Stable homology of general linear groups with polynomial coefficients

Notes of a course given to Tokyo's advanced students

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$\mathbb{k}$ denotes a commutative ground ring.

1 Main theorem and strategy of proof

Let us begin with a few classical definitions.

If $\mathcal{C}$ is a category, a *bifunctor* on $\mathcal{C}$ is a functor on the category $\mathcal{C}^e := \mathcal{C}^{op} \times \mathcal{C}$ (e stands for *enveloping*). We denote also by $\mathbf{F}(\mathcal{C})$ the category whose objects are arrows of $\mathcal{C}$ and whose morphisms from $a \xrightarrow{f} b$ to $a' \xrightarrow{f'} b'$ are pairs of morphisms $(a \xrightarrow{u} a', b' \xrightarrow{v} b)$ of $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle u}\downarrow & & \uparrow{\scriptstyle v} \\ a' & \xrightarrow{f'} & b' \end{array}$$

commutes ($\mathbf{F}(\mathcal{C})$ is Quillen's category of factorizations [1]). We have a canonical functor $\pi_{\mathcal{C}} : \mathbf{F}(\mathcal{C}) \to \mathcal{C}^e$ sending an object $a \xrightarrow{f} b$ on (b, a) and an arrow $(a \xrightarrow{u} a', b' \xrightarrow{v} b)$ on (v, u).

Let us introduce also the category $\mathbf{S}(\mathcal{C})$ with the same objects as $\mathcal{C}$ and morphisms $a \to b$ pairs $(a \xrightarrow{u} b, b \xrightarrow{v} a)$ of morphisms of $\mathcal{C}$ such that $v \circ u = \mathrm{Id}_a$ (in all these categories, composition is induced by the one of $\mathcal{C}$). We have a canonical functor $\mathbf{S}(\mathcal{C}) \to \mathbf{F}(\mathcal{C})$ which maps an object a on Id_a and is the identity on arrows. This functor is fully faithful.

If $\mathcal{C}$ is a small category, we define *Hochschild homology* of $\mathcal{C}$ with coefficients into a bifunctor B of $\mathcal{F}(\mathcal{C}^e; \mathbb{k})$ as follows. In degree 0, $HH_0(\mathcal{C}; B)$ is by definition the coend of B. As it defines a right-exact functor in B, we can left-derive it, to get our homological functor $HH_*(\mathcal{C}; -)$. Another equivalent way to define it is:

$$HH_*(\mathcal{C}; B) = H_*(\mathbf{F}(\mathcal{C}); \pi_{\mathcal{C}}^* B).$$

When B is a *separable* bifunctor $F \boxtimes G$, $HH_*(\mathcal{C}; B)$ is naturally isomorphic to $\mathrm{Tor}_*^{\mathcal{C}}(F, G)$ if F or G takes $\mathbb{k}$-flat values.

The aim of this talk is to sketch the proof of the following important result, proven in [2].

Theorem 1.1 (Scorichenko). *Let R be a ring, $\mathbf{P}(R)$ be the category of left R-modules R^n for $n \in \mathbb{N}$ and F be a polynomial functor in $\mathcal{F}(\mathbf{P}(R)^e; \Bbbk)$. Then one has a natural isomorphism*

$$H_*(GL_\infty(R); F_\infty) \simeq HH_*(GL_\infty(R) \times \mathbf{P}(R); F)$$

of graded $\Bbbk$-modules, where $GL_\infty(R)$ acts trivially on F on the right-hand side.

The first step is the following consequence of the general framework that we gave in a previous talk.

Proposition 1.2. *Let R a ring and F an object of $\mathcal{F}(\mathbf{S}(R); \Bbbk)$. One has a natural isomorphism*

$$H_*(GL_\infty(R); F_\infty) \simeq H_*(GL_\infty(R) \times \mathbf{S}(R); F)$$

of graded $\Bbbk$-modules, where $GL_\infty(R)$ acts trivially on F on the right-hand side.

Theorem 1.1 is a corollary of this result and the following one, whose proof will take the remaining time of this lecture.

Theorem 1.3. *Let $\mathcal{A}$ be a small additive category and F be a polynomial functor in $\mathcal{F}(\mathcal{A}^e; \Bbbk)$. Then the canonical functor $\iota : \mathbf{S}(\mathcal{A}) \to \mathbf{F}(\mathcal{A})$ induces an isomorphism with coefficients into F (precomposed with the canonical functor $\mathbf{F}(\mathcal{A}) \to \mathcal{A}^e$).*

2 Scorichenko's criterium

The beautiful idea of Scorichenko to deal with homological cancellation with polynomial coefficients (on a small additive category $\mathcal{A}$) is to change the definition of cross-effects. We get so *natural transformations* which are very useful.

Let E be a finite set. Let us denote by t_E the endofunctor $a \mapsto a^{\oplus E}$ of $\mathcal{A}$. Note that, as $\mathcal{A}$ is additive, t_E is self-dual. So, precomposition by t_E is self-dual and induces isormorphisms between extension and torsion groups for functors on $\mathcal{A}$.

If I is a subset of E, let us introduce natural transformations $u_I : Id \to t_E$ and $p_I : t_E \to Id$ whose components (evaluated on an object A) are identity $A \to A$ for indices belongin to I and 0 elsewhere. If F is an object of $\mathcal{F}(\mathcal{A}; \Bbbk)$, we denote by $T_E(F) = t_E^* F (= F \circ t_E)$; assuming that E is a finite *pointed* set, one has natural transformations

$$cr_E^{\mathcal{A},dir}(F) = \sum_{I \in \mathcal{P}(E)} (-1)^{|I|} F(u_I) : F \to T_E(F)$$

(direct (pointed) cross-effect; exponents will be sometimes omitted) and

$$cr_E^{\mathcal{A},inv}(F) = \sum_{I \in \mathcal{P}(E)} (-1)^{|I|} F(p_I) : T_E(F) \to F$$

(inverse (pointed) cross-effect; exponents will also be sometimes omitted).

We have a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Tor}_*^{\mathcal{A}}(X, T_E(F)) & \xrightarrow{\simeq} & \mathrm{Tor}_*^{\mathcal{A}}(T_E(X), F) \\ {\scriptstyle \mathrm{Tor}_*^{\mathcal{A}}(X, cr_E^{\mathcal{A},inv}(F))}\downarrow & \swarrow {\scriptstyle \mathrm{Tor}_*^{\mathcal{A}}(cr_E^{\mathcal{A}^{op},inv}(X), F)} & \\ \mathrm{Tor}_*^{\mathcal{A}}(X, F) & & \end{array}$$

for X and F in $\mathcal{F}(\mathcal{A}^{op}; \Bbbk)$ and $\mathcal{F}(\mathcal{A}; \Bbbk)$ respectively.

Fact. *If F is a polynomial functor of $\mathcal{F}(\mathcal{A}; \Bbbk)$ of degree $\leq d$, then $cr_E^{\mathcal{A},dir}(F)$ and $cr_E^{\mathcal{A},inv}(F)$ are zero for all finite pointed set E with cardinality $\geq d + 2$.*

Theorem 2.1 (Scorichenko)**.** *Let X be a functor of $\mathcal{F}(\mathcal{A}^{op}; \Bbbk)$, $d \in \mathbb{N}$, E a finite pointed set of cardinality $d + 2$. Suppose that the cross-effect $cr_E^{\mathcal{A},inv}(X)$ is a weakly split epimorphism. By weakly split, we mean that the restriction of the natural transformation $cr_E^{\mathcal{A}^{op},inv}(X)$ to the (opposite category of the) subcategory $\mathbf{M}(\mathcal{A})$ of split monomorphisms of $\mathcal{A}$ (the splitting being not given in the structure) is a split epimorphism.*

Then $\mathrm{Tor}_*^{\mathcal{A}}(X, F) = 0$ for each polynomial functor F in $\mathcal{F}(\mathcal{A}; \Bbbk)$ with degree $\leq d$.

Sketch of proof. Define κ_E as the endofunctor of $\mathcal{F}(\mathcal{A}^{op}; \Bbbk)$ kernel of the natural transformation $cr_E^{\mathcal{A}^{op},inv}$ and $\mathcal{K}_d$ to be the class of functors X of $\mathcal{F}(\mathcal{A}^{op}; \Bbbk)$ such that $cr_E^{\mathcal{A}^{op},inv}(X)$ is an epimorphism. One checks, by induction on homological degree, that $\mathrm{Tor}_*^{\mathcal{A}}(X, F) = 0$ for each polynomial functor F in $\mathcal{F}(\mathcal{A}; \Bbbk)$ with degree $\leq d$ if the following condition is satisfied: for each integer $n \geq 0$, $\kappa_E^n(X)$ (κ_E^n denoting the n-th iteration of the endofunctor κ_E) belongs to $\mathcal{K}_d$. (For this, use the long exact sequence associated to the short exact sequence $0 \to \kappa_E(X) \to T_E(X) \xrightarrow{cr_E^{\mathcal{A}^{op},inv}(X)} X \to 0$, if X belongs to $\mathcal{K}_d$, when applying $\mathrm{Tor}_*^{\mathcal{A}}(-, F)$. Note that the morphisms induced by $cr_E^{\mathcal{A}^{op},inv}(X)$ in this long exact sequence are 0 for F polynomial of degree $\leq d$, because of the self-adjunction of cross-effects and the equality $cr_E^{\mathcal{A},inv}(F) = 0$.)

After that, it is enough to show that the assumption on X is preserved when applying the functor κ_E. It is an exercise from the following observations.

1. One has a natural isomorphism $T_E \circ T_E \simeq T_{E \times E}$ and, if γ denotes the involution of this functor swaping the two factors in $E \times E$, the diagram

$$\begin{array}{ccc} T_E \circ T_E & & \\ {\scriptstyle \gamma}\downarrow & \searrow^{T_E(cr_E^{\mathcal{A}^{op},inv})} & \\ T_E \circ T_E & \xrightarrow[cr_E^{\mathcal{A}^{op},inv}(T_E)]{} & T_E \end{array}$$

   commutes.

2. The natural transformation $cr_E^{\mathcal{A}^{op},inv}$ is defined from the natural transformations p_I (for I a pointed subset of E) on $\mathcal{A}^{op}$, which are the same as the natural transformations u_I of $\mathcal{A}$, which are split monomorphisms in $\mathcal{A}$ (as I is non-empty).

□

3 Application to derived Kan extension of $\iota : \mathbf{S}(\mathcal{A}) \to \mathbf{F}(\mathcal{A})$

Let us denote by $\Omega : \mathcal{F}(\mathbf{F}(\mathcal{A})^{op}; \Bbbk) \to \mathcal{F}((\mathcal{A}^e)^{op}; \Bbbk)$ the functor given on objects by

$$\Omega(X)(a,b) = \bigoplus_{f \in \mathcal{A}(b,a)} X(f).$$

This functor is exact and is the left Kan extension along the canonical functor $\mathbf{F}(\mathcal{A})^{op} \to (\mathcal{A}^e)^{op}$.

Thanks to formal results about derived Kan extensions that we saw in a previous talk, the following statement implies Theorem 1.3.

Proposition 3.1. *Let us denote by $L_\bullet$ the graded functor $f \mapsto \tilde{H}_\bullet(\iota \setminus f; \Bbbk)$ in $\mathcal{F}(\mathbf{F}(\mathcal{A})^{op}; \Bbbk)$. Then $\mathrm{Tor}_*^{\mathcal{A}^e}(\Omega(L_\bullet), F) = 0$ for each polynomial functor F of $\mathcal{F}(\mathcal{A}^e; \Bbbk)$.*

Proof. We need several qualitative properties of the functors L_q (whose explicit computation seems completely out of reach).

First of all, note that the direct sum on $\mathcal{A}$ induces a monoidal symmetric structure on $\mathbf{F}(\mathcal{A})$ (and the functor ι is strictly monoidal), whose unit $0 \to 0$ is an initial object. Moreover, one sees easily that, for all objects f of $\mathbf{F}(\mathcal{A})$ and x of $\mathbf{S}(\mathcal{A})$, the map $L_\bullet(f \oplus \iota(x)) \to L_\bullet(f)$ induced by the canonical arrow $f \to f \oplus \iota(x)$ is an *isomorphism*, because the functor $\iota \setminus f \to \iota \setminus (f \oplus \iota(x))$ induced by $- \oplus x$ induces an equivalence in homology (when all idempotents split in $\mathcal{A}$, this functor is an equivalence; the general case goes back to this one by an easy Morita equivalence argument), and there is a natural transformation from the identity to the composition of this functor followed by the functor $\iota \setminus (f \oplus \iota(x)) \to \iota \setminus f$ induced by $f \to f \oplus \iota(x)$.

Note also that $L_\bullet(\iota(x)) = 0$ because the category $\iota \setminus \iota(x)$ has an initial object (given by the identity of x in $\mathbf{S}(\mathcal{A})$).

We will need the following elementary lemma to finish the proof. □

Lemma 3.2. *Let a and b be two objects of $\mathcal{A}$. There exists an object x of $\mathbf{S}(\mathcal{A})$ such that:*

1. *For each arrow $f \in \mathcal{A}(b,a)$, there is a morphism $\alpha : f \to \iota(x)$ in $\mathbf{F}(\mathcal{A})$ whose image in $\mathcal{A}^e$ (through the canonical functor $\mathbf{F}(\mathcal{A}) \to \mathcal{A}^e$) belongs to $\mathbf{M}(\mathcal{A}^e)$ (that is, is a split monomorphism).*

2. *Moreover, for $f = 0$, one can choose α such that each morphism from $b \xrightarrow{0} a$ to an object in the essential image of ι whose underlying morphism of $\mathcal{A}^e$ belongs to $\mathbf{M}(\mathcal{A}^e)$ factorizes through α.*

3. *For each morphism $(a,b) \xrightarrow{\Phi} (r,s)$ of $\mathbf{M}(\mathcal{A}^e)$ and $\psi : (s \xrightarrow{0} r) \to \iota(t)$ of $\mathbf{F}(\mathcal{A})$, there exists a factorization as follows.*

$$\begin{array}{ccccc} f & \xrightarrow{\begin{pmatrix} 1 \\ \Phi \end{pmatrix}} & f \oplus (s \xrightarrow{0} r) & \xrightarrow{\mathrm{Id} \oplus \psi} & f \oplus \iota(t) \\ & \searrow & & \nearrow & \\ & & \iota(x) & & \end{array}$$

Proof. Take $x = a \oplus b$ and

$$\alpha = (b \xrightarrow{\begin{pmatrix} f \\ 1 \end{pmatrix}} a \oplus b, a \oplus b \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} a).$$

□

End of the proof of Proposition 3.1. Let (E, e) a finite pointed set. Let us define, for each object (a, b) of $\mathcal{A}^e$, a map

$$\Omega L_\bullet(a,b) = \bigoplus_{f \in \mathcal{A}(b,a)} L_\bullet(f) \to \Omega L_\bullet(a^{\oplus E}, b^{\oplus E}) = \bigoplus_{g \in \mathcal{A}(b^{\oplus E}, a^{\oplus E})} L_\bullet(g)$$

as follows. For $f \in \mathcal{A}(b, a)$, its restriction to $L_\bullet(f)$ is

$$L_\bullet(f) \simeq L_\bullet(f \oplus \iota(x)) \xrightarrow{L_\bullet(\mathrm{Id} \oplus \alpha)} L_\bullet\big(f \oplus (b^{\oplus E \setminus e} \xrightarrow{0} a^{\oplus E \setminus e})\big)$$

$$\cdots = L_\bullet(b^{\oplus E} \xrightarrow{\pi^* f} a^{\oplus E}) \hookrightarrow \Omega L_\bullet(a^{\oplus E}, b^{\oplus E})$$

where $\pi = p_e : (b,a)^{\oplus E} \to (b,a)$ is the arrow of $\mathcal{A}^e$ given by the projection on the factor labelled by e and $\alpha : (b^{\oplus E \setminus e} \xrightarrow{0} a^{\oplus E \setminus e}) \to \iota(x)$ is an is Lemma 3.2. One can check (using this same lemma) that it does not depend on the choice of α and that it is functorial with respect to split monomorphisms of $\mathcal{A}^e$.

Moreover, the composite of this morphism with the cross-effect $cr_E^{(\mathcal{A}^e)^{op}, inv}(\Omega L_\bullet)(a, b)$ is the identity. Indeed, its composite with $\Omega L_\bullet(u_I)(a, b)$, where I is a subset of E containing e, is the identity for $I = \{e\}$ and 0 else. To see this, we look at the composition (in $\mathbf{F}(\mathcal{A})$)

$$f \xrightarrow{u_I} f \oplus (b^{\oplus E \setminus e} \xrightarrow{0} a^{\oplus E \setminus e}) \xrightarrow{\mathrm{Id} \oplus \alpha} f \oplus \iota(x).$$

- For $I = \{e\}$, this composition is simply the canonical inclusion $f \to f \oplus \iota(x)$, so that the corresponding composition (after applying $\Omega L_\bullet$) gives the identity as wished;
- if I contains strictly $\{e\}$, then $u_I = \begin{pmatrix} 1 \\ u_{I \setminus e} \end{pmatrix}$ and $u_{I \setminus e}$ is a split monomorphism, so that we can apply the last statement of Lemma 3.2 to get that our composition induces 0.

Theorem 2.1 gives then the conclusion. □

References

[1] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[2] Alexander Scorichenko. *Stable K-theory and functor homology over a ring.* PhD thesis, Evanston, 2000.