Fields Institute Communications
Volume **44**, 2004

Convex Rearrangements of Random Elements

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Dedicated to Miklós Csörgő on the occasion of his 70th birthday

Abstract. We survey results concerning convex rearrangements, subsequently called convexifications, of stochastic processes. In particular, we discuss consistency, laws of the iterated logarithm, weak convergence of convexifications in various settings including real-valued and multivariate stochastic processes with compact and infinite time intervals. We also survey results concerning functionals of convexifications such as indices of fluctuations of stochastic processes. Throughout the paper we note connections of convexifications and their functionals with various areas of research such as Mathematics and Economics. In particular, we discuss relationships of convexifcations with the operators of monotone and convex rearrangements in Functional Analysis as well as with the generalized Lorenz curves in Econometrics. Finally, we briefly discuss results concerning closely related topics such as the Vervaat process and Baxter-type theorems.

1 Introduction

In this paper we survey results concerning convex rearrangements of stochastic processes and other closely related topics. Convexifications are obtained by arranging a finite number of increments of a process in ascending order, then considering partial sums of the ordered values as the y-coordinates of two dimensional points with equally spaced x-coordinates, and finally connecting the points using straight

2000 *Mathematics Subject Classification.* Primary 60G10, 60G15; Secondary 62E20, 62G30.
Youri Davydov was supported in part by the RFBR Grant 99-01-00112.
Ričardas Zitikis was supported in part by an NSERC of Canada individual research grant.

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lines. Since the y-coordinates are nondecreasing, the resulting curve is convex and, therefore, is called a convex rearrangement or, shortly, a convexification of the process under consideration.

It is of theoretical and practical interest to investigate asymptotic properties of convexifications when the number of the aforementioned x-coordinates tends to infinity. In this paper we discuss laws of large numbers (LLN), laws of the iterated logarithm (LIL), and weak convergence of convexifications in various settings such as real-valued and multivariate stochastic processes with compact and infinite time intervals. It should be noted that in the case of infinite time intervals the x-coordinates form a sequence, whereas in the case of compact time intervals the x-coordinates form a triangular array. Therefore, even though we restrict ourselves to stochastic process with stationary increments, investigations of strong (i.e. almost surely) limit theorems for convexifications usually require the use of different approaches in the two cases. One of the reasons for such differences is the fact that the use of the Birkhoff-Khintchin Ergodic Theorem becomes troublesome in the case of triangular arrays whereas it is well applicable in the case of sequences.

In this paper we also survey results concerning various functionals of convexifications. The functionals can be used, for example, for constructing estimators of parameters of stochastic processes and for measuring the fluctuations of the processes.

Throughout the paper we also note connections between convexifications and other areas of research such as Mathematics and Economics. Specifically, we discuss relationships of convexifcations with the operators of monotone and convex rearrangements in Functional Analysis, as well as with the generalized Lorenz curves in Econometrics. We note in passing that the indices of fluctuations of stochastic processes are related to indices of economic inequality and, particularly, to the E-Gini index of Econometrics.

We end the paper with brief discussions of closely to the topic of this paper related results concerning the Vervaat process and Baxter-type theorems.

2 One-dimensional convexifications: definition

In this section we define convexifications of real-valued processes with discrete and continuous time. We also demonstrate that the convexifcations can be represented as generalized Lorenz curves, a fact that we frequently use in subsequent sections.

2.1 Discrete time processes. Let $\xi = (\xi_0, \xi_1, \xi_2, \dots)$ be a discrete time stochastic process such that $\xi_0 = 0$ almost surely. For any $i = 1, 2, \dots$, let X_i be the increment

$$X_i = \xi_i - \xi_{i-1}.$$

Given n, let $X_{1:n} \le \cdots \le X_{n:n}$ be the order statistics of $X_1, \dots, X_n$. We connect the points $(0,0)$, $(k, \sum_{i=1}^{k} X_{i:n})$, $k = 1, 2, \dots, n$ using straight lines and obtain the curve defined by the following formula

$$\mathsf{C}_n(x) := \begin{cases} \sum_{i=1}^{[x]} X_{i:n} + (x - [x])\, X_{[x]+1:n}, & 0 \le x < n, \\ \sum_{i=1}^{n} X_{i:n} \ \left[= \sum_{i=1}^{n} X_i = \xi_n\right], & x = n. \end{cases} \tag{2.1}$$

The function C_n is called the convexification of ξ on the interval $[0, n]$ since it is, indeed, convex on the interval.

Since both the function C_n and its domain of definition $[0,n]$ depend on n, it is natural and convenient to redefine C_n as follows:

$$\mathsf{C}_n^0(t) := \mathsf{C}_n(nt). \tag{2.2}$$

The function C_n^0 is defined on the interval $[0,1]$ and possesses the geometric properties of the original one, C_n. We note in passing that the redefinition of C_n as C_n^0 on $[0,1]$ could be geometrically interpreted as the curve obtained by connecting the points $(0,0)$, $(\frac{k}{n}, \sum_{i=1}^{k} X_{i:n})$, $k = 1,2,\ldots,n$, using straight lines. We call C_n^0 the convexification of ξ on $[0,1]$. If the need arises to specify the dependence of C_n^0 on ξ, we then use the notation $\mathsf{C}_n^0[\xi]$ instead of C_n^0.

2.2 Continuous time processes. Let ξ be a stochastic process on $[0,\infty)$ such that $\xi(0) = 0$ almost surely. For any $i = 1,2,\ldots,n$, let X_i be the increment

$$X_i = \xi(i) - \xi(i-1).$$

Then C_n of (2.1) defines the convexification of ξ on $[0,n]$. Furthermore, equation (2.2) defines the convexification C_n^0 of ξ on $[0,1]$.

If ξ is a stochastic process on $[0,1]$ such that $\xi(0) = 0$ almost surely, then instead of the above defined X_i's we use the increments

$$X_{i,n} = \xi\Big(\frac{i}{n}\Big) - \xi\Big(\frac{i-1}{n}\Big)$$

to define both C_n and C_n^0. The important difference between the present situation and the previous one is that the random variables $X_{i,n}$ are now dependent on n.

When needed, in both $[0,\infty)$ and $[0,1]$ cases we use the notation $\mathsf{C}_n^0[\xi]$ to indicate the dependence of C_n^0 on ξ.

2.3 Generalized Lorenz curves. Given random variables $X_1,\ldots X_n$, let F_n and F_n^{-1} be the empirical and quantile functions (cf., e.g., Csörgő [7]). The corresponding generalized Lorenz curve (GLC) GL_n is defined on $[0,1]$ by the formula (cf. Gastwirth [35])

$$\mathsf{GL}_n(t) := \int_0^t F_n^{-1}(s)ds.$$

It is easy to check that the convexification C_n^0 based on $X_1,\ldots X_n$ satisfies the equation

$$\mathsf{C}_n^0 = n\mathsf{GL}_n. \tag{2.3}$$

We shall use formula (2.3) to reduce investigations of the asymptotic behaviour of C_n^0 to those of the more conveniently tractable GL_n.

3 General one dimensional convexifications: definition

In the previous section we defined convexifications by connecting (arranged) points using straight lines. Instead of the latter ones we could use other lines to connect the points and still have the resulting curve convex. In general, this can be done using (abstract) operators of monotone and convex rearrangements. We shall define and discuss the two operators in the second and third subsections below. The following, first, subsection is designed to demonstrate how the general (abstract) idea of convexifications works in the case of the appropriately reformulated C_n^0.

3.1 Convexification C_n^0 reformulated. Given $X_1, \dots X_n$, let D_n^0 be a piecewise linear function on $[0,1]$ defined by the formula:

$$\mathsf{D}_n^0(t) := \begin{cases} \sum_{i=1}^{[nt]} X_i + (nt - [nt])\, X_{[nt]+1}, & 0 \le t < 1, \\ \sum_{i=1}^{n} X_i, & t = 1. \end{cases}$$

Note that the difference between the curves D_n^0 and C_n^0 is that in the definition of D_n^0 we are summing up the random variables X_i and not their order statistics $X_{i:n}$. This could be interpreted as connecting the points $(0,0)$, $(\frac{k}{n}, \sum_{i=1}^{k} X_k)$, $k = 1, 2, \dots, n$, using straight lines. The function D_n^0 is absolutely continuous with the derivative $\partial \mathsf{D}_n^0(t) = nX_k$ on the interval $(\frac{k-1}{n}, \frac{k}{n})$ for any $k = 1, \dots, n$. Define the operator T by the equality

$$T(\partial \mathsf{D}_n^0) = nF_n^{-1}, \tag{3.1}$$

where F_n^{-1} is the quantile function corresponding to $X_1, \dots X_n$. Using the operator T and equality (2.3), we obtain the following representation

$$\mathsf{C}_n^0(t) = \int_0^t T(\partial \mathsf{D}_n^0)(s)ds \tag{3.2}$$

for the convexification C_n^0 on $[0,1]$. In what follows we use formula (3.2) with an abstract operator T of monotone rearrangement to define a general convexificationion of a (deterministic or random) function f.

3.2 General monotone rearrangements. Let f be a measurable real function on $[0,1]$. Then a function T_f is called *monotone rearrangement* of f if it is measurable, non-decreasing, and such that its distribution with respect to the Lebesgue measure is the same as that of f. The function T_f defines an operator T of montontone rearrangement,

$$T(f) := T_f,$$

on the space of all measurable functions f. To ensure the uniqueness of T, we assume, in addition to the properties above, that the function T_f is left-continuous.

The operator T can also be defined explicitly. Namely, given a measurable function f on $[0,1]$, let λf^{-1} be the distribution of f with respect to the Lebesgue measure λ. Let F_f be the distribution function corresponding to λf^{-1}. Furthermore, let F_f^{-1} be the left-continuous inverse of F_f. Then, the (unique) operator T of monotone rearrangement is defined by the formula

$$T(f) := F_f^{-1}.$$

Note that the equality of the distributions of $T(f)$ and f follows from the fact that $F_f^{-1}(t) \le x$ if and only if $t \le F_f(x)$.

The operator T has been widely used in various areas of Mathematics. For example, Lindenstrauss and Tzafriri [46], Kreĭn, Petunin, and Semenov [45], Stein and Weiss [56] discuss applications of monotone rearrangements in Analysis. Talagrand [57] formulates necessary and sufficient conditions for the continuity of Gaussian processes in terms of monotone rearrangements of covariances. Applications of monotone rearrangements in Probability Theory along with further references on the subject can be found, in particular, in Egorov [31], [32], Zhukova [61], [62], Thilly [58].

3.3 General convex rearrangements. Using the above defined operator T of monotone rearrangement, for any absolutely continuous functions f on $[0,1]$ with $f(0)=0$ we define a function V_f on $[0,1]$ by the formula

$$V_f(t) := \int_0^t T(\partial f)(s)ds.$$

Since V_f is convex, it is called *convex rearrangement* of f. Note also the equality $V_f(t)=f(t)$ when $t=0$ and $t=1$.

Obviously, the function V_f defines the operator V of convex rearrangement,

$$V(f) := V_f,$$

on the space of all absolutely continuous functions such that $f(0)=0$. Note that when $f=\mathsf{D}_n^0$, then $Vf=\mathsf{C}_n^0$, which is equality (3.2). The operator V is not linear but, nevertheless, is such that $V(cf)=cV(f)$ for any positive constant $c>0$. Thus, investigations of the asymptotic behaviour of $b_n^{-1}\mathsf{C}_n^0$ with appropriate normalizing constants $b_n>0$ is equivalent to that of $V(f_n)$ with $f_n=b_n^{-1}\mathsf{D}_n^0$. These and other properties of the operator V were discussed and used by Zhukova [61], Nevzorov and Zhukova [49], Thilly [58].

4 General one-dimensional convexifications: consistency

Davydov [19] demonstrated that convergence of $V(f_n)$ can be reformulated in terms of (a possibly different mode of) convergence of f_n. We now formulate a result by Davydov [19] as the following theorem where for a given function h, we use $\|h\|_\infty$ to denote the supremum of $|h(x)|$ over all x from the domain of definition of h.

Theorem 4.1 (Davydov [19]) *Let (f_n) be a sequence of absolutely continuous functions on $[0,1]$. Then there exists a convex function f such that, when $n\to\infty$,*

$$\|V(f_n)-f\|_\infty \to 0 \tag{4.1}$$

if and only if the following two assumptions hold: 1) *The limit $a := \lim_{n\to\infty} f_n(0)$ exists and is finite.* 2) *There exists a function g on $[0,1]$ such that*

$$\lambda(\partial f_n)^{-1} \Rightarrow \lambda g^{-1} \tag{4.2}$$

and

$$\int_0^1 (\partial f_n)_\pm d\lambda \to \int_0^1 g_\pm d\lambda. \tag{4.3}$$

Furthermore, under the above assumptions the equality $a=f(0)$ holds and one can choose $g=\partial f$.

We note in passing that Davydov [19] formulated the pointwise version of statement (4.1). However, in the context of Theorem 4.1 the uniform and pointwise modes of convergence coincide.

In the special case of convexification C_n^0, the verification of assumptions (4.2) and (4.3) can be reduced to LLN-type results for appropriately normalized increments of the process under consideration. This can easily be seen in the case of assumption (4.3), whereas in the case of assumption (4.2) one should first reformulate (4.2) in terms of the Fourier transforms of the measures and then consider the convergence of the real and imaginary parts of the transforms separately. This approach was used by Davydov [19], Thilly [58], Davydov and Thilly [26].

5 Processes with unbounded time: consistency

In this section we discuss strong consistency of convexifications C_n^0 in the case of either discrete time $\xi = (\xi_0, \xi_1, \xi_2, \dots)$ or continuous time $\xi = \{\xi(x), x \geq 0\}$ processes. We already know from (2.3) that in both cases the convexification C_n^0 can be expressed in terms of the corresponding generalized Lorenz curve GL_n.

Assuming that $X_1, X_2, \dots$ are identically distributed random variables with $\mathbf{E}(|X_1|) < \infty$, it is natural to expect that GL_n converges to the generalized Lorenz curve $\mathsf{GL}_F = \{\mathsf{GL}_F(t) : 0 \leq t \leq 1\}$ defined by the formula (cf. Gastwirth [35])

$$\mathsf{GL}_F(t) := \int_0^t F^{-1}(s)ds,$$

where F is the distribution function of X_1 and F^{-1} is the corresponding quantile function (cf., e.g., Csörgő [7]).

Theorem 5.1 (Goldie [36], Gail and Gastwirth [34]) *Let $(X_1, X_2, \dots)$ be a sequence of independent and identically distributed random variables such that $\mathbf{E}(|X_1|) < \infty$. Then*

$$\|\mathsf{GL}_n - \mathsf{GL}_F\|_\infty \to_{a.s.} 0.$$

We note that Goldie [36] proved strong uniform consistency of GL_n as formulated in Theorem 5.1 above whereas Gail and Gastwirth [34] independently proved strong pointwise consistency of $\mathsf{GL}_n(t)$ for any t in $[0,1]$. Note, however, that the limiting generalized Lorenz curve GL_F is convex and continuous, and thus the notions of pointwise and uniform strong convergencies coincide in this case.

When $X_1, X_2, \dots$ are increments of the Wiener process W on $[0, \infty)$, then Theorem 5.1 holds with $F = \Phi$, where Φ denotes the standard Gaussian distribution function. That is,

$$\|n^{-1}\mathsf{C}_n^0[W] - \mathsf{GL}_\Phi\|_\infty \to_{a.s.} 0. \tag{5.1}$$

Note the following formula

$$\mathsf{GL}_\Phi = -\phi \circ \Phi^{-1},$$

where ϕ is the standard Gaussian density function and $\circ$ denotes the composition of two functions.

In the case of processes with dependent increments such as the fractional Brownian motion we need to use more elaborate results than Theorem 5.1.

Theorem 5.2 (Csörgő and Yu [13]) *Let $(X_1, X_2, \dots)$ be a sequence of random variables with the same marginal distribution function F assumed to be continuous and such that $\mathbf{E}(|X_1|) < \infty$. Furthermore, assume that*

$$\|F_n - F\|_\infty \to_{a.s.} 0,$$

and let the statement

$$\frac{1}{n}\sum_{i=1}^n h(X_i) \to_{a.s.} \mathbf{E}(h(X_1)) \tag{5.2}$$

hold for any function $h \in \{g_\beta, h_\beta : 0 < \beta \leq 1\}$, where $g_\beta(x) := x\mathbf{I}\{x \leq F^{-1}(\beta)\}$ and $h_\beta(x) := x\mathbf{I}\{x \geq F^{-1}(1-\beta)\}$. Then

$$\|\mathsf{GL}_n - \mathsf{GL}_F\|_\infty \to_{a.s.} 0.$$

We note that Csörgő and Yu [13] defined the empirical generalized Lorenz curve GL_n in a slightly different way than that in the present paper. One can easily show, however, that the two definitions are asymptotically equivalent. Furthermore, Csörgő and Yu [13] assumed that the random variables $X_1, X_2, \ldots$ are non-negative, and this is the reason why statement (2.13) of Theorem 2.1 on p. 702 of Csörgő and Yu [13] was formulated under assumption (5.2) only for the functions h from the class $\{h_\beta : 0 < \beta \leq 1\}$. We note, however, that the non-negativity assumption on the random variables $X_1, X_2, \ldots$ is usually assumed because of tradition and not of necessity. Indeed, Lorenz curves have been used in Econometrics to describe distributions of incomes, which are usually non-negative.

The following theorem was proved, but not explicitly formulated, by Davydov and Zitikis [28].

Theorem 5.3 (Davydov and Zitikis [28]**)** *Let $(X_1, X_2, \ldots)$ be a sequence of random variables with the same marginal distribution function F such that $\mathbf{E}(|X_1|) < \infty$. Furthermore, assume that*

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i) \to_{a.s.} \mathbf{E}(h(X_1)) \tag{5.3}$$

for any measurable function h such that $\mathbf{E}(|h(X_1)|) < \infty$. Then

$$||\mathsf{GL}_n - \mathsf{GL}_F||_\infty \to_{a.s.} 0.$$

If the stochastic process $(X_1, X_2, \ldots)$ is strictly stationary and ergodic, then (5.3) holds due to the well known Birkhoff-Khintchin Ergodic Theorem. This is a useful observation especially in the case of Gaussian processes since in this case the strict stationarity of $(X_1, X_2, \ldots)$ follows from the weak stationarity, which is usually easy to verify, and the ergodicity assumption follows from

$$r(n) := \mathrm{Cov}(X_1, X_{1+n}) \to 0, \; n \to \infty, \tag{5.4}$$

which is also usually easy to verify. For example, in the case of the fractional Brownian motion W_α on $[0, \infty)$, statement (5.4) holds for any Hurst parameter $0 < \alpha < 1$. Thus, we have the following result.

Theorem 5.4 (Davydov and Zitikis [28]**)** *Let $W_\alpha = \{W_\alpha(x), x \geq 0\}$ be the fractional Brownian motion on $[0, \infty)$ with the Hurst parameter $0 < \alpha < 1$. Then*

$$||n^{-1}\mathsf{C}_n^0[W_\alpha] - \mathsf{GL}_\Phi||_\infty \to_{a.s.} 0.$$

6 Processes with deterministic noise: consistency

Let $\xi = \{\xi(x), x \geq 0\}$ be a stochastic process such that $\xi(0) = 0$ almost surely, and let $a : [0, \infty) \to \Re$ be a deterministic function. Define a new stochastic process $\eta = \{\eta(x), x \geq 0\}$ by the formula

$$\eta(x) := a(x) + \xi(x).$$

In this section we survey results concerning strong consistency of the convexification $\mathsf{C}_n^0[\eta]$. Note that when divided by n, $\mathsf{C}_n^0[\eta]$ is the generalized Lorenz curve

GL_n^{a+X} defined by the formula

$$\mathsf{GL}_n^{a+X}(t) := \int_0^t (F_n^{a+X})^{-1}(s)ds,$$

where F_n^{a+X} is the empirical distribution function corresponding to the random variables $a_i + X_i$, $i = 1, 2, \ldots, n$ with $a_i = a(i) - a(i-1)$ and $X_i = \xi(i) - \xi(i-1)$.

Given a sequence $(F_0, F_1, F_2, \ldots)$ of distribution functions, we write

$$F_n \Rightarrow_{a.s.} F_0$$

if the Lévy metric between F_n and F_0 converges to 0 almost surely when $n \to \infty$.

Theorem 6.1 (Davydov and Zitikis [29]) *Let $(a_1, a_2, \ldots)$ be a deterministic sequence such that*

$$\sup_n \Big(\frac{1}{n}\sum_{i=1}^n |a_i|^{1+\delta}\Big) < \infty$$

for some $\delta > 0$. Furthermore, let $(X_1, X_2, \ldots)$ be a strictly stationary and ergodic sequence such that $\mathbf{E}(|X_1|) < \infty$, and let there exists a distribution function F_0 such that $\int_{-\infty}^{\infty} |x| dF_0(x) < \infty$ and

$$F_n^{a+X} \Rightarrow_{a.s.} F_0. \tag{6.1}$$

Then

$$||\mathsf{GL}_n^{a+X} - \mathsf{GL}_{F_0}||_\infty \to_{a.s.} 0.$$

Sufficient assumptions for the validity of (6.1) were discussed in detail by Davydov and Zitikis [29], [30]. In particular, it is demonstrated there that if the sequence $(a_1, a_2, \ldots)$ has an asymptotic distribution function G and if $(X_1, X_2, \ldots)$ is a strictly stationary and ergodic sequence with the marginal distribution function F, then (6.1) holds with

$$F_0 = G * F.$$

Interestingly enough, the latter note implies that the influence of the additive deterministic noise $(a_1, a_2, \ldots)$ is exactly the same as in the case of the white noise.

We conclude this section with the note that Davydov and Zitikis [29] also discuss results concerning strong consistency of generalized Lorenz curves GL_n^{aX} based on the coordinatewise product of deterministic $(a_1, a_2, \ldots)$ and strictly stationary ergodic $(X_1, X_2, \ldots)$ sequences.

7 Gaussian processes on $[0,1]$: consistency

Let $\xi = \{\xi(t), 0 \le t \le 1\}$ be a stochastic process with stationary increments such that $\xi(0) = 0$ almost surely, and let $\mathbf{E}(\xi(t)) = 0$ for any $0 \le t \le 1$. Construct C_n^0 using the increments

$$X_{i,n} = \xi\Big(\frac{i}{n}\Big) - \xi\Big(\frac{i-1}{n}\Big),\ i = 1, \ldots, n.$$

Convergence of C_n^0 in probability can be derived from the corresponding results concerning processes with noncompact time intervals since the distributions of the appropriately normalized increments $X_{i,n}$ do not depend on n. For example, this observation and statement (5.1) immediately imply the following theorem of Nevzorov and Zhukova [49], which was originally proved using a different method.

Theorem 7.1 (Nevzorov and Zhukova [49]) *Let* $W = \{W(t), 0 \le t \le 1\}$ *be the standard Wiener process on* $[0,1]$. *Then*

$$||n^{-1/2}\mathsf{C}_n^0[W] - \mathsf{GL}_\Phi||_\infty \to_P 0.$$

The normalizing factor $n^{-1/2}$, and not n^{-1} as in the case of processes on $[0,\infty)$, in the statement of Theorem 7.1 is natural and can be explained as follows. Let GL_n^* be the generalized Lorenz curve based on the *normalized* random variables $X_{i,n}/\sigma_n$, $i = 1, \ldots, n$, where $\sigma_n^2 := \mathbf{E}(X_{1,n}^2) = 1/n$. Then the statement of Theorem 7.2 takes the following form: $||\mathsf{GL}_n^* - \mathsf{GL}_\Phi||_\infty \to_P 0$.

Since the increments $X_{i,n}$ depend on n, investigations of the strong convergence of C_n^0 is not a straightforward extension of techniques used for deriving results of previous sections. For example, in the case of triangular arrays $(X_1, \ldots, X_n)_{n=1}^\infty$ we cannot directly apply the Birkhoff-Khintchin Ergodic Theorem. Thus, the following theorem is a non-trivial generalization of Theorem 7.1.

Theorem 7.2 (Davydov and Vershik [27]) *Let* $W = \{W(t), 0 \le t \le 1\}$ *be the standard Wiener process on* $[0,1]$. *Then*

$$||n^{-1/2}\mathsf{C}_n^0[W] - \mathsf{GL}_\Phi||_\infty \to_{a.s.} 0. \tag{7.1}$$

Statement (7.1) is an analogue of (5.1) in the case of the Wiener process on $[0,1]$. Note the different normalizing constants: n^{-1} in the case of W on $[0,\infty)$, and $n^{-1/2}$ in the case of W on $[0,1]$.

The following theorem extends Theorem 7.2 to general Gaussian processes ξ on $[0,1]$. Let the increments $X_{i,n}$, $i = 1, \ldots, n$, be as above, and let

$$\sigma_n^2 := \mathbf{E}(X_{1,n}^2),$$
$$\rho_{n,k} := \mathbf{E}(X_{1,n}X_{1+k,n})/\sigma_n^2.$$

Theorem 7.3 (Davydov and Thilly [25]) *Let* $\xi = \{\xi(t), 0 \le t \le 1\}$ *be a Gaussian process with stationary increments. Assume that* $\xi(0) = 0$ *almost surely and* $\mathbf{E}(\xi(t)) = 0$ *for any* $0 \le t \le 1$. *If there exist* $c > 0$ *and* $\delta > 0$ *such that*

$$|\rho_{n,k}| \le \min\{1, c/(\ln k)^{1+\delta}\} \tag{7.2}$$

for any k *such that* $3 \le k \le n-1$, *then*

$$||(n\sigma_n)^{-1}\mathsf{C}_n^0[\xi] - \mathsf{GL}_\Phi||_\infty \to_{a.s.} 0.$$

In the case of the fractional Brownian motion W_α on $[0,1]$, the correlation coefficient $\rho_{n,k}$ equals $2^{-1}(v_\alpha(k-1) + v_\alpha(k+1)) - v_\alpha(k)$ with $v_\alpha(t) = t^{2\alpha}$, which is the variance of $W_\alpha(t)$. Thus, $|\rho_{n,k}| \le ck^{2\alpha-2}$, which implies (7.2) for any $0 < \alpha < 1$. Consequently, we have the following theorem.

Theorem 7.4 (Davydov and Thilly [25]) *Let* $W_\alpha = \{W_\alpha(t), 0 \le t \le 1\}$ *be the fractional Brownian motion on* $[0,1]$ *with the Hurst parameter* $0 < \alpha < 1$. *Then*

$$||n^{-(1-\alpha)}\mathsf{C}_n^0[W_\alpha] - \mathsf{GL}_\Phi||_\infty \to_{a.s.} 0.$$

8 Stable processes on $[0,1]$: consistency

Let $S_p = \{S_p(t), 0 \le t \le 1\}$ be a strictly p-stable random process on $[0,1]$ with $1 < p < 2$. That is, S_p is a process with independent and homogeneous increments and such that

$$S_p(t_2) - S_p(t_1) =_d (t_2 - t_1)^{1/p} S_p(1)$$

for any $0 \le t_1 < t_2 \le 1$.

Theorem 8.1 (Nevzorov and Zhukova [49]) *Let S_p be a strictly p-stable random process on $[0,1]$ with $1 < p < 2$. Then*

$$||n^{-(1-1/p)} \mathsf{C}_n^0[S_p] - \mathsf{GL}_{F_p}||_\infty \to_P 0,$$

where F_p is the distribution function of $S_p(1)$.

The almost surely version of Theorem 8.1 was proved by Davydov [19].

Theorem 8.2 (Davydov [19]) *Let S_p be a strictly p-stable random process on $[0,1]$ with $1 < p < 2$. Then*

$$||n^{-(1-1/p)} \mathsf{C}_n^0[S_p] - \mathsf{GL}_{F_p}||_\infty \to_{a.s.} 0.$$

When $p < 1$, then the first absolute moment of the increments of the p-stable process S_p is infinite and thus strong uniform convergence does not hold. For this reason, in the context of two dimensional random walks with increments in the domain of attraction of a p-stable law with $p < 1$, Davydov and Vershik [27] suggested to investigate weak, instead of strong, convergence of convexifications. These ideas were subsequently refined and extended to more general situations by Davydov and Egorov [20], [21].

9 Diffusion processes on $[0,1]$: consistency

Let a and b be two stochastic processes on $[0,1]$ such that $\int_0^1 |a(s)|ds < \infty$ and $\int_0^1 b^2(s)ds < \infty$ almost surely. Furthermore, let b be adapted to the natural filtration $\{\mathcal{F}_t, t \ge 0\}$ where the σ-algebra $\mathcal{F}_t := \sigma(W(s), s \le t)$ is generated by the Wiener process W on $[0,1]$. Under these assumptions we define the Itô-Wiener process $\xi = \{\xi(t), 0 \le t \le 1\}$ by the formula

$$\xi(t) = \int_0^t a(s)ds + \int_0^t b(s)dW(s). \tag{9.1}$$

Theorem 9.1 (Davydov and Thilly [26]) *Let a and b be two stochastic processes on $[0,1]$ as above, and let $\xi = \{\xi(t), 0 \le t \le 1\}$ be the Itô-Wiener process on $[0,1]$ defined in* (9.1). *Furthermore, assume that $\int_0^1 \mathbf{E}(|b(s)|^{4+\delta})ds < \infty$ for some $\delta > 0$. Then*

$$||n^{-1/2} \mathsf{C}_n^0[\xi] - \mathsf{GL}_{F_b}||_\infty \to_{a.s.} 0,$$

where

$$F_b(x) = \int_0^1 \Phi_{b(s)}(x)ds$$

with $\Phi_{b(s)}$ denoting the Gaussian distribution function with mean 0 and variance $b^2(s)$.

Note that when $a(s) \equiv 0$ and $b(s) \equiv 1$, then Theorem 9.1 reduces to Theorem 7.2. The most striking difference between the result of Theorem 9.1 and those of previous sections is the fact that when b is random then the limiting generalized Lorenz curve GL_{F_b} is random. In general, it would be interesting to know how much the assumption $\int_0^1 \mathbf{E}(|b(s)|^{4+\delta})ds < \infty$ could be relaxed and statement of Theorem 9.1 would still be true.

10 Multivariate convexifications: definition

Using the notions of zonotopes and zonoids, Koshevoy [40], [41], Koshevoy and Mosler [42], [43], [44], and Mosler [48] introduced and investigated (generalized) Lorenz curves for multivariate random variables. Davydov and Vershik [27], Thilly [58] used zonotopes and zonoids to define convexifications of multivariate stochastic processes and developed the first asymptotic theory for them. Davydov, Paulauskas and Račkauskas [24], Davydov and Paulauskas [23] generalized those ideas further to situations such as stable random elements in Banach spaces. We now proceed with brief descriptions of these developments.

Given two subsets $A, B \subset \Re^d$, the Minkowski sum $A + B$ is defined as the set $\{\mathbf{x} + \mathbf{y} : \ \mathbf{x} \in A, \ \mathbf{y} \in B\}$. The closed interval $[\mathbf{x}, \mathbf{y}]$ between two points $\mathbf{x}$ and $\mathbf{y}$ of $\Re^d$ is the set $\{\alpha\mathbf{x} + (1-\alpha)\mathbf{y} : \ 0 \le \alpha \le 1\}$. Furthermore, a set is called a zonotope if it is the Minkowski sum of a finite number of closed intervals. The convergence of zonotopes Z_n to a convex closed set $Z \subseteq \Re^d$ is defined in terms of the Hausdorff metric

$$\rho(K_1, K_2) := \max\left\{ \sup_{\mathbf{x}\in K_1} \inf_{\mathbf{y}\in K_2} \|\mathbf{x}-\mathbf{y}\|_2, \inf_{\mathbf{x}\in K_1} \sup_{\mathbf{y}\in K_2} \|\mathbf{x}-\mathbf{y}\|_2 \right\}$$

on the space of all convex closed subsets of $\Re^d$, where $\|\cdot\|_2$ is the Euclidian distance in $\Re^d$. The set Z is a zonoide. Indeed, a set Z is called a zonoid if there is a sequence of zonotopes Z_n such that $\rho(Z_n, Z) \to 0$ when $n \to \infty$. For more information on zonotopes, zonoides, and other related convex sets we refer, for example, to Bolker [3], [4], and Schneider [55].

10.1 Convexification C_n^0 reformulated. We shall now demonstrate that the convexification C_n^0 is one half of the boundary of a zonotope in $\Re^2$ (cf. Koshevoy [40], [41], Koshevoy and Mosler [42], [44], Mosler [48], Davydov and Vershik [27], Thilly [58]).

Given $X_1, \ldots, X_n$, let $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be two dimensional vectors defined as follows:

$$\mathbf{Y}_i := \Big(\frac{1}{n}, X_i\Big).$$

Define a zonotope Z_n in $\Re^2$ by the formula

$$Z_n := \sum_{i=1}^{n} [0, \mathbf{Y}_i].$$

Let ∂Z_n be the boundary of Z_n, and let L_0 be the straight line going through the points $(0,0)$ and $(1, \sum_{i=1}^n X_i)$. Define by L_n^+ the part of ∂Z_n which is on or above the line L_0, and by L_n^- the part of ∂Z_n which is on or below the line L_0. It is easy to check that

$$\mathsf{C}_n^0 = L_n^-.$$

Note also that the two half boundaries L_n^+ and L_n^- of Z_n are in one-to-one correspondence with each other. In fact, they are symmetric with respect to the point $(\frac{1}{2}, \frac{1}{2}\sum_1^n X_i)$. This, in turn, implies that the convexification C_n^0 and the zonotope Z_n are in one-to-one correspondence. Consequently, the information about the asymptotic behaviour of C_n^0 can be derived from the corresponding information about Z_n, and vice versa. Note, however, that contrary to the definition of C_n^0 the definition of Z_n can readily be extended to any multidimensional space. This is the topic of the next subsection.

10.2 Multivariate convexifications. Let $\boldsymbol{\xi}$ be a d-dimensional stochastic process on $[0,\infty)$ with stationary increments and such that $\boldsymbol{\xi}(0) = \mathbf{0}$ almost surely. Let

$$\mathbf{X}_i := \boldsymbol{\xi}(i) - \boldsymbol{\xi}(i-1),$$
$$\mathbf{Y}_{i,n} := \Big(\frac{1}{n}, \frac{1}{b_n}\mathbf{X}_i\Big),$$

where the normalizing constants $b_n > 0$ are chosen depending on the process $\boldsymbol{\xi}$. The convexifcation of $\boldsymbol{\xi}$ is defined as a lower part of the boundary of the zonotope

$$Z_n := \sum_{i=1}^n [0, \mathbf{Y}_{i,n}]. \tag{10.1}$$

If, however, $\boldsymbol{\xi}$ is a d-dimensional stochastic process on $[0,1]$ with stationary increments and such that $\boldsymbol{\xi}(0) = \mathbf{0}$ almost surely, then we define

$$\mathbf{X}_{i,n} := \boldsymbol{\xi}\Big(\frac{i}{n}\Big) - \boldsymbol{\xi}\Big(\frac{i-1}{n}\Big),$$
$$\mathbf{Y}_{i,n} := \Big(\frac{1}{n}, \frac{1}{b_n}\mathbf{X}_{i,n}\Big),$$

where the normalizing constants $b_n > 0$ are chosen depending on the process $\boldsymbol{\xi}$. With these notations, the convexifcation of $\boldsymbol{\xi}$ is defined as a lower part of the boundary of the zonotope Z_n in (10.1).

11 Multivariate processes on $[0,1]$: consistency

Let $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d\}$ be an orthonormal basis in the space $\Re^d$. Considering $\Re^d$ as a subspace of $\Re^{d+1}$, let $\mathbf{e}_0$ be a vector having norm 1 and orthogonal to all the vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d$. Denote $S := \mathrm{span}(\mathbf{e}_0)$ which, in the case of real-valued processes, plays the role of the time interval, $[0,1]$ or $[0,\infty)$. For an element $\mathbf{l} \in \Re^d$, let $H(\mathbf{l})$ denote the two dimensional subspace generated by the $S \times \mathbf{l}$. Finally, given a subspace $J \subseteq \Re^{d+1}$, denote the orthogonal projection from $\Re^{d+1}$ onto J by P_J.

Theorem 11.1 (Thilly [58]) *Let $\boldsymbol{\xi} = \{\boldsymbol{\xi}(t), 0 \le t \le 1\}$ be a d-dimensional stochastic process with stationary increments, $\boldsymbol{\xi}(0) = \mathbf{0}$ almost surely, and $\mathbf{E}(\boldsymbol{\xi}(t)) = \mathbf{0}$ for all $0 \le t \le 1$. Let $b_n > 0$ be positive numbers such that, for any $\mathbf{l} \in \Re^d$, there exists a convex subset $Z(\mathbf{l}) \subset \Re^2$ such that*

$$P_{H(\mathbf{l})}(Z_n) \to_\rho Z(\mathbf{l}) \quad a.s.,$$

where ρ is the Hausdorff metric in two dimensions. Then there exists a unique zonoid $Z \in \Re^{d+1}$ such that $P_{H(\mathbf{l})}(Z) = Z(\mathbf{l})$ for any $\mathbf{l} \in \Re^d$ and such that

$$Z_n \to_\rho Z \quad a.s.,$$

where ρ is the Hausdorff metric in $d+1$ dimensions.

Theorem 11.1 demonstrates, in particular, that the asymptotic behaviour of the zonotope Z_n can be described in terms of the asymptotic behaviours of its two dimensional projections. For the latter ones, the already discussed results of previous sections can be applied.

12 Multivariate Gaussian processes on $[0,1]$: consistency

Let $\boldsymbol{\xi} = \{\boldsymbol{\xi}(t), 0 \le t \le 1\}$ be a d-dimensional Gaussian stochastic process on $[0,1]$ with stationary increments. Assume that $\boldsymbol{\xi}(0) = \mathbf{0}$ almost surely and $\mathbf{E}(\boldsymbol{\xi}(t)) = \mathbf{0}$ for all $0 \le t \le 1$. Let $\mathbf{V}(h) := (v_{ij}(t))_{i,j=1}^{d}$ be the covariance matrix of $\boldsymbol{\xi}(t) = (\xi^1(t), \dots, \xi^d(t))$ with $v_{ij}(t) = \mathbf{E}(\xi^i(t)\xi^j(t))$ for all $1 \le i, j \le d$.

Theorem 12.1 (Thilly [58]) *Assume that the matrix* $\mathbf{V}$ *has the following asymptotic representation*

$$\mathbf{V}(h) \sim \mathbf{K}h^{2\alpha}$$

when $h \downarrow 0$, *where* $0 < \alpha < 1$ *is a parameter and* $\mathbf{K}$ *is a* $d \times d$ *symmetric and positive-definite matrix. If* $0 < \alpha < 1/2$, *then let there eixist a constant* c_1 *such that*

$$\mathbf{V}''(h) \sim c_1 \mathbf{K} h^{2\alpha - 2}$$

when $h \downarrow 0$. *If* $1/2 \le \alpha < 1$, *then let there exists* $\beta \in [0, 2\alpha - 1]$ *and* $c_2 < \infty$ *such that*

$$\mathbf{V}'(h) \sim c_2 \mathbf{K} h^{\beta}$$

when $h \downarrow 0$. *Then with* $b_n = n^{-(\alpha-1)}$ *we have that*

$$Z_n \to_{\rho} Z \quad a.s.,$$

where the zonoide Z *is defined by the formula*

$$Z := \left\{(t, \mathbf{x}) \in [0,1] \times \Re^d : \ \langle K^{-1}\mathbf{x}, \mathbf{x}\rangle \le \mathsf{GL}_{\Phi}^2(t)\right\},$$

with $\langle \cdot, \cdot \rangle$ *denoting the inner product in* $\Re^d$.

Let $\mathbf{W}_{\boldsymbol{\sigma},\boldsymbol{\alpha}} = \{\mathbf{W}_{\boldsymbol{\sigma},\boldsymbol{\alpha}}(t), 0 \le t \le 1\}$ be a d-dimensional Gaussian process on $[0,1]$ defined by the formula

$$\mathbf{W}_{\boldsymbol{\sigma},\boldsymbol{\alpha}}(t) = \big(\sigma_1 W_{\alpha_1}(t), \sigma_2 W_{\alpha_2}(t), \dots, \sigma_d W_{\alpha_d}(t)\big),$$

where $\sigma_1, \dots, \sigma_d > 0$ are some constants and $W_{\alpha_1}, \dots, W_{\alpha_d}$ are independent fractional Brownian motions with Hurst parameters $0 < \alpha_i < 1$. If all the parameters α_i are equal, then the corresponding matrix $\mathbf{K}$ is diagonal and the zonoide Z is expressed by the formula

$$Z = \left\{(t, \mathbf{x}) \in [0,1] \times \Re^d : \ \sum_{i=1}^{d} \frac{x_i^2}{\sigma_i^2} \le \mathsf{GL}_{\Phi}^2(t)\right\}.$$

If, however, the parameters α_i are not all equal, then the situation becomes more involved. Let $1 \le m \le d$ be the integer such that $\alpha_1 = \alpha_2 = \cdots = \alpha_m = \alpha$ for some $0 < \alpha < 1$ with the remaining α_i, $i \ge m+1$, being strictly smaller than α. Then the zonoide Z is expressed by the formula

$$Z = \left\{(t, \mathbf{x}) \in [0,1] \times \Re^d : \ \sum_{i=1}^{m} \frac{x_i^2}{\sigma_i^2} \le \mathsf{GL}_{\Phi}^2(t) \text{ and } x_i = 0, \ i > m+1\right\}.$$

13 Processes with unbounded time: LIL

In this section we discuss laws of the iterated logarithm for the convexification C_n^0 in the case of either discrete time $\xi = (\xi_0, \xi_1, \xi_2, \dots)$ and continuous time $\xi = \{\xi(x), x \geq 0\}$ processes with independent increments.

Let $X_1, X_2, \dots$ be independent random variables with the same distribution function F and finite first moment. Under these and further assumptions, Csörgő, Csörgő and Horváth [8] established a strong invariance principle for the generalized Lorenz process $\mathsf{GL}_n - \mathsf{GL}_F$ and used it to obtain the rate of uniform strong consistency. Their result can be considered as the first LIL-type result for the generalized Lorenz process.

Theorem 13.1 (Csörgő, Csörgő and Horváth [8]) *Let $(X_1, X_2, \dots)$ be a sequence of independent and identically distributed random variables. Furthermore, let the distribution function F of X_1 be absolutely continuous, and let the density f of F be positive on the interior of the support F. Assume that*

$$\sup_{0<t<1} t^\alpha (1-t)^\beta / f \circ F^{-1}(t) < \infty \tag{13.1}$$

for some $0 \leq \alpha, \beta < 3/2$. Then there exists a constant $c < \infty$ such that

$$\limsup_{n\to\infty} (n/(2\log\log n))^{1/2} \, ||\mathsf{GL}_n - \mathsf{GL}_F||_\infty \leq c \quad a.s.$$

Csörgő, Csörgő and Horváth [8] assumed that the random variables $X_1, X_2, \dots$ are non-negative. We already mentioned earlier that this assumption is traditional in the context of Lorenz curves but is unnatural in the context of the present paper. We have therefore removed the non-negativity assumption from Theorem 13.1, as well as from the following theorems.

We continue with results by Rao and Zhao [51] where the authors proved the first law of the iterated logoarithm in the Strassen's form. To formulate their result we need additional notations.

Let $\mathcal{H}$ denote the Finkelstein (cf. [33]) class consisting of all real-valued and absolutely continuous functions h on $[0,1]$ such that $h(0) = 0 = h(1)$ and $\int_0^1 (h'(s))^2 ds \leq 1$. Furthermore, let

$$\mathcal{L}_{0,F} := \{\ell_{h,F} : h \in \mathcal{H}\},$$

where the function $\ell_{h,F}$ on $[0,1]$ is defined by the formula

$$\ell_{h,F}(t) := -\int_{-\infty}^{F^{-1}(t)} h \circ F(x) dx. \tag{13.2}$$

Theorem 13.2 (Rao and Zhao [51]) *Let $(X_1, X_2, \dots)$ be a sequence of independent and identically distributed random variables. Furthermore, let the distribution function F of X_1 be absolutely continuous, and let the density f be positive on the interior of the support F. Assume that* (13.1) *holds for some $0 \leq \alpha, \beta < 3/2$. Then*

$$(n/(2\log\log n))^{1/2} \, (\mathsf{GL}_n - \mathsf{GL}_F) \rightrightarrows \mathcal{L}_{0,F} \quad a.s. \tag{13.3}$$

with respect to the uniform metric on $D[0,1]$.

The notation $\rightrightarrows$ used in Theorem 13.2 above means that, for almost all elementary events ω, the set

$$\mathcal{L}_F := \Big\{(n/(2\log\log n))^{1/2} \, (\mathsf{GL}_n(\cdot,\omega) - \mathsf{GL}_F(\cdot,\omega)) : n \geq 1\Big\}$$

is relatively compact in the space $D[0,1]$ equipped with the sup-norm $||\cdot||_\infty$, and that the set of all limit points of $\mathcal{L}_F$ is $\mathcal{L}_{0,F}$.

Rao and Zhao [51] also proved another theorem stating (13.3) under somewhat different assumptions than those of Theorem 13.2. Both theorems of Rao and Zhao [51], however, are special cases of the following one.

Theorem 13.3 (Csörgő and Zitikis [14]) *Let* $(X_1, X_2, \dots)$ *be a sequence of independent and identically distributed random variables such that* $\mathbf{E}(|X_1|^{2+\epsilon}) < \infty$ *for some* $\epsilon > 0$. *Furthermore, let* F *and* F^{-1} *be continuous on* $\Re$ *and* $(0,1)$, *respectively. Then*

$$(n/(2\log\log n))^{1/2}\,(\mathsf{GL}_n - \mathsf{GL}_F) \rightrightarrows \mathcal{L}_{0,F} \quad a.s.$$

with respect to the uniform metric on $D[0,1]$.

When $X_1, X_2, \dots$ are increments of the Wiener process W on $[0,\infty)$, then the assumptions of Theorem 13.3 are satisfied with $F = \Phi$. Consequently,

$$(n/(2\log\log n))^{1/2}\,(n^{-1}\mathsf{C}_n^0[W] - \mathsf{GL}_\Phi) \rightrightarrows \mathcal{L}_{0,\Phi} \quad \text{a.s.}$$

14 Processes with unbounded time: weak convergence

Consider discrete time $\xi = (\xi_0, \xi_1, \dots)$ and continuous time $\xi = \{\xi(x), x \geq 0\}$ processes. In both cases, equality (2.3) reduces the problem of weak convergence of C_n^0 to the corresponding one concerning the appropriately normalized stochastic process $\mathsf{GL}_n - \mathsf{GL}_F$ based on the first n coordinates of the random sequence $(X_1, X_2, \dots)$. We assume throughout that all the coordinates of $(X_1, X_2, \dots)$ have the same distribution function F.

14.1 Independent case. Here we consider the case when $(X_1, X_2, \dots)$ is a sequence of independent random variables.

Theorem 14.1 (Goldie [36]) *Let* $(X_1, X_2, \dots)$ *be a sequence of independent and identically distributed random variables such that* $\mathbf{E}(X_1^2) < \infty$. *Assume that the distribution function* F *of* X_1 *is continuous, has support* $S \subseteq (0,\infty)$ *such that* $\sup(S) = \infty$, *and satisfies the following condition*

$$\lim_{x\to\infty} c(1 - F(cx))/(1 - F(x)) < 1 \tag{14.1}$$

with some constant $c > 1$. *If the support* S *is connected, then*

$$\sqrt{n}\,(\mathsf{GL}_n - \mathsf{GL}_F) \Rightarrow \ell_{B,F} \tag{14.2}$$

in the space $D[0,1]$ *equipped with the uniform metric, where the Gaussian process* $\ell_{B,F}$ *is defined by formula* (13.2) *with a Brownian bridge* B *on* $[0,1]$. *If, however,* S *is not connected, then* (14.2) *holds in the space* $D[0,1]$ *equipped with the* M_1 *topology.*

The assumption of non-negativity of random variables $X_1, X_2, \dots$ in Theorem 14.1 is due to the fact that the notion of Lorenz curve originates in Econometrics (cf. Lorenz [47]) where the random variables are interpreted as incomes. It is easy to see that the non-negativity assumption can be removed by, for example, adding an analog of (14.1) in a neighbourhood of $-\infty$. We note, however, that Csörgő, Csörgő, and Horváth [8] proved statement (14.2) without assuming (14.1), and we shall now formulate their result without the therein stated non-negativity assumption on $X_1, X_2, \dots$.

Theorem 14.2 (Csörgő, Csörgő, and Horváth [8]) *On a suitably constructed probability space, the following holds. Let* $(X_1, X_2, \dots)$ *be a sequence of independent and identically distributed random variables with the same marginal distribution function* F *and such that* $\mathbf{E}(X_1^2) < \infty$. *Furthermore, let* F *be continuous on* $\Re$, *and let the quantile function* F^{-1} *be continuous on* $(0,1)$. *Then there exists a sequence* $(B_1, B_2, \dots)$ *of Brownian bridges on* $[0,1]$ *such that*

$$\|\sqrt{n}\,(\mathsf{GL}_n - \mathsf{GL}_F) - \ell_{B_n,F}\|_\infty \to_P 0. \tag{14.3}$$

Obviously, statement (14.3) implies (14.2). Further elaborations and related references on statement (14.3) can be found in Csörgő, Gastwirth, and Zitikis [11]. We conclude the discussion by noting that when $X_1, X_2, \dots$ are increments of the Wiener process W on $[0,\infty)$, then Theorem 14.2 implies that

$$\sqrt{n}\,\big(n^{-1}\mathsf{C}_n^0[W] - \mathsf{GL}_\Phi\big) \Rightarrow \ell_{B,\Phi} \tag{14.4}$$

in the space $D[0,1]$ equipped with the uniform metric.

14.2 Dependent case. Extending (14.4) to the case of the fractional Brownian motion requires an analogue of Theorem 14.2 for dependent random variables.

Let $(X_1, X_2, \dots)$ be a strictly stationary sequence, and let F be the distribution function of X_1. Furthermore, let F_n be the empirical distribution function based on the first n coordinates of the sequence, and let e_n be the empirical process on $[0,1]$ defined by the equation

$$e_n \circ F := \sqrt{n}(F_n - F). \tag{14.5}$$

Let $B^* = \{B^*(t), 0 \le t \le 1\}$ be a mean zero Gaussian process with the covariance kernel $s \wedge t - st + K^*(s,t)$, where

$$K^*(s,t) := \sum_{k=1}^{\infty} \Big(\mathrm{Cov}(\mathbf{I}_{\{X_1 \le F^{-1}(s)\}}, \mathbf{I}_{\{X_{1+k} \le F^{-1}(t)\}}) + \mathrm{Cov}(\mathbf{I}_{\{X_{1+k} \le F^{-1}(s)\}}, \mathbf{I}_{\{X_1 \le F^{-1}(t)\}})\Big). \tag{14.6}$$

Theorem 14.3 (Csörgő and Yu [13]) *Let* $(X_1, X_2, \dots)$ *be a strictly stationary sequence. Furthermore, let the distribution function* F *and its inverse* F^{-1} *be continuous on* $\Re$ *and* $(0,1)$, *respectively. Assume that the series* $K_F(s,t)$ *converges absolutely for any* $s, t \in (0,1)$. *Furthermore, assume that*

$$e_n/q \Rightarrow B^*/q \tag{14.7}$$

in the space D *equipped with the* J_1 *topology, where* $q : (0,1) \to (0,\infty)$ *is a continuous function such that for some* $c_0 > 0$, $0 < \nu \le 1/2$, *and* $\beta > 1/2$ *the bound*

$$q(u) \ge c_0 u^\nu (1-u)^\nu \left(-\log\left(u(1-u)\right)\right)^\beta =: c_0 q_{\nu,\beta}(u)$$

holds for all $0 < u < 1$. *Finally assume that the distribution function* F *is such that, with* ν *and* β *specified above,*

$$\int_{-\infty}^{\infty} q_{\nu,\beta} \circ F(x)dx < \infty.$$

Then

$$\sqrt{n}(\mathsf{GL}_n - \mathsf{GL}_F) \Rightarrow \ell_{B^*,F}$$

in the space $D[0,1]$ *equipped with the* J_1 *topology.*

The verification of assumption (14.7) is a challenging problem, and we shall now discuss it in some detail. When $X_1, X_2, \dots$ are independent random variables, then $K^*(s,t) = 0$, which in turn implies that B^* is a Brownian bridge. In this case optimal assumptions for the validity of (14.7) were obtained by Csörgő, Csörgő, Horváth, and Mason [10] (cf. also the monograph by Csörgő and Horváth [12] on the topic). In the case of strictly stationary sequence $(X_1, X_2, \dots)$, sufficient assumption for the validity of (14.7) were obtained by Shao and Yu [54]. In the Gaussian case, and in the context of convexifications of stochastic processes, the problem was investigated by Davydov and Zitikis [28]. We shall discuss the latter situation in the following section.

15 Gaussian processes with unbounded time: weak convergence

Throughout this section we assume that the (strictly) stationary sequence $(X_1, X_2, \dots)$ is Gaussian. Furthermore, we assume that the mean μ of X_1 is finite and the variance σ^2 is positive. Since the joint (Gaussian) densities of (X_1, X_{1+k}) and (X_{1+k}, X_1) coincide, we have the following equality

$$K^*(s,t) = 2\sum_{k=1}^{\infty} \mathrm{Cov}(\mathbf{I}_{\{X_1 \le F^{-1}(s)\}}, \mathbf{I}_{\{X_{1+k} \le F^{-1}(t)\}}) \tag{15.1}$$

for the kernel $K^*(s,t)$ in (14.6).

15.1 Short-range dependent case. It is easy to check that the sum in (15.1) converges absolutely if

$$\sum_{k=1}^{\infty} |\mathrm{Cov}(X_1, X_{1+k})| < \infty. \tag{15.2}$$

In turn, if $X = (X_1, X_2, \dots)$ is such that (15.2) holds, then the sequence $(X_1, X_2, \dots)$ is called short-range dependent.

Let I be the function on $[0,1]$ defined by the formula

$$\mathsf{I}(t) = t.$$

Theorem 15.1 (Davydov and Zitikis [28]) *Let $(X_1, X_2, \dots)$ be a Gaussian and short-range dependent sequence. Then*

$$\sqrt{n}(\sigma^{-1}(\mathsf{GL}_n - \mu\mathsf{I}) - \mathsf{GL}_\Phi) \Rightarrow \ell_{B^*,\Phi}$$

in the space $D[0,1]$ equipped with the uniform topology.

Theorem 15.1 is applicable in the case of the fractional Brownian motion W_α on $[0,\infty)$ with the Hurst parameter $0 < \alpha \le 1/2$. More generally, consider the fractional Brownian motion $W_{\alpha,\mu,\sigma} = \{W_{\alpha,\mu,\sigma}(x), x \ge 0\}$ on $[0,\infty)$ with finite drift $\mu \in \Re$ and positive variance $\sigma^2 > 0$, that is,

$$W_{\alpha,\mu,\sigma}(x) = \mu x + \sigma W_\alpha(x), \ x \ge 0.$$

When $0 < \alpha \le 1/2$, then the sequence $(X_1, X_2, \dots)$ of the increments $X_i = W_{\alpha,\mu,\sigma}(i) - W_{\alpha,\mu,\sigma}(i-1)$ is short-range dependent. Thus Theorem 15.1 implies that

$$\sqrt{n}\left(\sigma^{-1}\left(n^{-1}\mathsf{C}_n^0[W_{\alpha,\mu,\sigma}] - \mu\mathsf{I}\right) - \mathsf{GL}_\Phi\right) \Rightarrow \ell_{B^*,\Phi} \tag{15.3}$$

in the space $D[0,1]$ equipped with the uniform topology.

15.2 Long-range dependent case. If the sum in (15.2) diverges, then the stationary sequence $(X_1, X_2, \dots)$ is called long-range dependent. When working with long-range dependent sequences, a stronger assumption than the divergence of the sum is usually imposed. Namely, let there exist a constant $D \in (0,1)$ such that

$$\mathrm{Cov}(X_1, X_{1+k}) = k^{-D} L(k), \ k = 1, 2, \dots, \tag{15.4}$$

where L is a slowly varying at infinity function such that $L(k) > 0$ for all sufficiently large k. Assumption (15.4) is usually considered as the definition of long-range dependence. We shall follow this tradition in the present paper as well.

Let $Y_i := \sigma^{-1}(X_i - \mu)$ for all $i \geq 1$, and let

$$d_n^2 := 1 + \frac{2}{n} \sum_{k=1}^{n-1} (n-k) \mathrm{Cov}(Y_1, Y_{1+k}).$$

Furthermore, let $\Gamma_\Phi = \{\Gamma_\Phi(t), 0 \leq t \leq 1\}$ be defined by the formula

$$\Gamma_\Phi(t) := t N_{0,1},$$

where $N_{0,1}$ is a standard Gaussian random variable

Theorem 15.2 (Davydov and Zitikis [28]**)** *Let $(X_1, X_2, \dots)$ be a Gaussian and long-range dependent sequence with assumption* (15.4) *satisfied. Then*

$$d_n^{-1} \sqrt{n} (\sigma^{-1} (\mathsf{GL}_n - \mu \mathsf{I}) - \mathsf{GL}_\Phi) \Rightarrow \Gamma_\Phi$$

in the space $D[0,1]$ equipped with the uniform topology.

In the case of the fractional Brownian motion $W_{\alpha,\mu,\sigma} = \{W_{\alpha,\mu,\sigma}(x), x \geq 0\}$ with drift $\mu \in \Re$ and variance $\sigma^2 > 0$, the sequence $(X_1, X_2, \dots)$ of the increments $X_i = W_{\alpha,\mu,\sigma}(i) - W_{\alpha,\mu,\sigma}(i-1)$ is long-range dependent if the Hurst parameter $1/2 < \alpha < 1$. In this case we have $d_n^2 = n^{2\alpha - 1}$. Consequently, Theorem 15.2 implies that

$$n^{1-\alpha} \left(\sigma^{-1} \left(n^{-1} \mathsf{C}_n^0 [W_{\alpha,\mu,\sigma}] - \mu \mathsf{I} \right) - \mathsf{GL}_\Phi \right) \Rightarrow \Gamma_\Phi \tag{15.5}$$

in the space $D[0,1]$ equipped with the uniform topology.

16 The fBm on $[0,1]$: weak convergence and characterization

16.1 Weak convergence. Consider the normalized increments

$$Y_{k,n} = n^\alpha \left(W_\alpha\Big(\frac{k}{n}\Big) - W_\alpha\Big(\frac{k-1}{n}\Big) \right), \ k = 1, 2, \dots, n,$$

of the fractional Brownian motion W_α on the interval $[0,1]$. The mean of $Y_{k,n}$ is 0, the variance is 1, and the covariance

$$\mathrm{Cov}(Y_{1,n}, Y_{1+k,n}) = \frac{1}{2}((k+1)^{2\alpha} + (k-1)^{2\alpha}) - k^{2\alpha}.$$

The sequence $\{Y_{k,n}, k = 1, 2, \dots, n\}$ is Gaussian. Using these properties, we obtain that, for any fixed n, the distribution of the generalized Lorenz curve based on $\{Y_{k,n}, k = 1, 2, \dots, n\}$ coincides with the distribution of the generalized Lorenz curve based on a single sequence of Gaussian random variables $\{Y_k, k = 1, 2, \dots, n\}$ exhibiting the same dependence structure as the sequence $\{Y_{k,n}, k = 1, 2, \dots, n\}$. Thus, the distribution of the generalized Lorenz process based on the triangular array $\{Y_{k,n}, k = 1, 2, \dots, n\}$ coincides with the distribution of the generalized Lorenz process based on the single sequence $\{Y_k, k = 1, 2, \dots, n\}$. For the latter process,

we can use statements (15.3) and (15.5) with $\mu = 0$ and $\sigma^2 = 1$ to derive weak convergence of the convexifications $\mathsf{C}_n^0[W_\alpha]$ when $n \to \infty$.

Theorem 16.1 (Davydov and Zitikis [28]) *Let $W_\alpha = \{W_\alpha(t), 0 \le t \le 1\}$ be the fractional Brownian motion on $[0,1]$ with the Hurst parameter $0 < \alpha < 1$. Then in the space $D[0,1]$ equipped with the uniform topology we have the following three statements:*

1) *If $0 < \alpha < 1/2$, then $\sqrt{n}\left(n^{-(1-\alpha)}\mathsf{C}_n^0[W_\alpha] - \mathsf{GL}_\Phi\right) \Rightarrow \ell_{B^*,\Phi}$.*
2) *If $\alpha = 1/2$, then $\sqrt{n}\left(n^{-(1-\alpha)}\mathsf{C}_n^0[W_\alpha] - \mathsf{GL}_\Phi\right) \Rightarrow \ell_{B,\Phi}$.*
3) *If $1/2 < \alpha < 1$, then $n^{1-\alpha}\left(n^{-(1-\alpha)}\mathsf{C}_n^0[W_\alpha] - \mathsf{GL}_\Phi\right) \Rightarrow \Gamma_\Phi$.*

16.2 Characterization. The idea of utilizing asymptotic results concerning processes on $[0,\infty)$ for deriving the corresponding ones for processes on compact intervals such as $[0,1]$ can only be applied in the case of the fractional Brownian motion. A rigorous formulation of this statement is given in Statement 16.2 below.

Let $\{\xi(t), 0 \le t \le 1\}$ be a zero-mean Gaussian process with stationary increments. Let

$$Y_{k,n} := \frac{n}{v_n}\left(\xi\Big(\frac{k}{n}\Big) - \xi\Big(\frac{k-1}{n}\Big)\right), \ k = 1, 2, \ldots, n,$$

where v_n is defined by the condition $\mathbf{E}(Y_{k,n}^2) = 1$.

Statement 16.2 *Among all mean-squared-error continuous Gaussian processes $\xi = \{\xi(t), 0 \le t \le 1\}$ with stationary increments and such that $\xi(0) = 0$ almost surely, the distribution of the sequence $\{Y_{k,n}, k = 1, 2, \ldots, n\}$ does not depend on n, for any n, if and only if the process ξ is the fractional Brownian motion $W_\alpha = \{W_\alpha(t), 0 \le t \le 1\}$.*

Indeed, let $v(t) := \mathbf{E}(\xi(t)^2)$ and let $\psi(t) := v(t)/v(1)$. It is easy to check that the functional equation $\psi(ts) = \psi(t)\psi(s)$ holds for all rational numbers $s, t \in [0,1]$. Due to the mean-squared-error continuity, the equation $\psi(ts) = \psi(t)\psi(s)$ also holds for all real s and t. The latter equation implies Statement 16.2.

17 Functionals of convexifications: estimating the Hurst parameter

The limiting Gaussian process Γ_Φ in statement (15.5) does not depend on $1/2 < \alpha < 1$. This fact can be used for constructing estimators of the Hurst parameter $1/2 < \alpha < 1$. For example, integrating both sides of (15.5) with respect to the Lebesgue measure on $(0,1)$ and then multiplying them by 2, we obtain the following statement.

Statement 17.1 (Davydov and Zitikis [28]) *Let $W_{\alpha,\mu,\sigma}$ be the fractional Brownian motion on $[0,\infty)$ with drift $\mu \in \Re$ and variance $\sigma^2 > 0$. Then, for any $1/2 < \alpha < 1$, the asymptotic distribution of*

$$2n^{1-\alpha}\left(\frac{1}{n\sigma}\int_0^1 \mathsf{C}_n^0[W_{\alpha,\mu,\sigma}](t)dt - \frac{\mu}{2\sigma} + \frac{1}{2\sqrt{\pi}}\right)$$

is standard Gaussian.

Other approaches for estimating the Hurst parameter α were considered by a number of authors. For more details we refer, for example, to Istas and Lang [38], Coeurjolly [17], Thilly [58], Philippe and Thilly [50], as well as to the references therein.

18 Functionals of convexifications: indices of fluctuations

The convexification $\mathsf{C}_n[\xi]$ and the process ξ itself have the same values at the two endpoints of the interval $[0,n]$. Thus, the area between the convexification $\mathsf{C}_n[\xi]$ and the straight line $t \mapsto t\mathsf{C}_n[\xi](n)$ can be considered as an index of fluctuations of ξ on $[0,n]$. In general, the L_p-norm between the two curves can be considered as an index of fluctuations (cf. Davydov, Khoshnevisan, Shi, and Zitikis [22]). A rigorous formulation of the idea now follows.

Let $(X_1, X_2, \dots)$ be a sequence of random variables with the same marginal distribution function F and $\mathbf{E}(|X_1|) < \infty$. Let $p \geq 1$ be any number, and let $||h||_p$ be the usual $L_p([0,1],\Lambda)$ norm with Λ denoting a positive measure on $[0,1]$. Assume that the measure Λ is such that

$$0 < \Lambda([0,1]) < \infty. \tag{18.1}$$

Following the idea above, we define the empirical index of fluctuations by the formula

$$||\mathsf{I}_n - \mathsf{GL}_n||_p,$$

where I_n is a function on $[0,1]$ defined by the formula

$$\mathsf{I}_n(t) := t\mathsf{GL}_n(1).$$

Note that $\mathsf{GL}_n(1)$ is the sample mean of $X_1, \dots, X_n$. The theoretical index of fluctuations is defined, analogously, by the formula

$$||\mathsf{I}_F - \mathsf{GL}_F||_p,$$

where

$$\mathsf{I}_F(t) := t\mathsf{GL}_F(1).$$

Note that $\mathsf{GL}_F(1) = \mathbf{E}(X_1)$. Therefore, $\mathsf{I}_F(t) \equiv 0$ whenever $\mathbf{E}(X_1) = 0$.

The idea for measuring the fluctuations of stochastic processes as described above originates in Econometrics (cf., e.g., Chakravarty [6]) where similar indices are used for measuring economic inequality. Indeed, most of the (empirical) indices of economic inequality are related to $||\mathsf{I}_n - \mathsf{GL}_n||_p$, where $\Lambda(dt) = w(t)dt$ with a function w determined by some Econometric considerations. For further results and references on the subject we refer, for example, to the papers by Barrett and Donald [1], Zitikis [64], [65].

19 Indices of fluctuations: consistency

Barrett and Donald [1] developed a complete asymptotic theory for $||\mathsf{I}_n - \mathsf{GL}_n||_p$ and other related indices of Econometrics when the random variables $X_1, X_2, \dots$ are independent.

Theorem 19.1 (Barrett and Donald [1]) *Let $(X_1, X_2, \dots)$ be a sequence of independent and identically distributed random variables. Assume that the distribution function F is twice differentiable, has support in a bounded interval $[x_l, x_u]$ for some constants $0 < x_l < x_u < \infty$, and the density function f of F is such that*

$$0 < \inf_{x_l \leq x \leq x_u} f(x) \leq \sup_{x_l \leq x \leq x_u} f(x) < \infty.$$

Furthermore, assume that $\Lambda(ds) = c \cdot ds$ for a positive constant $c > 0$. Then, for any $p \geq 1$,

$$||\mathsf{I}_n - \mathsf{GL}_n||_p \to_P ||\mathsf{I}_F - \mathsf{GL}_F||_p.$$

Naturally, it is of interest and importance to relax assumptions of Theorem 19.1. The assumption of non-negativity of random variables is due to tradition and can easily be dropped. The removal of assumptions on the density f, including the existence of f itself, is not, however, a straightforward task.

Theorem 19.2 (Zitikis [65]) *Let* $(X_1, X_2, \dots)$ *be a sequence of independent and identically distributed random variables such that* $\mathbf{E}(|X_1|) < \infty$. *Furthermore, assume that* $\Lambda(ds) = c \cdot ds$ *for a positive constant* $c > 0$. *Then, for any* $p \geq 1$,

$$||\mathsf{I}_n - \mathsf{GL}_n||_p \to_{a.s.} ||\mathsf{I}_F - \mathsf{GL}_F||_p.$$

Theorem 19.2 is applicable for measuring the fluctuations of the Wiener process W. Indeed, when $(X_1, X_2, \dots)$ is the sequence of increments of W on $[0, \infty)$, then Theorem 19.2 implies that

$$||\mathsf{I}_n - n^{-1}\mathsf{C}_n^0[W]||_p \to_{a.s.} ||\mathsf{GL}_\Phi||_p$$

for any $p \geq 1$. When Λ is the Lebesgue measure, we have the formula

$$||\mathsf{GL}_\Phi||_p = \frac{1}{2\pi(p+1)^{1/(2p)}}. \quad [\Lambda(ds) = ds.] \tag{19.1}$$

In order to consider the case of the fractional Brownian motion W_α on $[0, \infty)$, we need to extend Theorem 19.2 to dependent random variables $X_1, X_2, \dots$.

Theorem 19.3 (Davydov, Khoshnevisan, Shi, and Zitikis [22]) *Let* $(X_1, X_2, \dots)$ *be a strictly stationary and ergodic sequence, and let* $\mathbf{E}(|X_1|) < \infty$. *Then, for any* $p \geq 1$ *and any positive measure* Λ *on* $[0, 1]$ *satisfying* (18.1),

$$||\mathsf{I}_n - \mathsf{GL}_n||_p \to_{a.s.} ||\mathsf{I}_F - \mathsf{GL}_F||_p.$$

The sequence $(X_1, X_2, \dots)$ of increments of the fractional Brownian motion W_α on $[0, \infty)$ is Gaussian, strongly stationary, and also ergodic due to the fact that $\mathrm{Cov}(X_1, X_k) \to 0$ when $k \to \infty$ for any $0 < \alpha < 1$. Consequently, Theorem 19.3 implies that, for any $0 < \alpha < 1$ and $p \geq 1$,

$$||\mathsf{I}_n - n^{-1}\mathsf{C}_n^0[W_\alpha]||_p \to_{a.s.} ||\mathsf{GL}_\Phi||_p.$$

20 Indices of fluctuations: weak convergence

In this section we discuss weak convergence of the appropriately centered and normalized index $||\mathsf{I}_n - \mathsf{GL}_n||_p$.

20.1 Independent case. This section is devoted to the case when $(X_1, X_2, \dots)$ is a sequence of independent random variables. In the formulation of the following theorem we use the function $\ell_{h,F}$ defined in (13.2), a Brownian bridge B, and a standard Gaussian random variable $N_{0,1}$.

Theorem 20.1 (Barrett and Donald [1]) *Let* $(X_1, X_2, \dots)$ *be a sequence of independent and identically distributed random variables. Furthermore, let all the assumptions of Theorem* 19.1 *be satisfied. Then, for any* $p \geq 1$,

$$n^{1/2}\left(||\mathsf{I}_n - \mathsf{GL}_n||_p - ||\mathsf{I}_F - \mathsf{GL}_F||_p\right) \to_d ||\mathsf{I}_F - \mathsf{GL}_F||_p^{1-p}\sigma_0 N_{0,1},$$

where σ_0^2 *is the variance of*

$$\int_0^1 (t\mathsf{GL}_F(1) - \mathsf{GL}_F(t))^{p-1} (\ell_{B,F}(t) - t\ell_{B,F}(1)) \Lambda(dt). \tag{20.1}$$

The assumptions on the density f of F in Theorem 20.1 were removed by Zitikis [65]. We formulate this result as the following theorem.

Theorem 20.2 (Zitikis [65]) *Let* $(X_1, X_2, \dots)$ *be a sequence of independent and identically distributed random variables such that* $\mathbf{E}(X_1^2) < \infty$. *Assume that* $\Lambda(ds) = c \cdot ds$ *for a positive constant* $c > 0$. *Then, for any* $p \geq 1$,

$$n^{1/2} \left(||\mathsf{I}_n - \mathsf{GL}_n||_p - ||\mathsf{I}_F - \mathsf{GL}_F||_p \right) \to_d ||\mathsf{I}_F - \mathsf{GL}_F||_p^{1-p} \sigma_0 N_{0,1}.$$

Theorem 20.2 is applicable when, for example, $(X_1, X_2, \dots)$ is the sequence of increments of the Wiener process W on $[0, \infty)$. Thus, the following result:

$$n^{1/2} \left(||\mathsf{I}_n - n^{-1}\mathsf{C}_n^0[W]||_p - ||\mathsf{GL}_\Phi||_p \right) \to_d ||\mathsf{GL}_\Phi||_p^{1-p} \sigma_0 N_{0,1},$$

where σ_0^2 is the variance of (20.1) with F replaced everywhere by Φ. Recall also that when Λ is the Lebesgue measure, then formula (19.1) is valid for $||\mathsf{GL}_\Phi||_p$. In order to cover the fractional Brownian motion W_α on $[0, \infty)$, we need, however, to extend Theorem 20.2 to dependent random variables. This is the topic of the following two subsections.

20.2 Short-range dependent case. This section is devoted to the case when $(X_1, X_2, \dots)$ is a short-range dependent sequence. In the formulation of the following theorem we use a Gaussian process B^* with mean zero and the covariance kernel defined in (14.6).

Theorem 20.3 (Davydov, Khoshnevisan, Shi, and Zitikis [22]) *Let* $(X_1, X_2, \dots)$ *be a Gaussian, stationary, and short-range dependent sequence. Then, for any* $p \geq 1$ *and any positive measure* Λ *on* $[0,1]$ *satisfying* (18.1),

$$n^{1/2} \left(||\mathsf{I}_n - \mathsf{GL}_n||_p - ||\mathsf{I}_F - \mathsf{GL}_F||_p \right) \to_d ||\mathsf{I}_F - \mathsf{GL}_F||_p^{1-p} \sigma_1 N_{0,1}$$

where σ_1^2 *is the variance of*

$$\int_0^1 (t\mathsf{GL}_F(1) - \mathsf{GL}_F(t))^{p-1} (\ell_{B^*,F}(t) - t\ell_{B^*,F}(1)) \Lambda(dt). \tag{20.2}$$

The sequence $(X_1, X_2, \dots)$ in the case of increments of the fractional Brownian motion W_α on $[0, \infty)$ is short-range dependent when $\alpha \leq 1/2$. Thus, Theorem 20.3 is applicable and we have the following result

$$n^{1/2} \left(||\mathsf{I}_n - n^{-1}\mathsf{C}_n^0[W_\alpha]||_p - ||\mathsf{GL}_\Phi||_p \right) \to_d ||\mathsf{GL}_\Phi||_p^{1-p} \sigma_1 N_{0,1},$$

for any $p \geq 1$, where σ_1^2 is the variance of (20.2) with F replaced everywhere by Φ.

20.3 Long-range dependent case. Investigation of weak convergence of $||\mathsf{I}_n - \mathsf{GL}_n||_p$ in the case of long-range dependent sequences is complex. In this section we therefore restrict ourselves to the increments

$$X_i = W_\alpha(i) - W_\alpha(i-1),\ i = 1, 2, \dots$$

of the fractional Brownian motion W_α on $[0, \infty)$.

Let G be a function on $\Re$ defined by the formula

$$G(y) := \int_0^1 (-\mathsf{GL}_\Phi(s))^{p-1} g(y, s) \Lambda(ds),$$

where

$$g(y, s) := -s \int_{-\infty}^{\infty} \left(\mathbf{1}_{\{y \le x\}} - \Phi(x)\right) dx + \int_{-\infty}^{\Phi^{-1}(s)} \left(\mathbf{1}_{\{y \le x\}} - \Phi(x)\right) dx.$$

Theorem 20.4 (Davydov, Khoshnevisan, Shi, and Zitikis [22]**)** *Let* Λ *be a positive measure on* $[0, 1]$ *satisfying* (18.1). *Then, for any* $1/2 < \alpha < 3/4$ *and* $p \ge 2$,

$$n^{1/2} \left(||\mathsf{I}_n - n^{-1} \mathsf{C}_n^0[W_\alpha]||_p - ||\mathsf{GL}_\Phi||_p \right) \to_d ||\mathsf{GL}_\Phi||_p^{1-p} \sigma_3 N_{0,1},$$

where

$$\sigma_3^2 := \sum_{k=1}^{\infty} \mathbf{E}(G(X_1) G(X_{1+k})). \tag{20.3}$$

The infinite series on the right-hand side of (20.3) converges absolutely if and only if $\alpha < 3/4$. Thus, Theorem 20.4 as it stands cannot be extended beyond the values $\alpha < 3/4$. Nevertheless, Theorem 20.5 below demonstrates that the case $\alpha = 3/4$ is still fairly close to that concerning the values $\alpha < 3/4$.

Theorem 20.5 (Davydov, Khoshnevisan, Shi, and Zitikis [22]**)** *Let* Λ *be a positive measure on* $[0, 1]$ *satisfying* (18.1). *Then, for* $\alpha = 3/4$ *and any* $p \ge 2$,

$$\frac{n^{1/2}}{\sigma_{4,n}} \left(||\mathsf{I}_n - n^{-1} \mathsf{C}_n^0[W_\alpha]||_p - ||\mathsf{GL}_\Phi||_p \right) \to_d ||\mathsf{GL}_\Phi||_p^{1-p} N_{0,1},$$

where

$$\sigma_{4,n}^2 := \sum_{k=1}^{n} \mathbf{E}(G(X_1) G(X_{1+k})).$$

In order to cover the remaining case $3/4 < \alpha < 1$, we need additional notations. Namely, let $\mathcal{R}$ denote a Rosenblatt random variable, that is, such that the equality

$$\mathbf{E}\left(e^{z\mathcal{R}}\right) = \exp\left(\frac{1}{2} \sum_{k=2}^{\infty} \frac{(2z)^k}{k} \mathfrak{S}_k\right)$$

holds for any sufficiently small $z \in C$, where

$$\mathfrak{S}_k := \int_0^1 \dots \int_0^1 \prod_{\ell=1}^{k-1} |y_{\ell-1} - y_\ell|^{-2(1-\alpha)} \cdot |y_k - y_1|^{-2(1-\alpha)} \, dy_1 \dots dy_k.$$

Theorem 20.6 (Davydov, Khoshnevisan, Shi, and Zitikis [22]) *Let Λ be a positive measure on $[0,1]$ satisfying (18.1). If $p=2$, then we additionally assume that, for any $\nu > 0$, the measure $\widetilde{\Lambda}(ds) := s^{-\nu}(1-s)^{-\nu}\Lambda(ds)$ is finite on $(0,1)$. Then, for any $3/4 < \alpha < 1$ and $p \geq 2$,*

$$n^{2(1-\alpha)} \left(\| \mathsf{I}_n - n^{-1}\mathsf{C}_n^0[W_\alpha] \|_p - \| \mathsf{GL}_\Phi \|_p \right) \to_d \| \mathsf{GL}_\Phi \|_p^{1-p} (\kappa_p/2)(N_{0,1}^2 + 3\mathcal{R}),$$

where

$$\kappa_p := \int_0^1 (\phi \circ \Phi^{-1}(s))^p \Lambda(ds),$$

and $N_{0,1}$ and $\mathcal{R}$ are independent standard Gaussian and Rosenblatt random variables, respectively.

21 Induced GLC: definition and consistency

21.1 Definition. Let $X_1, X_2, \ldots$ be real valued random variables, and let $\mathbf{Y}_1, \mathbf{Y}_2, \ldots$ be random elements in $\Re^d$. Furthermore, let $X_{1:n} \leq \cdots \leq X_{n:n}$ be the order statistics of $X_1, \ldots, X_n$, and let $\mathbf{Y}_1^{(n)}, \ldots, \mathbf{Y}_n^{(n)}$ be the values, called induced order statistics, of $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ corresponding to the order statistics $X_{1:n}, \ldots, X_{n:n}$. The (empirical) induced generalized Lorenz curve $\mathsf{GL}_n^{IND} = \{\mathsf{GL}_n^{IND}(t) : 0 \leq t \leq 1\}$ is defined by the formula

$$\mathsf{GL}_n^{IND}(t) := \begin{cases} \frac{1}{n}\sum_{i=1}^{[nt]} \mathbf{Y}_i^{(n)} + \left(t - \frac{[nt]}{n}\right) \mathbf{Y}_{[nt]+1}^{(n)}, & 0 \leq t < 1, \\ \frac{1}{n}\sum_{i=1}^{n} \mathbf{Y}_i^{(n)} \ \left[= \frac{1}{n}\sum_{i=1}^{n} \mathbf{Y}_i\right], & t = 1. \end{cases}$$

Note that when $Y_i = X_i$ for any $i = 1, 2, \ldots$, then $\mathsf{GL}_n^{IND} = \mathsf{GL}_n$.

21.2 Consistency. Let X_1 and Y_1 be two real-valued random variables. Assume that $\mathbf{E}(|Y_1|) < \infty$ and denote the distribution function of X_1 by F. Furthermore, define the regression function m of Y_1 on X_1 by the formula

$$m(x) := \mathbf{E}(Y_1 | X_1 = x). \tag{21.1}$$

With these notations, let $\mathsf{GL}_{m,F}^{IND}$ be the (theoretical) induced generalized Lorenz curve defined by the formula

$$\mathsf{GL}_{m,F}^{IND}(t) := \int_0^t m \circ F^{-1}(s) ds.$$

Theorem 21.1 (Rao and Zhao [52]) *Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be independent and identically distributed bivariate random vectors such that $\mathbf{E}(|Y_1|) < \infty$. Then*

$$\| \mathsf{GL}_n^{IND} - \mathsf{GL}_{m,F}^{IND} \|_\infty \to_{a.s.} 0.$$

When $X_1 = Y_1$ almost surely, then $m(x) = x$ and thus $\mathsf{GL}_{m,F}^{IND} = \mathsf{GL}_F$.

22 Induced GLC: law of the iterated logoarithm

22.1 Univariate case. Let X_1 and Y_1 be two real-valued random variables. Assume that $\mathbf{E}(|Y_1|) < \infty$ and denote the distribution function of X_1 by F. Furthermore, let

$$v(x) := \mathbf{E}((Y_1 - m(X_1))^2 | X_1 = x), \tag{22.1}$$
$$\sigma_1^2 := \mathbf{E}(Y_1 - m(X_1))^2.$$

Let $\mathcal{K}$ be the class consisting of all real-valued and absolutely continuous functions k on $[0,1]$ such that $k(0) = 0$ and $\int_0^1 (k'(s))^2 ds \leq 1$. Furthermore, let $\mathcal{H}$ be the Finkelstein (cf. [33]) class of functions h already defined in this paper above. With these notations, let

$$\mathcal{G}_{m,v,F} := \{g_{k,h,F} : k \in \mathcal{K},\ h \in \mathcal{H}\},$$

where the function $g_{k,h,F}$ on $[0,1]$ is defined by the formula

$$g_{k,h,F}(t) := \sigma_1 k\left(\frac{1}{\sigma_1^2} \mathsf{GL}_{v,F}^{IND}(t)\right) - \int_0^t h(s) dm \circ F^{-1}(s).$$

Theorem 22.1 (Rao and Zhao [53]) *Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be independent and identically distributed bivariate random vectors, and let $\mathbf{E}(|Y_1|^{2+\epsilon}) < \infty$ for some $\epsilon > 0$. With x_F and y_F denoting the two (possibly infinite) endpoints of the support of F, let F have a continuous and positive density on (x_F, y_F), and let the derivative m' of the regression function m be continuous on (x_F, y_F). Finally, assume that one of the following two conditions holds:*

- C_0: *The quantile regression function $r := m \circ F^{-1}$ is bounded on $[0,1]$.*
- C_1: *In a neighbourhood of 0, the function r is nonincreasing and $t \mapsto \sqrt{t} r(t)$ nondecreasing, whereas in a neighbourhood of 1, the function r is nondecreasing and $t \mapsto \sqrt{1-t}\, r(t)$ nonincreasing. Furthermore, let there exist two constants $c < \infty$ and $\tau < 1$ such that*

$$\left|\frac{r(t_1)}{r(t_2)}\right| \leq c \left(\frac{(t_1 \vee t_2)(1 - t_1 \wedge t_2)}{(t_1 \wedge t_2)(1 - t_1 \vee t_2)}\right)^{\tau}.$$

for any t_1, t_2 in a neighbourhood of 0, as well as for any t_1, t_2 in a neighbourhood of 1.

Then

$$(n/(2 \log\log n))^{1/2} (\mathsf{GL}_n^{IND} - \mathsf{GL}_F^{IND}) \rightrightarrows \mathcal{G}_{m,v,F} \quad a.s. \tag{22.2}$$

with respect to the uniform metric on $D[0,1]$.

22.2 Multivariate case. Let $\mathbf{Y}_1$ be a random element in $\Re^d$. Assume that $\mathbf{E}(\|\mathbf{Y}_1\|_2^2) < \infty$, where $\|\cdot\|_2$ denotes the Euclidian norm in $\Re^d$. Denote

$$\mathbf{m}(x) := \mathbf{E}(\mathbf{Y}_1 | X_1 = x), \tag{22.3}$$
$$\mathbf{V}(x) := \mathbf{E}\left((\mathbf{Y} - \mathbf{m}(X))(\mathbf{Y} - \mathbf{m}(X))^T | X = x\right), \tag{22.4}$$

and let $\mathbf{V}^{1/2}(x)$ be the positive definite matrix defined by the equation

$$\mathbf{V}^{1/2}(x) \mathbf{V}^{1/2}(x)^T = \mathbf{V}(x). \tag{22.5}$$

Furthermore, let

$$\mathbf{D}_F(t,s) := \mathbf{V}^{1/2} \circ F^{-1}(s) I_{\{s<t\}},$$

$$\mathbf{e}_F(t,s) := \left(\mathbf{m} \circ F^{-1}(s) - \mathbf{m} \circ F^{-1}(t)\right) I_{\{s<t\}} + \int_0^t u d\mathbf{m} \circ F^{-1}(u).$$

For the sake of notational simplicity, denote the space $L^2[0,1]$ by L, and let L^d be the space of vectors $\mathbf{f} = (f_1, \ldots, f_d)$ with all the components $f_i \in L$. With these notations, define the class of functions

$$\mathcal{S}_{\mathbf{m},\mathbf{V},F} := \left\{ \mathbf{s}_{\mathbf{f},g} : \ \mathbf{f} \in L^d, g \in L, ||\mathbf{f}||_{L^d}^2 + ||g||_L^2 \le 1 \right\},$$

where

$$\mathbf{s}_{\mathbf{f},g}(t) := \mathbf{D}_F(t,\cdot)\mathbf{f}^T + \mathbf{e}_F(t,\cdot)^T g$$

with the product of two elements in L defined as their inner product.

Theorem 22.2 (Davydov and Egorov [20]) *Let* $(X_1, \mathbf{Y}_1), (X_2, \mathbf{Y}_2), \ldots$ *be independent and identically distributed random vectors, and let* $\mathbf{E}(\|\mathbf{Y}_1\|_2^2) < \infty$. *Furthermore, let the distribution function* F *of* X_1 *be continuous, and let there exist two constants* $c < \infty$ *and* $0 < a < 1/2$ *such that*

$$||\mathbf{m} \circ F^{-1}(t)||_2 \le c t^{-a} (1-t)^{-a} \tag{22.6}$$

for all $0 < t < 1$. *Then*

$$(n/(2\log\log n))^{1/2} \left(\mathsf{GL}_n^{IND} - \mathsf{GL}_F^{IND}\right) \rightrightarrows \mathcal{S}_{\mathbf{m},\mathbf{V},F} \quad a.s. \tag{22.7}$$

with respect to the uniform metric on $D^d[0,1]$.

23 Induced GLC: weak convergence

23.1 Univariate case. Let m and v be defined in (21.1) and (22.1), respectively. Furthermore, let $A = \{A(t) : 0 \le t \le 1\}$ be a Gaussian process with mean 0 and the covariance kernel

$$\mathbf{E}(A(s)A(t)) = \int_0^{s \wedge t} v \circ F^{-1}(u) du,$$

and let $\Gamma_{m,v,F}$ be a mean zero Gaussian process on $[0,1]$ defined by the formula

$$\Gamma_{m,v,F}(t) := A(t) - \int_0^t B(s) dm \circ F^{-1}(s)$$

with B denoting a Brownian bridge on $[0,1]$ independent of A.

Theorem 23.1 (Rao and Zhao [52]) *Let* $(X_1, Y_1), (X_2, Y_2), \ldots$ *be independent and identically distributed bivariate random vectors, and let* $\mathbf{E}(Y_1^2) < \infty$. *Furthermore, let* F *be continuous on the real line, and let* $m \circ F^{-1}$ *be continuous on the interval* $(0,1)$. *Finally, assume that either assumption* C_0 *or* C_1 *of Theorem* 22.1 *holds. Then*

$$n^{1/2}(\mathsf{GL}_n^{IND} - \mathsf{GL}_F^{IND}) \Rightarrow \Gamma_{m,v,F} \tag{23.1}$$

in the space $D[0,1]$ *equipped with the uniform topology.*

23.2 Multivariate case. Consider the case when $\mathbf{Y}_1$ is a random element in $\Re^d$. Let $\mathbf{m}(x)$, $\mathbf{V}(x)$, and $\mathbf{V}^{1/2}(x)$ be the same as in (22.3), (22.4), and (22.5), respectively. Define a mean zero Gaussian process $\mathbf{\Gamma}_{\mathbf{m},\mathbf{V},F}$ on $[0,1]$ by the formula

$$\mathbf{\Gamma}_{\mathbf{m},\mathbf{V},F} := \int_0^t \mathbf{V}^{1/2} \circ F^{-1}(s) dW(s) - \int_0^t B(s) d\mathbf{m} \circ F^{-1}(s),$$

where W is a Wiener process on $[0,1]$, and B is a Brownian bridge on $[0,1]$ independent of W

Theorem 23.2 (Davydov and Egorov [20]) *Let $(X_1,\mathbf{Y}_1),(X_2,\mathbf{Y}_2),\ldots$ be independent and identically distributed random vectors, and let $\mathbf{E}(\|\mathbf{Y}_1\|_2^2) < \infty$. Furthermore, let the distribution function F of X_1 be continuous, and let assumption (22.6) be satisfied. Then*

$$n^{1/2}(\mathsf{GL}_n^{IND} - \mathsf{GL}_F^{IND}) \Rightarrow \mathbf{\Gamma}_{\mathbf{m},\mathbf{V},F} \tag{23.2}$$

in the space $D^d[0,1]$ equipped with the uniform topology.

Davydov and Egorov [21] also considered weak convergence of the appropriately normalized process $\mathsf{GL}_n^{IND} - \mathsf{GL}_F^{IND}$ when the random vectors $\mathbf{Y}_1,\mathbf{Y}_2,\ldots$ are in the domain of attraction of a stable law.

24 GLC under random censorship: definition and consistency

Let $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$ be two sets of independent and non-negative random variables. Denote the distribution function of X_1 by F and that of Y_1 by H. Furthermore, let $Z_i := X_i \wedge Y_i$ and $\delta_i := \mathbf{I}_{\{X_i \le Y_i\}}$ for any $i = 1,\ldots,n$. Assuming that only the pairs $(Z_1,\delta_1),\ldots,(Z_n,\delta_n)$ are observed, we construct the Kaplan-Meier [39] product-limit estimator F_n^{KM} for F. The corresponding (empirical) generalized Lorenz curve GL_n is defined by the formula:

$$\mathsf{GL}_n^{KM}(t) := \int_0^t (F_n^{KM})^{-1}(s) ds.$$

With the notation

$$T_H := \inf\{x : H(x) = 1\} \tag{24.1}$$

we have the following theorem.

Theorem 24.1 (Csörgő, Csörgő, and Horváth [9]) *Let $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$ be independent and non-negative random variables. Then, for any fixed $t_0 < F(T_H)$,*

$$\sup_{0 \le t \le t_0} |\mathsf{GL}_n^{KM}(t) - \mathsf{GL}_F(t)| \to_{a.s.} 0.$$

25 GLC under random censorship: weak convergence

Csörgő, Csörgő, and Horváth [9] established a strong approximation result for the process $\sqrt{n}(\mathsf{GL}_n - \mathsf{GL}_F)$. To formulate it, let T_H be the same as in (24.1), and let $d_{H,F}$ be a function on $[0,1]$ defined by the formula

$$d_{H,F}(t) := \int_0^t \frac{1}{(1-s)^2(1 - H \circ F^{-1}(s))} ds.$$

Theorem 25.1 (Csörgő, Csörgő, and Horváth [9]) *On a suitably constructed probability space, the following holds. Let* $X_1, \dots, X_n$ *and* $Y_1, \dots, Y_n$ *be independent and non-negative random variables. Furthermore, let* t_0 *be any constant such that* $t_0 < F(T_H)$, *and let the quantile function* F^{-1} *be continuous on* $(0, t_0]$. *Then there exists a sequence* $(W_1, W_2, \dots)$ *of standard Wiener processes on* $[0, \infty)$ *such that*

$$\sup_{0 \le t \le t_0} |\sqrt{n}(\mathsf{GL}_n^{KM}(t) - \mathsf{GL}_F(t)) - \ell_{A_n,F}(t)| \to_P 0, \tag{25.1}$$

where A_n *is a Gaussian process defined by the formula*

$$A_n(t) := (1-t) W_n \circ d_{H,F}(t).$$

Theorem 25.1 implies that

$$\sqrt{n}(\mathsf{GL}_n^{KM} - \mathsf{GL}_F) \Rightarrow \ell_{A,F}$$

in the space $D[0, t_0]$ equipped with the uniform metric, where A is defined by the formula $A(t) := (1-t)W \circ d_{H,F}(t)$ with a standard Wiener process W on $[0, \infty)$. Note that when all the censoring random variables Y_i are equal to $+\infty$ almost surely, then $d_{H,F}(t) = 1/(1-t)$ and thus A is a Brownian bridge B on $[0, 1]$.

26 Related results: Vervaat process

Let $X_1, \dots X_n$ be identically distributed random variables, and let F be the distribution function of X_1. Furthermore, let F_n and F_n^{-1} be the corresponding empirical and quantile functions, and let e_n be the emprical process on $[0, 1]$ defined in (14.5). The (general) Vervaat process $\mathsf{V}_n = \{\mathsf{V}_n : 0 \le t \le 1\}$ is defined (cf. Csörgő and Zitikis [14]) on $[0, 1]$ by the equation:

$$\sqrt{n}(\mathsf{GL}_n - \mathsf{GL}_F) = \ell_{e_n,F} + \sqrt{n}\mathsf{V}_n, \tag{26.1}$$

where $\ell_{e_n,F}$ is defined by formula (13.2).

The process V_n was first defined for the uniform on $[0, 1]$ distribution function by Vervaat [59], [60] where he used V_n to demonstrate that, no matter how normalized, the Bahadur-Kiefer process does not converge to any non-degenerate process.

The (general) Vervaat process V_n for any distribution function F was introduced by Csörgő and Zitikis [14] (cf. Horváth [37] for similar ideas) where they demonstrated, assuming the independence of $X_1, \dots X_n$ and appropriate moment assumptions, that $\sqrt{n}\mathsf{V}_n$ is asymptotically smaller than $\ell_{e_n,F}$. This observation played a pivotal role in Csörgő and Zitikis [14] for deriving the LIL in Strassen's form for $\sqrt{n}(\mathsf{GL}_n - \mathsf{GL}_F)$ since the problem was reduced to the corresponding one for the relatively easy process $\ell_{e_n,F}$. A similar approach was used by Csörgő, Gastwirth, and Zitikis [11] to prove (weighted) weak convergence of the process $\sqrt{n}(\mathsf{GL}_n - \mathsf{GL}_F)$, and by Zitikis [65] to derive asymptotic properties of the empirical E-Gini index under minimal assumptions. A number of results were summarized in two survey papers by Zitikis [63], and Csörgő and Zitikis [15]. Further results concerning the Vervaat and Vervaat-error processes in the case of independent random variables $X_1, \dots X_n$ were obtained by Csörgő and Zitikis [16], Csáki, Csörgő, Földes, Shi and Zitikis [5]. In the case of dependent random variables $X_1, \dots X_n$, the Vervaat process V_n was investigated by Davydov and Zitikis [28], Davydov, Khoshnevisan, Shi and Zitikis [22].

27 Related results: Baxter-type theorems

When investigating various asymptotic properties of convexifications of stochastic processes, Davydov and Thilly [26] noticed that their methods can be used to obtain Baxter-type results for the processes (cf. Baxter [2]). In particular, with the notations and assumptions of Theorem 7.3, Davydov and Thilly [26] obtained the following result.

Theorem 27.1 (Davydov and Thilly [26]) *Let* $\xi = \{\xi(t), 0 \le t \le 1\}$ *be a mean zero Gaussian process with stationary increments, and let* $\xi(0) = 0$ *almost surely. Assume that* (7.2) *holds. Then*

$$\frac{1}{n\sigma_n^p} \sum_{k=1}^{n} \left| \xi\Big(\frac{k}{n}\Big) - \xi\Big(\frac{k-1}{n}\Big) \right|^p \to_{a.s.} \mathbf{E}(|N_{0,1}|^p) \tag{27.1}$$

when $n \to \infty$, *where* σ_n^2 *is the variance of* $\xi(1/n) - \xi(0)$.

In the case of the fractional Brownian motion W_α for any $0 < \alpha < 1$, Theorem 27.1 generalizes results of Baxter [2] obtained when $p = 2$ and $\alpha = 1/2$ (the Wiener process case), as well as those of Dasgupta and Kallianpur [18] obtained when p is even. Note that when $p = 1/\alpha$, then statement (27.1) in the case of $\xi = W_\alpha$ reduces to the following one:

$$\sum_{k=1}^{n} \left| W_\alpha\Big(\frac{k}{n}\Big) - W_\alpha\Big(\frac{k-1}{n}\Big) \right|^{1/\alpha} \to_{a.s.} \mathbf{E}(|N_{0,1}|^{1/\alpha}). \tag{27.2}$$

In particular, (27.2) proves that, for any $0 < \alpha < 1$, the fractional Brownian motion W_α has finite variation of order $1/\alpha$.

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