# On the role played by extreme summands when a sum of i.i.d. random vectors is asymptotically $\alpha$-stable

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**Abstract**

In the focus of our attention is the limit distribution of the sum of i.i.d. random vectors from which all the extreme summands are removed. The problem is rather trivial if the summands are ordered by their norms. It is of much more interest when as extremes the vertices of the convex hull generated by the vectors are taken.

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# 1 Introduction

It is convenient to begin with the one-dimensional case. Let $\xi, \xi_1, \xi_2, \dots$ be i.i.d. random variables and

$$\zeta_n = \xi_1 + \xi_2 + \cdots + \xi_n.$$

Arranging the summands in ascending order we obtain the so-called order statistics $\xi_{(1)}, \xi_{(2)}, \dots, \xi_{(n)}$. Denote by $\xi'_{(1)}, \xi'_{(2)}, \dots, \xi'_{(n)}$ the summands ordered by absolute value. Thus,

$$\xi_{(1)} \le \xi_{(2)} \le \cdots \le \xi_{(n)} \quad \text{and} \quad |\xi'_{(1)}| \le |\xi'_{(2)}| \le \cdots \le |\xi'_{(n)}|.$$

Truncate the sum $\zeta_n$ in the following ways

$$\zeta_n^{(k)} = \sum_{j=1}^{n-k} \xi'_{(j)}, \quad \zeta_n^{(m,k)} = \sum_{j=m+1}^{n-k} \xi_{(j)} \tag{1.1}$$

where $k \ge 1, m \ge 1$ are fixed.

The question is: *what is the limit distribution of the truncated sums?* Obviously, if $\mathrm{E}\xi^2 < \infty$ then the truncation does not affect the limit distribution i.e. $\zeta_n$, $\zeta_n^{(k)}$ and $\zeta_n^{(m,k)}$ being equally centered and normalized converge in distribution to a random variable having the standard normal distribution. The situation is quite different when the distribution of $\xi$ belongs to the domain of attraction of a non-gaussian stable law. In this case the tail distribution function $\mathrm{P}\{|\xi| > x\}$ is of regular variation as $x \to \infty$ i.e.

$$\mathrm{P}\{|\xi| > x\} = x^{-\alpha} l(x) \tag{1.2}$$

where the so-called characteristic exponent $\alpha \in (0, 2)$ and $l(x)$ slowly varies as $x \to \infty$. Furthermore, there exists

$$\lim_{x \to \infty} \frac{\mathrm{P}\{\xi < -x\}}{\mathrm{P}\{|\xi| > x\}} = c_1 \in [0, 1] \tag{1.3}$$

(see e.g. Ibragimov and Linnik (1971, Ch. 2) or Feller (1971, Ch. 17)).

In order to avoid purely technical complications we assume for time being that $\alpha \in (0, 1) \cup (1, 2)$. Moreover, when $1 < \alpha < 2$ we assume that $\mathrm{E}\xi = 0$.

Define

$$b_n = \inf\{t \mid \ t^{-\alpha} l(t) \le 1/n\}.$$

It is well known (see e.g. Ibragimov and Linnik (1971, Ch. 2)) that under such a choice of $b_n$ the variable $b_n^{-1}\zeta_n$ converges in distribution to a random variable whose characteristic function $\phi(t)$ is such that

$$\ln \phi(t) = \begin{cases} \alpha c_1 \int_{-\infty}^0 \frac{e^{\imath ts} - 1}{|s|^{\alpha+1}} \mathrm{d}s + \alpha c_2 \int_0^\infty \frac{e^{\imath ts} - 1}{s^{\alpha+1}} \mathrm{d}s & \text{if} \quad 0 < \alpha < 1 \\ \\ \alpha c_1 \int_{-\infty}^0 \frac{e^{\imath ts} - 1 - \imath ts}{|s|^{\alpha+1}} \mathrm{d}s + \alpha c_2 \int_0^\infty \frac{e^{\imath ts} - 1 - \imath ts}{s^{\alpha+1}} \mathrm{d}s & \text{if} \quad 1 < \alpha < 2 \end{cases} \tag{1.4}$$

where $c_1$ is as in (1.3) and $c_2 = 1 - c_1$.

Consider the distribution function

$$G_k(z) = \frac{1}{(k-1)!} \int\limits_{z^{-\alpha}}^{\infty} r^{k-1} e^{-r} \mathrm{d}r. \tag{1.5}$$

Let $\min(c_1, c_2) > 0$. It is easy to prove that as $n \to \infty$

$$\mathrm{P}\{-\xi_{(m)} < ub_n,\ \xi_{(n-k+1)} < vb_n\} \to G_m^{(1)}(u) G_k^{(2)}(v)$$

uniformly in $u,\ u \geq 0$, and $v,\ v \geq 0$. Here

$$G_k^{(i)}(u) = G_k(c_i^{-1/\alpha} u),\ i = 1, 2.$$

If, say, $c_1 = 0$ then $b_n^{-1} \xi_{(m)}$ converges in probability to zero while

$$\mathrm{P}\{\xi_{(n-k+1)} < vb_n\} \to G_k^{(2)}(v).$$

It means that the extreme summands are of the same order as the whole sum and, therefore, their role is essential.

That role was studied by many authors (see e.g. Darling, (1952), Bobrov and Arov (1960), Teugels (1981)). In particular, it was established that for $0 < \alpha < 1$

$$\lim_{n\to\infty} \mathrm{P}\{b_n^{-1} \zeta_n^{(k)} < x,\ b_n^{-1} |\xi'_{(n-k+1)}| < u\} = \int\limits_0^u F(x; z) \mathrm{d}G_k(z), \tag{1.6}$$

where $x \to F(x; z)$ is the distribution function of the infinitely divisible law such that the logarithm of its characteristic function is of the form

$$h(t; z) = \alpha c_1 \int_{-z}^{0} \frac{e^{\imath ts} - 1}{|s|^{\alpha+1}} \mathrm{d}s + \alpha c_2 \int_0^z \frac{e^{\imath ts} - 1}{s^{\alpha+1}} \mathrm{d}s \tag{1.7}$$

But if $1 < \alpha < 2$ then instead of (1.7) we have

$$h(t; z) = \imath t (c_2 - c_1) \frac{\alpha}{\alpha - 1} z^{-\alpha} + \alpha c_1 \int_{-z}^{0} \frac{e^{\imath ts} - 1 - \imath ts}{|s|^{\alpha+1}} \mathrm{d}s + \alpha c_2 \int_0^z \frac{e^{\imath ts} - 1 - \imath ts}{s^{\alpha+1}} \mathrm{d}s. \tag{1.8}$$

Thus, the limit distribution for $\zeta_n^{(k)}$ is a mixture of of infinitely divisible laws. Furthermore, the mixing parameter is the level on which the Poisson spectrum of the stable law is truncated while $G_k(z)$ serves as a weight function.

The second way of truncation was considered in Khamdamov and Nagaev (2002). In particular, the authors of the referred paper established the following fact. If $\min(c_1, c_2) > 0$ then as $n \to \infty$

$$\begin{aligned} &\mathrm{P}\{\zeta_n^{(m,k)} < xb_n,\ -\xi_{(m)} < ub_n,\ \xi_{(n-k+1)} < vb_n\} \\ &= \textstyle\int_0^u \int_0^v F(x; y, z) \mathrm{d}G_m^{(1)}(y) \mathrm{d}G_k^{(2)}(z) + o(1) \end{aligned} \tag{1.9}$$

uniformly in $u,\ u \geq 0$, and $v,\ v \geq 0$. Here $F(x; y, z),\ y > 0,\ z > 0$, is the two-parameters family of infinitely divisible laws such that the logarithms of the characteristic functions are of the form

$$h(t; y, z) = \begin{cases} \alpha c_1 \int_{-y}^{0} \frac{e^{\imath ts}-1}{|s|^{\alpha+1}} \mathrm{d}s + \alpha c_2 \int_0^z \frac{e^{\imath ts}-1}{s^{\alpha+1}} \mathrm{d}s & \text{if} \quad 0 < \alpha < 1 \\ \imath t \frac{\alpha}{\alpha-1}(c_2 z^{-\alpha} - c_1 y^{-\alpha}) + \alpha c_1 \int_{-y}^{0} \frac{e^{\imath ts}-1-\imath ts}{|s|^{\alpha+1}} \mathrm{d}s & \\ \qquad + \alpha c_2 \int_0^z \frac{e^{\imath ts}-1-\imath ts}{s^{\alpha+1}} \mathrm{d}s & \text{if} \quad 1 < \alpha < 2. \end{cases}$$

Here in contrast to (1.7) and (1.8) the Poisson measure of the stable law determined by (1.4) is truncated in an asymmetric way. Intuitively, we expect that similar phenomenon could be observed in higher dimensions. The basic goal of the present paper is to verify this conjecture. It should be noted that the results established in Kalinauskaitė (1973) testify to that conjecture.

The paper is organized as follows. In Section 2 the main results are stated. They are proved in Sections 4 and 5. Section 3 contains auxiliary facts from the point processes theory. Concluding remarks are given in Section 6.

# 2 Main results

In what follows we adhere the following notation. The inner product in $I\!R^d$ is denoted by $\langle \cdot, \cdot \rangle$. The unit vector $|x|^{-1}x$ is denoted by $e_x$. By $B(\mathbf{0}, r)$ and $B^c(\mathbf{0}, r)$ we denote, respectively, the closed ball of radius $r$ centered at origin and its complement.

Consider i.i.d. $d$-dimensional random vectors $\xi$, $\xi^{(1)}, \xi^{(2)}, \ldots$. Assume that the common distribution belongs to the domain of attraction of an $\alpha$-stable law.

Let, first, $\alpha \in (0,1) \cup (1,2)$. It is well known that the distribution of $\xi$ belongs to the domain of attraction of a non-gaussian $\alpha$-strictly stable distribution if the sequence of measures

$$P_n(A) = \mathrm{P}\left\{ b_n^{-1}(\xi^{(1)} + \cdots + \xi^{(n)}) \in A \right\}, \quad A \in \mathcal{B}_{I\!R^d},$$

weakly converges to a non-degenerate limit distribution. In this case $b_n = n^{1/\alpha} h(n)$, where $h(r)$ slowly varies as $r \to \infty$ and the limiting distribution $S_\alpha$ is determined by the characteristic function $\varphi(t)$ which admits the following representation

$$\ln \varphi(t) = \begin{cases} \int\limits_{I\!R^d} (e^{\imath\langle t,s\rangle} - 1) Q_\alpha(\mathrm{d}s) & \text{if} \quad \alpha \in (0,1) \\ \\ \int\limits_{I\!R^d} (e^{\imath\langle t,s\rangle} - 1 - \imath\langle t,s\rangle) Q_\alpha(\mathrm{d}s) & \text{if} \quad \alpha \in (1,2). \end{cases} \tag{2.10}$$

Here $Q_\alpha = \tau \times \theta$ is the product measure ($I\!R^d$ is identified with $S^{d-1} \times (0,\infty)$) such that $\tau$ is a bounded measure defined on the $\sigma$-algebra $\mathcal{B}_{S^{d-1}}$ of the Borel subsets of $S^{d-1}$ while $\theta$ is defined on the Borel subsets of $(0,\infty)$ and

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}(r) = \alpha r^{-\alpha-1}.$$

More precisely,

$$Q_\alpha \left\{ u \in R^d \mid \ |u| > r,\ e_u \in E \subset S^{d-1} \right\} = \tau(E) r^{-\alpha}, \quad E \in \mathcal{B}_{S^{d-1}}. \tag{2.11}$$

By $\lambda^d,\ d \geq 1$, we denote the Lebesgue measure in $I\!R^d$ setting for simplicity $\lambda^1 = \lambda$. The measure $\tau$ is frequently called the *Poisson spectral* measure. We remind that $e_u = |u|^{-1}u$.

Within the framework of the considered problems the general form (2.10) of the Levy-Khinchine representation seems to be more convenient than the usually utilized one (see e.g. Samorodnitsky and Taqqu (1994, Th. 2.3.1)).

Denote by $S_\tau$ the support of $\tau$. It should be noted that a stable law is strictly $d$-dimensional if and only if $S_\tau$ spans $I\!R^d$.

It is known that $\xi$ belongs to the domain of attraction of $S_\alpha$ if and only if (see e.g. Rvacheva (1962))

$$\mathrm{P}\{|\xi| > r\} = r^{-\alpha} l(r), \tag{2.12}$$

where $l(r)$ slowly varies as $r \to \infty$ and the measures

$$\nu_r(E) = \mathrm{P}\{e_\xi \in E \mid |\xi| > r\}, \quad E \in \mathcal{B}_{S^{d-1}}, \tag{2.13}$$

weakly converge as $r \to \infty$ to a measure $\nu$.

It is worth noting also that the measures in (2.11) and (2.13) relate in the following way

$$\nu(E) = \left(\tau(S^{d-1})\right)^{-1} \tau(E).$$

With no loss of generality we may assume that $\tau(S^{d-1}) = 1$.

So, the strict $d$-dimensionality of the limit stable law is determined by the tail behaviour of the underlying distribution.

Let us choose

$$b_n = \inf\{r \mid \ r^{-\alpha} l(r) \leq 1/n\}. \tag{2.14}$$

Denote
$$\xi_*^{(n)} = \max(|\xi^{(1)}|, \ldots, |\xi^{(n)}|).$$
It is easily seen that (cf. e.g. Resnick (1987, Ch.5)) as $n \to \infty$

$$\mathrm{P}\{b_n^{-1}\xi_*^{(n)} < r\} \to \exp\left(-Q_\alpha(B^c(\mathbf{0}, r))\right) = \exp(-r^{-\alpha})$$

uniformly in $r > 0$.

Thus, as in the one-dimensional case the normalizing sequences for $\zeta^{(n)} = \xi^{(1)} + \cdots + \xi^{(n)}$ and $\xi_*^{(n)} \max(|\xi^{(1)}|, \ldots, |\xi^{(n)}|)$ coincide. In other words the maximal summand is comparable with the whole sum. Furthermore, the measures

$$\nu_n^*(E) = \mathrm{P}(e_{\xi_*^{(n)}} \in E), \;\; E \in \mathcal{B}_{S^{d-1}},$$

weakly converge as $n \to \infty$ to $\nu$. Loosely speaking, it implies that asymptotically the points of maximal norm lie only in the cone $K(S_\tau)$ generated by $S_\tau$.

Note that extremity of summands may be understood in various ways. Here we focus on a way concerned with the following peeling procedure corresponding to the second type of truncation in (1.1). Within the framework of such an approach to truncating the following tail property of the underlying distribution plays a crucial role.

**Definition 2.1** *Let the distribution* $\mathrm{P}\{\xi \in A\}$ *belong to the domain of attraction of a strictly d-dimensional stable law. We say that the distribution and corresponding to it Poisson measure are* non-unilateral *if the minimal closed cone* $\bar{K}(S_\tau)$ *containing* $S_\tau$ *coincides with* $\mathbb{R}^d$.

Obviously, the condition $\min(c_1, c_2) > 0$ in (1.9) implies that the corresponding one-dimensional underlying distribution is non-unilateral.

Now define the set $\Xi_n$ as

$$\Xi_n = \Xi_{n,0} := \{b_n^{-1}\xi^{(1)}, \ldots, b_n^{-1}\xi^{(n)}\} = \{\hat{\xi}^{(1)}, \ldots, \hat{\xi}^{(n)}\}.$$

Denote by $C_{n,0}$ the convex hull generated by $\Xi_{n,0}$ and let $V_{n,0} = \{\hat{\xi}^{(j)} \mid \hat{\xi}^{(j)} \in \partial C_{n,0}\}$. Obviously, $C_{n,0}$ is a multidimensional analog of the interval $b_n^{-1}(\xi_{(1)}, \xi_{(n)})$. Put $\Xi_{n,1} = \Xi_{n,0} \setminus V_{n,0}$. Let $C_{n,1}$ be the convex hull generated by $\Xi_{n,1}$ while $V_{n,1} = \{\hat{\xi}^{(j)} \mid \hat{\xi}^{(j)} \in \partial C_{n,1}\}$. Then $C_{n,1}$ stands for an analog of $b_n^{-1}(\xi_{(2)}, \xi_{(n-1)})$. Note that the cardinalities $\#(V_{n,0})$ and $\#(V_{n,1})$ are random. Having implemented $k$ steps of this procedure we arrive to a set $\Xi_{n,k}$. Define

$$\zeta^{(n,k)} = \begin{cases} b_n^{-1} \sum\limits_{\hat{\xi}^{(j)} \in \Xi_{n,k}} \xi^{(j)} & \text{if} \quad \Xi_{n,k} \neq \emptyset \\ \\ 0 & \text{otherwise.} \end{cases} \tag{2.15}$$

This sum serves as an analogue of $\zeta_n^{(k,k)}$ in (1.1).

If the underlying distribution $\mathrm{P}\{\xi \in A\}$ is not unilateral then for any fixed $k$ and all sufficiently large $n$

$$\mathbf{0} \in \text{int } C_{n,k}, \qquad \inf\{|x| \mid \ x \in \partial C_{n,k}\} \text{ a.s.}$$

and as $n \to \infty$

$$\#(V_{n,i}) = O_p(1), \ i = 0, 1, \ldots, k.$$

If the measure $\mathrm{P}\{\xi \in A\}$ is absolutely continuous with respect to $\lambda^d$ or, at least, it has no atoms then a.s. $V_{n,i}$ contains only the vertices of $C_{n,i}, \ i = 0, 1, \ldots, k$.

It is worth emphasizing that peeling implies removing of a random number of summands. In contrast to the one-dimensional case where the number of the removed summands is fixed.

In order to state our main result we have to introduce a Poisson point process (p.p.p.) $\pi$ with the intensity measure $Q_\alpha$ defined by (2.11).

Let $D_\pi$ denote the set of points of $\pi$ or, in other words, $D_\pi$ supports the random measure $\pi$. We may apply to $D_\pi$ the same peeling procedure as in the case of the finite set $\Xi_{n,0}$. As a result we obtain the sequence of sets $D_0 = D_\pi, \ D_1, \ D_2, \ldots,$ the sequence of their convex hulls $C_0, \ C_1, \ C_2, \ldots$ and the sets of boundary points $V_k = \{x \mid \ x \in \partial C_k \cap D_\pi\}, \ k = 0, 1, 2, \ldots$ . Furthermore, $D_i = D_{i-1} \setminus V_{i-1}$. If $\tau$ is non-unilateral then each $C_k$ a.s. contains the origin and is a polyhedron.

Since $Q_\alpha = \tau \times \theta$ where $\theta$ is absolutely continuous $V_k$ is a.s. the set of vertices of $C_k$.

For any convex polyhedron $L$ we denote by $V_L$ and $\nu_L$, respectively, the set and the total number of its vertices i.e. $\nu_L = \#(V_L)$. Let

$$\mathcal{L}_k = \Big\{L \mid \ L \text{ is a convex polyhedron}, \ \nu_L = k, \ \lambda^d(L) > 0\Big\}$$

and

$$\mathcal{L} = \bigcup_{k=d+1}^{\infty} \mathcal{L}_k.$$

Denote by $\mathcal{K}$ the space of all compact convex subsets of $\mathbb{R}^d$ equipped with the Hausdorff metric. Consider the closed set

$$F_m = \{L \mid \ L \text{ is a convex polyhedron}, \ \nu_L \leq m, \}$$

and the open set

$$F = \Big\{A \in \mathcal{K} \mid \ \lambda^d(A) > 0\Big\}.$$

Obviously,

$$\mathcal{L} = \cap_{m=1}^{\infty} F_m \cup F.$$

It implies that $\mathcal{L}$ is a borelian subset of $\mathcal{K}$. Furthermore, $\mathcal{L}$ is equipped in the usual way with the induced topology and the induced measurable structure.

It is worth comparing the following statement with (1.9).

**Theorem 2.2** *If the underlying distribution is not unilateral then as $n \to \infty$ the distribution of $\zeta^{(n,k)}$ converges weekly to the measure $N_k$ which admits the following representation*

$$N_k(A) = \int_{\mathcal{L}} G_L(A) H_{k-1}(\mathrm{d}L), \quad A \in \mathcal{B}_{I\!R^d},$$

*where $G_L$ is infinitely divisible distribution whose characteristic function $\varphi_L(t)$ is such that*

$$\ln \varphi_L(t) = \begin{cases} \int_L (e^{\imath\langle t,s\rangle} - 1) Q_\alpha(\mathrm{d}s) & \text{if} \quad \alpha \in (0,1) \\ \\ \int_L (e^{\imath\langle t,s\rangle} - 1 - \imath\langle t,s\rangle) Q_\alpha(\mathrm{d}s) - \imath \int_{L^c} \langle t,s\rangle Q_\alpha(\mathrm{d}s) & \text{if} \quad \alpha \in (1,2) \end{cases}$$

*and $H_k$ is the distribution of $C_k$.*

From Theorem 2.2 and Lemmas 3.4 and 3.5 below it follows that the limit distribution of $\zeta^{(n,k)}$ is the mixture of infinitely divisible distributions in which the mixing "parameter" is a convex polyhedron $L$ and the weight measure $H_{k-1}$ is the distribution of the $k$-th convex hull $C_{k-1}$. Unfortunately, it is not clear how to find a closed form for this distribution.

Now let $\alpha = 1$. The limit stable law is strictly $\alpha$-stable if and only if the Poisson measure is symmetric i.e. $\tau(-E) = \tau(E),\ E \in \mathcal{B}_{S^{d-1}}$. The distribution of $\xi$ belongs to the domain of attraction of such a limit law if the sequence of measures

$$P_n(A) = \mathrm{P}\left\{ b_n^{-1}(\xi^{(1)} + \cdots + \xi^{(n)} - a^{(n)}) \in A \right\}, \quad A \in \mathcal{B}_{I\!R^d},$$

weakly converges to a non-degenerate limit distribution. In this case $b_n = nh(n),\ \ h(r)$ slowly varies as $r \to \infty$ and

$$a^{(n)} = \int_{B(\mathbf{0},1)} s\mathrm{P}\{b_n^{-1}\xi \in \mathrm{d}s\}.$$

The limiting distribution $S_1$ is determined by the characteristic function $\varphi(t)$ which admits the following representation (cf. (2.10)

$$\ln \varphi(t) = \int_{I\!R^d} (e^{\imath\langle t,s\rangle} - 1 - \imath\delta(\langle t,s\rangle)) Q_1(\mathrm{d}s) \tag{2.16}$$

where

$$\delta(z) = \begin{cases} 1 & \text{if} \quad |z| \le 1 \\ \\ 0 & \text{if} \quad |z| > 1. \end{cases}$$

In contrast to (2.15) we have to deal with a centered sum

$$\zeta_1^{(n,k)} = \begin{cases} b_n^{-1} \sum\limits_{\hat{\xi}^{(j)} \in \Xi_{n,k}} \xi^{(j)} - \gamma_{n,k} & \text{if} \quad \Xi_{n,k} \neq \emptyset \\ \\ 0 & \text{otherwise} \end{cases} \tag{2.17}$$

where

$$\gamma_{n,k} = \#(\Xi_{n,k-1}) \int\limits_{C_{n,k-1}} s\mathrm{P}\{b_n^{-1}\xi \in \mathrm{d}s\}.$$

**Theorem 2.3** *Let $\alpha = 1$ and $\tau$ is symmetric. If the underlying distribution is d-dimensional then as $n \to \infty$ the distribution of $\zeta_1^{(n,k)}$ converges weekly to the measure $N_k$ which admits the following representation*

$$N_k(A) = \int\limits_{\mathcal{L}} \bar{G}_L(A) H_{k-1}(\mathrm{d}L), \quad A \in \mathcal{B}_{\mathbb{R}^d},$$

*where $\bar{G}_L$ is infinitely divisible distribution whose characteristic function $\bar{\varphi}_L(t)$ is such that*

$$\ln \bar{\varphi}_L(t) = \int\limits_L (e^{\imath\langle t,s\rangle} - 1 - \imath\langle t,s\rangle) Q_1(\mathrm{d}s)$$

*and $H_k$ is the distribution of $C_k$.*

It is easily seen that the centering variable $\gamma_{n,k}$ can be replaced by

$$\gamma_n = n \int\limits_{C_{n,k-1}} s\mathrm{P}\{b_n^{-1}\xi \in \mathrm{d}s\}.$$

# 3 Point processes

In order to prove our basic result we have to make use of some notions and facts of the general theory of point processes (see e.g. Resnick (1987)).

As a phase space in which the points are scattered we take $\mathbb{E} = \hat{\mathbb{R}}^d \setminus \{\mathbf{0}\}$ where $\hat{\mathbb{R}}^d$ is the compactification of $\mathbb{R}^d$ by $\{\infty\}$. The space is equipped with the relative topology. Denote by $\mathcal{B}_{\mathbb{E}}$ the $\sigma$-algebra of the Borel subsets of $\mathbb{E}$. A subset $A$ is relatively compact if and only if $A \cap B(\mathbf{0}, r) = \emptyset$ for some $r > 0$.

Denote by $\mathcal{M}(\mathbb{E})$ the family of simple point Radon measures i.e. of those having the form

$$\mu = \sum_{x \in D} \delta_x$$

where $\delta_x$ is the delta-measure concentrated at $x$ while $D$ is such that for any $r > 0$ the set $D \cap B^c(\mathbf{0}, r)$ is finite. Obviously, $D$ supports $\mu$ and we write $D = \text{supp } \mu$. Denote by $\mathcal{H}(\mathbb{E})$ the set of supports which corresponds to $\mathcal{M}(\mathbb{E})$. We call the elements of $\mathcal{H}(\mathbb{E})$ *configurations.*

Denote by $\mathcal{C}^+ = \mathcal{C}^+_{\mathcal{K}}(\mathbb{E})$ the class of all continuous compactly supported positive functions defined on $\mathbb{E}$. We say that $\mu_n$ converges vaguely to $\mu$ if for any $f \in \mathcal{C}^+$

$$\int f d\mu_n \to \int f d\mu.$$

Obviously, $\mathcal{M}(\mathbb{E})$ equipped with the vague topology becomes a topological space. It is well known that this space is metrizable and, moreover, is a Polish space.

Note that $\mathcal{B}_{\mathcal{M}(\mathbb{E})}$, the $\sigma$-algebra of Borel subsets of $\mathcal{M}(\mathbb{E})$, coincides with the $\sigma$-algebra generated by the mappings

$$\mu \to \mu(A), \quad A \in \mathcal{B}_{\mathbb{E}}.$$

The canonical one-to-one mapping $I$ of $\mathcal{M}(\mathbb{E})$ onto $\mathcal{H}(\mathbb{E})$, $I(\mu) = \text{ supp } \mu$, transfers the topology of $\mathcal{M}(\mathbb{E})$ over $\mathcal{H}(\mathbb{E})$ and makes the latter a Polish space. It is easily seen that the vague convergence of $\mu_n$ to $\mu$ can be restated in terms of the corresponding supports in the following way. Let $D = \{x^{(1)}, x^{(2)}, \dots\}$ be a configuration with somehow labeled points. The vague convergence implies that the points of $D_n = \{x_n^{(1)}, x_n^{(2)}, \dots\}$ can be labeled so that $x_n^{(k)} \to x^{(k)}$ as $n \to \infty$, for all $k$.

A simple point process $\eta$ is defined as a measurable mapping of the underlying probability space $(\Omega, \mathcal{F}, \mathcal{P})$ to $\left(\mathcal{M}(\mathbb{E}), \mathcal{B}_{\mathcal{M}(\mathbb{E})}\right)$. The measure $P_\eta = \mathcal{P}\eta^{-1}$ defined on $\mathcal{B}_{\mathcal{M}(\mathbb{E})}$ is called the distribution of $\eta$.

The weak convergence $\eta_n \overset{w}{\Rightarrow} \eta$ of a sequence of points processes is defined as the weak convergence $P_{\eta_n} \overset{w}{\Rightarrow} P_\eta$ of their distributions.

Thus, the metrization of $\mathcal{M}(\mathbb{E})$, mentioned above, provides a power tool for solving the problem of weak convergence of point processes. It enables one to apply the well-developed theory of weak convergence of probability measures in Polish spaces.

For the sake of simplicity assume that the $\sigma$-algebra $\mathcal{B}_{\mathcal{M}(\mathbb{E})}$ is completed with respect to the measure $P_\eta$.

Consider the mapping $C_k : \ \mathcal{H}(\mathbb{E}) \to \mathcal{K}$. It maps a configuration $D$ to its $k$-th convex hull which appeared in the way described in Section 2. It is well defined because $\forall \varepsilon > 0$ the set $D \cap B^c(\mathbf{0}, \varepsilon)$ is finite. It is easily verified that the mapping $C_0$ is continuous.

Recall that $Q_\alpha = \tau \times \theta$ where $\theta$ is absolutely continuous. It implies that that our point process $\eta$ is such that with probability one for any $k$ points $x^{(1)}, \dots, x^{(k)}$ from its support we have

$$x^{(k)} \notin \text{span}\{x^{(1)}, \dots, x^{(k-1)}\}.$$

Then for any $k$ the mapping $C_k$ is $P_\eta$-a.s. continuous and thus measurable. Therefore, $\eta_n \overset{w}{\Rightarrow} \eta$ implies the weak convergence of all the convex hulls, that is as $n \to \infty$

$$C_k\left(I(\eta_n)\right) \overset{w}{\Rightarrow} C_k\left(I(\eta)\right), \quad k = 0, 1, \dots .$$

In the sequel we omit the letter $I$ and use the same notation both for the p.p. and for its configuration.

Let us turn to our concrete scheme. Assume that the underlying distribution is absolutely continuous with a density $p(x)$ i.e.

$$\mathrm{P}\{\xi \in A\} = \int_A p(x)\lambda^d(\mathrm{d}x), \quad A \in \mathcal{B}_{I\!R^d}. \tag{3.18}$$

Within the framework of Theorem 2.2 such an assumption leads to no loss of generality but essentially simplifies arguments. In particular, from (3.18) it follows that the random set

$$\Xi_n = \{b_n^{-1}\xi^{(1)}, \dots, b_n^{-1}\xi^{(n)}\}$$

contains a.s. exactly $n$ points. We call any such set $D = D_n \subset I\!R^d$, $\#(D) = n$, the $n$-configuration.

Define the random measure

$$\beta_n(A) = \#(A \cap \Xi_n), \; A \in \mathcal{B}_{I\!R^d},$$

which is called a binomial point process (b.p.p.). Obviously,

$$\mathrm{P}\{\beta_n(A) = k\} = \tbinom{n}{k}\,\mathrm{P}^k\{b_n^{-1}\xi \in A\}\mathrm{P}^{n-k}\{b_n^{-1}\xi \notin A\}, \quad k = 0, 1, \dots, n.$$

Under the conditions of Theorem 2.2, as $n \to \infty$, $\beta_n$ weakly converges to $\pi$ (see e.g. Resnick (1987, Prop. 3.21)) The joint density of the random vector $b_n^{-1}(\xi^{(1)}, \dots, \xi^{(n)})$ has the form $\prod_{j=1}^{n} \bar{p}(x^{(j)})$ where $\bar{p}(x) = b_n p(b_n x), \; x \in I\!R^d$. It is convenient to define the density of $\Xi_n$. Note that a $d \times n$-matrix

$$X_n = (x^{(1)}, \dots, x^{(n)}), \; x^{(j)} \in I\!R^d, \; j = 1, 2, \dots, n,$$

can be regarded as an element of $I\!R^{nd}$. Let $\mathcal{X} \subset \mathcal{B}_{I\!R^{nd}}$ be such that for any $X_n \in \mathcal{X}$

$$|x^{(1)}| < |x^{(2)}| < \cdots < |x^{(n)}|.$$

Define the measure $\bar{\lambda}^{nd} = \bar{\lambda}^{nd}(A \cap \mathcal{X}), \; A \in \mathcal{B}_{I\!R^{nd}}$. Obviously, it is supported by $\mathcal{X}$. Arranging the elements of $\Xi_n$ in the ascending order of their norms yields

$$\bar{\Xi}_n = \{b_n^{-1}\bar{\xi}^{(1)}, \dots, b_n^{-1}\bar{\xi}^{(n)}\}$$

where

$$|\bar{\xi}^{(1)}| < |\bar{\xi}^{(2)}| < \cdots < |\bar{\xi}^{(n)}| \text{ a.s. .}$$

It is worth emphasizing that $\Xi_n$ and $\bar{\Xi}_n$ considered as subsets of $I\!R^d$ are identical.

Now we define the distribution density of $\Xi_n$ or, equally, of $\bar{\Xi}_n$ as

$$\mathbf{p}_n(D_n) = n! \prod_{j=1}^{n} \bar{p}(x^{(j)}) = n! \prod_{x \in D_n} \bar{p}(x).$$

Let, as before, $V_{n,j}$ be the set of the vertices of the j-th convex hull $C_{n,j} = C_j(\Xi_n)$ which is a result of peeling. The proof of the following statement is evident.

**Lemma 3.1** *Let (3.18) be fulfilled. Then conditional density of the $(n-m)$-configuration $\Xi_n \setminus \bigcup_{j=0}^{k-1} V_{n,j}$ at a 'point' $D_{n-m}$ given that*

$$\sum_{j=0}^{k-1} \#(V_{n,j}) = m,\ C_{n,k-1} = L \in \mathcal{L}$$

*is of the form*

$$\mathbf{p}_{n-m}(D_{n-m}) = \begin{cases} (n-m)! \prod\limits_{x \in D_{n-m}} (\bar{p}(x)/\mathrm{P}\{b_n^{-1}\xi \in L\}) & \text{if } D_{n-m} \subset \operatorname{int} L \\ \\ 0 & \text{otherwise.} \end{cases}$$

From Lemma 3.1 we immediately come to the following statement.

**Corollary 3.2** *Let $\alpha \neq 1$,*

$$\varphi_{n,L}(t) \stackrel{def}{=} \frac{\int_L e^{\imath\langle t,s\rangle} \mathrm{P}\{b_n^{-1}\xi \in \mathrm{d}s\}}{\mathrm{P}\{b_n^{-1}\xi \in L\}}, \quad L \in \mathcal{L},$$

*and $\zeta^{(n,k)}$ be defined as in (2.15). If $\Xi_{n,k} \neq \emptyset$ then*

$$\mathrm{E}\ \{\exp(\imath\langle t, \zeta^{(n,k)}\rangle) \mid\ C_{n,0} \in \mathcal{L}_{n_0}, \ldots, C_{n,k-2} \in \mathcal{L}_{n_{k-2}}, C_{n,k-1} = L \in \mathcal{L}_{n_{k-1}}\} = \varphi_{n,L}^{n-m}(t)$$

*where $m = n_0 + \cdots + n_{k-1}$.*

**Corollary 3.3** *Let $\alpha = 1$ and*

$$\bar{\varphi}_{n,L}(t) \stackrel{def}{=} \frac{e^{-\imath\langle\gamma_L,t\rangle} \int_L e^{\imath\langle t,s\rangle} \mathrm{P}\{b_n^{-1}\xi \in \mathrm{d}s\}}{\mathrm{P}\{b_n^{-1}\xi \in L\}}, \quad L \in \mathcal{L},$$

*where*
$$\gamma_L = \int\limits_L s\mathrm{P}\{b_n^{-1}\xi \in \mathrm{d}s\}.$$

*Let further* $\zeta_1^{(n,k)}$ *be defined as in (2.17). If* $\Xi_{n,k} \neq \emptyset$ *then*

$$\mathrm{E}\left\{\exp(\imath\langle t, \zeta_1^{(n,k)}\rangle) \mid C_{n,0} \in \mathcal{L}_{n_0}, \ldots, C_{n,k-2} \in \mathcal{L}_{n_{k-2}}, C_{n,k-1} = L \in \mathcal{L}_{n_{k-1}}\right\} = \varphi_{n,L}^{n-m}(t)$$

*where* $m = n_0 + \cdots + n_{k-1}$.

We emphasize that utilization of Lemma 3.1 and Corollaries 3.2 and 3.3 makes the proofs of Theorems 2.2 and 2.3 much simpler.

Let $\pi$ be the p.p.p. with the control measure $Q_\alpha$ described earlier. It is not difficult to establish the weak convergence
$$\Xi_n \Rightarrow \pi$$
(see e.g. Resnick (1987, Prop. 3.21)).

The next two statements play an important role in what follows.

**Lemma 3.4** *Let* $\bar{H}_{n,k}$ *be the joint distribution of* $(C_{n,0}, \ldots, C_{n,k}) = (C_0(\Xi_n), \ldots, C_k(\Xi_n))$ *and* $\bar{H}_k$ *be that of* $(C_0, \ldots, C_k) = (C_0(\pi), \ldots, C_k(\pi))$. *As* $n \to \infty$

$$\bar{H}_{n,k} \overset{w}{\Rightarrow} \bar{H}_k.$$

**Proof.** Since $\Xi_n \overset{w}{\Rightarrow} \pi$ the statement immediately follows from the $P_\pi$-a.s. continuity of the mappings $C_j,\ j = 0, \ldots, k$.

**Lemma 3.5** *The b.p.p. restricted to* $C_{n,k}$ *weakly converges to the p.p.p.* $\pi$ *restricted to* $C_k$.

**Proof.** Consider the mapping $f:\ \mathcal{H}(E) \to \mathcal{H}(E)$ defined as

$$f(D) = C_k \cap D.$$

It is easily seen that $f$ is $P_\pi$-a.s. continuous. It implies that

$$f(\Xi_n) \overset{w}{\Rightarrow} f(\pi).$$

The lemma is proved.

# 4 Proof of Theorem 2.2

It is convenient to give in the very beginning the general scheme of the proof.

In order to apply Lemma 3.1 and Corollary 3.2 we, first, smoothen the underlying distribution. It helps us to avoid superfluous technical complications. Then, on the basis of Corollary 3.2 we make use of the method of characteristic functions. It is applied to the conditional distribution of the truncated sum given the first $k$ convex hulls $C_{n,0}, \dots, C_{n,k-1}$. The convergence in distribution of those convex hulls follows from the general facts of the theory of point processes discussed in Section 3 (see Lemmas 3.4 and 3.5).

Consider a random vector $\varepsilon$ which is independent on $\xi$. Assume that the distribution of $\varepsilon$ is absolutely continuous. Then, obviously, the distribution of $\tilde{\xi} = \xi + \varepsilon$ is also absolutely continuous. If, additionally, $\mathrm{E}\varepsilon = \mathbf{0},\ \mathrm{E}|\varepsilon|^2 < \infty$ then $\tilde{\xi}$ and $\xi$ belong to the domain of attraction of the same stable law. The simplest way of smoothening is to take $\varepsilon$ having the uniform distribution in the unit ball $B(\mathbf{0}, 1)$.

Let $\tilde{\xi}, \tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}, \dots$ be independent copies of $\tilde{\xi} = \xi + \varepsilon$ and $\tilde{\zeta}^{(n,k)}$ be built by them in the same way as $\zeta^{(n,k)}$. It is clear that

$$\tilde{\zeta}^{(n,k)} = \zeta^{(n,k)} + o_p(1).$$

It implies that one may assume, with no loss of generality, that the underlying distribution is absolutely continuous and, therefore, Lemma 3.1 and Corollary 3.2 may be applied. From the statements it follows that the conditional distribution of $\zeta^{(n,k)}$ given $C_{n,0}, \dots, C_{n,k-1}$ is that of the sum of i.i.d. random vectors having common distribution $\mathrm{P}\{\xi \in A \mid \xi \in L\}$ where $L$ is the "value" of $C_{n,k-1}$.

Now we are going to establish the limit distribution of such a sum.

Keeping this in mind consider the sequence of measures defined as (see (2.14))

$$Q^{(n)}(A) = n \int_A \mathrm{P}\{b_n^{-1}\xi \in \mathrm{d}s\}, \qquad A \in \mathcal{B}_{\mathbb{R}^d}.$$

It is well known (see e.g. Resnick (1987) Proposition 5.17, (iii) and (iv)) that for any continuous compactly supported function $f(u)$ defined on $\mathbb{R}^d$ and such that $\mathbf{0} \notin \operatorname{supp} f$

$$\int_{\mathbb{R}^d} f(s) Q^{(n)}(\mathrm{d}s) \to \int_{\mathbb{R}^d \setminus \{\mathbf{0}\}} f(s) Q_\alpha(\mathrm{d}s) \tag{4.19}$$

where $Q_\alpha = \tau \times \theta$ is as in (2.10).

We need the following fact which formally is not covered by (4.19).

**Lemma 4.1** *Let*

$$f_t^{(\alpha)}(s) = \begin{cases} e^{\imath\langle t,s\rangle} - 1 & \text{if } \alpha \in (0,1) \\ e^{\imath\langle t,s\rangle} - 1 - \imath\langle t, s\rangle & \text{if } \alpha \in (1,2). \end{cases}$$

*Then for any convex polyhedron $L$ as $n \to \infty$*

$$\int_L f_t^{(\alpha)}(s) Q^{(n)}(\mathrm{d}s) \to \int_L f_t^{(\alpha)}(s) Q_\alpha(\mathrm{d}s)$$

*uniformly in $t$, $|t| < T$.*

**Proof.** It is evident that the functions $f_t^{(\alpha)}(s)$, $|t| < T$, are equicontinuous for any $T < \infty$. So, the statement is obvious if $\mathbf{0} \notin L$. But if $\mathbf{0} \in L$ then the situation requires additional arguments since $Q_\alpha(B(\mathbf{0}, \delta) \setminus \{\mathbf{0}\}) = \infty$ for any $\delta > 0$.

Let us denote by $\omega(t)$ any function such that $\lim\limits_{t \to 0} \omega(t) = 0$.

We have to show that for any $T > 0$ and $0 < \delta < \Delta$

$$\overline{\lim_{n\to\infty}} \sup_{|t|<T} \int\limits_{\delta<|s|<\Delta} |f_t^{(\alpha)}(s)| Q^{(n)}(\mathrm{d}s) \le \omega(\Delta). \tag{4.20}$$

The relation (4.20) implies integability of $|f_t^{(\alpha)}(s)|$ which is uniform with respect to both $n$ and $t$.

First note that

$$\sup_{|t|<T} |f_t^{(\alpha)}(s)| \le \begin{cases} T|s| & \text{if } \alpha \in (0,1) \\ \\ T^2|s|^2 & \text{if } \alpha \in (1,2). \end{cases} \tag{4.21}$$

Consider the functions

$$h_n(r) = n\mathrm{P}\{b_n^{-1}|\xi| > r\}, \ r \ge 0,$$

and (cf. (2.11))

$$h_\alpha(r) = Q_\alpha\left(B^c(\mathbf{0}, r)\right) = r^{-\alpha}, \ r > 0.$$

According to (2.12) and (2.14) as $n \to \infty$

$$h_n(r) \to h_\alpha(r)$$

uniformly in any $[r_0, \infty) \subset (0, \infty)$.

Obviously, for any $0 < \delta < \Delta < 1$

$$-\int_\delta^\Delta r^k \mathrm{d}h_n(r) = \int\limits_{\delta<|s|<\Delta} |s|^k Q^{(n)}(\mathrm{d}s).$$

Integrating by parts yields

$$\int_\delta^\Delta r^k \mathrm{d}h_n(r) = -\Delta^k h_n(\Delta) + \delta^k h_n(\delta) + k\int_\delta^\Delta r^{k-1} h_n(r)\mathrm{d}r.$$

In view of (2.12)

$$h_n(r) = n(rb_n)^{-\alpha} l(rb_n)$$

where $l(r)$ is of slow variation as $r \to \infty$. If $r > 1,\ rb_n \geq r_0$ then for any $\eta > 0$

$$l(rb_n) \leq 2l(b_n) r^{-\eta}$$

provided $r_0$ is sufficiently large. From (2.14) it follows that

$$nb_n^{-\alpha} l(b_n) \to 1.$$

Therefore, for all sufficiently large $n$

$$\delta^k h_n(\delta) \leq 2\delta^{k-\alpha-\eta}$$

provided $\delta b_n \geq r_0$. Similarly,

$$\int\limits_{r_0/b_n}^{\Delta} r^{k-1} h_n(r) \mathrm{d}r \leq nb_n^{-\alpha} l(b_n) \int\limits_{0}^{\Delta} r^{k-\alpha-\eta_1} \mathrm{d}r \leq c\Delta^{k-\alpha-\eta}$$

provided $k \geq \alpha + \eta$.

If $\delta b_n < r_0$ then

$$h_n(\delta) \leq n\delta^k = O(nb_n^{-k})$$

and

$$\int\limits_{\delta}^{r_0/b_n} r^{k-1} h_n(r) \mathrm{d}r = O(nb_n^{-k}).$$

Choosing $k = [\alpha] + 1$ and $\eta$ sufficiently small we conclude that

$$\overline{\lim_{n\to\infty}} \Big| \int\limits_{\delta}^{\Delta} r^k \mathrm{d}h_n(r) \Big| \leq \omega(\Delta)$$

and, therefore,

$$\overline{\lim_{n\to\infty}} \int\limits_{\delta<|s|<\Delta} |s|^k Q^{(n)}(\mathrm{d}s) \leq \omega(\Delta) \tag{4.22}$$

uniformly in $\delta \in (0, \Delta)$.

On the other hand

$$\int\limits_{\delta}^{\Delta} r^k \mathrm{d}h_\alpha(r) = \int\limits_{\delta<|s|<\Delta} |s|^k Q_\alpha(\mathrm{d}s) \leq \omega(\Delta). \tag{4.23}$$

In view of (4.20), (4.21), (4.22) and (4.23) the lemma follows.

Consider a sequence $\{L_n\}$ of convex polyhedrons converging to $L$ in the Hausdorff metric $\rho$. Set

$$g_n(s) = f_t^{(\alpha)}(s)\mathbf{1}_{L_n}(s)\left(1 - \mathbf{1}_{B(\mathbf{0},\Delta)}(s)\right).$$

Obviously, for any $s \in (\text{int } L) \cap B^c(\mathbf{0}, \Delta)$ we have

$$g_n(s_n) \to g(s) = f_t^{(\alpha)}(s)\mathbf{1}_L(s)\left(1 - \mathbf{1}_{B(\mathbf{0},\Delta)}(s)\right)$$

for any sequence $s_n \to s$. By Theorem 5.1 Billingsley (1968) in view of (4.22) and (4.23) we obtain

$$\int\limits_{L_n} f_t^{(\alpha)}(s)Q^{(n)}(\mathrm{d}s) \to \int\limits_{L} f_t^{(\alpha)}(s)Q_\alpha(\mathrm{d}s) \tag{4.24}$$

for any fixed $t$.

**Lemma 4.2** *Let (cf. Corollary 3.2)*

$$\varphi_L(t) = \begin{cases} \exp\{\int\limits_L f_t^{(\alpha)}(s)Q_\alpha(\mathrm{d}s)\} & \text{if } \alpha \in (0,1) \\ \\ \exp\{\int\limits_L f_t^{(\alpha)}(s)Q_\alpha(\mathrm{d}s) - \imath \int\limits_{L^c} \langle t, s\rangle Q_\alpha(\mathrm{d}s)\} & \text{if } \alpha \in (1,2). \end{cases}$$

*If* $\mathbf{0} \in \text{int } L$ *then as* $n \to \infty$

$$\varphi_{n,L}^n(t) \stackrel{def}{=} \left(\frac{\int\limits_L e^{\imath\langle t,s\rangle}\mathrm{P}\{b_n^{-1}\xi \in \mathrm{d}s\}}{\mathrm{P}\{b_n^{-1}\xi \in L\}}\right)^n \to \varphi_L(t).$$

**Proof.** Since $\mathbf{0}$ is an interior point of $L$ we have $\mathrm{P}\{b_n^{-1}\xi \in L\} \to 1$ as $n \to \infty$.

In the case $\alpha \in (0,1)$ the lemma immediately follows from Lemma 4.1 and the evident representation

$$\varphi_{n,L}(t) = 1 + \frac{\int\limits_L (e^{\imath\langle t,s\rangle} - 1)Q^{(n)}(\mathrm{d}s)}{n\mathrm{P}\{b_n^{-1}\xi \in L\}}.$$

But if $\alpha \in (1,2)$ then one has to make use of the representation

$$\varphi_{n,L}(t) = 1 + \frac{\int\limits_L (e^{\imath\langle t,s\rangle} - 1 - \imath\langle t, s\rangle)Q^{(n)}(\mathrm{d}s)}{n\mathrm{P}\{b_n^{-1}\xi \in L\}} + \imath\,\frac{\int\limits_L \langle t, s\rangle Q^{(n)}(\mathrm{d}s)}{n\mathrm{P}\{b_n^{-1}\xi \in L\}}.$$

Taking into account that

$$\int\limits_{\mathbb{R}^d} \langle t, s\rangle Q^{(n)}(\mathrm{d}s) \equiv 0$$

we may rewrite the representation for $\phi_{n,L}(t)$ as follows

$$\varphi_{n,L}(t) = 1 + \frac{\int\limits_L (e^{\imath\langle t,s\rangle} - 1 - \imath\langle t,s\rangle) Q^{(n)}(\mathrm{d}s)}{n\mathrm{P}\{b_n^{-1}\xi \in L\}} - \imath \frac{\int\limits_{L^c} \langle t,s\rangle Q^{(n)}(\mathrm{d}s)}{n\mathrm{P}\{b_n^{-1}\xi \in L\}}.$$

If

$$\int\limits_{L^c} \langle t,s\rangle Q^{(n)}(\mathrm{d}s) \to \int\limits_{L^c} \langle t,s\rangle Q_\alpha(\mathrm{d}s) \tag{4.25}$$

then in view of Lemma 4.1 the desired statement follows. In order to justify (4.25) it is sufficient to note that according to (2.14) for any $r_0 > 0$

$$\int\limits_{|s|>r_0} |s| Q^{(n)}(\mathrm{d}s) = \int\limits_{r>r_0} r \mathrm{d}h_n(r) \le r_0 h_n(r_0) \to r_0^{-\alpha}$$

and

$$\int\limits_{|s|>r_0} |s| Q_\alpha(\mathrm{d}s) = \int\limits_{r>r_0} r dh_\alpha(r) = r_0^{-\alpha}.$$

Thus (4.25) is established. The lemma is proved.

Let $\psi_{n,k}(t),\ k \ge 1$, be the characteristic function of $\zeta^{(n,k)}$. Represent it as

$$\psi_{n,k}(t) = \sum_{\bar n \in Z_{n,k-1}} \mathrm{E}\{\exp(\imath\langle t, \zeta^{(n,k)}\rangle);\ C_{n,0} \in \mathcal{L}_{n_0}, \ldots, C_{n,k-1} \in \mathcal{L}_{n_{k-1}}\}$$

$$+\mathrm{P}\{\sum_{l=0}^{k-1} \#(V_{n,l}) = n\}$$

where

$$Z_{n,k-1} = \{\bar n = (n_0, \ldots, n_{k-1}) \mid\ n_i \ge d+1,\ i = 0, \ldots, k-1,\ n_0 + \cdots + n_{k-1} \le n\}.$$

By Lemmas 3.4 and 3.5 for any fixed $k \ge 1$

$$\sum_{l=0}^{k-1} \#(V_{n,l}) = O_p(1).$$

It implies that as $n \to \infty$

$$\mathrm{P}\{\sum_{l=0}^{k-1} \#(V_{n,l}) = n\} = o(1).$$

Consider the event

$$X_{n,m} = \{\omega \in \Omega \mid\ \sum_{l=0}^{k-1} \#(V_{n,l}) \le m\}$$

where $m$ is fixed. Obviously,

$$\overline{\lim_{n\to\infty}}\ \mathrm{P}(X^c_{n,m}) \le \omega(1/m).$$

Therefore, for all sufficiently large $n$

$$\psi_{n,k}(t) = \sum_{\bar n \in Z_{m,k-1}} \mathrm{E}\{\exp(\imath\langle t, \zeta^{(n,k)}\rangle);\ C_{n,0} \in \mathcal{L}_{n_0}, \ldots, C_{n,k-1} \in \mathcal{L}_{n_{k-1}}\} + \Theta\omega(1/m).$$

By $\Theta$ we denote any variable taking values in $[-1, 1]$. Let $A_{k-2}$ be the event $\{C_{n,0} \in \mathcal{L}_{n_0}, \ldots, C_{n,k-2} \in \mathcal{L}_{n_{k-2}}\}$. In view of Corollary 3.2

$$\mathrm{E}\{\exp(\imath\langle t, \zeta^{(n,k)}\rangle);\ A_{k-2},\ C_{n,k-1} \in \mathcal{L}_{n_{k-1}}\}$$

$$= \int_{\mathcal{L}_{n_{k-1}}} \mathrm{E}\{\exp(\imath\langle t, \zeta^{(n,k)}\rangle)\ |\ \ A_{k-2},\ C_{n,k-1} = L\}\bar H_{n,k-1}(\mathcal{L}_{n_0}, \ldots, \mathcal{L}_{n_{k-2}}, \mathrm{d}L)$$

$$= \int_{\mathcal{L}_{n_{k-1}}} \varphi^{n-|\bar n|}_{n,L}(t)\bar H_{n,k-1}(\mathcal{L}_{n_0}, \ldots, \mathcal{L}_{n_{k-2}}, \mathrm{d}L).$$

Here $|\bar n| = n_0+\cdots+n_{k-1}$. We remind that $\bar H_{n,k-1}$ is the joint distribution of $(C_{n,0}, \ldots, C_{n,k-1})$.

If $|\bar n| \le m$ then by Lemma 4.2 as $n \to \infty$

$$\varphi^{n-|\bar n|}_{n,L}(t) \to \varphi_L(t)$$

uniformly in $t,\ |t| < T$. On the other hand by Lemma 3.4

$$\bar H_{n,k-1} \overset{w}{\Rightarrow} \bar H_{k-1}.$$

Thus,

$$\psi_{n,k}(t) = \sum_{\bar n \in Z_k} \int_{\mathcal{L}_{n_{k-1}}} \varphi^{n-|\bar n|}_{n,L}(t)\bar H_{n,k-1}(\mathcal{L}_{n_0}, \ldots, \mathcal{L}_{n_{k-2}}, \mathrm{d}L)$$

$$= \sum_{n_{k-1}\ge d+1} \int_{\mathcal{L}_{n_{k-1}}} \varphi^{n-|\bar n|}_{n,L}(t)H_{k-1}(\mathrm{d}L) + \Theta\omega(1/m)$$

where

$$Z_k = \{\bar n = (n_0, \ldots, n_{k-1})\ |\ \ n_i \ge d+1,\ i = 0, \ldots, k-1\}$$

and $H_{k-1}$ is the distribution of $C_{k-1}$.

Note that

$$\sum_{n_{k-1}\ge d+1} \int_{\mathcal{L}_{n_{k-1}}} \varphi^{n-|\bar n|}_{n,L}(t)H_{k-1}(\mathrm{d}L) = \int_{\mathcal{L}} \varphi^{n-|\bar n|}_{n,L}(t)H_{k-1}(\mathrm{d}L)$$

and $m$ is arbitrary. This together with Lemma 4.1 and (4.24) completes the proof. Theorem 2.2 is proven.

# 5 Remarks on the proof of Theorem 2.3

Note, first, that Lemma 4.1 and (4.24) remain valid with $f_t^{(\alpha)}(s)$ being replaced with

$$f_t^{(1)}(s) = e^{\imath\langle t,s\rangle} - 1 - \imath\langle t,s\rangle.$$

Instead of Lemma 4.2 the following statement holds.

**Lemma 5.1** *Let (cf. Corollary 3.3)*

$$\bar\varphi_L(t) = \exp\{\int\limits_L f_t^{(1)}(s)Q_1(\mathrm{d}s)\}.$$

*If* $\mathbf{0} \in \mathrm{int}\ L$ *then as* $n \to \infty$

$$\bar\varphi_{n,L}^n(t) \overset{def}{=} \left(\frac{e^{-\imath\langle\gamma_L,t\rangle}\int\limits_L e^{\imath\langle t,s\rangle}\mathrm{P}\{b_n^{-1}\xi \in \mathrm{d}s\}}{\mathrm{P}\{b_n^{-1}\xi \in L\}}\right)^n \to \bar\varphi_L(t).$$

**Proof.** The proof follows immediately from the evident representation. Since $\mathbf{0}$ is an interior point of $L$ we have

$$\bar\varphi_{n,L}(t) = \left(1 + \frac{\int\limits_L (e^{\imath\langle t,s\rangle} - 1 - \imath\langle t,s\rangle)Q^{(n)}(\mathrm{d}s)}{n\mathrm{P}\{b_n^{-1}\xi \in L\}} + \imath\,\frac{\langle\gamma_L, t\rangle}{n\mathrm{P}\{b_n^{-1}\xi \in L\}}\right)e^{-\imath\langle\gamma_L,t\rangle}.$$

Now it remains only to replace in the proof of Theorem 2.2 $\zeta^{(n,k)}$ by $\zeta_1^{(n,k)}$ and $\varphi_{n,L}(t)$ by $\bar\varphi_{n,L}(t)$.

# 6 Concluding remarks

**5.1** Arrange $\xi^{(1)}, \dots, \xi^{(n)}$ in ascending order of the absolute values. Denote the so ordered vectors $\bar\xi^{(1)}, \dots, \bar\xi^{(n)}$ i.e.

$$|\bar\xi^{(1)}| \le \dots \le |\bar\xi^{(n)}|.$$

From (2.12) and (2.14) it easily follows that

$$\mathrm{P}\{b_n^{-1} \mid \bar\xi^{(n-k+1)}| < r\} \to G_k(r),$$

where $G_k(r)$ is as in (1.5).

Consider the random vector

$$\bar\zeta^{(n,k)} = \sum_{j=1}^{n-k} \bar\xi^{(j)}$$

which is a multivariate analog of $\zeta_n^{(k)}$ in (1.1). The proof of the following statement directly follows the lines of the proof of Theorem 1 in Teugels (1981).

**Theorem 6.1** *For any fixed $k \geq 1$ as $n \to \infty$ the distribution of $\bar{\zeta}^{(n,k)}$ converges weekly to the measure $M_k$ which admits the following representation*

$$M_k(A) = \int_0^\infty S_r(A) \mathrm{d}G_k(r), \qquad A \in \mathcal{B}^d,$$

*where $S_r$ is the infinitely divisible distribution with the characteristic function*

$$\ln \varphi_r(t) = \begin{cases} \int\limits_{B(\mathbf{0},r)} (e^{\imath\langle t,s\rangle} - 1) Q_\alpha(\mathrm{d}s) & \textit{if} \quad \alpha \in (0,1) \\ \\ \int\limits_{B(\mathbf{0},r)} (e^{\imath\langle t,s\rangle} - 1 - \imath\langle t,s\rangle) Q_\alpha(\mathrm{d}s) - \imath \int\limits_{B^c(\mathbf{0},r)} \langle t,s\rangle Q_\alpha(\mathrm{d}s) & \textit{if} \quad \alpha \in (1,2). \end{cases}$$

So, we arrive to a multivariate analogue of (1.6).

**5.2** It looks very tempting to deduce Theorem 2.2 from the weak convergence of the considered above binomial point process to the limiting Poisson point process. This is really feasible but only in the case $\alpha \in (0,1)$. Here the functions

$$f_k(D) = \sum_{x \in \mathrm{int}\ C_k(D) \cap D} x, \qquad k = 0,1,2,\ldots; \qquad D \in \mathcal{H}(I\!\!E),$$

are $P_\pi$-a.s. continuous.

Therefore as $n \to \infty$

$$\zeta^{(n,k)} \Rightarrow \zeta^{(k)} = \sum_{x \in D_\pi \cap\ \mathrm{int} C_k} x.$$

Furthermore, (cf. Theorem 2.2)

$$\mathrm{P}\{\zeta^{(k)} \in A\} = \int_{\mathcal{L}} \mathrm{P}\{\sum_{x \in D_\pi \cap\, \mathrm{int} L} x \in A\} H_{k-1}(\mathrm{d}L), \qquad A \in \mathcal{B}_{I\!\!R^d},$$

where, we remind, $H_k$ is the distribution of $C_k$.

However, if $\alpha \in (1,2)$ then a.s.

$$f_k(D) = \infty,\ k = 0,1,2,\ldots\ .$$

**5.3** The next remark aims at explaining why we assume that the Poisson measure $\tau$ is non-unilateral. Denote by $\mathrm{conv}(A)$ the convex hull generated by $A \subset I\!\!R^d$. It is easily seen that the underlying distribution is non-unilateral if and only if support $S_\tau \subset S^{d-1}$ of $\tau$ is such that

$$\mathbf{0} \in \mathrm{int\ conv}\ (S_\tau). \tag{6.26}$$

Obviously

$$\mathrm{int\ conv}\ (S_\tau) \subset B(\mathbf{0},1).$$

Consider a simple example. Assume that $\alpha \in (0,1),\ d \geq 2$ and the spectral measure $\tau$ is discrete and supported by $\{e^{(1)}, \ldots, e^{(d+1)}\}$. These vectors must span $I\!R^d$ otherwise $\pi$ is not $d$-dimensional. If $\mathbf{0}$ is an interior point of $\mathrm{conv}(S_\tau)$ then each $C_k,\ k = 0, 1, 2, \ldots$ has exactly $d+1$ vertices lying on the rays

$$\gamma_i = \{x \mid \ x = te^{(i)},\ t > 0\},\ i = 1, 2, \ldots, d+1.$$

Obviously, $\mathcal{L} = \mathcal{L}_{d+1}$ and the statement of Theorem 2.2 becomes much simpler.

But if $\mathbf{0} \notin \mathrm{conv}(S_\tau)$ then a.s. $C_0$ has infinite number of vertices. Furthermore, if all $e^{(1)}, \ldots, e^{(d+1)}$ lie on the boundary of the cone generated by these vectors then int $C_0$ contains no points of $\pi$. Therefore,

$$f_k(D) = \sum_{x \in \mathrm{int}\ C_k(D) \cap D} x \equiv \mathbf{0}, \qquad k = 0, 1, 2, \ldots \ .$$

So, the statement of Theorem 2.2 is not valid.

Apparently, the number of vertices of $C_0$ is always infinite when $\alpha \in (0,1)$ and (6.26) is not fulfilled. Such cases require quite different techniques. The authors are going to discuss them elsewhere.

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