Polynomial behaviour for stable homology of congruence groups

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An ideal in a (unital) ring is the same as a ring without unit: such a (non-unital) ring I can be seen as the two-sided ideal given by the kernel of the augmentation $\mathbb{Z} \ltimes I \twoheadrightarrow \mathbb{Z}$, where $\mathbb{Z} \ltimes I$ is the unital ring obtained by formally adding a unit to I. The framework in the preprint [Dja], on which this talk is reporting, is more general, but most of the ideas and applications are already in this classical setting.

The *congruence groups* associated to I are defined by

$$\Gamma_n(I) := \operatorname{Ker}\left(GL_n(\mathbb{Z} \ltimes I) \twoheadrightarrow GL_n(\mathbb{Z})\right).$$

We look for qualitative properties of the homology of these groups. As in the case of usual linear groups, we have obvious stabilisation maps $H_*(\Gamma_n(I);\mathbb{Z}) \to H_*(\Gamma_{n+1}(I);\mathbb{Z})$: we will deal only with *stable* properties (as in algebraic K-theory), that is, properties of the colimit of this sequence of graded abelian groups. But we have also a richer structure: $H_*(\Gamma_n(I);\mathbb{Z})$ in endowed with a natural action of $GL_n(\mathbb{Z})$ (induced by the conjugation action) which is generally not trivial (even stably). We will later express these structures (and their compatibility properties) in a functorial setting.

Earlier known results

Suslin [Sus95] proved the following striking Theorem (which improves the rational result that he got with Wodzicki in [SW92], with a different method).

Theorem 1 (Suslin 1995). *Let $d > 0$ be an integer*

1. *The following statements are equivalent.*
   - *(a) Stably in n, the action of $GL_n(\mathbb{Z})$ on $H_i(\Gamma_n(I);\mathbb{Z})$ is trivial for $i < d$;*
   - *(b) I is excisive for algebraic K-theory in homological degree $< d$;*
   - *(c) $\operatorname{Tor}_i^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z}) = 0$ for $0 < i < d$.*
2. *There is a natural map $H_d(\Gamma_n(I);\mathbb{Z}) \to \mathfrak{gl}_n\big(\operatorname{Tor}_d^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z})\big)$ (where $\mathfrak{gl}_n(M)$ denotes the matrices $n\times n$ with entries in M) which is $GL_n(\mathbb{Z})$-equivariant, compatible with stabilisation in n, and whose kernel and cokernel bear a trivial $GL_n(\mathbb{Z})$-action stably in n if the previous conditions are fulfiled.*

(Note that $\operatorname{Tor}_1^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z}) \simeq I/I^2$, so the conditions are only seldom fulfiled for $d > 1$; for $d = 1$, the last statement is classical and not hard.)

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Other known results give informations on $H_*(\Gamma_n(I))$ for each homological degree, but only for particular non-unital rings I.

In [Cal15], Calegari proved the following asymptotic polynomial behaviour for homology of classical congruence groups.

Theorem 2 (Calegari 2015). *Let p be a prime number and k, i be non-negative integers. Then*

$$\dim H_k(\Gamma_n(p^i\mathbb{Z});\mathbb{F}_p) = \frac{n^{2k}}{k!} + O(n^{2k-2})$$

Another important recent result (whose methods are completely independent from the ones used to prove both previous Theorems) is due to Putman [Put15], with an input given by an older work by Charney [Cha84]. This result was quickly improved by the systematic use of functorial methods that we will remind now.

Statements in terms of polynomial functors

Let $(\mathcal{C},+,0)$ be a small symmetric monoidal category whose unit 0 is an initial object. For convenience we will assume that the objects of $\mathcal{C}$ are the natural integers and that $+$ is the usual sum on objects. The precomposition by $-+1$ is an exact endofunctor, denoted by τ, of the category $\mathcal{C}$-**Mod** of functors from $\mathcal{C}$ to abelian groups; with Vespa we studied in [DV] the quotient category $\mathbf{St}(\mathcal{C}\text{-}\mathbf{Mod})$ of $\mathcal{C}$-**Mod** obtained by killing the functors which are stably zero, that is, by quotienting out the localising subcategory of $\mathcal{C}$-**Mod** generated by functors F such that the canonical map $F \to \tau(F)$ is zero (equivalently, a functor F is stably zero if and only if $\operatorname*{colim}_{n\in\mathbb{N}} F(n) = 0$).

We introduced two notions of polynomial functor of degree d: a strong one, which captures also unstable phenomena, and a weak one, which depends only of the isomorphism class of the functor in $\mathbf{St}(\mathcal{C}\text{-}\mathbf{Mod})$. For example, a functor in $\mathcal{C}$-**Mod** is weakly polynomial of degree ≤ 0 if and only if it is isomorphic in $\mathbf{St}(\mathcal{C}\text{-}\mathbf{Mod})$ to a constant functor. For the definition of strongly and weakly polynomial functors and properties, we refer to [DV] or to the talk by Vespa in this meeting. Weakly polynomial functors of (weak) degree $\leq d$ (or more precisely, their images in $\mathbf{St}(\mathcal{C}\text{-}\mathbf{Mod})$) form a localising subcategory of $\mathbf{St}(\mathcal{C}\text{-}\mathbf{Mod})$ denoted by $\mathcal{P}ol_d(\mathcal{C}\text{-}\mathbf{Mod})$. For example, $\mathfrak{gl}_\bullet(M)$ is a strongly polynomial functor of degree 2 in $\mathbf{S}(\mathbb{Z})$-**Mod** (where $\mathbf{S}(\mathbb{Z})$ is defined just below), for any abelian group M.

We are interested here in the following monoidal categories $\mathcal{C}$ with the previous properties: the category **FI** for which $\mathbf{FI}(n,m)$ is the set of injections from $\mathbf{n} := \{1,\dots,n\}$ to $\mathbf{m}$ (the monoidal structure being given by disjoint union) and the category $\mathbf{S}(R)$, where R is a unital ring, for which

$$\mathbf{S}(R)(n,m) := \{(f,g)\in \operatorname{Hom}_R(R^n,R^m)\times\operatorname{Hom}_R(R^m,R^n)\,|\,g\circ f=\operatorname{Id}\}$$

(the monoidal structure being given by direct sum). These categories are also *homogeneous categories* in the sense of Randal-Williams and Wahl [RWW] (a very general framework which is related to the one used at the beginning of [DV10]).

For any unital ring R, $n\mapsto GL_n(R)$ defines a functor $GL_\bullet(R)$ from $\mathbf{S}(R)$ to the category of groups. If I is a non-unital ring, $n\mapsto \Gamma_n(I)$ is a subfunctor of $GL_\bullet(\mathbb{Z}\ltimes I)$. By taking the homology, we get a functor $H_d(\Gamma_\bullet(I))$ in $\mathbf{St}(\mathbf{S}(\mathbb{Z}\ltimes I)\text{-}\mathbf{Mod})$ for each d, which lives indeed in $\mathbf{S}(\mathbb{Z})$-**Mod** (because inner automorphisms act trivially in homology). By restricting it along the canonical monoidal functor $\mathbf{FI}\to\mathbf{S}(\mathbb{Z})$, several authors, improving Putman [Put15], showed that $H_d(\Gamma_\bullet(I))$ is strongly polynomial for each d if the ring I is nice enough — see Church-Ellenberg-Farb-Nagpal [CEFN14] of Church-Ellenberg [CE17]. Recently, Church-Miller-Nagpal-Reinhold [CMNR] got the following result, always by using **FI**-modules.

Theorem 3 (Church-Miller-Nagpal-Reinhold, preprint 2017)**.** *If I is an ideal in a unital ring R satisfying Bass condition (SR_{r+2}), then for each non-negative integer d, $H_d(\Gamma_\bullet(I);\mathbb{Z})$ is a weakly polynomial functor of (weak) degree $\leq 2d+r$.*

In [Dja], the following stronger result is proven.

Theorem 4. *Let I be a ring without unit and $e>0$ an integer such that $\mathrm{Tor}_i^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z})=0$ for $0<i<e$ (for example, $e=1$).*

1. *For each integers $r,d\geq 0$ and each object F in $\mathcal{P}ol_r(\mathbf{S}(\mathbb{Z}\ltimes I)\text{-}\mathbf{Mod})$, the functor $H_d(\Gamma_\bullet(I);F)$ belongs to $\mathcal{P}ol_{2[d/e]+r}(\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod})$ (where the brackets denote the floor function).*

2. *If e is odd (respectively even), then for each integer $n\geq 0$, $H_{ne}(\Gamma_\bullet(I);F)$ is isomorphic in the quotient category $\mathcal{P}ol_{2n}(\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod})/\mathcal{P}ol_{2n-2}(\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod})$ to $\Lambda^n(\mathfrak{gl}_\bullet(\mathrm{Tor}_e^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z})))$ (resp. $S^n(\mathfrak{gl}_\bullet(\mathrm{Tor}_e^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z})))$), where Λ^n (resp. S^n) denotes the n-th exterior (resp. symmetric) power (over the integers).*

For $n=1$, the second part of this theorem is equivalent to Suslin's Theorem 1.

Ingredients of the proof

The input of the proof of Theorem 4 is a version in degree 0 with *twisted* coefficients: one has an (easy) stable natural isomorphism $H_0(\Gamma_\bullet(I);F)\simeq\Phi_*(F)$ in $\mathbf{St}(\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod})$ for any functor F in $\mathbf{S}(\mathbb{Z}\ltimes I)\text{-}\mathbf{Mod}$, where $\Phi:\mathbf{S}(\mathbb{Z}\ltimes I)\to\mathbf{S}(\mathbb{Z})$ denotes the reduction modulo the ideal I and Φ_* the left Kan extension along Φ.

One can then derive this isomorphism (even in a quite more general framework) to get a stable spectral sequence

$$E^2_{i,j}=\mathbf{L}_i\big((-\underset{\oplus}{\otimes}H_j(\Gamma_\bullet(I);\mathbb{Z}))\circ\Phi_*\big)(F)\Rightarrow H_{i+j}(\Gamma_\bullet(I);F)$$

where $\underset{\oplus}{\otimes}:(\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod})\times(\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod})\to\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod}$ is the composition of the external tensor product with the left Kan extension along the direct sum functor $\mathbf{S}(\mathbb{Z})\times\mathbf{S}(\mathbb{Z})\to\mathbf{S}(\mathbb{Z})$.

When F factorises through $\Phi:\mathbf{S}(\mathbb{Z}\ltimes I)\to\mathbf{S}(\mathbb{Z})$, the abutment $H_*(\Gamma_\bullet(I);F)$ of the spectral sequence can be expressed simply from F and $H_*(\Gamma_\bullet(I);\mathbb{Z})$, thanks to the universal coefficients exact sequence for group homology. So the spectral sequence gives informations on $H_*(\Gamma_\bullet(I);\mathbb{Z})$. One needs several steps to show the wished result with this program, especially:

- a comparison theorem of stable (in the sense of categories **St** introduced above!) derived categories of $\mathbf{S}(\mathbb{Z})$-**Mod** and $\mathbf{F}(\mathbb{Z})$-**Mod**, where $\mathbf{F}(\mathbb{Z})$ denotes Quillen's category of factorizations of free abelian groups of finite rank, on (weakly) polynomial functors. This is inspired by Scorichenko's thesis [Sco00].

- A study of the left derived functors of Φ_* on polynomial functors (using the first step);

- a study of the tensor product $\underset{\oplus}{\otimes}$ and its left derivatives on polynomial functors (also using the first step);

- a concrete argument of triangular groups inspired by Suslin-Wodzicki [SW92];

- some functorialities of the above spectral sequence.

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