On the stable homology of orthogonal groups over finite fields with twisted coefficients

(talk at the Skye conference on Algebraic Topology, Group Theory and Representation Theory)

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Aim : to compute homology groups of the following form :

$$H_i(O_{n,n}(\mathbb{F}_q); F(\mathbb{F}_q^{2n})) \quad \text{for } n \text{ big enough}$$

where F is a suitable functor between $\mathbb{F}_q$-vector spaces.

(Here q is a power of an *odd* prime.)

Stable homology of orthogonal groups

Let k be a field. We say that an endofunctor F of the category of finite-dimensional k-vector spaces is *polynomial* if the function $n \mapsto \dim F(k^n)$ is polynomial. (Standard examples : tensor, extorior, divided or symmetric powers.)

Theorem (Charney). *Let F be such a polynomial functor. For each integer i, the canonical map*
$H_i(O_{n,n}(k); F(k^{2n}) \to H_i(O_{n+1,n+1}(k); F(k^{2n+2}))$
is an isomorphism for n big enough.

These groups are called *stable homology of orthogonal groups with coefficients in F* and denoted by

$$H_i(O_\infty(k); F_\infty)$$

(because they are isomorphic to
$H_i(\operatorname{colim}_n O_{n,n}(k); \operatorname{colim}_n F(k^{2n}))$.

Case of untwisted coefficients
(for a finite field)

Before our work, the only known result seemed to be the case of constant coefficients :

Theorem (Fiedorowicz-Priddy). *Suppose that k is a finite field. Then $H_i(O_\infty(k); k) = 0$ for all integer $i > 0$.*

An example : divided powers

Even the degree 0-case (computation of coinvariants) is not completely trivial. Let us denote by $\Gamma^d = (T^d)^{\Sigma_d}$ (invariants of the d-th tensor power under the action of the symmetric group).

Suppose that q is an odd prime power. The following proposition is an exercise, which does not require any powerful tool but is not totally obvious.

Proposition. *$H_0(O_\infty(\mathbb{F}_q); \Gamma^\bullet_\infty)$ can be endowed with a graded Hopf algebra structure isomorphic to $\Gamma(V_q)$, where the graded vector space V_q has dimension 1 in degree $q^s + 1$ (for all integer $s \geq 0$), 0 elsewhere.*

With our method, you can show the following theorem :

Theorem (Djament-Vespa). $H_*(O_\infty(\mathbb{F}_q); \Gamma^\bullet_\infty)$ *can be endowed with a bigraded Hopf algebra structure isomorphic to* $\Gamma(E_q)$, *where the bigraded vector space* E_q *has dimension* 1 *in bidegree* $(2q^s m, q^s + 1)$ *(for all integers* $s \geq 0$ *and* $m \geq 0$ *; the first degree is the homological one),* 0 *elsewhere.*

(Dually, it says that the stable cohomology of orthogonal groups of $\mathbb{F}_q$ with coefficients in a polynomial algebra is an explicit polynomial algebra.)

An inspiring precedent : the stable homology of general linear groups

Let k be a finite field. We denote by $\mathcal{F}(k)$ be the category of functors from finite-dimensional k-vector spaces to k-vector spaces.

Theorem (Betley). *Let $F \in \mathcal{F}(k)$ be a polynomial functor. Then $H_i(GL_\infty(k); F_\infty)$ is naturally isomorphic to $F(0)$ for $i = 0$ and is 0 for $i > 0$.*

The following result is a (dual) generalisation of Betley's theorem. It was extended in a suitable form to any ring by Scorichenko.

Theorem (Betley, Suslin). *Let F and G be polynomial functors in $\mathcal{F}(k)$. Then the canonical map*

$$\mathrm{Ext}^*_{\mathcal{F}(k)}(F, G) \to \mathrm{Ext}^*_{k[GL_\infty(k)]}(F_\infty, G_\infty)$$

is an isomorphism.

Our main result

We suppose always that k is a finite field (possibly of characteristic 2.)

Theorem (Djament-Vespa). *Let $F \in \mathcal{F}(k)$ be a polynomial functor. There is a natural (graded) isomorphism*

$$H_*(O_\infty(k); F_\infty) \simeq \mathrm{Tor}_*^{\mathcal{E}^f(k)}(V \mapsto k[S^2(V^*)], F).$$

Notations used in the theorem : V^* denotes the dual of V, $S^2(V^*)$ the second symetric power on V^* (that is the vector space of quadratic forms on V !). If E is a set, we denote by $k[E]$ the k-vector space with basis E.

$\mathcal{E}^f(k)$ is the (essentially) small category of finite-dimensional k-vector spaces. So if you don't want to speak of torsion groups on small categories, dualize the assertion : the dual of the torsion group of the theorem identifies naturally with $\mathrm{Ext}^*_{\mathcal{F}(k)}(F, V \mapsto k[S^2(V^*)]^*)$.

Why are these torsion groups computable (for F a nice polynomial functor and $char(k) \neq 2$) ?

Suppose that $E^\bullet$ is a *graded exponential functor* in $\mathcal{F}(k)$, what means that E^i preserves finite-dimensional vector spaces for all i and there exists a graded natural isomorphism

$$E^\bullet(U \oplus V) \simeq E^\bullet(U) \otimes E^\bullet(V).$$

(The divided — or symmetric, or exterior — powers satisfy this property.)

Then it is easy to see that the extension group

$$\mathrm{Ext}^*_{\mathcal{F}(k)}(E^\bullet, V \mapsto k[T^2(V^*)]^*)$$

is isomorphic to

$$\mathrm{Ext}^*_{\mathcal{F}(k)}(E^\bullet, DE^\bullet),$$

where D is the duality functor of $\mathcal{F}(k)$ given by $D(F)(V) = F(V^*)^*$. (D exchanges divided and symmetric powers.)

The extension groups $\mathrm{Ext}^*_{\mathcal{F}(k)}(\Gamma^i, S^j)$ have been computed (with all their structure) by Franjou-Friedlander-Scorichenko-Suslin (*Annals of math.* 1999). So

$$\mathrm{Ext}^*_{\mathcal{F}(k)}(\Gamma^\bullet, V \mapsto k[T^2(V^*)]^*)$$

is known.

Now, if you suppose that k has odd characteristic, the second tensor power T^2 splits as $S^2 \oplus \Lambda^2$. In particular, $k[S^2(V^*)]^*$ is an explicit functorial direct factor of $k[T^2(V^*)]^*$. This allows to compute

$$\mathrm{Ext}^*_{\mathcal{F}(k)}(\Gamma^\bullet, V \mapsto k[S^2(V^*)]^*)$$

and so the stable homology of orthogonal groups with coefficients in divided powers.

Ingredients of the proof of the main theorem (I)

Let us introduce first some auxiliary (essentially) small categories.

- $\mathcal{E}^f_{inj}$ is the subcategory of $\mathcal{E}^f(k)$ with the same objects and injections as morphisms.

- $\mathcal{E}^{deg}_q$ is the category finite-dimensional k-quadratic spaces (possibly degenerate) with linear *injections* preserving quadratic structures as morphisms.

- $\mathcal{E}_q$ is the full subcategory of $\mathcal{E}^{deg}_q$ of non-degenerate quadratic spaces.

Ingredients of the proof of the main theorem (II) : the three steps

1. There is a natural isomorphism

$$H_*(O_\infty(k); F_\infty) \simeq H_*(\mathcal{E}_q; F).$$

   It is not too difficult (rather formal form Witt's theorem).

2. The inclusion of categories $\mathcal{E}_q \to \mathcal{E}_q^{deg}$ induces an isomorphism

$$H_*(\mathcal{E}_q; F) \simeq H_*(\mathcal{E}_q^{deg}; F)$$

   if F is polynomial. This is the hardest part of the proof (see next slides).

3. We have natural isomorphisms

$$H_*(\mathcal{E}_q^{deg}; F) \simeq \mathrm{Tor}_*^{\mathcal{E}^f_{inj}}(V \mapsto k[S^2(V^*)], F)$$

   (easy adjunction argument)

$$\simeq \mathrm{Tor}_*^{\mathcal{E}^f}(V \mapsto k[S^2(V^*)], F)$$

(this is a corollary of a key step of Suslin's proof of his result on stable homology of general linear groups).

Ingredients of the proof of the main theorem (III) : overview of the second step

For formal reasons, there is a convergent spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{\mathcal{E}_q^{deg}}(L_q, F) \Rightarrow H_{p+q}(\mathcal{E}_q; F)$$

where L_q is the value on the constant functor k of the q-th derived functor of an adjoint to the precomposition by the inclusion functor $\mathcal{E}_q \to \mathcal{E}_q^{deg}$.

The second step is showed by proving that

$\mathrm{Tor}_p^{\mathcal{E}_q^{deg}}(L_q, F) = 0$ for $q > 0$ and F polynomial.

To make this, we use the two following observations :

1. $L_q(0) = 0$ for $q > 0$;

2. L_q sends any inclusion $V \to V \perp H$, where H is *non-degenerate*, on an isomorphism.

Proposition. *The fraction category obtained from $\mathcal{E}_q^{deg}$ by inverting the arrows of the previous sort is equivalent to the category $Sp(\mathcal{E}_{inj}^f)$; an equivalence is given by the radical.*

Définition (Bénabou). The category $Sp(\mathcal{C})$ of *spans* on a category $\mathcal{C}$ with fibered products is the category with the same objects as $\mathcal{C}$, where a morphism from X to Y is an equivalence class of diagram of $\mathcal{C}$

$$[X \leftarrow A \rightarrow Y] = \begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \\ X & & \end{array}$$

(the equivalence relation identifying two diagrams $[X \leftarrow A \rightarrow Y]$ and $[X \leftarrow A' \rightarrow Y]$ when there is an isomorphism from A to A' making the obvious diagram to commute) ; the composition of $[X \leftarrow A \rightarrow Y]$ and $[Y \leftarrow B \rightarrow Z]$ is $[X \leftarrow A \underset{Y}{\times} B \rightarrow Z]$.

(So a (non-linear) Mackey functor from $\mathcal{C}$ is an ordinary functor from $Sp(\mathcal{C})$.)

After this, we conclude by using the following decomposition result, which translates well-known facts on the structure of Mackey functors :

Proposition. *If the field k is finite, the category of functors from $Sp(\mathcal{E}^f_{inj})$ to k-vector spaces is equivalent (in an explicit way) to the product of the categories of $k[GL_n(k)]$-modules for all integers $n \geq 0$.*