Delocalization for Schrödinger operators with random Dirac masses

Henrik Ueberschär

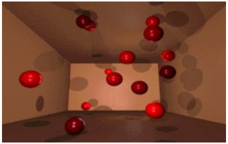
Institut de Mathématiques de Jussieu - Paris Rive Gauche

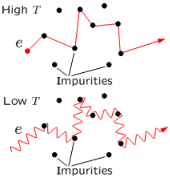
CEMPI Scientific Day

Lille, 10 February 2017

Disordered Systems

- E.g. box containing *N* nuclei
- Motion of electron in this system
- High temperature :
  classical Drude model (1900)
- Low temperature : quantum effects,
  interference : Anderson model (1958)





Random Schrödinger Operators

- Simplification : ignore interactions, one body Hamiltonian, Dirac masses
- We consider the random Schrödinger operator

$$H_\Omega = -\Delta + \sum_{\omega \in \Omega} \delta(x - \omega), \quad x \in \mathbb{R}^d, d = 2, 3$$

  where Ω is a stochastic process on $\mathbb{R}^d$
- Rigorous realization of H_Ω : theory of self-adjoint extensions

Random displacement model

- Consider

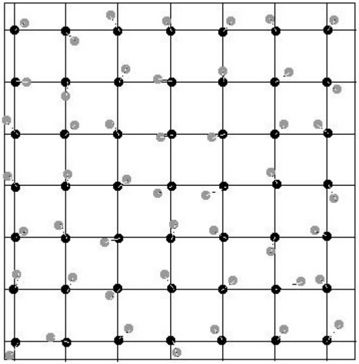
$$H_\Omega = -\Delta + \sum_{\xi \in \mathbb{Z}^d} \delta(x - \xi - \omega_\xi)$$

where ω_ξ are i. i. d. r. v.

- Probability density $\mathbb{P}(x) = P(|x|)$, where $P \in C_c^\infty(\mathbb{R}_+)$, $P(0) > 0$, decreasing.

- Disorder parameter :

$$\delta = \mathbb{E}_\mathbb{P}(|X|)$$



Anderson localization

- Anderson 1958 : strong disorder $\Rightarrow$ localized eigenfunctions

Definition

Let $E_2 > E_1 > 0$. We say that H_Ω is **exponentially localized** *on $I = [E_1, E_2]$, if H_Ω has a.s. pure point spectrum on I and the eigenfunctions ψ_λ satisfy the bound :*

$$\left|\psi_\lambda(x)\psi_\lambda(y)\right| \leq C_\omega e^{-|x-y|/L_{loc}}$$

where the localization length L_{loc} depends on the choice of I and $\omega \in \Omega$ is a sample of the stochastic process.

- **Mathematical results :** Goldsheid-Molchanov-Pastur, Fröhlich-Spencer, Simon, Bourgain-Kenig, Klopp, A. Boutet de Monvel, Germinet, ...

Anderson transition : physical motivation

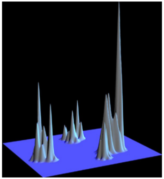
- Localization for strong disorder
- Delocalization for weak disorder ?
- Scaling Theory (Abrahams, Anderson, Licciardello et Ramakrishnan, 1979) :

  $d = 1$: localization
  $d = 2$: critical case
  $d \geq 3$: transition expected

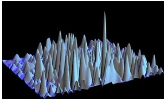
- Delocalized regime :

  absolutely continuous spectrum
  extended eigenstates

intensity $|\psi_\lambda|^2 dx$



localized

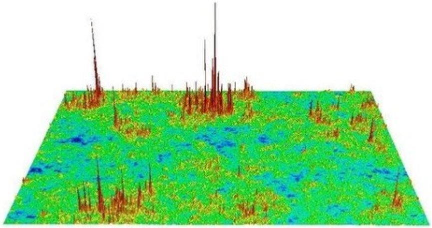


delocalized

Pictures : D. Wiersma et al.

Geometry of eigenfunctions

- Multifractal structure of eigenfunctions near transition point
- Delocalization : $L_{loc} \to \infty$ as $E \to E_c$
- Want to prove lower bound for L_{loc} $\to$ test on compact spaces



Picture : A. Mirlin
(Karlsruhe Institute of Technology)

Seba's billiard

- D – rectangular billiard
- In 1990 Petr Seba studied the singular Hamiltonian

$$H_\alpha = -\Delta + \alpha\delta(x - x_0), \quad \alpha \in \mathbb{R}, x_0 \in int(D)$$

with Dirichlet boundary conditions on ∂D

- Consider the restricted Laplacian

$$H_0 = -\Delta|_{C_c^\infty(R - x_0)}$$

which admits family of self-adjoint extensions $-\Delta_\varphi$, $\varphi \in (-\pi, \pi)$

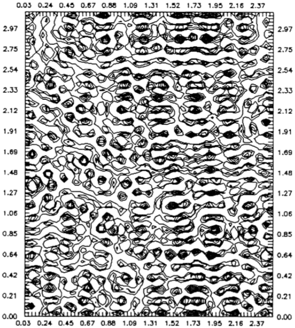


FIG. 2. The topography of the positive part of the eigenfunction corresponding to the 411 eigenvalue of H_α with $\alpha = 100$. The point scatterer is placed at the point $(0.55\pi/a, 0.65\pi)$.

The perturbed spectrum

- New eigenvalues are solutions to the equation (square torus $\mathbb{R}^2/2\pi\mathbb{Z}^2$)

$$\sum_{\xi\in\mathbb{Z}^2}\left\{\frac{1}{|\xi|^2-\lambda}-\frac{|\xi|^2}{|\xi|^4+1}\right\}=C_0\tan\left(\frac{\varphi}{2}\right)$$

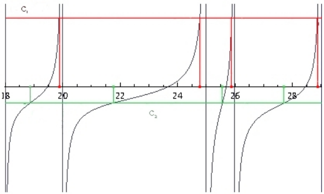
- Eigenfunction

$$G_\lambda(x,x_0)=(-\Delta-\lambda)^{-1}\delta(x-x_0)$$

has convenient L^2-representation

$$G_\lambda(x,x_0)=\frac{1}{4\pi^2}\sum_{\xi\in\mathbb{Z}^2}\frac{e^{i\langle x-x_0,\xi\rangle}}{|\xi|^2-\lambda}$$

- A plot of the LHS of the spectral equation as a function of λ



- Singularities at the old eigenvalues

Random impurities on the torus

- Consider the torus $\mathbb{T}_L^d = \mathbb{R}^d / L\mathbb{Z}^d$, $d = 2, 3$, $L \gg 1$, and the operator

$$H_{\Omega,L} = -\Delta + \sum_{j=1}^{N} \delta(x - x_j)$$

  where x_1 is fixed and $x_2, \cdots, x_N$ are i. i. d. uniformly distributed random variables on $\mathbb{T}_L^d$.

- Can realize the formal operator above as self-adjoint extensions of the restricted Laplacian

$$H_0 = -\Delta|_{C_c^\infty(\mathbb{T}_L^d - \mathcal{O})}, \quad \mathcal{O} = \{x_1, \cdots, x_N\}$$

- Operator H_0 is positive symmetric with deficiency indices (N, N), extensions parametrized by $U(N)$

Spectrum and eigenfunctions

- Fix $U = e^{i\varphi}\,\mathrm{Id}_N$, $\varphi \in (-\pi, \pi)$, denote corresponding extension by $-\Delta_\varphi$
- Green's function :

$$G_\lambda(x,y) = \frac{1}{-\Delta - \lambda}\delta(x-y), \quad \lambda \notin \sigma(-\Delta)$$

- We have the spectral equation

$$\det A_\lambda^\varphi = 0$$

where

$$(A_\lambda^\varphi)_{kl} = G_\lambda(x_k, x_l) - \Re G_i(x_k, x_l) - \tan(\frac{\varphi}{2})\Im G_i(x_k, x_l)$$

- Two types of eigenfunctions : old and new
- At most N new eigenvalues per Laplacian eigenspace which interlace with distinct Laplacian eigenvalues
- New eigenfunctions are given by superpositions of Green's functions :

$$\psi_\lambda(x) = \sum_{k=1}^{N} v_k G_\lambda(x, x_j), \quad v \in \ker A_\lambda^\varphi$$

- Note : the v_k are functions of the random variables $x_2, \cdots, x_N \in \mathbb{T}_L^d$

Random displacement models

- Let $B_L = [-L, L]^d$ and $L \gg 1$. Consider

$$H_{\Omega,L} = -\Delta + \sum_{\xi \in \mathbb{Z}^d \cap B_L} \delta(x - \xi - \omega_\xi)$$

with Dirichlet BCs on ∂B_L

- Fix small $\epsilon > 0$. Let $\chi \in C^\infty(B_1)$, $\chi \geq 0$, with supp $\chi \subset B(0, \epsilon)$ and $\|\chi\|_1 = 1$. Denote $\chi_L = L^{-d}\chi(\cdot/L)$.

- Let $H_{\Omega,L}\psi_\lambda^L = \lambda\psi_\lambda^L$, $\|\psi_\lambda^L\|_2 = 1$.
We define the smoothed L^2-density

$$\Psi_\lambda^L(x) := \left(\int_{B_L} \chi_L(x' - x) |\psi_\lambda^L(x')|^2 dx' \right)^{1/2}$$

for any $x \in B_{(1-\epsilon)L}$.

Definition

Let $F \in C^0(\mathbb{R}_+)$ be strictly decreasing and $L \gg 1$. We say that $H_{\Omega,L}$ is F-localized on an interval $\mathcal{I} = [E_1, E_2]$ if we have for any $a \in \mathcal{I}$ and $\lambda = \min(\sigma(H_{\Omega,L}) \cap [a, E_2])$

$$\forall x, y \in B_{(1-\epsilon)L}, |x-y| \asymp L : \ \mathbb{E}\left(\Psi_\lambda^L(x)\Psi_\lambda^L(y)\right) \lesssim F(|x-y|).$$

- Set $F(\tau) = e^{-\tau/L_{loc}}$ for exponential localization, where L_{loc} depends on the choice of interval $\mathcal{I}$
- For a given spectral window $\mathcal{I}$ consider boxes of size $L \gg L_{loc}(\mathcal{I})$

Delocalization for random displacement models

- If $\delta \ll 1$, we can show that for E, L large enough $H_{\Omega,L}$ fails to be exponentially localized
- In fact we can rule out any decay but a certain polynomial one

Theorem (H.U. 2016)

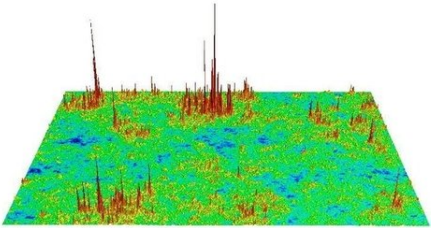
Let $d = 2, 3$ and

$$F(\tau) = \frac{1}{1 + \tau^{\alpha_d + \epsilon}}.$$

There exist $\alpha_d > 0$, $E_0 > 0$ and $L_0 = L_0(E_0)$ such that for any interval $I = [E_1, E_2]$, $E_1 > E_0$ and any $L \geq L_0$, $H_{\Omega,L}$ fails to be F-localized.

Outlook

- Delocalization for random Schrödinger operators with generic potentials
- Explicit construction of extended states
- Level statistics near transition point



Multifractal wave function near Anderson transition. A. Mirlin (Karlsruhe)

Thank you for your attention !