an operad P together with an operad morphism $\eta : \mathsf{K} \to \mathsf{P}$. Usually, we omit abusively the morphism $\eta : \mathsf{K} \to \mathsf{P}$ in the notation of an operad under K and we identify an object of $\mathcal{O}_0 \setminus \mathsf{K}$ with a non-unitary operad P endowed with a morphism $\eta : \mathsf{K} \to \mathsf{P}$ given with P.

If P is an operad under K, then the category of P-algebras is equipped with a canonical restriction functor $\eta^* : {}_{\mathsf{P}}\mathcal{E} \to {}_{\mathsf{K}}\mathcal{E}$ associated to the morphism $\eta : \mathsf{K} \to \mathsf{P}$. As a consequence, the bar complex restricts naturally to a functor on the category of P-algebras, for all operads $\mathsf{P} \in \mathcal{O}_0 \setminus \mathsf{K}$. Formally, this functor is given by the composite

$$ {}_{\mathsf{P}}\mathcal{E} \xrightarrow{\eta^*} {}_{\mathsf{K}}\mathcal{E} \xrightarrow{B} \mathcal{E} . $$

Observations of §1.1.2 imply that we recover the usual bar complex of associative algebras in the case where P is the associative operad A together with the canonical augmentation morphism $\epsilon : \mathsf{K} \xrightarrow{\sim} \mathsf{A}$.

1.3.2. On E_∞-operads as operads under Stasheff's operad

By definition, an E_∞-operad is an operad E equipped with a weak-equivalence of operads $\epsilon : \mathsf{E} \xrightarrow{\sim} \mathsf{C}$, called the augmentation of E, where C denotes the (non-unitary) commutative operad, the operad associated to the category of (non-unital) associative and commutative algebras. In the literature, an E_∞-operad is usually assumed to be Σ_*-cofibrant and we take this convention as well. Observe that the augmentation $\epsilon : \mathsf{E} \xrightarrow{\sim} \mathsf{C}$ is automatically a fibration because C is an operad in $\Bbbk$-modules, equipped with a trivial differential.

In the introduction of this part, we mention that any E_∞-operad E forms an operad under Stasheff's operad K. Recall that we have an operad morphism $\alpha : \mathsf{A} \to \mathsf{C}$ so that the restriction functor $\alpha^* : {}_{\mathsf{C}}\mathcal{E} \to {}_{\mathsf{A}}\mathcal{E}$ represents the embedding from the category of associative and commutative algebras to the category of all associative algebras. We simply fix a lifting

$$\begin{array}{ccc} \mathsf{K} & \overset{\eta}{\dashrightarrow} & \mathsf{E} \\ \downarrow \sim & & \downarrow \sim \\ \mathsf{A} & \xrightarrow{\alpha} & \mathsf{C} \end{array}$$

in order to obtain an operad morphism $\eta : \mathsf{K} \to \mathsf{E}$ such that the restriction functor $\eta^* : {}_{\mathsf{E}}\mathcal{E} \to {}_{\mathsf{K}}\mathcal{E}$ extends the standard category embedding $\alpha^* : {}_{\mathsf{C}}\mathcal{E} \hookrightarrow {}_{\mathsf{A}}\mathcal{E}$ from commutative algebras to associative algebras. Observe that $\eta : \mathsf{K} \to \mathsf{E}$ is uniquely determined up to homotopy only. Therefore, in this article, we assume tacitely that such a morphism $\eta : \mathsf{K} \to \mathsf{E}$ is fixed for any given E_∞-operad E.

By observations of §1.3.1, we obtain that the bar complex restricts to a functor on the category of E-algebras. In addition, since we have a commutative

diagram of restriction functors

$$\begin{array}{ccccc} \mathcal{E} & \prec\!\!\cdots^{B} & {}_{\mathsf{K}}\mathcal{E} & \prec^{\eta^*} & {}_{\mathsf{E}}\mathcal{E}\ , \\ & & \uparrow & & \uparrow \\ & & {}_{\mathsf{A}}\mathcal{E} & \prec^{\alpha^*} & {}_{\mathsf{C}}\mathcal{E} \end{array}$$

we obtain that the bar complex of E-algebras extends the usual bar complex on the category of associative and commutative algebras.

1.3.3. Remark

In our construction, we mention that the morphism $\eta : \mathsf{K} \to \mathsf{E}$ is unique up to homotopy. Indeed, as usual in a model category, all morphisms $\eta_0, \eta_1 : \mathsf{K} \to \mathsf{E}$ that lift the classical operad morphism $\alpha : \mathsf{A} \to \mathsf{C}$ are connected by a left homotopy in the category of dg-operads. By [15, Theorem 5.2.2], the existence of such a left homotopy implies the existence of a natural weak-equivalence between the composite functors

$$ {}_{\mathsf{E}}\mathcal{E} \overset{\eta_0^*}{\underset{\eta_1^*}{\rightrightarrows}} {}_{\mathsf{K}}\mathcal{E} \xrightarrow{B} \mathcal{E}\ . $$

To conclude, we have a well-defined bar complex functor $B : {}_{\mathsf{E}}\mathcal{E} \to \mathcal{E}$ once the E_∞-operad E is provided with a fixed operad morphism $\eta : \mathsf{K} \to \mathsf{E}$ that lifts the classical operad morphism $\alpha : \mathsf{A} \to \mathsf{C}$. Otherwise the bar complex functor $B : A \mapsto B(A)$ is uniquely determined up to homotopy only.

1.4. The bar module

By definition, the bar construction of a K-algebra in right R-modules $N \in {}_{\mathsf{K}}\mathcal{M}_{\mathsf{R}}$ returns a right R-module $B(N)$, and this right R-module determines a functor $S_{\mathsf{R}}(B(N)) : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$. For our purpose, we note:

Proposition 1.4.1. *Let $\mathcal{E} = \mathcal{C}$, the category of dg-modules, or $\mathcal{E} = \mathcal{M}_{\mathsf{S}}$, the category of right modules over an operad S. Let N be any K-algebra in right R-modules. The bar complex of N in right R-modules satisfies the relation*

$$S_{\mathsf{R}}(B(N), A) = B(S_{\mathsf{R}}(N, A)),$$

for all $A \in {}_{\mathsf{R}}\mathcal{E}$, where on the right-hand side we consider the bar complex of the K-algebra $S_{\mathsf{R}}(N, A) \in {}_{\mathsf{K}}\mathcal{E}$ associated to $A \in {}_{\mathsf{R}}\mathcal{E}$ by the functor $S_{\mathsf{R}}(N) : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{K}}\mathcal{E}$ defined by N.

Proof. Since the functor $M \mapsto S_{\mathsf{R}}(M)$ preserves internal tensor products of the category of right R-modules and external tensor products over dg-modules, we obtain

$$S_{\mathsf{R}}(T^c(\Sigma N), A) = T^c(S_{\mathsf{R}}(\Sigma N, A)) = T^c(\Sigma\, S_{\mathsf{R}}(N, A)).$$

The map $\partial : S_{\mathsf{R}}(T^c(\Sigma N), A) \to S_{\mathsf{R}}(T^c(\Sigma N), A)$ induced by the bar coderivation of $B(N)$ can also be identified with the bar coderivation of $B(S_{\mathsf{R}}(N, A))$. This identification is tautological as the action of K on $S_{\mathsf{R}}(N, A)$ is induced by the action of K on N and, hence, the operations $\mu_r : S_{\mathsf{R}}(N, A)^{\otimes r} \to S_{\mathsf{R}}(N, A)$ are the maps induced by the operations $\mu_r : N^{\otimes r} \to N$ on N. □

Recall that an operad R forms an algebra over itself in the category of right modules over itself. If R comes equipped with a morphism $\eta : \mathsf{K} \to \mathsf{R}$ and forms an operad under Stasheff's operad K, then R also defines an algebra over K in right modules over itself by restriction of structure on the left. The bar module B_{R} is the bar complex $B_{\mathsf{R}} = B(N)$ of this K-algebra $N = \eta^* \mathsf{R}$. First, we check that this object fulfils the requirement of Proposition 1.A:

Proposition 1.4.2. *Let $\mathcal{E} = \mathcal{C}$, the category of dg-modules, or $\mathcal{E} = \mathcal{M}_{\mathsf{S}}$, the category of right modules over an operad S. The functor $S_{\mathsf{R}}(B_{\mathsf{R}}) : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ associated to the bar module B_{R} is naturally isomorphic to the bar construction $A \mapsto B(A)$ on the category of R-algebras in $\mathcal{E}$.*

Proof. According to Proposition 1.4.1, we have $S_{\mathsf{R}}(B_{\mathsf{R}}, A) = B(S_{\mathsf{R}}(\eta^* \mathsf{R}, A))$, where $\eta^* \mathsf{R}$ is the K-algebra in right R-modules defined by the operad R. Recall that $S_{\mathsf{R}}(\mathsf{R}) : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{R}}\mathcal{E}$ represents the identity functor of the category of R-algebras. Moreover, we have an identity $S_{\mathsf{R}}(\eta^* N, A) = \eta^* S_{\mathsf{R}}(N, A)$ for all R-algebras N in right R-modules (see recollections of §0.4.3). Hence the object $S_{\mathsf{R}}(\eta^* \mathsf{R}, A)$ represents the K-algebra associated to $A \in {}_{\mathsf{R}}\mathcal{E}$ by restriction of structure and we obtain finally $S_{\mathsf{R}}(B_{\mathsf{R}}, A) = B(S_{\mathsf{R}}(\eta^* \mathsf{R}, A)) = B(A)$. □

For our purpose, we examine the functoriality of this construction with respect to the operad R. For this aim, we use the following formal observation:

Observation 1.4.3. *Let $\psi : \mathsf{R} \to \mathsf{S}$ be a morphism of operads under K.*

(1) *The map $\psi : \mathsf{R} \to \mathsf{S}$ defines a morphism $\psi_\sharp : \mathsf{R} \to \mathsf{S}$ in the category of K-algebras in right R-modules, where we use restrictions of structure on the left to make R (respectively, S) into a K-algebra and restrictions of structure on the right to make S into a right R-module.*

(2) *The morphism of K-algebras in right S-modules $\psi_\flat : \mathsf{R} \circ_{\mathsf{R}} \mathsf{S} \to \mathsf{S}$ adjoint to $\psi_\sharp : \mathsf{R} \to \mathsf{S}$ forms an isomorphism.*

From this observation and Observation 1.2.1, we deduce that a morphism of operads under K gives rise to a morphism $\psi_\sharp : B_{\mathsf{R}} \to B_{\mathsf{S}}$, in the category of right R-modules and to an isomorphism $\psi_\flat : B_{\mathsf{R}} \circ_{\mathsf{R}} \mathsf{S} \xrightarrow{\simeq} B_{\mathsf{S}}$, which is obviously adjoint to $\psi_\sharp$. Since we assume that weak-equivalences (respectively, fibrations) are created by forgetful functors, we obtain that $\psi_\sharp : \mathsf{R} \to \mathsf{S}$ defines a weak-equivalence (respectively, a fibration) in the category of K-algebras in right R-modules if ψ is a weak-equivalence (respectively, a fibration) of operads. Hence, Proposition 1.2.2 and Proposition 1.2.6 return:

Proposition 1.4.4. *The morphism $\psi_\sharp : B_{\mathsf{R}} \to B_{\mathsf{S}}$ defines a fibration in the category of right R-modules if $\psi : \mathsf{R} \to \mathsf{S}$ is a fibration of operads under K.*

The morphism $\psi_\sharp : B_{\mathsf{R}} \to B_{\mathsf{S}}$ defines a weak-equivalence in the category of right R-modules if $\psi : \mathsf{R} \to \mathsf{S}$ is a weak-equivalence of operads under K and the underlying collections of the operads R and S consist of cofibrant dg-modules $\mathsf{R}(n), \mathsf{S}(n) \in \mathcal{C}$, $n \in \mathbb{N}$. □

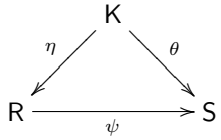
The isomorphism $\psi_\flat : B_{\mathsf{R}} \circ_{\mathsf{R}} \mathsf{S} \xrightarrow{\simeq} B_{\mathsf{S}}$ has a natural interpretation at the functor level. In §0.4.3, we recall that the functor $S_{\mathsf{S}}(M \circ_{\mathsf{R}} \mathsf{S}) : {}_{\mathsf{S}}\mathcal{E} \to \mathcal{E}$, where $M \circ_{\mathsf{R}} \mathsf{S}$ is the extension of structure of a right R-module M, is isomorphic to the composite

$$ {}_{\mathsf{S}}\mathcal{E} \xrightarrow{\psi^*} {}_{\mathsf{R}}\mathcal{E} \xrightarrow{S_{\mathsf{R}}(M)} \mathcal{E}, $$

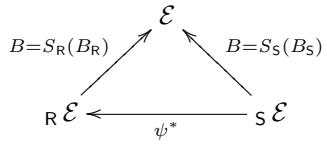
where $\psi^* : {}_{\mathsf{S}}\mathcal{E} \to {}_{\mathsf{R}}\mathcal{E}$ is the restriction functor associated to $\psi : \mathsf{R} \to \mathsf{S}$. For an operad under Stasheff's operad, the bar complex functor $B : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ is defined precisely by a composite of this form:

$$ {}_{\mathsf{R}}\mathcal{E} \xrightarrow{\eta^*} {}_{\mathsf{K}}\mathcal{E} \xrightarrow{B} \mathcal{E}, $$

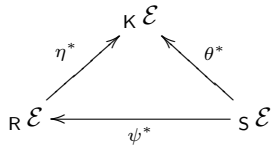
where we assume again $\mathcal{E} = \mathcal{C}$ or $\mathcal{E} = \mathcal{M}_{\mathsf{S}}$. Now suppose given a diagram



so that $\psi : \mathsf{R} \to \mathsf{S}$ is a morphism of operads under K. The diagram of functors



commutes just because the relation $\theta = \psi\eta$ implies that the diagram of restriction functors

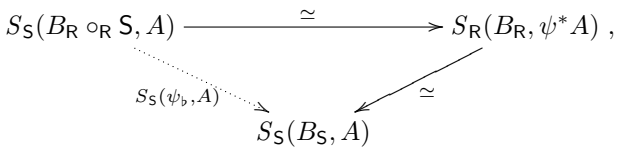


commutes. Thus, for a morphism $\psi : \mathsf{R} \to \mathsf{S}$ in $\mathcal{O}_0 \setminus \mathsf{K}$, we have a natural isomorphism $S_{\mathsf{R}}(B_{\mathsf{R}}, \psi^* A) \simeq S_{\mathsf{S}}(B_{\mathsf{S}}, A)$, for all $A \in {}_{\mathsf{R}}\mathcal{E}$. Moreover:

Proposition 1.4.5. *Let $\psi : \mathsf{R} \to \mathsf{S}$ be any morphism of operads under K. The natural isomorphism*

$$ \psi_\flat : B_{\mathsf{R}} \circ_{\mathsf{R}} \mathsf{S} \xrightarrow{\simeq} B_{\mathsf{S}} $$

induces an isomorphism of functors $S_{\mathsf{S}}(\psi_*) : S_{\mathsf{S}}(B_{\mathsf{R}} \circ_{\mathsf{R}} \mathsf{S}) \to S_{\mathsf{S}}(B_{\mathsf{S}})$ *that fits a commutative diagram*



for all $A \in {}_{\mathsf{S}}\mathcal{E}$.

PROOF. The proposition is a formal consequence of coherence properties between distribution isomorphisms $(M \otimes N) \circ_{\mathsf{R}} \mathsf{S} \simeq (M \circ_{\mathsf{R}} \mathsf{S}) \otimes (N \circ_{\mathsf{R}} \mathsf{S})$ and the functor isomorphisms $S_{\mathsf{R}}(M \otimes N) \simeq S_{\mathsf{R}}(M) \otimes S_{\mathsf{R}}(N)$. □

In particular, for the initial morphism $\eta : \mathsf{K} \to \mathsf{R}$ of an operad $\mathsf{R} \in \mathcal{O}_0 \setminus \mathsf{K}$, the isomorphism $\eta_{\flat} : B_{\mathsf{K}} \circ_{\mathsf{K}} \mathsf{R} \simeq B_{\mathsf{R}}$ reflects the definition of $S_{\mathsf{R}}(B_{\mathsf{R}}) : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ as the restriction of a functor $B : {}_{\mathsf{K}}\mathcal{E} \to \mathcal{E}$.

To complete our results, observe that the operad R defines a cofibrant object in the category of right modules over itself. Accordingly, Proposition 1.2.4 implies:

Proposition 1.4.6. *The module* B_{R} *forms a cofibrant object in the category of right* R*-modules.* □

2. The multiplicative structure of the bar construction

Introduction

In this section, we prove the existence and uniqueness of algebra structures on the bar module of E_∞-operads. Then we use the correspondence between right modules and functors to obtain the existence and uniqueness of functorial algebra structures on the bar construction itself $B(A)$, for all algebras A over a given E_∞-operad E.

To prove the existence of algebra structures on the bar module B_{E} the idea is to use endomorphism operads of right modules over operads. Recall briefly that the endomorphism operad of an object M in a category $\mathcal{E}$ is a universal operad in dg-modules End_M such that the structure of a P-algebra on M is equivalent to an operad morphism $\nabla : \mathsf{P} \to \mathrm{End}_M$. In this section, we may specify P-algebra structures by pairs (M, ∇), where $\nabla : \mathsf{P} \to \mathrm{End}_M$ is the operad morphism that determines the P-algebra structure of M, because we deal with objects which are not endowed with a natural internal P-algebra structure.

In §2.1.2, we observe that the bar module B_{C} of the commutative operad C can be equipped with the structure of a commutative algebra, like the bar complex of any commutative algebra. This structure is represented by an operad morphism $\nabla_c : \mathsf{C} \to \mathrm{End}_{B_{\mathsf{C}}}$, from the commutative operad C to the endomorphism operad of B_{C}.

Our main existence theorem, proved in §2.1, reads:

Theorem 2.A. *Let* E *be any* E_∞*-operad. Let* Q *be any cofibrant operad augmented over the commutative operad* C*. Let* $\epsilon : \mathsf{E} \to \mathsf{C}$ *and* $\phi : \mathsf{Q} \to \mathsf{C}$ *denote the respective augmentations of these operads.*

There is an operad morphism $\nabla_\epsilon : \mathsf{Q} \to \mathrm{End}_{B_\mathsf{E}}$ *which equips the bar module* B_E *with a left* Q*-action so that:*

(1) *The bar module* B_E *forms a* Q*-algebra in right* E*-modules.*
(2) *The natural isomorphism of right* C*-modules* $B_\mathsf{E} \circ_\mathsf{E} \mathsf{C} \simeq B_\mathsf{C}$ *defines an isomorphism in the category of* Q*-algebras in right* C*-modules*

$$(B_\mathsf{E}, \nabla_\epsilon) \circ_\mathsf{E} \mathsf{C} \simeq \phi^*(B_\mathsf{C}, \nabla_c),$$

where the Q*-algebra structure of* B_C *is obtained by restriction of its* C*-algebra structure through the augmentation of* Q*.*

The interpretation of this theorem at the level of the bar construction is straightforward and is also established in §2.1.

To prove Theorem 2.A, we observe that condition (2) is equivalent to a lifting problem in the category of operads, for which axioms of model categories imply immediately the existence of a solution.

In §2.2, we check that the isomorphism $(B_\mathsf{E}, \nabla_\epsilon) \circ_\mathsf{E} \mathsf{C} \simeq \phi^*(B_\mathsf{C}, \nabla_c)$ of condition (2) implies, by adjunction, the existence of a weak-equivalence in the category of Q-algebras in right E-modules

$$(B_\mathsf{E}, \nabla_\epsilon) \xrightarrow{\sim} \phi^*(B_\mathsf{C}, \nabla_c).$$

From this assertion we conclude immediately that all solutions of the existence Theorem 2.A yield equivalent objects in the homotopy category of Q-algebras in right E-modules. Then we apply the homotopy invariance theorems of [14, §15] to obtain that all solutions of the existence Theorem 2.A give homotopy equivalent structures on the bar construction. This gives our uniqueness result.

2.1. The existence theorem

This subsection is devoted to the existence part of our theorems.

To begin with, we examine the structure of the bar construction of commutative algebras in the context of right modules over operads – we check that the bar module B_C of the commutative operad C forms naturally a commutative algebra in right C-modules. Then we describe constructions which make Theorem 2.A equivalent to a lifting problem in the category of operads and we solve this lifting problem by arguments of homotopical algebra.

Recollection 2.1.1 (The shuffle product). The tensor coalgebra $T^c(\Sigma A)$ can be equipped with a product $\smile : T^c(\Sigma A) \otimes T^c(\Sigma A) \to T^c(\Sigma A)$ defined componentwise by sums of tensor permutations

$$(\Sigma A)^{\otimes m} \otimes (\Sigma A)^{\otimes n} \xrightarrow{\sum_w w_*} (\Sigma A)^{\otimes m+n}$$

where w ranges over the set of (m,n)-shuffles in Σ_{m+n}. This product is naturally associative and commutative. For an associative and commutative algebra A, the bar coderivation $\partial : T^c(\Sigma A) \to T^c(\Sigma A)$ defines a derivation with respect to $\smile$. Hence, in this case, we obtain that the bar complex $B(A) = (T^c(\Sigma A), \partial)$ is still an associative and commutative algebra.

Clearly, this standard construction for commutative algebras in dg-modules can be extended to algebras in a category of right modules over an operad R – just use the symmetry isomorphism of the tensor product of right R-modules in the definition of the shuffle product. Then we obtain that the bar complex $B(N)$ comes equipped with the structure of a commutative algebra in right R-modules if N is so.

Recall that the map $S_{\mathsf{R}}(N) : A \mapsto S_{\mathsf{R}}(N, A)$ defines a functor from R-algebras to commutative algebras if N is a commutative algebra in right R-modules. As the map $S_{\mathsf{R}} : M \mapsto S_{\mathsf{R}}(M)$ defines a functor of symmetric monoidal categories (check recollections in §0.3.2), we obtain that the shuffle product

$$(\Sigma N)^{\otimes m} \otimes (\Sigma N)^{\otimes n} \xrightarrow{\sum_w w_*} (\Sigma N)^{\otimes m+n}$$

corresponds to the shuffle product

$$S_{\mathsf{R}}(\Sigma N, A)^{\otimes m} \otimes S_{\mathsf{R}}(\Sigma N, A)^{\otimes n} \xrightarrow{\sum_w w_*} S_{\mathsf{R}}(\Sigma N, A)^{\otimes m+n}$$

at the functor level. Hence we obtain finally:

Observation 2.1.2. *Let $\mathcal{E}$ be the category of dg-modules $\mathcal{E} = \mathcal{C}$, or any category of right modules over an operad $\mathcal{E} = \mathcal{M}_{\mathsf{S}}$. For a commutative algebra in right R-modules N, the bar complex $B(N)$ comes equipped with the structure of a commutative algebra in right R-modules so that the isomorphism of functors $S_{\mathsf{R}}(B(N), A) \simeq B(S_{\mathsf{R}}(N, A))$ defines an isomorphism of commutative algebras, for all $A \in {}_{\mathsf{R}}\mathcal{E}$.*

We apply this observation to the commutative algebra in right C-modules formed by the commutative operad itself, for which we have $S_{\mathsf{C}}(\mathsf{C}) = \mathrm{Id}$, the identity functor on the category of commutative algebras. We obtain that the standard commutative algebra structure of the bar construction is realized by the structure of a commutative algebra in right C-modules on the bar module B_{C}.

Recall that, for any morphism $\psi : \mathsf{R} \to \mathsf{S}$ in $\mathcal{O}_0 \setminus \mathsf{K}$, we have a natural isomorphism $\psi_\flat : B_{\mathsf{R}} \circ_{\mathsf{R}} \mathsf{S} \simeq B_{\mathsf{S}}$ and this relation reflects the definition of the bar complex $B : {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ by the restriction of a functor $B : {}_{\mathsf{K}}\mathcal{E} \to \mathcal{E}$ (see Proposition 1.4.5). In particular, for an E_∞-operad E, equipped with an augmentation morphism $\epsilon : \mathsf{E} \xrightarrow{\sim} \mathsf{C}$, we have an isomorphism $\epsilon_\flat : B_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \simeq B_{\mathsf{C}}$.

Our aim is to lift the structure of the bar module B_{C} of the commutative operad C to the bar module B_{E} of any E_∞-operad E. Before proving our result, we recall the definition of an endomorphism operad and we give an interpretation of extensions and restrictions of structure in terms of morphisms on endomorphism operads.

Recollection 2.1.3 (Endomorphism operads). The endomorphism operad of an object M in a category $\mathcal{E}$ is defined by the hom-objects

$$\mathrm{End}_M(n) = \mathrm{Hom}_{\mathcal{E}}(M^{\otimes n}, M),$$

where the symmetric groups operate by tensor permutations on the source and the operad structure of End_M is deduced from the composition operation of enriched symmetric monoidal categories. For a P-algebra A, the operad morphism $\nabla : \mathsf{P} \to \mathrm{End}_A$, equivalent to the P-algebra structure of A, is defined simply by the morphisms

$$\nabla : \mathsf{P}(n) \to \mathrm{Hom}_{\mathcal{E}}(A^{\otimes n}, A)$$

adjoint to the evaluation morphisms $\lambda : \mathsf{P}(n) \otimes A^{\otimes n} \to A$. We refer to [23] or to [14, §3.4,§6.3] for an explicit definition of End_M in the context of right modules over operads. In the sequel, we only use general properties of End_M arising from the abstract definition of hom-objects $\mathrm{Hom}_{\mathcal{E}}(-,-)$.

Note that endomorphism operads End_M have a 0-term

$$\mathrm{End}_M(0) = M \neq 0$$

in contrast to our conventions on operads, but this apparent contradiction does not create any difficulty: in our constructions, one can replace any endomorphism operad End_M by a suboperad $\overline{\mathrm{End}}_M \in \mathcal{O}_0$ such that

$$\overline{\mathrm{End}}_M(n) = \begin{cases} 0, & \text{if } n = 0, \\ \mathrm{End}_M(n), & \text{otherwise,} \end{cases}$$

because any operad morphism $\nabla : \mathsf{P} \to \mathrm{End}_M$, where $\mathsf{P} \in \mathcal{O}_0$, factors through $\overline{\mathrm{End}}_M$.

Recollection 2.1.4 (Endomorphism operads and extension functors). Recall that an operad morphism $\psi : \mathsf{R} \to \mathsf{S}$, gives rise to a functor of extension of structure on the right

$$\psi_! : {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}} \to {}_{\mathsf{P}}\mathcal{M}_{\mathsf{S}}$$

(see recollections in §0.4). One can also observe that the operad morphism $\psi : \mathsf{R} \to \mathsf{S}$ induces a morphism of endomorphism operads:

$$\psi_! : \mathrm{End}_M \to \mathrm{End}_{M \circ_{\mathsf{R}} \mathsf{S}},$$

for all $M \in \mathcal{M}_{\mathsf{R}}$, essentially because the extension functor $\psi_! : M \mapsto M \circ_{\mathsf{R}} \mathsf{S}$ preserves tensor products (see [14, Proposition 9.4.4] and recollections in §0.4). For a P-algebra in right R-modules represented by a pair (N, ∇), where $N \in \mathcal{M}_{\mathsf{R}}$ and $\nabla : \mathsf{P} \to \mathrm{End}_N$, we obtain that the P-algebra in right S-modules $\psi_!(N, \nabla) = (N, \nabla) \circ_{\mathsf{R}} \mathsf{S}$, obtained from (N, ∇) by extension of structure on the right, is represented by the pair $\psi_!(N, \nabla) = (\psi_! N, \psi_! \nabla) = (N \circ_{\mathsf{R}} \mathsf{S}, \nabla \circ_{\mathsf{R}} \mathsf{S})$, where $\psi_! \nabla = \nabla \circ_{\mathsf{R}} \mathsf{S}$ is the composite

$$\mathsf{P} \xrightarrow{\nabla} \mathrm{End}_N \xrightarrow{\psi_!} \mathrm{End}_{N \circ_{\mathsf{R}} \mathsf{S}}.$$

This assertion is proved by a formal verification (we refer to [14, §3.4,§9.4]).

The functor of restriction of structure on the left

$$\phi^* : {}_{\mathsf{Q}}\mathcal{M}_{\mathsf{R}} \to {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}},$$

where $\phi : \mathsf{P} \to \mathsf{Q}$ is an operad morphism, has an obvious simpler description in terms of operad morphisms. Namely, for any Q-algebra in right R-modules represented by a pair (N, ∇), where $N \in \mathcal{M}_{\mathsf{R}}$ and $\nabla : \mathsf{Q} \to \mathrm{End}_N$, the P-algebra in right R-modules $\phi^*(N, \nabla)$, obtained from (N, ∇) by restriction of structure on the left, is represented by the pair $\phi^*(N, \nabla) = (N, \nabla\phi)$, where $\nabla\phi$ is the composite

$$\mathsf{P} \xrightarrow{\phi} \mathsf{Q} \xrightarrow{\nabla} \mathrm{End}_N .$$

For bar modules, we have an isomorphism $\psi_\flat : B_{\mathsf{R}} \circ_{\mathsf{R}} \mathsf{S} \xrightarrow{\simeq} B_{\mathsf{S}}$ and hence an isomorphism of endomorphism operads

$$\mathrm{End}_{B_{\mathsf{R}} \circ_{\mathsf{R}} \mathsf{S}} \simeq \mathrm{End}_{B_{\mathsf{S}}} .$$

Accordingly, we obtain that any morphism $\psi : \mathsf{R} \to \mathsf{S}$ in the category $\mathcal{O}_0 \setminus \mathsf{K}$ of operads under Stasheff's operad K gives rise to a morphism

$$\mathrm{End}_{B_{\mathsf{R}}} \xrightarrow{\psi_*} \mathrm{End}_{B_{\mathsf{S}}} .$$

One checks readily that the map $\psi \mapsto \psi_*$ preserves composites and identities so that the map $\mathsf{R} \mapsto \mathrm{End}_{B_{\mathsf{R}}}$ defines a functor on $\mathcal{O}_0 \setminus \mathsf{K}$. In addition, we have:

Observation 2.1.5. *Let E be any E_∞-operad, equipped with an augmentation $\epsilon : \mathsf{E} \xrightarrow{\sim} \mathsf{C}$. Let Q be any operad together with an augmentation $\phi : \mathsf{Q} \to \mathsf{C}$.*

Let $\nabla_c : \mathsf{C} \to \mathrm{End}_{B_{\mathsf{C}}}$ be the morphism of dg-operads determined by the commutative algebra structure of the bar module B_{C}. Let $\nabla_\epsilon : \mathsf{Q} \to \mathrm{End}_{B_{\mathsf{E}}}$ be an operad morphism which provides the bar module B_{E} with the structure of a Q-algebra in right E-modules.

The natural isomorphism $\epsilon_\flat : B_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \xrightarrow{\simeq} B_{\mathsf{C}}$ defines an isomorphism in the category of Q-algebras in right C-modules

$$\epsilon_\flat : (B_{\mathsf{E}}, \nabla_\epsilon) \circ_{\mathsf{E}} \mathsf{C} \xrightarrow{\simeq} \phi^*(B_{\mathsf{C}}, \nabla_c)$$

if and only if ∇_ϵ fits a commutative diagram

$$\begin{array}{ccc} \mathsf{Q} & \overset{\nabla_\epsilon}{\dashrightarrow} & \mathrm{End}_{B_{\mathsf{E}}} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\epsilon_*} \\ \mathsf{C} & \xrightarrow[\nabla_c]{} & \mathrm{End}_{B_{\mathsf{C}}} \end{array} .$$

To solve the lifting problem arising from this assertion, we prove:

Lemma 2.1.6. *The functor* $\mathsf{R} \mapsto \mathrm{End}_{B_{\mathsf{R}}}$ *preserves fibrations and acyclic fibrations between operads* $\mathsf{R} \in \mathcal{O}_0 \setminus \mathsf{K}$ *whose underlying collection* $\mathsf{R}(n)$, $n \in \mathbb{N}$, *consists of cofibrant dg-modules.*

Proof. In this proof, we prefer to use the notation of the functor $\psi_! : \mathcal{M}_{\mathsf{R}} \to \mathcal{M}_{\mathsf{S}}$ to denote extensions of structure of right modules over operads rather than the equivalent relative composition product $\psi_! M = M \circ_{\mathsf{R}} \mathsf{S}$. Similarly, we use the notation of the functor $\psi^* : \mathcal{M}_{\mathsf{R}} \to \mathcal{M}_{\mathsf{S}}$ to denote the restriction of structure of right modules over operads.

In general, the morphism of endomorphism operads $\psi_! : \mathrm{End}_M \to \mathrm{End}_{\psi_! M}$ induced by an operad morphism $\psi : \mathsf{R} \to \mathsf{S}$ consists of morphisms

$$\mathrm{Hom}_{\mathsf{R}}(M^{\otimes r}, M) \xrightarrow{\psi_!} \mathrm{Hom}_{\mathsf{S}}(\psi_! M^{\otimes r}, \psi_! M)$$

formed by using that $\psi_! : \mathcal{M}_{\mathsf{R}} \to \mathcal{M}_{\mathsf{S}}$ defines a functor of symmetric monoidal categories over dg-modules. We use the adjunction between extension and restriction functors $\psi_! : \mathcal{M}_{\mathsf{R}} \rightleftarrows \mathcal{M}_{\mathsf{S}} : \psi^*$ to identify these morphisms with composites

$$\mathrm{Hom}_{\mathsf{R}}(M^{\otimes r}, M) \xrightarrow{\eta(M)_*} \mathrm{Hom}_{\mathsf{R}}(M^{\otimes r}, \psi^* \psi_! M) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{S}}(\psi_! M^{\otimes r}, \psi_! M),$$

where $\eta(M)_*$ refers to the morphism on hom-objects induced by the adjunction unit $\eta(M) : M \to \psi^* \psi_!(M)$.

For endomorphism operads of bar modules, we obtain that the morphism $\psi_* : \mathrm{End}_{B_{\mathsf{R}}} \to \mathrm{End}_{B_{\mathsf{S}}}$, induced by a morphism $\psi : \mathsf{R} \to \mathsf{S}$ in $\mathcal{O}_0 \setminus \mathsf{K}$, can be defined by composites in diagrams of the form:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathsf{R}}(B_{\mathsf{R}}^{\otimes r}, B_{\mathsf{R}}) & \xrightarrow{\eta(B_{\mathsf{R}})_*} & \mathrm{Hom}_{\mathsf{R}}(B_{\mathsf{R}}^{\otimes r}, \psi^* \psi_! B_{\mathsf{R}}) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathsf{S}}(\psi_! B_{\mathsf{R}}^{\otimes r}, \psi_! B_{\mathsf{R}}) \\
& \searrow^{(\psi_\sharp)_*} & \downarrow{\scriptstyle \psi^*(\psi_\flat)_* \simeq} & & \downarrow{\scriptstyle \simeq} \\
& & \mathrm{Hom}_{\mathsf{R}}(B_{\mathsf{R}}^{\otimes r}, \psi^* B_{\mathsf{S}}) & \xrightarrow[\simeq]{} & \mathrm{Hom}_{\mathsf{S}}(\psi_! B_{\mathsf{R}}^{\otimes r}, B_{\mathsf{S}}) \\
& & & & \uparrow{\scriptstyle \simeq} \\
& & & & \mathrm{Hom}_{\mathsf{S}}(B_{\mathsf{S}}^{\otimes r}, B_{\mathsf{S}})
\end{array}$$

Accordingly, to prove our lemma, we are reduced to check that the morphism

$$\mathrm{Hom}_{\mathsf{R}}(B_{\mathsf{R}}^{\otimes r}, B_{\mathsf{R}}) \xrightarrow{(\psi_\sharp)_*} \mathrm{Hom}_{\mathsf{R}}(B_{\mathsf{R}}^{\otimes r}, \psi^* B_{\mathsf{S}})$$

induced by $\psi_\sharp : B_{\mathsf{R}} \to \psi^* B_{\mathsf{S}}$ forms a fibration (respectively, an acyclic fibration) of dg-modules if $\psi : \mathsf{R} \to \mathsf{S}$ is so.

In Lemma 1.4.6, we prove that B_{R} forms a cofibrant right R-module. In Proposition 1.4.4, we record that $\psi_\sharp : B_{\mathsf{R}} \to \psi^* B_{\mathsf{S}}$ forms a fibration (respectively, an acyclic fibration) of right R-modules if ψ is a fibration (respectively,

an acyclic fibration) of operads. By axioms of symmetric monoidal model categories enriched over dg-modules, we can conclude from these assertions that the morphism

$$\mathrm{Hom}_{\mathsf{R}}(B_{\mathsf{R}}^{\otimes r}, B_{\mathsf{R}}) \xrightarrow{(\psi_\sharp)_*} \mathrm{Hom}_{\mathsf{R}}(B_{\mathsf{R}}^{\otimes r}, \psi^* B_{\mathsf{S}})$$

forms a fibration (respectively, an acyclic fibration) if $\psi : \mathsf{R} \to \mathsf{S}$ is so and this proves the lemma. □

By axioms of model categories, Lemma 2.1.6 implies immediately:

Lemma 2.1.7. *Let* E *be any* E_∞*-operad, equipped with an augmentation* $\epsilon : \mathsf{E} \xrightarrow{\sim} \mathsf{C}$*. Let* Q *be any operad together with an augmentation* $\phi : \mathsf{Q} \to \mathsf{C}$*. If* Q *is cofibrant, then the lifting problem*

$$\begin{array}{ccc} \mathsf{Q} & \overset{\nabla_\epsilon}{\dashrightarrow} & \mathrm{End}_{B_{\mathsf{E}}} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\epsilon_*} \\ \mathsf{C} & \xrightarrow[\nabla_c]{} & \mathrm{End}_{B_{\mathsf{C}}} \end{array}$$

has a solution. □

From which we conclude:

Theorem 2.1.A (Claim of Theorem 2.A). *Assume that* Q *is cofibrant. Then there is a morphism* $\nabla_\epsilon : \mathsf{Q} \to \mathrm{End}_{B_{\mathsf{E}}}$*, which provides the bar module* B_{E} *with the structure of a* Q*-algebra in right* E*-modules, and so that the natural isomorphism of right* C*-modules* $\epsilon_\flat : B_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \xrightarrow{\simeq} B_{\mathsf{C}}$ *defines an isomorphism*

$$\epsilon_\flat : (B_{\mathsf{E}}, \nabla_\epsilon) \circ_{\mathsf{E}} \mathsf{C} \xrightarrow{\simeq} \phi^*(B_{\mathsf{C}}, \nabla_c)$$

in the category of Q*-algebras in right* C*-modules.* □

The proof of Theorem 2.A is now achieved. □

Theorem 2.1.A gives as a corollary:

Theorem 2.1.B. *Suppose we have a morphism* $\nabla_\epsilon : \mathsf{Q} \to \mathrm{End}_{B_{\mathsf{E}}}$ *so that* B_{E} *forms a* Q*-algebra in right* E*-modules as asserted in Theorem 2.1.A.*

Then the bar complex $B(A) = S_{\mathsf{E}}(B_{\mathsf{E}}, A)$*,* $A \in {}_{\mathsf{E}}\mathcal{E}$*, becomes equipped with an induced* Q*-algebra structure such that:*

(1) *The operad* Q *acts on* $B(A)$ *functorially in* A*.*
(2) *If* A *is a commutative algebra, then the action of* Q *on* $B(A)$ *reduces to the standard action of the commutative operad on* $B(A)$*, the action determined by the shuffle product of tensors.*

PROOF. To obtain assertion (1), we use that the structure of a Q-algebra in right E-modules gives rise to a Q-algebra structure at the functor level. Explicitly, according to recollections of §0.3.3, the evaluation morphism $\mathsf{Q}(n) \otimes B_{\mathsf{E}}^{\otimes n} \to B_{\mathsf{E}}$ gives rise to an evaluation morphism at the functor level

$$\mathsf{Q}(n) \otimes \underbrace{S_{\mathsf{E}}(B_{\mathsf{E}}, A)}_{=B(A)}{}^{\otimes n} \to \underbrace{S_{\mathsf{E}}(B_{\mathsf{E}}, A)}_{=B(A)}$$

so that the map $A \mapsto B(A)$ determines a functor from the category of E-algebras to the category of Q-algebras.

To obtain assertion (2), we use the relationship, recalled in §0.4.3, between extensions and restrictions at the module and functor levels. In the context of the theorem, for a commutative algebra A, we have a natural isomorphism in the category of Q-algebras

$$\underbrace{S_{\mathsf{C}}((B_{\mathsf{E}}, \nabla_\epsilon), \epsilon^* A)}_{=(B(A), \nabla_\epsilon)} \simeq S_{\mathsf{C}}((B_{\mathsf{E}}, \nabla_\epsilon) \circ_{\mathsf{E}} \mathsf{C}, A)$$

where the Q-algebra structure on the left-hand side comes from the bar module B_{E}. On the other hand, we have a natural isomorphism

$$\underbrace{\phi^*(S_{\mathsf{C}}((B_{\mathsf{C}}, \nabla_c), A))}_{=\phi^*(B(A), \nabla_c)} \simeq S_{\mathsf{C}}(\phi^*(B_{\mathsf{C}}, \nabla_c), A),$$

where ∇_c represents the standard commutative algebra structure of the bar complex of A. Hence, if $(B_{\mathsf{E}}, \nabla_\epsilon) \circ_{\mathsf{E}} \mathsf{C} \simeq \phi^*(B_{\mathsf{C}}, \nabla_c)$, then we have a natural isomorphism of Q-algebras $(B(A), \nabla_\epsilon) \simeq \phi^*(B(A), \nabla_c)$, for all $A \in {}_{\mathsf{C}}\mathcal{E}$. □

2.2. The uniqueness theorem

In this subsection, we prove that all solutions of the existence Theorem 2.A yield equivalent objects in the homotopy category of Q-algebras in right E-modules, as well as equivalent structures on the bar construction at the functor level.

For this aim, we use the morphism of right E-modules $\epsilon_\sharp : B_{\mathsf{E}} \to B_{\mathsf{C}}$, adjoint to the natural isomorphism $\epsilon_\flat : B_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \xrightarrow{\simeq} B_{\mathsf{C}}$ considered in Theorem 2.A. By Proposition 1.4.4 this morphism $\epsilon_\sharp : B_{\mathsf{E}} \to B_{\mathsf{C}}$ defines an acyclic fibration since the augmentation of an E_∞-operad $\epsilon : \mathsf{E} \to \mathsf{C}$ forms itself an acyclic fibration in the category of operads. Furthermore:

Lemma 2.2.1. *Suppose that the bar module B_{E} is equipped with the structure of a Q-algebra in right E-modules so that the natural isomorphism $\epsilon_\flat : B_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \xrightarrow{\simeq} B_{\mathsf{C}}$, defines an isomorphism of Q-algebras in right C-modules*

$$\epsilon_\flat : (B_{\mathsf{E}}, \nabla_\epsilon) \circ_{\mathsf{E}} \mathsf{C} \xrightarrow{\simeq} (B_{\mathsf{C}}, \nabla_c),$$

as asserted in Theorem 2.A.

Then the morphism of right E*-modules* $\epsilon_\sharp : B_\mathsf{E} \to B_\mathsf{C}$*, adjoint to* $\epsilon_\flat : B_\mathsf{E} \circ_\mathsf{E} \mathsf{C} \xrightarrow{\simeq} B_\mathsf{C}$*, defines a morphism of* Q*-algebras in right* E*-modules*

$$\epsilon_\sharp : (B_\mathsf{E}, \nabla_\epsilon) \to (B_\mathsf{C}, \nabla_c),$$

and, hence, forms an acyclic fibration in that category.

Proof. In §0.4.2, we recall that the extension and restriction functors $\psi_! : \mathcal{M}_\mathsf{R} \rightleftarrows \mathcal{M}_\mathsf{S} : \psi^*$ associated to any operad morphism $\psi : \mathsf{R} \to \mathsf{S}$ restrict to functors on P-algebras, for any operad P, so that we have an adjunction relation:

$$\psi_! : {}_\mathsf{P}\mathcal{M}_\mathsf{R} \rightleftarrows {}_\mathsf{P}\mathcal{M}_\mathsf{S} : \psi^*.$$

The lemma is an immediate corollary of this proposition. □

This lemma gives immediately:

Theorem 2.2.A. *Suppose we have operad morphisms* $\nabla_0, \nabla_1 : \mathsf{Q} \to \mathrm{End}_{B_\mathsf{E}}$ *that provide the bar module* B_E *with the structure of a* Q*-algebra in accordance with requirements (1-2) of Theorem 2.A.*

The algebras (B_E, ∇_0) *and* (B_E, ∇_1) *are connected by weak-equivalences*

$$(B_\mathsf{E}, \nabla_0) \xrightarrow{\sim} \cdot \xleftarrow{\sim} (B_\mathsf{E}, \nabla_1)$$

in the category of Q*-algebras in right* E*-modules.* □

As usual in a model category, the weak-equivalences

$$(B_\mathsf{E}, \nabla_0) \xrightarrow{\sim} \cdot \xleftarrow{\sim} (B_\mathsf{E}, \nabla_1)$$

can be replaced by a chain of weak-equivalences of Q-algebras in right E-modules

$$(B_\mathsf{E}, \nabla_0) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} (B_\mathsf{E}, \nabla_1)$$

in which all intermediate objects are cofibrant as Q-algebras in right E-modules, and hence as right E-modules since any cofibrant algebra over a (cofibrant) operad Q forms a cofibrant object in the underlying category (by [7, Corollary 5.5], [14, Proposition 12.3.2]). Recall that the bar module B_E forms itself a cofibrant E-module by Proposition 1.4.6.

In [14, §15], we prove that the natural transformation

$$S_\mathsf{E}(f, A) : S_\mathsf{E}(M, A) \xrightarrow{\sim} S_\mathsf{E}(N, A)$$

induced by a weak-equivalence $f : M \xrightarrow{\sim} N$ such that M, N are cofibrant right E-modules forms a weak-equivalence for all E-algebras A which are cofibrant in the underlying category (see Theorem 15.1.A in *loc. cit.*). Accordingly, in our context, we obtain:

Theorem 2.2.B. *Suppose we have morphisms* $\nabla_0, \nabla_1 : \mathsf{Q} \to \mathrm{End}_{B_{\mathsf{E}}}$*, as in Theorem 2.1.A, that yield functorial* Q*-algebra structures on the bar construction* $B(A)$ *as in Theorem 2.1.B.*

The Q*-algebras* $(B(A), \nabla_0)$ *and* $(B(A), \nabla_1)$ *can be connected by morphisms of* Q*-algebras*

$$(B(A), \nabla_0) \overset{\sim}{\leftarrow} \cdot \overset{\sim}{\rightarrow} \cdots \overset{\sim}{\rightarrow} (B(A), \nabla_1),$$

functorially in A*, and these morphisms are weak-equivalences whenever the* E*-algebra* A *defines a cofibrant object in the underlying category* $\mathcal{E}$*.* □

3. The categorical bar module

Introduction

In the next section we prove that the bar construction $B(A)$, equipped with the algebra structure of Theorem 2.1.B, defines a model of the suspension in the homotopy category of E_∞-algebras. For this aim we use a model of the suspension, defined in the general setting of pointed simplicial model categories and yielded by a categorical version of the bar construction.

The purpose of this section is to recall the definition of this categorical bar construction $C(A)$ in the context of algebras over an operad R and to define an R-algebra in right R-modules C_{R} such that $C(A) = S_{\mathsf{R}}(C_{\mathsf{R}}, A)$. The plan of this section parallels the plan of §1 on the bar module B_{R}. In §3.1, we recall the definition of the categorical bar construction $C(A)$ in the context of algebras over operads, where we take either the category of dg-modules $\mathcal{E} = \mathcal{C}$ or a category of right modules over an operad $\mathcal{E} = \mathcal{M}_{\mathsf{S}}$ as an underlying symmetric monoidal category; in §3.2, we study the categorical bar construction of algebras in right modules over operads; in §3.3, we observe that the required R-algebra in right R-modules C_{R} is returned by the categorical bar construction of the R-algebra in right R-modules formed by the operad itself. Then we examine the functoriality of the construction $\mathsf{R} \mapsto C_{\mathsf{R}}$ and the homotopy invariance of the categorical bar module C_{R}.

3.1. Recollections: the categorical bar construction

The categorical bar complex $C(A)$ is defined by the realization of a simplicial construction $\underline{C}(A)$ whose definition makes sense in any pointed category (explicitly, in any category equipped with a zero object $*$). For our purpose, we recall this definition in the context of algebras over a non-unitary operad P, assumed to satisfy $\mathsf{P}(0) = 0$, and where the underlying category $\mathcal{E}$ is either the category of dg-modules itself $\mathcal{E} = \mathcal{C}$ or a category of right modules over an operad $\mathcal{E} = \mathcal{M}_{\mathsf{S}}$. Note simply that the zero object of $\mathcal{E}$ is equipped with a P-algebra structure if P is a non-unitary operad and defines obviously a zero object in ${}_{\mathsf{P}}\mathcal{E}$. Thus the category of P-algebras in $\mathcal{E}$, where P is any non-unitary operad, is tautologically pointed.

Recall that $\mathcal{O}_0$ denotes the category of non-unitary operads.

3.1.1. The simplicial categorical bar complex

To define the categorical bar construction $C_{\mathsf{P}}(A)$ of an algebra $A \in {}_{\mathsf{P}}\mathcal{E}$, we form first a simplicial P-algebra $\underline{C}(A)$ such that

$$\underline{C}(A)_n = A^{\vee n},$$

where $\vee$ denotes the categorical coproduct in the category of P-algebras in $\mathcal{E}$. The faces and degeneracies of $\underline{C}(A)$ are defined explicitly by formulas

$$d_i = \begin{cases} 0 \vee A^{\vee n-1}, & \text{for } i = 0, \\ A^{\vee i-1} \vee \nabla \vee A^{\vee n-i-1}, & \text{for } i = 1, \ldots, n-1, \\ A^{\vee n-1} \vee 0, & \text{for } i = n, \end{cases}$$
$$s_j = A^{\vee j} \vee 0 \vee A^{\vee n-j}, \quad \text{for } j = 0, \ldots, n,$$

where $\nabla : A \vee A \to A$ denotes the codiagonal of A.

3.1.2. On normalized complexes

In the context of dg-modules $\mathcal{E} = \mathcal{C}$, we use the standard normalized chain complex to associate a dg-module $N_*(\underline{C})$ to any simplicial dg-modules $\underline{C}$.

For a simplicial Σ_*-module $\underline{C}$, the collection of normalized chain complexes $N_*(\underline{C}(n))$, $n \in \mathbb{N}$, defines a Σ_*-module $N_*(\underline{C})$ naturally associated to $\underline{C}$. For a simplicial right R-module $\underline{C}$, we have an obvious isomorphism

$$N_*(\underline{C}) \circ \mathsf{R} \xrightarrow{\simeq} N_*(\underline{C} \circ \mathsf{R}),$$

so that $N_*(\underline{C})$ inherits the structure of a right R-module and defines an object of $\mathcal{M}_{\mathsf{R}}$.

In our constructions, we use the classical Eilenberg-Mac Lane equivalence, which gives a natural morphism

$$N_*(\underline{C}) \otimes N_*(\underline{D}) \xrightarrow{EM} N_*(\underline{C} \otimes \underline{D}),$$

for all simplicial dg-modules $\underline{C}, \underline{D}$. In the context of a category of right modules over an operad $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$, we have termwise Eilenberg-Mac Lane morphisms

$$\Sigma_r \otimes_{\Sigma_s \times \Sigma_t} N_*(\underline{C}(s)) \otimes N_*(\underline{D}(t)) \xrightarrow{EM} N_*(\Sigma_r \otimes_{\Sigma_s \times \Sigma_t} \underline{C}(s) \otimes \underline{D}(t)),$$

inherited from dg-modules, which assemble to give an Eilenberg-Mac Lane morphism in $\mathcal{M}_{\mathsf{R}}$

$$N_*(\underline{C}) \otimes N_*(\underline{D}) \xrightarrow{EM} N_*(\underline{C} \otimes \underline{D}),$$

and similarly as regards the external tensor product in $\mathcal{M}_{\mathsf{R}}$.

In all cases, if $\underline{C} \equiv C$ is a constant simplicial object, then the Eilenberg-Mac Lane morphism is identified with a natural isomorphism

$$C \otimes N_*(\underline{D}) \simeq N_*(C \otimes \underline{D}).$$

3.1.3. The normalized categorical bar construction

The categorical bar construction $C(A)$ is defined by the normalized chain complex

$$C(A) = N_*(\underline{C}(A)).$$

This object is equipped with the structure of a P-algebra, like the normalized chain complex of any simplicial algebra over an operad. Formally, we have evaluation products

$$\mathsf{P}(n) \otimes C(A)^{\otimes n} \to C(A)$$

defined by the composite of the Eilenberg-Mac Lane equivalences

$$N_*(\underline{C}(A))^{\otimes n} \xrightarrow{EM} N_*(\underline{C}(A)^{\otimes n})$$

with the morphisms

$$\mathsf{P}(n) \otimes N_*(\underline{C}(A)^{\otimes n}) = N_*(\mathsf{P}(n) \otimes \underline{C}(A)^{\otimes n}) \to N_*(\underline{C}(A))$$

induced by the evaluation product of $\underline{C}(A)$.

3.2. The categorical bar construction of algebras in right modules over operads

In this section, we study the categorical bar construction of P-algebras in right modules over an operad R. In this context, the categorical bar construction $N \mapsto C(N)$ returns a P-algebra in right R-modules. As in §1.4.1, we determine the functor $S_{\mathsf{R}}(C(N)) : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{P}}\mathcal{E}$ associated to this object $C(N) \in {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}}$.

In the context of the standard bar construction, we use that the functor $S_{\mathsf{R}} : \mathcal{M}_{\mathsf{R}} \to \mathcal{F}_{\mathsf{R}}$ preserves tensor products to identify the functor $A \mapsto S_{\mathsf{R}}(B(N), A)$ associated to the bar complex of a K-algebra in right R-modules with the bar complex $B(S_{\mathsf{R}}(N, A))$ of the K-algebra $S_{\mathsf{R}}(N, A) \in {}_{\mathsf{K}}\mathcal{E}$. Similarly, as the functor $S_{\mathsf{R}} : {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}} \to {}_{\mathsf{P}}\mathcal{F}_{\mathsf{R}}$ preserves colimits of P-algebras (see §0.3.3), we obtain:

Lemma 3.2.1. *Let ${}_{\mathsf{P}}\mathcal{E}^{\Delta}$ be the category of simplicial P-algebras. Let N be a P-algebra in right R-modules.*

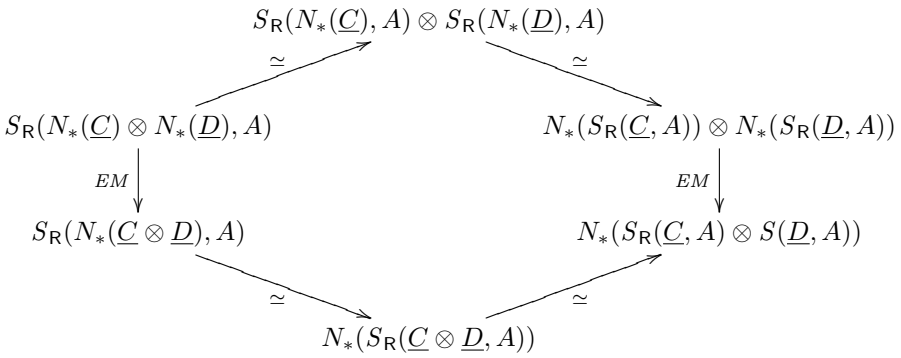
The functor $S_{\mathsf{R}}(\underline{C}(N)) : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{P}}\mathcal{E}^{\Delta}$ associated to the simplicial categorical bar construction of N satisfies the identity

$$S_{\mathsf{R}}(\underline{C}(N), A) = \underline{C}(S_{\mathsf{R}}(N, A)),$$

for all $A \in {}_{\mathsf{R}}\mathcal{E}$, where on the right-hand side we consider the simplicial categorical bar complex of the P-algebra $S_{\mathsf{R}}(N, A)$ associated to $A \in {}_{\mathsf{R}}\mathcal{E}$ by the functor $S_{\mathsf{R}}(N) : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{P}}\mathcal{E}$ defined by N. □

As the normalized chain complex $N_*(\underline{C})$ of a simplicial object $\underline{C}$ is defined by a cokernel and the functor $M \mapsto S_{\mathsf{R}}(M)$ preserve colimits in right R-modules, we have a natural isomorphism $S_{\mathsf{R}}(N_*(\underline{M}), A) \simeq N_*(S_{\mathsf{R}}(\underline{M}, A))$, for all $A \in {}_{\mathsf{R}}\mathcal{E}$.

This isomorphism commutes with Eilenberg-Mac Lane equivalences in the sense that the coherence diagram



commutes. As a consequence, if $\underline{C}$ is a simplicial P-algebra in right R-modules, then the functor identity $S_{\mathsf{R}}(N_*(\underline{C}), A) = N_*(S_{\mathsf{R}}(\underline{C}, A))$ holds in the category of P-algebras.

From these observations, we conclude:

Proposition 3.2.2. *Let N be a P-algebra in right R-modules.*

The functor $S_{\mathsf{R}}(C(N)) : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{P}}\mathcal{E}$ associated to the categorical bar construction of N satisfies the relation

$$S_{\mathsf{R}}(C(N), A) \simeq C(S_{\mathsf{R}}(N, A)),$$

for all $A \in {}_{\mathsf{R}}\mathcal{E}$, where on the right-hand side we consider the categorical bar complex of the P-algebra $S_{\mathsf{R}}(N, A)$ associated to $A \in {}_{\mathsf{R}}\mathcal{E}$ by the functor $S_{\mathsf{R}}(N) : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{P}}\mathcal{E}$ defined by N. □

Remark 3.2.3. In §1.2, we observe that the functor $N \mapsto B(N)$ commutes with extensions and restrictions of structure on the right. The same assertion holds for the functor $N \mapsto C(N)$ defined by the categorical bar construction just because both functors $\psi_! : {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}} \rightleftarrows {}_{\mathsf{P}}\mathcal{M}_{\mathsf{S}} : \psi^*$ preserve coproducts. The functor $N \mapsto C(N)$ also commutes with extensions of structure on the left, but not with restrictions of structure on the left since this latter operation does not preserve coproducts. Nevertheless, we still have a natural morphism $C(\phi^* N) \to \phi^* C(N)$ induced by the natural transformations $(\phi^* N)^{\vee n} \to \phi^*(N^{\vee n})$.

3.3. The categorical bar module

The categorical bar module of an operad R, like the bar module of §1.4, is the categorical bar construction of the R-algebra in right R-modules formed by the operad itself. For the sake of coherence, we use the notation C_{R} for this categorical bar module $C_{\mathsf{R}} = C(\mathsf{R})$ and we set similarly $\underline{C}_{\mathsf{R}} = \underline{C}(\mathsf{R})$.

In §0.3.3, we recall that $S_{\mathsf{R}}(\mathsf{R}) : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{R}}\mathcal{E}$ represents the identity functor on the category of R-algebras. Hence, Proposition 3.2.2 gives:

Proposition 3.3.A. *The functor $S_{\mathsf{R}}(C_{\mathsf{R}}) : {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{R}}\mathcal{E}$ associated to C_{R} is naturally isomorphic to the categorical bar construction $A \mapsto C(A)$ in the category of R-algebras.* □

As in §1.4, we examine the structure of C_{R} and the functoriality of the construction $\mathsf{R} \mapsto C_{\mathsf{R}}$.

The categorical bar module C_{R} does not form a cofibrant object in right R-modules, unlike the bar module B_{R}, but we prove that C_{R} is cofibrant as a Σ_*-module provided that the operad R is so (according to our usual convention, we say that C_{R} is Σ_*-cofibrant). Thus, we forget right module structures and we examine the R-algebra in Σ_*-modules underlying the categorical bar module C_{R}. For the simplicial categorical bar module $\underline{C}_{\mathsf{R}}$, we obtain:

Lemma 3.3.1. *We have an identity*

$$\underline{C}_{\mathsf{R}} = \mathsf{R}(\underline{C}(\mathsf{I})),$$

where $\mathsf{R}(\underline{C}(\mathsf{I}))$ represents the free R-algebra on the categorical bar construction of the unit Σ_*-module I in the category of Σ_*-modules.

Proof. By construction, the forgetful functor $U : \mathcal{M}_{\mathsf{R}} \to \mathcal{M}$ preserves enriched monoidal category structures. By [14, Proposition 3.3.3], this assertion implies that the forgetful functor $U : {}_{\mathsf{R}}\mathcal{M}_{\mathsf{R}} \to {}_{\mathsf{R}}\mathcal{M}$, from the category of R-algebras in right R-modules to the category of R-algebras in Σ_*-modules, preserves colimits. As a consequence, we obtain that $\underline{C}_{\mathsf{R}}$ agrees with the categorical bar construction of R in Σ_*-modules.

Observe that the operad R forms a free object in the category of R-algebras in Σ_*-modules: we have explicitly $\mathsf{R} = \mathsf{R} \circ \mathsf{I} = \mathsf{R}(\mathsf{I})$. By adjunction, a coproduct of free objects satisfies the relation $\mathsf{R}(M) \vee \mathsf{R}(N) = \mathsf{R}(M \oplus N)$, for all $M, N \in \mathcal{M}$. Hence, we obtain readily:

$$(\underline{C}_{\mathsf{R}})_n = \mathsf{R}^{\vee n} = \mathsf{R}(\mathsf{I})^{\vee n} = \mathsf{R}(\mathsf{I}^{\oplus n}),$$

for all $n \in \mathbb{N}$. The determination of faces and degeneracies of the categorical bar construction is also formal from the universal property of free objects, so that we obtain the conclusion of the lemma. □

As a byproduct, we obtain:

Proposition 3.3.2. *The categorical bar module C_{R} is Σ_*-cofibrant if the operad R is so.*

Proof. The assumption about the operad R implies that simplicial R-algebras form a model category (see references of §0.2.6). Lemma 3.3.1 implies that the simplicial categorical bar module $\underline{C}_{\mathsf{R}}$ forms a cofibrant simplicial R-algebra in Σ_*-modules. By [7, Corollary 5.5], [14, Proposition 12.3.2], this assertion implies that $\underline{C}_{\mathsf{R}}$ is cofibrant in the underlying category of simplicial Σ_*-modules, and hence that the normalized chain complex $C_{\mathsf{R}} = N_*(\underline{C}_{\mathsf{R}})$ associated to $\underline{C}_{\mathsf{R}}$ is cofibrant as a Σ_*-module. □

Construction 3.3.3 (Functoriality of the categorical bar module). In §1.4, we observe that a morphism of operads under K gives rise to a morphism of right R-modules $\psi_\sharp : B_{\mathsf{R}} \to B_{\mathsf{S}}$. In this paragraph, we check that a morphism of operads gives rise to an analogous morphism of R-algebras in right R-modules

$$\psi_\sharp : C_{\mathsf{R}} \to \psi^* C_{\mathsf{S}},$$

where $\psi^* C_{\mathsf{S}}$ refers to the R-algebra in right R-modules obtained by a two-sided restriction of $C_{\mathsf{S}} \in {}_{\mathsf{S}}\mathcal{M}_{\mathsf{S}}$. We prove next that $\psi_\sharp$ defines a weak-equivalence (respectively, a fibration) if ψ is so.

Formally, we use that ψ determines a two-sided restriction functor $\psi^* : {}_{\mathsf{S}}\mathcal{M}_{\mathsf{S}} \to {}_{\mathsf{R}}\mathcal{M}_{\mathsf{R}}$ and the operad morphism $\psi : \mathsf{R} \to \mathsf{S}$ defines a morphism $\psi : \mathsf{R} \to \psi^* \mathsf{S}$ in the category of R-algebras in right R-modules. As a consequence, by functoriality of the categorical bar construction $N \mapsto C(N)$, we obtain that $\psi : \mathsf{R} \to \mathsf{S}$ induces a natural morphism of R-algebras in right R-modules

$$C(\mathsf{R}) \xrightarrow{C(\psi)} C(\psi^* \mathsf{S}).$$

On the other hand, for any algebra $N \in {}_{\mathsf{S}}\mathcal{M}_{\mathsf{S}}$, we have a morphism $\underline{C}(\psi^* N) \to \psi^* \underline{C}(N)$ induced by the natural transformation $(\psi^* N)^{\vee n} \to \psi^*(N^{\vee n})$. As a consequence, we have a natural morphism

$$C(\psi^* N) \xrightarrow{\psi_\sharp} \psi^* C(N).$$

between the categorical bar complex of $N \in {}_{\mathsf{S}}\mathcal{M}_{\mathsf{S}}$ and the categorical bar complex of $\psi^* N \in {}_{\mathsf{R}}\mathcal{M}_{\mathsf{R}}$. Our morphism $\psi_\sharp : C_{\mathsf{R}} \to \psi^* C_{\mathsf{S}}$ is given by the composite:

$$C(\mathsf{R}) \xrightarrow{C(\psi)} C(\psi^* \mathsf{S}) \xrightarrow{\psi_\sharp} \psi^* C(\mathsf{S}).$$

If we forget right module structures, then we obtain readily:

Observation 3.3.4. *The morphism*

$$\underline{C}_{\mathsf{R}} \xrightarrow{\psi_\sharp} \psi^* \underline{C}_{\mathsf{S}}$$

associated to an operad morphism $\psi : \mathsf{R} \to \mathsf{S}$ is given dimensionwise by the natural morphism of free objects

$$\mathsf{R}(\underline{C}(\mathsf{I})) \xrightarrow{\psi(\underline{C}(\mathsf{I}))} \mathsf{S}(\underline{C}(\mathsf{I}))$$

induced by $\psi : \mathsf{R} \to \mathsf{S}$.

We use this observation to prove:

Lemma 3.3.5. *If $\psi : \mathsf{R} \to \mathsf{S}$ is a weak-equivalence (respectively a fibration) of operads, then the morphism*

$$\psi_\sharp : C_{\mathsf{R}} \to \psi^* C_{\mathsf{S}}$$

defines a weak-equivalence (respectively a fibration) in ${}_{\mathsf{R}}\mathcal{M}_{\mathsf{R}}$, for all non-unitary operads $\mathsf{R}, \mathsf{S} \in \mathcal{O}_0$.

PROOF. Since all forgetful functors create weak equivalences and fibrations, we can forget right module structures in the proof of this lemma and we can use the representation of Observation 3.3.4.

We deduce immediately from the form of the free R-algebra

$$\mathsf{R}(\underline{C}(\mathsf{I})) = S(\mathsf{R}, \underline{C}(\mathsf{I})) = \mathsf{R} \circ \underline{C}(\mathsf{I})$$

that the morphism of simplicial dg-modules $\psi(\underline{C}(\mathsf{I})) = \psi \circ \underline{C}(\mathsf{I}) : \mathsf{R} \circ \underline{C}(\mathsf{I}) \to \mathsf{S} \circ \underline{C}(\mathsf{I})$ induced by a surjective morphism of dg-operads $\phi : \mathsf{R} \to \mathsf{S}$ is surjective as well. As a byproduct, so is the morphism induced by $\psi(\underline{C}(\mathsf{I}))$ on normalized chain complexes. Thus we conclude that the morphism $\psi_\sharp : C_\mathsf{R} \to \psi^* C_\mathsf{S}$ induced by a fibration of dg-operads forms a fibration as well.

Recall that the composition product of Σ_*-modules $M \circ N$ preserves weak-equivalences in M, provided that $N(0) = 0$ and the modules $N(r)$, $r > 0$, are cofibrant in dg-modules (see [13, §2.3], see also [14, §11.6]). From this assertion, we deduce that the morphism of simplicial Σ_*-modules $\psi(\underline{C}(\mathsf{I})) = \psi \circ \underline{C}(\mathsf{I}) : \mathsf{R} \circ \underline{C}(\mathsf{I}) \to \mathsf{S} \circ \underline{C}(\mathsf{I})$ induced by a weak-equivalence of dg-operads $\psi : \mathsf{R} \to \mathsf{S}$ defines a weak-equivalence, and so does the morphism induced by $\psi(\underline{C}(\mathsf{I}))$ on normalized chain complexes. Hence, we conclude that the morphism $\psi_\sharp : C_\mathsf{R} \to \psi^* C_\mathsf{S}$ induced by a weak-equivalence of dg-operads $\psi : \mathsf{R} \xrightarrow{\sim} \mathsf{S}$ forms a weak-equivalence. □

4. The homotopy interpretation of the bar construction

Introduction

In this section, we prove that, for cofibrant algebras over E_∞-operads, the usual bar construction $B(A)$, equipped with the algebra structure given by Theorem 2.1.B, is equivalent to the categorical bar construction $C(A)$ as an E_∞-algebra. Then we use that the categorical bar construction $C(A)$ is equivalent to the suspension ΣA in the homotopy categories of algebras over an operad to conclude:

Theorem 4.A. *Suppose that* E *forms itself a cofibrant* E_∞*-operad and set* $\mathsf{Q} = \mathsf{E}$.

Assume that the bar complex $B(A)$ *is equipped with the structure of an* E*-algebra, for all* $A \in {}_\mathsf{E}\mathcal{E}$*, and that this structure is realized at the module level, as stated in Theorem 2.1.B. Then we have natural* E*-algebra equivalences*

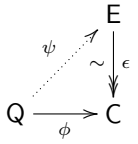
$$B(A) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} \Sigma A$$

that connect $B(A)$ *to the suspension of* A *in the model category of* E*-algebras, for all cofibrant* E*-algebras* A. □

This theorem can easily be generalized to include the case where the operad E is not itself cofibrant (see §4.2).

Again we realize the equivalence between $B(A)$ and $C(A)$ at the module level. To be explicit, let E be any E_∞-operad (possibly not cofibrant), let Q be

any cofibrant E_∞-operad, and assume that the bar module B_{E} is equipped with the structure of a Q-algebra in right E-modules, as asserted in Theorem 2.A. Recall that the categorical bar module B_{E} forms an E-algebra in right E-modules. Since Q is supposed to be cofibrant, we can pick an operad morphism in the lifting diagram



to make any E-algebra in right E-modules into a Q-algebra in right E-modules by restriction of structure. In §4.1, we check that B_{E} and C_{E} define equivalent objects in the homotopy category of Q-algebras in right E-modules. Thus we have a chain of weak-equivalences of Q-algebras in right E-modules

$$B_{\mathsf{E}} \overset{\sim}{\leftarrow} \cdot \overset{\sim}{\rightarrow} \cdots \overset{\sim}{\rightarrow} C_{\mathsf{E}}.$$

In §4.2, we use a theorem of [14, §15] to obtain that these weak-equivalences give rise to weak-equivalences at the functor level

$$\underbrace{S_{\mathsf{E}}(B_{\mathsf{E}}, A)}_{=B(A)} \overset{\sim}{\leftarrow} \cdot \overset{\sim}{\rightarrow} \cdots \overset{\sim}{\rightarrow} \underbrace{S_{\mathsf{E}}(C_{\mathsf{E}}, A)}_{=C(A)}$$

for all cofibrant E-algebras $A \in {}_{\mathsf{E}}\mathcal{E}$ and our conclusion follows.

4.1. The equivalence of bar constructions

First we prove the existence of equivalences between the bar modules B_{E} and C_{E} associated to an E_∞-operad E. This result is a consequence of the following observation:

Lemma 4.1.1. *For the commutative operad C, we have an identity of C-algebras in right C-modules $B_{\mathsf{C}} = C_{\mathsf{C}}$.*

Proof. This observation is a consequence of the definition of the coproduct in the category of non-unitary commutative algebras. Explicitly, for non-unitary commutative algebras in dg-modules, and more generally in any symmetric monoidal category, we have an identity: $A \vee B = A \oplus B \oplus A \otimes B$. As a consequence, for the simplicial categorical bar complex $\underline{C}(N)$ of any commutative algebra N in right C-modules, we obtain

$$\underline{C}(N)_n = (N \otimes \cdots \otimes N) \oplus (\text{degeneracies}).$$

Thus, at the level of normalized chain complexes, we obtain the relation $C(N) = N_*(\underline{C}(N)) = B(N)$. The case $N = \mathsf{C}$ gives the announced identity $B_{\mathsf{C}} = C_{\mathsf{C}}$. □

Roughly, for an E_∞-operad E, we lift the isomorphism of this lemma to a weak-equivalence of E_∞-algebras in right E-modules.

Suppose we have a cofibrant E_∞-operad Q together with an augmentation $\phi : \mathsf{Q} \xrightarrow{\sim} \mathsf{C}$. Assume that the bar module B_{E} is equipped with the structure of a Q-algebra in right E-modules as asserted in Theorem 2.A. In Proposition 2.2.1, we observe that the obtained Q-algebra B_{E} is endowed with a weak-equivalence

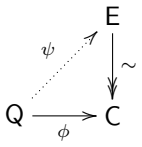
$$\epsilon_\sharp : B_{\mathsf{E}} \xrightarrow{\sim} \phi^* B_{\mathsf{C}}$$

in the category of Q-algebras in right E-modules.

On the other hand, we observe in Proposition 3.3.5, that the categorical bar module C_{E} is endowed with a weak-equivalence

$$\epsilon_\sharp : C_{\mathsf{E}} \xrightarrow{\sim} \epsilon^* C_{\mathsf{C}}$$

in the category of E-algebras in right E-modules. As explained in the introduction of this section, since Q is cofibrant, we can pick a lifting in the operad diagram



to obtain a morphism $\psi : \mathsf{Q} \to \mathsf{E}$ in $\mathcal{O}_0 \,/\, \mathsf{C}$. By restriction of structure, the equivalence $\epsilon_\sharp : C_{\mathsf{E}} \xrightarrow{\sim} \epsilon^* C_{\mathsf{C}}$ gives rise to an equivalence

$$\psi^*(\epsilon_\sharp) : \psi^* C_{\mathsf{E}} \xrightarrow{\sim} \psi^* \epsilon^* C_{\mathsf{C}} = \phi^* C_{\mathsf{C}}$$

in the category of Q-algebras in right E-modules.

Therefore, we obtain:

Theorem 4.1.2. *Assume that the bar module B_{E} is equipped with the structure of a Q-algebra in right E-modules as in Theorem 2.A. Then we have weak-equivalences*

$$B_{\mathsf{E}} \xrightarrow{\sim} \phi^* B_{\mathsf{C}} = \phi^* C_{\mathsf{C}} \xleftarrow{\sim} \psi^* C_{\mathsf{E}}$$

in the category of Q-algebras in right E-modules. □

Again, we can use model category structures to replace the weak-equivalences

$$B_{\mathsf{E}} \xrightarrow{\sim} \cdot \xleftarrow{\sim} \psi^* C_{\mathsf{E}}$$

by a chain of weak-equivalences

$$B_{\mathsf{E}} \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} \psi^* C_{\mathsf{E}}$$

in which all intermediate objects are cofibrant objects of the category of Q-algebras in right E-modules. Recall that a right E-module M is called Σ_*-cofibrant, like an operad, if M is cofibrant as a Σ_*-module. By [14, Proposition

14.1.1], any cofibrant right E-module is Σ_*-cofibrant since the E_∞-operad E is supposed to be Σ_*-cofibrant. Accordingly, the bar module B_{E} is Σ_*-cofibrant. The categorical bar module C_{E} is also Σ_*-cofibrant by Proposition 3.3.2. Since a cofibrant Q-algebra in right E-modules forms a cofibrant object in the underlying category of right E-modules by [7, Corollary 5.5], [14, Proposition 12.3.2], and hence a Σ_*-cofibrant module by [14, Proposition 14.1.1], we conclude that all objects in our chain of weak-equivalences are Σ_*-cofibrant. At the functor level, we obtain that these weak-equivalences give rise to:

Theorem 4.1.3. *The bar construction $B(A)$ is connected to the categorical bar construction $C(A)$ by natural weak-equivalences of Q-algebras*

$$B(A) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} \psi^* C(A),$$

for all cofibrant E-algebras A, where we use a restriction of structure to make the E-algebra $C(A)$ into a Q-algebra. □

Proof. In [14, §15], we prove that a weak-equivalence $\phi : M \xrightarrow{\sim} N$ between Σ_*-cofibrant right R-modules M and N induces a weak-equivalence at the functor level:

$$S_{\mathsf{R}}(\phi, A) : S_{\mathsf{R}}(M, A) \xrightarrow{\sim} S_{\mathsf{R}}(N, A),$$

for all cofibrant R-algebras A. Accordingly, the morphisms

$$B_{\mathsf{E}} \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} \psi^* C_{\mathsf{E}}$$

induce weak-equivalences of Q-algebras

$$S_{\mathsf{E}}(B_{\mathsf{E}}, A) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} S_{\mathsf{E}}(\psi^* C_{\mathsf{E}}, A).$$

for all cofibrant E-algebras A.

Recall that the functor $N \mapsto S_{\mathsf{E}}(N)$ commutes with restrictions of structure on the left. Therefore we have weak-equivalences between

$$B(A) = S_{\mathsf{E}}(B_{\mathsf{E}}, A)$$

and

$$\psi^* C(A) = \psi^* S_{\mathsf{E}}(C_{\mathsf{E}}, A) = S_{\mathsf{E}}(\psi^* C_{\mathsf{E}}, A)$$

as required. □

4.2. The equivalence with suspensions

The next assertion is proved in [25] (in the context of dg-modules, but the generalization to any category over dg-modules $\mathcal{E}$ is straightforward):

Fact 4.2.1 (See [25, §3, §14]). *Assume that P is a Σ_*-cofibrant operad in dg-modules so that the category of P-algebras in $\mathcal{E}$ forms a semi-model category.*

For every cofibrant P-algebra in $\mathcal{E}$, the P-algebra $C(A)$ is connected to ΣA, the suspension of A in the model category of P-algebras in $\mathcal{E}$, by weak-equivalences of P-algebras

$$C(A) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} \Sigma A,$$

functorially in A.

From this assertion and Theorem 4.1.3 we conclude:

Theorem 4.A (Claim of Theorem 4.A). *Suppose that* E *is a cofibrant* E_∞*-operad and set* $\mathsf{Q} = \mathsf{E}$.

Assume that the bar complex $B(A)$ *is equipped with the structure of an* E*-algebra, for all* $A \in {}_{\mathsf{E}}\mathcal{E}$*, and that this structure is realized at the module level, as stated in Theorem 2.A. Then we have natural* E*-algebra equivalences*

$$B(A) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} \Sigma A$$

that connect $B(A)$ *to the suspension of* A *in the model category of* E*-algebras, for all cofibrant* E*-algebras* A. □

To complete this result, recall that the extension and restriction functors

$$\phi_! : {}_{\mathsf{P}}\mathcal{E} \rightleftarrows {}_{\mathsf{Q}}\mathcal{E} : \phi^*.$$

associated to a weak-equivalence of Σ_*-cofibrant operads $\phi : \mathsf{P} \to \mathsf{Q}$ define Quillen adjoint equivalences of model categories (see [7] or [14, §16]). As a byproduct, Theorem 4.A can be generalized to cover the case where the E_∞-operad E is not cofibrant as an operad. In this context, we obtain weak-equivalences of Q-algebras

$$B(A) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} \psi^*(\Sigma A),$$

for all cofibrant E-algebras A, where ΣA is the suspension of A in the model category of E-algebras.

The iterated bar construction and iterated loop spaces

In this concluding part, we study applications of our results to cochain complexes of spaces and iterated loop spaces.

To fix our framework, a space X refers to a simplicial set and we consider the normalized cochain complex $N^*(X)$ with coefficients in the ground ring $\Bbbk$. One proves that $N^*(X)$ can be equipped with the structure of a (unitary) E-algebra, for some E_∞-operad E, for all $X \in \mathcal{S}$, so that the map $X \mapsto N^*(X)$ defines a functor from the category of simplicial sets $\mathcal{S}$ to the category of E-algebras ${}_{\mathsf{E}}\mathcal{C}$ (see [18] for a first proof of this result and [6, 27] for more combinatorial constructions). In the context of pointed spaces, we replace $N^*(X)$ by the reduced cochain complex $\bar{N}^*(X)$ to use objects without unit. Then we obtain that $\bar{N}^*(X)$ comes equipped with the structure of an E-algebra, for some non-unitary E_∞-operad E, in accordance with our conventions.

Let F_X be any cofibrant replacement of $\bar{N}^*(X)$ in the model category of E-algebras. According to results of [25], the suspension ΣF_X is equivalent to $\bar{N}^*(\Omega X)$ in the homotopy category of E-algebras provided that ΩX is connected and under standard finiteness and completeness assumptions on X (see [25, Theorem 1.2] and its proof in *loc. cit.*). For our needs, we also have to record

that the equivalence $\Sigma F_X \sim \bar{N}(\Omega X)$ is natural in the homotopy category of E-algebras.

Theorem 2.A implies the existence of a well-defined iterated bar complex $B^n(A)$ for all E-algebras A. Theorem 4.A implies that this iterated bar complex $B^n(A)$ is equivalent to the iterated suspension $\Sigma^n A$ if A is a cofibrant E-algebra. Thus, in the case of a cochain algebra $\bar{N}^*(X)$, we obtain equivalences $B^n(\bar{N}^*(X)) \sim B^n(F_X) \sim \Sigma^n F_X$ and we have $\Sigma^n F_X \sim \bar{N}^*(\Omega^n X)$ by an inductive application of the results of [25]. The assumptions which are made explicit in *loc. cit.* are reasonable for a single loop space ΩX, but give needlessly conditions in the case of higher iterated loop spaces, at least in the context where the ground ring is a finite primary field $\Bbbk = \mathbb{F}_p$. The actual purpose of this part is to review shortly the arguments of *loc. cit.* and to examine assumptions on the space X which ensure the equivalence $B^n(\bar{N}^*(X)) \sim \bar{N}^*(\Omega^n X)$.

One checks by a careful inspection of [25, §3, §5] that the suspension ΣF_X is equivalent to $\bar{N}^*(\Omega X)$ in the homotopy category of E-algebras as long as the cohomological Eilenberg-Moore spectral sequence of the path space fibration

$$E_2 = \mathrm{Tor}_*^{H^*(X,\Bbbk)}(\Bbbk, \Bbbk) \Rightarrow H^*(\Omega X, \Bbbk)$$

converges. By induction, we obtain that the n-fold suspension $\Sigma^n F_X$ is equivalent in the homotopy category of E-algebras to $\bar{N}^*(\Omega^n X)$, the cochain algebra of the n-fold iterated loop space of X, if the cohomological Eilenberg-Moore spectral sequence of the path space fibration converges for all loop spaces $\Omega^m X$, where $1 \leq m \leq n$. Record simply the next usual conditions which ensure this convergence in the context where the ground is either the rational field $\Bbbk = \mathbb{Q}$ or a finite primary field $\Bbbk = \mathbb{F}_p$:

Fact 1 (See [12, 32]). *Let $n \geq 1$. Suppose that:*

- $(1)_0$ *The homotopy groups $\pi_*(X)$ are trivial for all $* \leq n$ (case $\Bbbk = \mathbb{Q}$).*
- $(1)_p$ *The homotopy groups $\pi_*(X)$ are finite p-groups for all $* \leq n$ (case $\Bbbk = \mathbb{F}_p$).*

and

- (2) *The homotopy groups $\pi_*(X)$ are finitely generated in every degree $* > 0$.*

Then the cohomological Eilenberg-Moore spectral sequence

$$E_2 = \mathrm{Tor}_*^{H^*(\Omega^m X,\Bbbk)}(\Bbbk, \Bbbk) \Rightarrow H^*(\Omega^{m+1} X, \Bbbk)$$

converges for every $m < n$.

The finiteness assumptions on homotopy groups imply that $H_*(\Omega^m X, \Bbbk)$ forms a finitely generated $\Bbbk$-module, in every degree $* \geq 0$, for every $m \leq n$ (to check this folk claim, proceed by induction on the Postnikov tower of $\Omega^m X$). Therefore the assumptions of Fact 1 imply the convergence of the Eilenberg-Moore spectral sequence by [12], even if we deal with non-connected spaces by [32].

As a corollary, we obtain:

Fact 2. *In the situations of Fact 1, we have a natural equivalence* $\Sigma^n F_X \sim \bar{N}(\Omega^n X)$ *in the homotopy category of* E*-algebras.*

This assertion can also be proved by using models of Postnikov towers in the category of E-algebras (we also refer to [25] for the definition of this model). This finer argument would show that assumptions on lower homotopy groups $\pi_m(X)$, for $m < n$, are unnecessary and can be dropped. Thus the assertion of Fact 2 holds under the finiteness assumption (2) of Fact 1 as long as the group $\pi_n(X)$ is trivial in the case $\Bbbk = \mathbb{Q}$, a finite p-group in the case of a finite field $\Bbbk = \mathbb{F}_p$.

As regards the iterated bar complex, the existence of weak-equivalences $B^n(\bar{N}^*(X)) \sim B^n(F_X) \sim \Sigma^n F_X$ and Fact 2 imply:

Theorem 3. *Under the assumptions of Fact 1, we have a natural isomorphism*

$$H^*(B^n \bar{N}^*(X)) \simeq \bar{H}^*(\Omega^n X, \Bbbk),$$

for every $n \geq 1$. □

In the case $\Bbbk = \mathbb{F}_p$, one can use the classical Bousfield-Kan tower $\{R_s X\}$ to improve Theorem 3.

Recall simply that $\bar{H}^*(X, \mathbb{F}_p) \simeq \operatorname{colim}_s \bar{H}^*(R_s X, \mathbb{F}_p)$ (see [9, Proposition III.6.5] and [11]). Equivalently, the natural morphism $\operatorname{colim}_s \bar{N}^*(R_s X) \to \bar{N}^*(X)$ defines a weak-equivalence in the category of E-algebras. Note that the bar complex commutes with sequential colimits so that the natural morphism $\operatorname{colim}_s B^n(\bar{N}^*(R_s X)) \to B^n(\bar{N}^*(X))$ defines a weak-equivalence as well. Recall also that the spaces $R_s X$ satisfy assumptions $(1)_p$ and (2) of Fact 1 if the cohomology modules $H^*(X, \mathbb{F}_p)$ are degreewise finite (this folk assertion follows from a standard application of the spectral sequence of [9, §X.6]).

Hence Theorem 3 implies:

Theorem 4. *Let* X *be a pointed space whose cohomology modules* $H^*(X, \mathbb{F}_p)$ *are degreewise finite. Let* $R_s X$ *denote Bousfield-Kan' tower of* X *(where* $R = \mathbb{F}_p$*). Then we have a natural isomorphism*

$$H^*(B^n \bar{N}^*(X)) \simeq \operatorname{colim}_s \bar{H}^*(\Omega^n R_s X, \mathbb{F}_p),$$

for every $n \geq 1$. □

This result can be improved in good cases. For instance, if X is a nilpotent space whose homotopy groups are degreewise finitely generated, then theorems of [32] imply:

$$\operatorname{colim}_s \bar{H}^*(\Omega^n_0 R_s X, \mathbb{F}_p) = \bar{H}^*(\Omega^n_0 R_\infty X, \mathbb{F}_p),$$

where $R_\infty X$ refers to Bousfield-Kan' p-completion of X and $\Omega^n_0 Y$ denotes the connected component of the base point of $\Omega^n Y$, for any pointed space Y. Observe also that $\Omega^n Y \sim \pi_n(Y) \times \Omega^n_0 Y$ and note that

$$\operatorname{colim}_s \bar{H}^0(\Omega^n R_s X, \mathbb{F}_p) = \operatorname{colim}_s \mathbb{F}_p^{\pi_n(R_s X)} = \mathbb{F}_p^{\pi_n(R_\infty X)^\wedge_p},$$

where the notation $\mathbb{F}_p^{\pi_n(R_\infty X)^\wedge_p}$ refers to the module of maps $\alpha : \pi_n(R_\infty X) \to \mathbb{F}_p$ which are continuous with respect to the p-profinite topology (see [9, §§III-VI], see also [30] for a conceptual setting to do p-profinite topology).

Acknowledgements

I am grateful to several readers for many comments on the exposition of this work, at various stages of preparation of the article. Muriel Livernet and Haynes Miller motivated me to give a comprehensive survey of the background in order to make the paper more accessible and to improve the presentation of my results. I thank Geoffrey Powell for a careful reading of a preliminary version of the article and valuable observations on the writing. I thank David Chataur, Mark Hovey, Jean-Louis Loday, Daniel Tanré, for checking the final manuscript and further accurate observations which helped me to put the finishing touches.

References

[1] J.F. Adams, *On the cobar construction*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 409–412.

[2] J.F. Adams, P.J. Hilton, *On the chain algebra of a loop space*, Comment. Math. Helv. **30** (1956), 305–330.

[3] H. Baues, *The double bar and cobar constructions*, Compositio Math. **43** (1981), 331–341.

[4] H. Baues, *Geometry of loop spaces and the cobar construction*, Memoirs of the American Mathematical Society **25**, American Mathematical Society, 1980.

[5] H. Baues, *The cobar construction as a Hopf algebra*, Invent. Math. **132** (1998), 467–489.

[6] C. Berger, B. Fresse, *Combinatorial operad actions on cochains*, Math. Proc. Camb. Philos. Soc. **137** (2004), 135–174.

[7] C. Berger, I. Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. **78** (2003), 805–831.

[8] L. Billera, M. Kapranov, B. Sturmfels, *Cellular strings on polytopes*, Proc. Amer. Math. Soc. **122** (1994), 549–555.

[9] A.K. Bousfield, D.M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics **304**, Springer-Verlag, 1972.

[10] G. Carlsson, J. Milgram, *Stable homotopy and iterated loop spaces*, *in* "Handbook of algebraic topology", North-Holland (1995), 505–583.

[11] E. Dror, *Pro-nilpotent representation of homotopy types*, Proc. Amer. Math. Soc. **38** (1973), 657–660.

[12] W. Dwyer, *Strong convergence of the Eilenberg-Moore spectral sequence*, Topology **13** (1974), 255–265

[13] B. Fresse, *Koszul duality of operads and homology of partition posets*, *in* "Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory (Evanston, 2002)", Contemp. Math. **346**, Amer. Math. Soc. (2004), 115–215.

[14] B. Fresse, *Modules over operads and functors*, Lecture Notes in Mathematics **1967**, Springer-Verlag, 2009.

[15] B. Fresse, *Operadic cobar constructions, cylinder objects and homotopy morphisms of algebras over operads*, *in* "Alpine perspectives on algebraic topology (Arolla, 2008)", to appear.

[16] B. Fresse, *Iterated bar complexes of E-infinity algebras and homology theories*, preprint `arXiv:0810.5147` (2008).

[17] V. Hinich, *Homological algebra of homotopy algebras*, Comm. Algebra **25** (1997), 3291–3323.

[18] V. Hinich, V. Schechtman, *On homotopy limit of homotopy algebras*, *in* "K-theory, arithmetic and geometry (Moscow, 1984–1986)", Lecture Notes in Mathematics **1289**, Springer-Verlag (1987), 240–264.

[19] P. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs **99**, American Mathematical Society, 2003.

[20] M. Hovey, *Model categories*, Mathematical Surveys and Monographs **63**, American Mathematical Society, 1999.

[21] M. Hovey, *Monoidal model categories*, preprint `arXiv:math.AT/9803002` (1998).

[22] T. Kadeishvili, S. Saneblidze, *Iterating the bar construction*, Georgian Math. J. **5** (1998), 441–452.

[23] M. Kapranov, Y. Manin, *Modules and Morita theorem for operads*, Amer. J. Math. **123** (2001), 811–838.

[24] M. Karoubi, *Quasi-commutative cochains in algebraic topology*, preprint `arXiv:math.AT/0509268` (2005).

[25] M. Mandell, *E_∞-algebras and p-adic homotopy theory*, Topology **40** (2001), 43–94.

[26] P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics **271**, Springer-Verlag, 1972.

[27] J. McClure, J.H. Smith, *Multivariable cochain operations and little n-cubes*, J. Amer. Math. Soc. **16** (2003), 681–704

[28] J. Milgram, *Iterated loop spaces*, Annals of Math. **84** (1966), 386–403.

[29] H. Miller, *The sullivan conjecture on maps from classifying spaces*, Ann. Math. **120** (1984), 39–87.

[30] F. Morel, *Ensembles profinis simpliciaux et interprétation géométrique du foncteur T*, Bull. Soc. Math. France **124** (1996), 347–373.

[31] J. Rubio, F. Sergeraert, *A program computing the homology group of loop spaces*, SIGSAM Bulletin **25** (1991), 20–24.

[32] B. Shipley, *Convergence of the homology spectral sequence of a cosimplicial space*, Amer. J. Math. **118** (1996), 179–207.

[33] J.R. Smith, *Iterating the cobar construction*, Memoirs of the Amerian Mathematical Society **524**, Amerian Mathematical Society, 1994.

[34] J.R. Smith, *Operads and algebraic homotopy*, preprint `arXiv:math.AT/0004003` (2000).

[35] M. Spitzweck, *Operads, algebras and modules in general model categories*, preprint `arXiv:math.AT/0101102` (2001).