

Lifting Galois sections along torsors

NIELS BORNE, MICHEL EMSALEM, AND JAKOB STIX

Abstract — Associated to a smooth curve X/k and an effective étale divisor $D \subseteq X$, we construct torus torsors F_D and E_D over X . The torsor E_D is torsion if and only if D is a torsion packet on X . The fundamental group $\pi_1(F_D)$ agrees with the maximal cuspidally central extension $\pi_1^{cc}(U) \rightarrow \pi_1(X)$ of the complement $U = X \setminus D$. The obstruction to lifting Galois sections $s : \text{Gal}_k \rightarrow \pi_1(X)$ to $\pi_1(F_D)$ is controlled by the generalized first Chern class of the torsor. If the base field is $k = \mathbb{Q}$ and D is a union of torsion packets, we show unconditionally that every Galois section lifts to $\pi_1(F_D)$.

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1. INTRODUCTION

Let \bar{k} be a fixed separable closure of an arbitrary field k , and let $\text{Gal}_k = \text{Gal}(\bar{k}/k)$ be the absolute Galois group of k . For a variety X/k , let $X_{\bar{k}} = X \times_k \bar{k}$ be the base change of X to \bar{k} .

1.1. The lifting problem. The étale fundamental group $\pi_1(X, \bar{x})$ with base point \bar{x} is an extension

$$1 \rightarrow \pi_1(X_{\bar{k}}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \rightarrow \text{Gal}_k \rightarrow 1. \tag{1.1}$$

In this note we study Galois sections $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ of (1.1) with respect to their lifting properties. More precisely, we ask for a map $f : W \rightarrow X$ of geometric origin, whether s lifts along the induced map

$$f_* : \pi_1(W, \bar{w}) \rightarrow \pi_1(X, \bar{x}),$$

i.e., whether there is a Galois section $t : \text{Gal}_k \rightarrow \pi_1(W, \bar{w})$ such that $s = f_* \circ t$. The maps f in question are of two types:

- (1) the inclusion $j : U \subseteq X$ of a dense open, or
- (2) the fibration $h : E \rightarrow X$ by a torsor under a k -torus T .

If X/k is smooth and the Galois section $s = s_a$ comes by functoriality from a k -rational point $a : \text{Spec}(k) \rightarrow X$, then s_a lifts along $j_* : \pi_1(U, \bar{x}) \rightarrow \pi_1(X, \bar{x})$ for all open subschemes $U \subseteq X$ with (tangential) lift of base point \bar{x} . This is well known for curves X/k but holds in general for X/k smooth (see the converse direction of [Sti13a] Prop. 3, valid over general fields).

It turns out that there is *geometry* behind this lifting problem that at least governs the lifting to an intermediate step: the cuspidally central quotient introduced by Mochizuki

$$\pi_1(U, \bar{x}) \twoheadrightarrow \pi_1^{cc}(U, \bar{x}) \twoheadrightarrow \pi_1(X, \bar{x}),$$

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see Section §3.4. The alluded geometry is the main topic of this note: the torus torsor $F_D \rightarrow X$, see Definition 3, attached to the étale effective divisor $D = X \setminus U$ with respect to the k -torus $T_D = \mathbb{R}_{D/k} \mathbb{G}_m$. There is a natural isomorphism

$$\pi_1^{cc}(U, \bar{x}) \simeq \pi_1(F_D, \bar{y})$$

augmented over $\pi_1(X, \bar{x})$, see Proposition 30.

Grothendieck's section conjecture predicts that for a smooth hyperbolic curve X over a number field k , all Galois sections of (1.1) arise from rational points of X , see [Sti13b] for a survey. The section conjecture implies that every section admits lifts for case (1) — a secondary conjecture that we are fond of calling the *cuspidalization conjecture*. We refer to [Sti12] and [Saï12] for conditional results on the section conjecture assuming the cuspidalization conjecture.

The term *cuspidalization* is due to Mochizuki and his pioneering work [Moc07] that controls, for $U \subset X$ as above, the group cover $\pi_1(U, \bar{x}) \rightarrow \pi_1(X, \bar{x})$ from $\pi_1(X, \bar{x})$ to some extent (for the universal family of all such U). Since then the notion of cuspidalization has been linked by Mochizuki, Tamagawa and Hoshi to the relation between $\pi_1(X, \bar{x})$ and certain fundamental groups of configuration spaces for points on X . This is a different notion of cuspidalization from ours.

The cuspidalization conjecture in our sense was studied by Saïdi [Saï10] with respect to the maximal cuspidally abelian quotient $\pi_1^{c-ab}(U, \bar{x})$ and certain *good* Galois sections under restrictive assumptions on the base field: all open subgroups of Gal_k are required to have trivial center. The Galois section is *good* if it universally kills Chern classes of line bundles, a property that is directly linked to lifting to the maximal cuspidally central quotient $\pi_1^{cc}(U, \bar{x})$, see Lemma 40.

Below, the only assumption we have to make (mainly for convenience) is characteristic 0. Moreover, we focus on the torus torsor in the geometric category that governs the criterion for cuspidally central lifting.

1.2. Results. Starting in Section §3, we make the simplifying assumption that k has characteristic 0. Hence we discuss the results of the paper under this assumption below.

The main result of this note establishes unconditional lifting of Galois sections for curves X over \mathbb{Q} related to the notion of torsion packets $D \subseteq X$, see Definition 14.

Theorem A (Theorem 55). *Let X/\mathbb{Q} be a smooth projective curve of positive genus, and let $D \subset X$ be a union of torsion packets. Then every Galois section $s : \text{Gal}_{\mathbb{Q}} \rightarrow \pi_1(X, \bar{x})$ lifts to a section $\text{Gal}_{\mathbb{Q}} \rightarrow \pi_1(F_D, \bar{y})$.*

Let us emphasize, that unconditional results concerning the cuspidalization conjecture have been extremely rare so far. In order to illustrate the range of application of Theorem A, note that the subset $D = X(\mathbb{Q})$ of all \mathbb{Q} -rational points forms a union of torsion packets on X . The theme of torsion packets was already present in [BE13], but the results here are better, for details see Remark 43.

A related criterion describes lifting for the geometrically pro- Σ fundamental group with respect to the map $\pi_1(E_D, \bar{y}) \rightarrow \pi_1(X, \bar{x})$, see Theorem 42. Here $E_D = F_D/\mathbb{G}_m$ is another torus torsor on X , see Definition 3.

The proof of Theorem A is based on the identification, valid for a general torsor $E \rightarrow X$ over a $K(\pi, 1)$ space X under a k -torus T , of an extension class with a (generalized) first Chern class

$$[\pi_1(E/X)] = c_1(E),$$

under identifying $H^2(\pi_1(X, \bar{x}), \pi_1(T_k, 1)) \simeq H^2(X, \mathbb{T}(T))$, see Proposition 39. The existence of the extension $\pi_1(E/X)$ itself follows from Theorem 21: a criterion when a fibration $f : X \rightarrow S$ is good enough to yield a short exact sequence of fundamental groups

$$1 \rightarrow \pi_1(X_{\bar{s}}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \xrightarrow{\pi_1(f)} \pi_1(S, \bar{s}) \rightarrow 1.$$

The criterion asks S to be an algebraic $K(\pi, 1)$ in characteristic 0 and f to admit a proper flat relative compactification $f: \bar{X} \rightarrow S$, such that $\bar{X} \setminus X$ is a normal crossing divisor relative to S . Theorem 22 states that torsors under a torus admit such a good compactification.

A further ingredient for the proof of Theorem A connects the relative Brauer group, see Definition 17, to local divisibility of line bundles in neighbourhoods of a Galois section s , and crucially to the vanishing of certain restrictions of first Chern classes. Similar results are obtained by M. Saïdi, but under restrictive assumptions on the base field k . Our result works over any field in characteristic 0:

Theorem B (Theorem 51). *Let X/k be a smooth projective curve of positive genus, and let $s: \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ be a Galois section. Consider the following assertions.*

- (a) *All line bundles \mathcal{L} on X are locally divisible in neighbourhoods of s .*
- (b) *$s^* \circ c_1: \text{Pic}(X) \rightarrow H^2(k, \hat{\mathbb{Z}}(1))$ vanishes.*
- (c) *The relative Brauer group $\text{Br}(X/k)$ vanishes.*
- (c') *The relative Brauer group $\text{Br}(X'/k)$ vanishes for all neighbourhoods X' of s .*

Then the following implications hold: (c') \implies (a) \iff (b) \implies (c).

We also would like to mention a torsion criterion for E_D that illustrates the role played by the assumption on the divisor D being a torsion packet.

Theorem C (Theorem 16). *Let X be a smooth, proper, geometrically connected curve over $S = \text{Spec}(k)$. Then the torsor $E_D \rightarrow X$ is torsion if and only if D is a torsion packet.*

1.3. Notation and conventions. Let us fix some notation and conventions that are valid throughout the text.

By $X_{S'}$ we will denote the base change $X \times_S S'$ of an S scheme X by $S' \rightarrow S$. However, we would like to direct the kind reader's attention to the following exceptions. The notation S_D, T_D (resp. E_D, F_D) introduced in Section §2 denote a certain torus (resp. torus torsor) associated to a divisor D . And X_D , again in Section §2, denotes the curve X pinched along the divisor $D \subseteq X$.

When talking about torsors, one has to fix a topology in which the torsors trivialize locally. This will be the étale topology always without further mention.

There are natural mutually inverse translations between line bundles and \mathbb{G}_m torsors: the \mathbb{G}_m torsor L to a line bundle \mathcal{L} is

$$L = \mathbf{Isom}(\mathcal{O}, \mathcal{L}),$$

and the line bundle \mathcal{L} to a \mathbb{G}_m -torsor L is

$$\mathcal{L} = L \times^{\mathbb{G}_m} \mathcal{O}.$$

For a map $X \rightarrow S$, denote the relative Picard stack by $\mathcal{P}ic_{X/S}$, and denote by $\text{Pic}_{X/S}$ the relative Picard scheme. By definition an object of $\mathcal{P}ic_{X/S}$ over $T \rightarrow S$ is an invertible sheaf on $X \times_S T$. The natural map

$$\mathcal{P}ic_{X/S}(T) \rightarrow \text{Pic}_{X/S}(T)$$

that maps a line bundle to its isomorphism class will be denoted by $\mathcal{L} \mapsto [\mathcal{L}]$.

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2. TORSORS ASSOCIATED TO A DIVISOR ON A CURVE

Starting from a reduced effective divisor on a curve we are going to construct two torsors over the curve. We give two constructions: the first using Picard stacks is prepared in §2.1 and the second functorial but non-stacky construction in §2.2 to which the reader can leap forward.

2.1. Motivation via Picard stacks. We start with a reminder about the "pinching" operation in the category of schemes, following Ferrand (see [Fer03]). Let X be a scheme over an arbitrary base scheme S , D a closed subscheme of X , affine over S . A *pinching of X along D* is a scheme X_D fitting into a co-cartesian diagram:

$$\begin{array}{ccc} D & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \longrightarrow & X_D \end{array}$$

where $X \rightarrow X_D$ is affine and $S \rightarrow X_D$ is a closed immersion. Such a pinching exists when $D \rightarrow S$ is moreover finite and every finite subset of points of X (resp. S) is contained in an affine open subset ([Fer03], Théorème 5.4). We always assume in the sequel that this hypothesis holds. It does, for instance, when X is a curve over the spectrum of a field $S = \text{Spec}(k)$, and D is associated to an effective, reduced Cartier divisor. We recall the following theorem.

Theorem 1. *Assume that D is included in some open \mathcal{U} of X , affine over S . Then the following diagram is cartesian:*

$$\begin{array}{ccc} \mathcal{P}ic_{X_D/S} & \longrightarrow & \mathcal{P}ic_{S/S} = \mathbf{B}\mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathcal{P}ic_{X/S} & \longrightarrow & \mathcal{P}ic_{D/S}. \end{array}$$

Proof. Since the formation of X_D and of $\mathcal{P}ic_{X/S}$ commutes with flat base change, this is a local problem on S , so we can assume that S , D , and \mathcal{U} are affine. In particular, the pinching \mathcal{U}_D exists and is affine. Set $U = X \setminus D$, and note that $X \rightarrow X_D$ induces an isomorphism between U and $V = X_D \setminus S$. This shows that X_D can be interpreted as the glueing of \mathcal{U}_D and U along $\mathcal{U} \cap U$, and the result follows from the affine situation (see [Fer03], Théorème 2.2 iv). \square

Corollary 2. *In the notation above, we assume that $D \rightarrow S$ is finite and flat. Let S' be a S -scheme, \mathcal{L} an invertible sheaf on $X_{S'} = X \times_S S'$, and $S' \rightarrow \mathcal{P}ic_{X/S}$ the corresponding morphism. There is a natural cartesian diagram:*

$$\begin{array}{ccc} \mathbf{R}_{D_{S'}/S'}(\mathbf{Isom}_{D_{S'}}(\mathcal{L}|_{D_{S'}}, \mathcal{O}_{D_{S'}})) / \mathbb{G}_m & \longrightarrow & \mathcal{P}ic_{X_D/S} \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\quad \mathcal{L} \quad} & \mathcal{P}ic_{X/S}. \end{array}$$

where $\mathbf{R}_{D_{S'}/S'}$ denotes the Weil restriction, and \mathbb{G}_m acts via the diagonal $\mathbb{G}_m \rightarrow \mathbf{R}_{D/S}(\mathbb{G}_m)$.

Proof. According to Theorem 1, it is enough to show that the following diagram is cartesian:

$$\begin{array}{ccc} \mathbf{R}_{D_{S'}/S'}(\mathbf{Isom}_{D_{S'}}(\mathcal{L}|_{D_{S'}}, \mathcal{O}_{D_{S'}})) / \mathbb{G}_m & \longrightarrow & \mathcal{P}ic_{S/S} = \mathbf{B}\mathbb{G}_m \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\quad \mathcal{L}|_{D_{S'}} \quad} & \mathcal{P}ic_{D/S}. \end{array}$$

Let $S \rightarrow \mathcal{P}ic_{S/S}$ be the morphism corresponding to the structural sheaf \mathcal{O}_S (in other words $S \rightarrow \mathcal{P}ic_{S/S} = \mathbf{B}\mathbb{G}_m$ is the universal \mathbb{G}_m -torsor). Then, by the definition of the Picard stack,

the following diagram is cartesian:

$$\begin{array}{ccc}
\mathbf{R}_{D_{S'}/S'}(\mathbf{Isom}_{D_{S'}}(\mathcal{L}|_{D_{S'}}, \mathcal{O}_{D_{S'}})) & \longrightarrow & S \\
\downarrow & & \downarrow \\
S' & \xrightarrow{\mathcal{L}|_{D_{S'}}} & \mathcal{P}ic_{D/S} \\
& & \downarrow \\
& & \mathcal{P}ic_{S/S} = B\mathbb{G}_m
\end{array}$$

This is enough to conclude, since the action of \mathbb{G}_m on $\mathbf{R}_{D_{S'}/S'}(\mathbf{Isom}_{D_{S'}}(\mathcal{L}|_{D_{S'}}, \mathcal{O}_{D_{S'}}))$ is free, and the formation of the quotient stack commutes with base change. \square

2.2. Definition of torsors E_D and F_D . Here and in the rest of Section §2, let X be a smooth curve defined over $S = \text{Spec}(k)$, and $\Delta : X \rightarrow X \times_S X$ the diagonal embedding. This is a Cartier divisor, defining an invertible sheaf $\mathcal{O}_{X \times_S X}(\Delta)$. Let D be an effective, étale Cartier divisor on X . We will always assume that $D \neq 0$. We apply the constructions of Section §2.1 to the morphism $X \rightarrow \mathcal{P}ic_{X/S}$ associated with $\mathcal{L} = \mathcal{O}_{X \times_S X}(\Delta)$.

Definition 3. We define torsors

- (1) $F_D = \mathbf{R}_{D \times_S X/X}(\mathbf{Isom}_{D \times_S X}(\mathcal{O}_{X \times_S X}(\Delta)|_{D \times_S X}, \mathcal{O}_{D \times_S X})) \rightarrow X$ under the torus
$$T_D = \mathbf{R}_{D/S}(\mathbb{G}_m),$$
- (2) $E_D = F_D/\mathbb{G}_m \rightarrow X$ under the torus $S_D = T_D/\mathbb{G}_m$.

Remark 4. (1) The group T_D is indeed a torus, because we assumed $D \rightarrow S$ finite étale.
(2) Corollary 2 implies that the following diagram is cartesian

$$\begin{array}{ccc}
E_D & \longrightarrow & \mathcal{P}ic_{X_D/S} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mathcal{O}(\Delta)} & \mathcal{P}ic_{X/S}
\end{array} \tag{2.1}$$

and this leads to an alternative definition of E_D . Moreover it also follows from the proof of Corollary 2 that $F_D \rightarrow E_D$ is the \mathbb{G}_m -torsor associated to the natural morphism

$$E_D \rightarrow \mathcal{P}ic_{X_D/S} \rightarrow \mathcal{P}ic_{S/S} = B\mathbb{G}_m.$$

This defines a line bundle $\mathcal{N} = F_D \times^{\mathbb{G}_m} \mathcal{O}_{E_D}$ on E_D , that lifts canonically to a line bundle \mathcal{P} on $X_D \times_S E_D$.

- (3) It is immediate from the definition that the restrictions to $U = X \setminus D$ of the torsors $F_D \rightarrow X$ and $E_D \rightarrow X$ are trivial. More precisely, there is a canonical morphism

$$U \rightarrow F_D$$

over X corresponding to the isomorphism $\mathcal{O}_{D \times_S U} \simeq \mathcal{O}_{X \times_S X}(\Delta)|_{D \times_S U}$ obtained by restricting to $D \times_S U$ the canonical section:

$$\mathcal{O}_{X \times_S X} \rightarrow \mathcal{O}_{X \times_S X}(\Delta).$$

Remark 5. The group of characters $\text{Hom}(T_D, \mathbb{G}_m)$ of T_D is the permutation Gal_k -module

$$M_D = \bigoplus_{D(\bar{k})} \mathbb{Z},$$

and the group of characters $\text{Hom}(S_D, \mathbb{G}_m)$ of S_D is the submodule of elements of degree 0

$$M^0 = \left(\bigoplus_{D(\bar{k})} \mathbb{Z} \right)^{\Sigma=0}.$$

2.3. Functoriality properties. We keep the notation of Section §2.2. Let $\emptyset \neq D_1 \subset D_2 \subset X$ be two effective étale divisors on X . Then the inclusion of character modules $M_{D_1} \subseteq M_{D_2}$ (resp. $M_{D_1}^0 \subseteq M_{D_2}^0$) induce maps of tori $T_{D_2} \rightarrow T_{D_1}$ (resp. $S_{D_2} \rightarrow S_{D_1}$).

Proposition 6. *Let $\emptyset \neq D_1 \subset D_2 \subset X$ be two effective étale divisors on X . There are canonical isomorphisms*

- (1) $F_{D_1} \simeq F_{D_2} \times^{T_{D_2}} T_{D_1}$,
- (2) $E_{D_1} \simeq E_{D_2} \times^{S_{D_2}} S_{D_1}$.

Proof. (1) There is a $T_{D_2} \rightarrow T_{D_1}$ equivariant map $F_{D_2} \rightarrow F_{D_1}$ as can be seen from diagram (2.1) and the resulting alternative description of F_D . This leads to a T_{D_1} -equivariant map

$$F_{D_2} \times^{T_{D_2}} T_{D_1} \rightarrow F_{D_1}$$

which is necessarily an isomorphism of torsors. The proof of (2) is similar. \square

Let us now consider the particular situation where the divisor D on X is the disjoint union of two divisors D_1 and D_2 .

Proposition 7. *Let $D = D_1 \sqcup D_2$ be a disjoint union of non-empty étale effective divisors on X . Then*

- (1) $T_D \simeq T_{D_1} \times_k T_{D_2}$,
- (2) $F_D \simeq F_{D_1} \times_X F_{D_2}$.

Moreover there is an exact sequence $1 \rightarrow \mathbb{G}_m \rightarrow S_D \rightarrow S_{D_1} \times_k S_{D_2} \rightarrow 1$ and an isomorphism

- (3) $E_D/\mathbb{G}_m \simeq E_{D_1} \times_X E_{D_2}$.

Proof. Assertion (1) follows immediately by comparing the character groups of the tori involved. Using the inclusions $D_i \subseteq D$ and the corresponding projections $T_D \rightarrow T_{D_i}$ we deduce from Proposition 6 (1) that we have an equivariant map

$$F_D \rightarrow F_{D_1} \times_X F_{D_2}$$

that therefore is an isomorphism, hence (2).

The exact sequence is clear by passing to the corresponding dual sequence

$$0 \rightarrow M_{D_1}^0 \times M_{D_2}^0 \rightarrow M_D^0 \xrightarrow{\sum_{D_1, D_2} \Sigma} (\mathbb{Z} \oplus \mathbb{Z})^{\Sigma=0} \rightarrow 0$$

of character groups. Here \sum_{D_i} is the sum over the entries from $D_i(\bar{k})$. Assertion (3) then follows immediately from (2) by passing to quotients by \mathbb{G}_m on each factor. \square

2.4. Torsion criterion I. We keep the setup of Section §2.2 and give a criterion when E_D is torsion. For a nonnegative integer $N \geq 1$ let E_D^N be the N -th power of the torsor $E_D \rightarrow X$, i.e., the push forward by $N \cdot : S_D \rightarrow S_D$.

Proposition 8. *Let $N \geq 1$ be a nonnegative integer. The following are equivalent.*

- (1) $E_D^N \rightarrow X$ is trivial as an S_D -torsor.
- (2) There exists an invertible sheaf \mathcal{L} on X and an isomorphism

$$\text{pr}_2^* \mathcal{L} \simeq \mathcal{O}_{X \times X}(N \cdot \Delta)|_{D \times X},$$

where pr_2 denotes the second projection $\text{pr}_2 : D \times X \rightarrow X$.

(3) *There exists a dotted arrow making the following diagram commute:*

$$\begin{array}{ccccc} X & \xrightarrow{\mathcal{O}(\Delta)} & \mathcal{P}ic_{X/S} & \xrightarrow{(\cdot)^{\otimes N}} & \mathcal{P}ic_{X/S} \\ \uparrow & & & \nearrow \text{dotted} & \\ D & \longrightarrow & S & & \end{array}$$

Proof. The N -th power of E_D is given by two cartesian diagrams

$$\begin{array}{ccccc} E_D^N & \longrightarrow & \mathcal{P}ic_{X_D/S} & \longrightarrow & \mathcal{P}ic_{S/S} = \mathbf{B}\mathbb{G}_m \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\mathcal{O}(N \cdot \Delta)} & \mathcal{P}ic_{X/S} & \longrightarrow & \mathcal{P}ic_{D/S}. \end{array}$$

Assertion (1) is equivalent to the existence of a section $X \rightarrow E_D^N$, which is equivalent to a line bundle \mathcal{M} on $X_D \times_S X$

$$\mathcal{M} : X \rightarrow \mathcal{P}ic_{X_D/S}$$

lifting $\mathcal{O}(N \cdot \Delta)$ by the first square being cartesian, which in turn is equivalent to a line bundle \mathcal{L} on X

$$\mathcal{L} : X \rightarrow \mathbf{B}\mathbb{G}_m$$

lifting

$$\mathcal{O}_{X \times X}(N \cdot \Delta)|_{D \times X} : X \rightarrow \mathcal{P}ic_{D/S}$$

by the second square being cartesian by Theorem 1. Therefore (1) is equivalent to (2).

A dotted line as in (3) is a line bundle \mathcal{L} on X , and assertion (3) states that there is such an \mathcal{L} and an isomorphism

$$\mathrm{pr}_1^* \mathcal{L} \simeq \mathcal{O}_{X \times X}(N \cdot \Delta)|_{X \times D}$$

where $\mathrm{pr}_1 : X \times D \rightarrow X$ is the first projection. Since Δ is symmetric under the flip of factors in $X \times X$, this is clearly just a reformulation of assertion (2). \square

Remark 9. If in condition (3) of Proposition 8 we replace Picard stacks by Picard schemes, we get a weaker condition that was considered by the first two authors in [BE13], Théorème 13. From this observation, we deduce the following purity result.

Corollary 10. *Let k be a field of characteristic 0. Let k'/k be a finite Galois extension containing the fields of definition of all points in D , and let ℓ be a prime not dividing $[k' : k]$. If the torsor E_D is torsion of order prime to ℓ , then the Gal_k representation $\mathrm{H}^1(U_{\bar{k}}, \mathbb{Z}_\ell(1))$ is pure, i.e., the canonical short exact sequence*

$$0 \rightarrow \mathrm{H}^1(X_{\bar{k}}, \mathbb{Z}_\ell(1)) \rightarrow \mathrm{H}^1(U_{\bar{k}}, \mathbb{Z}_\ell(1)) \rightarrow \mathrm{H}_{D_{\bar{k}}}^2(X_{\bar{k}}, \mathbb{Z}_\ell(1))_0 \rightarrow 0$$

splits as a sequence of Gal_k representations.

Proof. By general limit arguments we may assume that k is a finitely generated extension of \mathbb{Q} . Note that we can in particular keep the assumption on the degree $[k' : k]$. Then combine Proposition 8 (3) and [BE13], Théorème 13. \square

Corollary 11.

- (1) *If $E_D^N \rightarrow X$ is trivial as a S_D -torsor, then for each degree zero Cartier divisor δ of X whose support is contained in the support of D , there is an isomorphism $\mathcal{O}_X(N \cdot \delta) \simeq \mathcal{O}_X$.*
- (2) *If D is totally split, these properties are equivalent.*

Proof. (1) One shows more generally, with notations of Proposition 8 (2): for each Cartier divisor δ of X whose support is contained in the support of D , there is an isomorphism

$$\mathcal{O}_X(N \cdot \delta) \simeq \mathcal{L}^{\otimes \deg \delta}.$$

One can reduce to the case where δ is reduced and irreducible, and this is then a direct consequence of Lemma 12 below.

(2) For each rational point x of D , the invertible sheaf $\mathcal{O}_{X \times X}(\Delta)|_{x \times X}$ identifies with the pullback of $\mathcal{O}_X(x)$ along the isomorphism $x \times X \simeq X$. So if D is totally split, the existence of an invertible sheaf \mathcal{L} verifying the second point of Proposition 8 is equivalent to the existence of an invertible sheaf \mathcal{L} so that for each rational point x of D , there exists an isomorphism $\mathcal{L} \simeq \mathcal{O}_X(N \cdot x)$. But the existence of such an invertible sheaf is clear from the hypothesis that every degree zero divisor with support in D is N -torsion. \square

Lemma 12. *Let $N(\cdot)$ denote the norm of an invertible sheaf along the morphism $\delta \times X \rightarrow X$. Then $N(\mathcal{O}_{X \times X}(-\Delta)|_{\delta \times X})$ and $\mathcal{O}_X(-\delta)$ coincide as ideals of \mathcal{O}_X .*

Proof. One can reduce to the case where X is an affine curve, which is straightforward. \square

For later use, we record here the following useful lemma.

Lemma 13. *Let $N : T_D \rightarrow \mathbb{G}_m$ be the norm map. Then there is a canonical isomorphism*

$$N_* F_D \simeq \mathbf{Isom}_X(\mathcal{O}_X, \mathcal{O}_X(D)).$$

Proof. Since the norm N is an epimorphism, it is enough to produce a morphism

$$F_D \rightarrow \mathbf{Isom}_X(\mathcal{O}_X, \mathcal{O}_X(D))$$

equivariant with respect to N . Let $X' \rightarrow X$ be a morphism and $f \in F_D(X')$. The data of f is equivalent to the data of its image g in $E_D(X')$, plus a trivialisation $\alpha : \mathcal{N}|_{X'} \simeq \mathcal{O}_{X'}$. The morphism g in turn is equivalent to the data of an invertible sheaf \mathcal{M} on $X_D \times_S X'$ lifting $\mathcal{O}_{X \times_S X}(\Delta)|_{X \times_S X'}$ along $X \times_S X' \rightarrow X_D \times_S X'$. By definition of \mathcal{N} , we have $\mathcal{M}|_{X'} \simeq \mathcal{N}|_{X'}$. So there is a given isomorphism

$$\mathcal{N}|_{D \times_S X'} \simeq \mathcal{O}_{X \times_S X}(\Delta)|_{D \times_S X'}.$$

Taking the norms along $D \times_S X' \rightarrow X'$ and using Lemma 12, one gets an isomorphism

$$\mathcal{N}|_{X'}^{\otimes \deg D} \simeq \mathcal{O}_X(D)|_{X'}.$$

Putting the isomorphism $\alpha : \mathcal{N}|_{X'} \simeq \mathcal{O}_{X'}$ again into the picture, we get the desired element of $\mathbf{Isom}_X(\mathcal{O}_X, \mathcal{O}_X(D))(X')$. \square

2.5. Torsion packets and torsion criterion II. Let X be a smooth, proper, geometrically connected curve over $S = \text{Spec}(k)$, where k is a field of characteristic 0.

Definition 14 ([BP01]). A (reduced) effective divisor D is a *torsion packet* if any degree 0 divisor on $X_{\bar{k}}$ with support in $D_{\bar{k}}$ is torsion.

Remark 15. (1) Any rational point defines a torsion packet.

(2) If D is a torsion packet, then any degree 0 divisor on X with support in D is torsion, because $\text{Pic } X \hookrightarrow \text{Pic } X_{\bar{k}}$ is injective.

(3) However, it is not true that if any degree 0 divisor on X with support in D is torsion, then D is a torsion packet. For instance, if D is irreducible, then any degree 0 divisor on X with support in D is even trivial. But D does not need to be a torsion packet. Indeed, according to [BP01], Corollary 3, if $\text{char } k = 0$ and X is of genus at least 2, the size of torsion packets is bounded. It is thus enough to choose D of degree strictly larger than this size, which is possible if k has an infinite absolute Galois group, to get a counterexample.

Theorem 16. *Let X be a smooth, proper, geometrically connected curve over $S = \text{Spec}(k)$.*

- (1) *The torsor $E_D \rightarrow X$ is torsion if and only if D is a torsion packet.*
- (2) *The torsor $F_D \rightarrow X$ is never torsion.*

Proof. (1) We first observe that the formation of E_D , as pinching and formation of Picard stacks, commutes with flat base change, thus $E_{D_{\bar{k}}} \simeq (E_D)_{\bar{k}}$. Hence Corollary 11 shows that $(E_D)_{\bar{k}}$ is torsion if and only if D is a torsion packet. The Hochschild-Serre spectral sequence gives the following exact sequence:

$$0 \rightarrow H^1(k, S_D) \rightarrow H^1(X, S_D) \rightarrow H^1(X_{\bar{k}}, S_D)^{\text{Gal}_k}$$

Moreover, the first group is torsion by Lemma 18 below, thus E_D is torsion if and only if $(E_D)_{\bar{k}}$ is torsion.

(2) Since the formation of F_D also commutes with base change, we can assume that k is algebraically closed. Since we assume D is non-zero, we can fix a rational point x in its support. It defines a character $T_D \rightarrow \mathbb{G}_m$, which by reduction of the structure group and functoriality of the formation of F_D in D , see Proposition 6, leads to the \mathbb{G}_m -torsor

$$F_x \simeq F_D \times^{T_D} \mathbb{G}_m.$$

By Definition 3, the corresponding invertible sheaf is $\mathcal{O}_X(-x)$, hence is non-torsion. So F_D is non-torsion as well. \square

Definition 17. (1) For any k -scheme Y , we define the relative Brauer group $\text{Br}(Y/k)$ as the kernel of the natural morphism $\text{Br}(k) \rightarrow \text{Br}(Y)$.

- (2) For any k -scheme Y of finite type, we define the index

$$\text{ind}(Y) = \gcd \{ \deg(y) ; y \text{ is a closed point of } Y \},$$

where $\deg(y) = [\kappa(y) : k]$ is the degree of the residue field extension at y .

Lemma 18. *We have*

$$H^1(k, S_D) = \text{Br}(D/k),$$

and, in particular, $H^1(k, S_D)$ is torsion and killed by $\text{ind}(D)$.

Proof. This follows from the Galois cohomology long exact sequence associated to the short exact sequence

$$0 \rightarrow \mathbb{G}_m \rightarrow T_D = R_{D/S}(\mathbb{G}_m) \rightarrow S_D \rightarrow 0, \quad (2.2)$$

Shapiro's Lemma and Hilbert's Theorem 90. If an extension k'/k has $x \in D(k')$, then the pullback

$$x^* : (R_{D/S}(\mathbb{G}_m))_{k'} \rightarrow \mathbb{G}_{m,k'}$$

splits (2.2) and so

$$\text{Br}(D/k) \hookrightarrow \text{Br}(k'/k)$$

which by a corestriction argument is killed by $[k' : k]$. Hence $\text{Br}(D/k)$ is killed by $\text{ind}(D)$. \square

3. EXTENSIONS OF FUNDAMENTAL GROUPS ASSOCIATED TO TORSORS

3.1. Algebraic $K(\pi, 1)$. Let $\acute{\text{E}}\text{t}(X)$ denote the étale homotopy type associated to a noetherian scheme X as in [AM69]. Let $X_{\text{fét}}$ denote the finite étale site of X . There is a natural continuous map $\gamma : X_{\text{ét}} \rightarrow X_{\text{fét}}$ of sites.

Definition 19. An **algebraic $K(\pi, 1)$** is a connected noetherian scheme X such that the canonical map

$$\acute{\text{E}}\text{t}(X) \rightarrow K(\pi_1(X, \bar{x}), 1)$$

is a weak equivalence. Equivalently, by [AM69] Theorem 4.3, an algebraic $K(\pi, 1)$ is a connected noetherian scheme X with geometric point \bar{x} , such that for any locally constant torsion sheaf

\mathcal{F} on the étale site $X_{\text{ét}}$, corresponding to the torsion $\pi_1(X, \bar{x})$ -module $F = \mathcal{F}_{\bar{x}}$, the natural morphism

$$\mathrm{H}^i(\pi_1(X, \bar{x}), F) \xleftarrow{\sim} \mathrm{H}^i(X_{\text{fét}}, \mathcal{F}) \xrightarrow{\gamma^*} \mathrm{H}^i(X, \mathcal{F}) \quad (3.1)$$

is an isomorphism for all $i \geq 0$.

Example 20.

- (1) The spectrum $\mathrm{Spec}(k)$ of a field k is an algebraic $\mathrm{K}(\pi, 1)$.
- (2) A smooth, connected curve $X \not\cong \mathbb{P}^1$ over an algebraically closed field k is an algebraic $\mathrm{K}(\pi, 1)$ ([Sti02], Proposition A.4.1 (1)).
- (3) It follows from (2) and the Hochschild-Serre spectral sequence that a smooth, geometrically connected curve X of positive genus over a field k is an algebraic $\mathrm{K}(\pi, 1)$.

3.2. Extensions of fundamental groups associated to fibrations. Let S be a connected noetherian scheme in characteristic 0. Let $\bar{f} : \bar{X} \rightarrow S$ be a proper flat map with geometrically connected and reduced fibers, and let $X \subseteq \bar{X}$ be the complement of a normal crossing divisor relative to S , and $f : X \rightarrow S$ the restriction of \bar{f} .

Let $\bar{x} \in X_{\bar{s}}$ be a geometric point in the fiber $f^{-1}(\bar{s}) = X_{\bar{s}}$ above the geometric point \bar{s} . Then by [SGA1] XIII Prop 4.1 and Example 4.4, the sequence induced by the natural maps

$$\pi_1(X_{\bar{s}}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \xrightarrow{\pi_1(f)} \pi_1(S, \bar{s}) \rightarrow 1 \quad (3.2)$$

is exact. The map on the left in (3.2) is also injective if in addition $X \rightarrow S$ is smooth and admits a section, see [SGA1] XIII Prop 4.3 and Example 4.4. We would like to give another criterion for this injectivity which applies to torsors under a torus.

Theorem 21. *Let S be an algebraic $\mathrm{K}(\pi, 1)$ in characteristic 0. Let $f : X \rightarrow S$ be a smooth surjective map with geometrically connected fibers. Further we assume that f admits a proper flat relative compactification $\bar{f} : \bar{X} \rightarrow S$, such that $\bar{X} \setminus X$ is a normal crossing divisor relative to S . Let $\bar{x} \in X$ be a geometric point with image $\bar{s} = f(\bar{x})$. Then*

$$1 \rightarrow \pi_1(X_{\bar{s}}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \xrightarrow{\pi_1(f)} \pi_1(S, \bar{s}) \rightarrow 1 \quad (3.3)$$

is exact.

Proof. It remains to show that $\pi_1(X_{\bar{s}}, \bar{x}) \rightarrow \pi_1(X, \bar{x})$ is injective. Let G be a finite group and let $\varphi : \pi_1(X_{\bar{s}}, \bar{x}) \rightarrow G$ be a connected G -torsor on $X_{\bar{s}}$. By [SGA1] XIII Corollaire 2.9, the non-abelian

$$\mathrm{R}^1 f_* G$$

is a locally constant constructible sheaf on $S_{\text{ét}}$.

Using the equivalence between locally constant sheaves of finite sets on $S_{\text{ét}}$ and $\pi_1(S, \bar{s})$ -finite sets, we may replace S by a finite étale connected cover such that $\mathrm{R}^1 f_* G$ becomes constant. Let

$$\Phi \in \mathrm{H}^0(S, \mathrm{R}^1 f_* G)$$

be the section that restricts to the isomorphism class of φ in the fiber above \bar{s} . This global section Φ can be represented on a certain étale covering $(U_i)_{i \in I}$ of S by G -torsors $Y_i \rightarrow X \times_S U_i$ whose restrictions to $U_i \times_S U_j$ are isomorphic for all $i, j \in I$, with the additional condition that, if U_{i_0} contains \bar{s} , the restriction of $Y_{i_0} \rightarrow X_{U_{i_0}}$ to the geometric fiber $X_{\bar{s}}$ is isomorphic to the connected G -cover represented by φ (here we use the fact that φ is surjective). There is no restriction to assume U_{i_0} to be connected, and the last condition implies that Y_{i_0} is connected, so $Y_{i_0} \rightarrow X_{U_{i_0}}$ is a G -cover. In particular $\mathrm{Aut}(Y_{i_0}/X_{U_{i_0}}) \simeq Z(G)$.

The obstruction for Φ to come from an actual G -torsor over X is the obstruction for the following gerbe \mathcal{G} bound by the center $Z(G)$ of G to be neutral: a section of \mathcal{G} over $u : U \rightarrow S$

is a G -torsor $Y \rightarrow X_U$ such that for any i , the restrictions of $Y \rightarrow X_U$ and $Y_i \rightarrow X_{U_i}$ are isomorphic over $X_{U \times_S U_i}$. The class of this gerbe \mathcal{G} lives in

$$H^2(S, Z(G))$$

where $Z(G)$ is the center of G .

This obstruction vanishes due to the $K(\pi, 1)$ assumption after passing to a further connected finite étale cover of S . Hence, putting everything together, for a suitable connected finite étale cover $S' \rightarrow S$ there exists a prolongation of φ as in the commutative diagram

$$\begin{array}{ccc} \pi_1(X'_{\bar{s}'}, \bar{x}) & \xrightarrow{\iota'} & \pi_1(X', \bar{x}') \\ \parallel & & \downarrow \tilde{\varphi} \\ \pi_1(X_{\bar{s}}, \bar{x}) & \xrightarrow{\varphi} & G. \end{array}$$

This shows that

$$\ker(\iota : \pi_1(X_{\bar{s}}, \bar{x}) \rightarrow \pi_1(X, \bar{x})) = \ker(\iota' : \pi_1(X'_{\bar{s}'}, \bar{x}') \rightarrow \pi_1(X', \bar{x}')) \subseteq \iota'^{-1}(\ker(\tilde{\varphi})) = \ker(\varphi)$$

and since this holds for all φ we conclude that $\pi_1(X_{\bar{s}}, \bar{x}) \rightarrow \pi_1(X, \bar{x})$ is injective as claimed. \square

Theorem 22. *Let X be a connected normal noetherian scheme, and let $E \rightarrow X$ be a torsor under an X -torus T . Then E admits a T -equivariant toric smooth projective compactification $\bar{E} \rightarrow X$ such that the complement $\bar{E} \setminus E$ is a normal crossing divisor relative to X .*

Proof. The proof is essentially a copy of [CTHS05] Corollaire 1. Let $\bar{x} \in X$ be a geometric point and consider the action $\rho : \pi_1(X, \bar{x}) \rightarrow \mathrm{GL}(M)$ on the character group $M = \mathrm{Hom}(T_{\bar{x}}, \mathbb{G}_m)$ of $T_{\bar{x}}$. Since X is normal, the X -torus T is isotrivial and so the image Γ of ρ is a finite group.

Let $X' \rightarrow X$ be the corresponding Γ -Galois finite étale cover. We denote by N the cocharacter group of $T_{\bar{x}}$ (that is, the dual of M). The pullback $T' = T \times_X X'$ carries a Γ -action on X' and comes by base change $X' \rightarrow \mathrm{Spec}(\mathbb{Z})$ from the Γ -equivariant \mathbb{Z} -torus

$$G = N \otimes_{\mathbb{Z}} \mathbb{G}_{m, \mathbb{Z}} \rightarrow \mathrm{Spec}(\mathbb{Z}).$$

The group N is a lattice in the \mathbb{R} -vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$ with a Γ -action. Théorème 1 of [CTHS05] shows the existence of a smooth, projective and Γ -equivariant fan inside $N \otimes_{\mathbb{Z}} \mathbb{R}$. We associate to this fan a Γ -equivariant, smooth, projective toric variety (by glueing for the respective monoids P the affine charts $\mathrm{Spec}(\mathbb{Z}[P])$ instead of $\mathrm{Spec}(K[P])$ like for traditional toric varieties over a field K)

$$\bar{G} \rightarrow \mathrm{Spec}(\mathbb{Z})$$

which is a smooth Γ -equivariant toric completion of G/\mathbb{Z} with boundary a normal crossing divisor relative to $\mathrm{Spec}(\mathbb{Z})$, see for example [Ful93] §2.1 for a version over \mathbb{C} . We conclude that T' has a Γ - T' -equivariant smooth, projective completion over X'

$$T' = G \times X' \subset \bar{G} \times X' =: \bar{T}'$$

with boundary a normal crossing divisor. The scheme \bar{T}' is endowed with a T' -equivariant invertible sheaf \mathcal{L}' that is ample relative X' . Replacing \mathcal{L}' by $\bigotimes_{\gamma \in \Gamma} \gamma^* \mathcal{L}'$, we can assume that \mathcal{L}' is equipped with a Γ -linearization. Since Γ acts via X -maps, we conclude that Γ -orbits are contained in affine open subschemes. Therefore the quotients by Γ exist already as schemes.

By passing to the quotients by Γ we obtain a T -equivariant, smooth, projective completion over X

$$T = T'/\Gamma \subset (\bar{G} \times X')/\Gamma =: \bar{T}$$

with boundary a normal crossing divisor. Indeed, the action of Γ on $\bar{G} \times X'$ is free (it is already free on the second factor), thus the quotient map is étale and preserves the étale local notions

of smoothness and normal crossing. The scheme \bar{T} is in turn endowed with a T -equivariant invertible sheaf \mathcal{L} that is ample relative X .

Finally, we define the desired good completion of E by the contraction

$$\bar{E} := E \times^T \bar{T}.$$

This is a projective scheme over X , since it is obtained by descent (see for example [Vis05], §4.4, §4.3.3) along $E \rightarrow X$ from $E \times_X \bar{T}$, equipped with the T -equivariant E -ample invertible sheaf $\text{pr}_2^* \mathcal{L}$, where $\text{pr}_2 : E \times_X \bar{T} \rightarrow \bar{T}$. \square

Corollary 23. *Let X be a normal algebraic $\mathbb{K}(\pi, 1)$ of characteristic 0 and let T be a torus over X and $E \rightarrow X$ a torsor under T . Let \bar{y} be a geometric point of E with image $\bar{x} \in X$. Then the sequence*

$$1 \rightarrow \pi_1(E_{\bar{x}}, \bar{y}) \rightarrow \pi_1(E, \bar{y}) \rightarrow \pi_1(X, \bar{x}) \rightarrow 1 \quad (3.4)$$

is exact.

Proof. This is an immediate consequence of Theorem 22 and Theorem 21. \square

Remark 24. (1) In the case $T = \mathbb{G}_m$, Corollary 23 has a proof independent of Theorem 21 by Mochizuki (see [Moc03], Lemma 4.4).

(2) The $\mathbb{K}(\pi, 1)$ assumption in Theorem 21 is indeed necessary. As an example, we consider an algebraically closed field \bar{k} and the \mathbb{G}_m -torsor

$$\mathbb{A}_{\bar{k}}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}_{\bar{k}}^n$$

associated to the line bundle $\mathcal{O}(1)$ and some $n \geq 1$. Then by Zariski-Nagata purity of the branch locus we have $\pi_1(\mathbb{A}_{\bar{k}}^{n+1} \setminus \{0\}) = 0$ and the sequence

$$1 \rightarrow \pi_1(\mathbb{G}_{m, \bar{k}}) \rightarrow \pi_1(\mathbb{A}_{\bar{k}}^{n+1} \setminus \{0\}) \rightarrow \pi_1(\mathbb{P}_{\bar{k}}^n) \rightarrow 1$$

is not exact on the left.

Remark 25. In the situation of Corollary 23, the choice of \bar{y} defines an isomorphism $T_{\bar{x}} \simeq E_{\bar{y}}$ by translation, and the group

$$\pi_1(T_{\bar{x}}, 1) \simeq \pi_1(E_{\bar{x}}, \bar{y})$$

is the fundamental group of an algebraic group (in characteristic 0, see [Sti13b] §13.1) and hence abelian (so that we can neglect base points). The conjugation action of $\pi_1(E, \bar{y})$ on $\pi_1(E_{\bar{x}}, \bar{y})$ thus defines a $\pi_1(X, \bar{x})$ -module structure on $\pi_1(E_{\bar{x}}, \bar{y})$. This module structure is functorial with respect to X -maps between torsors (not necessarily torsor maps).

Proposition 26. *Let X be an algebraic $\mathbb{K}(\pi, 1)$ of characteristic 0.*

- (1) *Let T be a torus over X , and let $E \rightarrow X$ be a torsor under T endowed with a geometric point $\bar{y} \in E$ with image $\bar{x} \in X$. The canonical isomorphism by translation*

$$\tau : \pi_1(T_{\bar{x}}, 1) \simeq \pi_1(E_{\bar{x}}, \bar{y})$$

is an isomorphism of $\pi_1(X, \bar{x})$ -modules.

- (2) *If, moreover, X is a geometrically connected variety over $S = \text{Spec}(k)$, and $T = T_X$ comes from a torsor T over S , then the $\pi_1(X, \bar{x})$ -action factors over the projection $\pi_1(X, \bar{x}) \rightarrow \text{Gal}_{\bar{k}}$ and translation is an isomorphism*

$$\tau : \mathbb{T}(T) \simeq \pi_1(T_{\bar{k}}, 1) = \pi_1(T_{X, \bar{x}}, 1) \simeq \pi_1(E_{\bar{x}}, \bar{y}).$$

Proof. (1) Let $a : E \times_X T \rightarrow E$ be the action map of the T -torsor. The map

$$(\text{pr}_1, a) : E \times_X T \rightarrow E \times_X E$$

is an isomorphism over X and compatible with the first projection to E . Note that $E \times_X E$ and $E \times_X T$ are $T \times_X T$ -torsors on X . By Remark 25 we have a commutative diagram of $\pi_1(X, \bar{x})$ -modules

$$\begin{array}{ccc} \pi_1(E_{\bar{x}} \times T_{\bar{x}}, (\bar{y}, 1)) & \xrightarrow{\text{pr}_1} & \pi_1(E_{\bar{x}}, \bar{y}) \\ \downarrow (\text{pr}_1, a) & & \downarrow \text{id} \\ \pi_1(E_{\bar{x}} \times E_{\bar{x}}, (\bar{y}, \bar{y})) & \xrightarrow{\text{pr}_1} & \pi_1(E_{\bar{x}}, \bar{y}) \end{array}$$

where the vertical maps are isomorphisms. The induced isomorphism of kernels of the horizontal maps is nothing but

$$\tau : \pi_1(T_{\bar{x}}, 1) \simeq \pi_1(E_{\bar{x}}, \bar{y})$$

by the Künneth formula. Therefore, τ is also an isomorphism of $\pi_1(X, \bar{x})$ -modules.

(2) The $\pi_1(X, \bar{x})$ -module structure on $\pi_1(T_{\bar{k}}, 1) = \pi_1(T_{X, \bar{x}}, 1)$ associated to $T_X \rightarrow X$ comes by functoriality from the action associated to $T \rightarrow \text{Spec}(k)$. The action thus factors through the projection $\pi_1(X, \bar{x}) \rightarrow \text{Gal}_k$. The identification $\mathbb{T}(T) \simeq \pi_1(T_{\bar{k}}, 1)$ as Gal_k -modules is classical, see for example [Sti13b] Section §13.1. \square

3.3. The fundamental group of F_D and functoriality in D . Here and in the rest of Section §3, let X be a smooth geometrically connected curve defined over $S = \text{Spec}(k)$ with k a field of characteristic 0. Let $D \neq 0$ be an effective, étale Cartier divisor on X . We assume that $X_{\bar{k}} \not\cong \mathbb{P}_{\bar{k}}^1$ so that X is an algebraic $\text{K}(\pi, 1)$ by Example 20 (2). Then Corollary 23 and Proposition 26 apply to F_D and E_D and yield a commutative diagram of π_1 -extensions

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(T_{D, \bar{k}}, 1) & \longrightarrow & \pi_1(F_D, \bar{y}) & \longrightarrow & \pi_1(X, \bar{x}) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \text{id} \\ 1 & \longrightarrow & \pi_1(S_{D, \bar{k}}, 1) & \longrightarrow & \pi_1(E_D, \bar{y}) & \longrightarrow & \pi_1(X, \bar{x}) \longrightarrow 1. \end{array} \quad (3.5)$$

We will need the following remarks that follow immediately from Proposition 6, Corollary 23 and the Künneth formula.

Lemma 27. *Let $D_1 \subset D_2 \subset X$ be two effective étale divisors on X . The canonical isomorphisms $F_{D_1} \simeq F_{D_2} \times^{T_{D_2}} T_{D_1}$ (resp. $E_{D_1} \simeq E_{D_2} \times^{S_{D_2}} S_{D_1}$) induce push out diagrams*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(T_{D_2, \bar{k}}, 1) & \longrightarrow & \pi_1(F_{D_2}, \bar{y}) & \longrightarrow & \pi_1(X, \bar{x}) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \text{id} \\ 1 & \longrightarrow & \pi_1(T_{D_1, \bar{k}}, 1) & \longrightarrow & \pi_1(F_{D_1}, \bar{y}) & \longrightarrow & \pi_1(X, \bar{x}) \longrightarrow 1, \end{array}$$

and respectively

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(S_{D_2, \bar{k}}, 1) & \longrightarrow & \pi_1(E_{D_2}, \bar{y}) & \longrightarrow & \pi_1(X, \bar{x}) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \text{id} \\ 1 & \longrightarrow & \pi_1(S_{D_1, \bar{k}}, 1) & \longrightarrow & \pi_1(E_{D_1}, \bar{y}) & \longrightarrow & \pi_1(X, \bar{x}) \longrightarrow 1. \end{array}$$

Lemma 28. *Let $D = D_1 \sqcup D_2$ be the disjoint union of two effective étale divisors $D_1, D_2 \subset X$. The projection maps $F_D \rightarrow F_{D_i}$, for $i = 1, 2$, induce the relative Künneth isomorphism*

$$\pi_1(F_D, \bar{y}) \xrightarrow{\sim} \pi_1(F_{D_1}, \bar{y}_1) \times_{\pi_1(X, \bar{x})} \pi_1(F_{D_2}, \bar{y}_2).$$

3.4. Comparison with the maximal cuspidally central quotient. The aim of this paragraph is to show the following interpretation of $\pi_1(F_D, \bar{y})$. Let $U = X \setminus D$ be the complement of the support of the divisor and set

$$N = \text{Ker}(\pi_1(U, \bar{x}) \rightarrow \pi_1(X, \bar{x})).$$

Recall the notion of the maximal **cuspidally central** quotient $\pi_1^{cc}(U, \bar{x})$ due to Mochizuki [Moc07] Definition 1.1(i): the biggest quotient $\pi_1^{cc}(U, \bar{x}) = \pi(U, \bar{x})/N_{cc}$ by a normal subgroup $N_{cc} \subseteq N$ such that one gets an exact sequence

$$1 \rightarrow N^{cc} \rightarrow \pi_1^{cc}(U, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \rightarrow 1$$

where $N^{cc} = N/N_{cc}$ is abelian, and the action of $\pi_1(X_{\bar{k}}, \bar{x})$ by conjugation on N^{cc} is trivial.

Lemma 29. *When D is the disjoint union of two divisors D_1 and D_2 defined over k and $U_i = X \setminus D_i$, then the canonical maps induce an isomorphism*

$$\pi_1^{cc}(U, \bar{x}) \simeq \pi_1^{cc}(U_1, \bar{x}) \times_{\pi_1(X, \bar{x})} \pi_1^{cc}(U_2, \bar{x}).$$

Proof. Lift to \mathbb{C} and use the known presentation of the topological fundamental group. \square

Proposition 30. *The canonical lift $U \rightarrow F_D$ of the inclusion $U \subset X$ induces an isomorphism*

$$\pi_1^{cc}(U, \bar{x}) \simeq \pi_1(F_D, \bar{x}).$$

Proof. By Corollary 23 and Proposition 26 the extension $\pi_1(F_D, \bar{y}) \rightarrow \pi_1(X, \bar{x})$ has abelian kernel and is central when restricted to $\pi_1(X_{\bar{k}}, \bar{x})$. Hence, the canonical lift $U \rightarrow F_D$ induces a morphism

$$\pi_1^{cc}(U, \bar{x}) \rightarrow \pi_1(F_D, \bar{y})$$

fitting in the following commutative diagram where the horizontal lines are exact by definition of $\pi_1^{cc}(U, \bar{x})$ and by Corollary 23:

$$\begin{array}{ccccccc} 1 & \longrightarrow & N^{cc} & \longrightarrow & \pi_1^{cc}(U, \bar{x}) & \longrightarrow & \pi_1(X, \bar{x}) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \text{id} \\ 1 & \longrightarrow & \pi_1(T_{D, \bar{k}}, 1) & \longrightarrow & \pi_1(F_D, \bar{y}) & \longrightarrow & \pi_1(X, \bar{x}) \longrightarrow 1. \end{array}$$

It remains to prove that the induced map $N^{cc} \rightarrow \pi_1(T_{D, \bar{k}}, 1)$ is an isomorphism. For that we may assume that $k = \bar{k}$. By Lemma 28 and Lemma 29 we may reduce to the case that $D = \{x\}$ has degree 1. Then both N^{cc} and $\pi_1(T_{D, \bar{k}}, 1)$ are isomorphic to $\hat{\mathbb{Z}}(1)$, so that it suffices to show that $N^{cc} \rightarrow \pi_1(T_{D, \bar{k}}, 1)$ is surjective. This in turn is equivalent to the image of $\pi_1(U, \bar{x}) \rightarrow \pi_1(F_D, \bar{y})$ containing the image of $\pi_1(T_{D, \bar{k}}, 1)$.

Denote by $R = \mathcal{O}_{X, x}^{\text{sh}}$ the strict henselization of X at x , by K the fraction field of R , and put: $\hat{X}_x = \text{Spec}(R)$, $\hat{U}_x = \text{Spec}(K)$, $\hat{F}_{D_x} = F_D \times_X \hat{X}_x$. We choose a geometric point of $\hat{F}_{D_x} \times_{\hat{X}_x} \hat{U}_x$. The commutative diagram of fundamental groups based at this point

$$\begin{array}{ccccccc} \pi_1(T_{D, \bar{K}}, 1) & \longrightarrow & \pi_1(\hat{F}_{D_x}, \bar{y}) & & & & \\ & & \downarrow & & & & \\ & & \simeq & & & & \\ 1 & \longrightarrow & \pi_1(T_{D, \bar{k}}, 1) & \longrightarrow & \pi_1(F_D, \bar{y}) & \longrightarrow & \pi_1(X, \bar{x}) \longrightarrow 1 \end{array}$$

shows that it is enough to prove that $\pi_1(\hat{U}_x, \bar{x}) \rightarrow \pi_1(\hat{F}_{D_x}, \bar{y})$ is surjective. Definition 3 gives that

$$\hat{F}_{D_x} = \mathbf{Isom}_{\hat{X}_x}(\mathcal{O}(x), \mathcal{O}).$$

The choice of a uniformizing parameter π at x induces an isomorphism $\widehat{F}_{Dx} \simeq \mathbb{G}_{m, \widehat{X}_x}$ over \widehat{X}_x so that the morphism

$$\pi : \text{Spec}(K) = \widehat{U}_x \rightarrow \widehat{F}_{Dx} \simeq \mathbb{G}_{m, \widehat{X}_x} = \text{Spec}(R[V, V^{-1}])$$

corresponds to $\pi \in \mathbb{G}_m(K)$.

Since $\pi_1(\widehat{X}_x) = 1$ and \widehat{X}_x is an algebraic $\mathbb{K}(\pi, 1)$, Corollary 23 yields an isomorphism $\pi_1(T_{D, \bar{K}}, 1) \simeq \pi_1(\widehat{F}_{Dx}, \bar{y})$. Thus a cofinal system of connected étale covers of $\text{Spec}(R[V, V^{-1}])$ is given by

$$\text{Spec}(R[V, V^{-1}, W]/(W^n - V)) \rightarrow \text{Spec}(R[V, V^{-1}]),$$

where $n \in \mathbb{N}$. The pullback along π to $\text{Spec}(K)$ is the tame extension

$$\text{Spec}(K[W]/(W^n - \pi)) \rightarrow \text{Spec}(K)$$

which is connected according to Eisenstein's criterion. So according to [SGA1] V Proposition 6.9, the morphism $\pi_1(\widehat{U}_x, \bar{x}) \rightarrow \pi_1(\widehat{F}_{Dx}, \bar{y})$ is surjective. \square

3.5. The fundamental group of E_D . In this section we describe $\pi_1(E_D, \bar{y})$ in terms of the surjection $\pi_1(U, \bar{x}) \twoheadrightarrow \pi_1(X, \bar{x})$. Let $\pi_1^{[\text{ab}]}(U, \bar{x})$ be the pushout of $\pi_1(U, \bar{x})$ along the maximal abelian quotient $\pi_1(U_{\bar{k}}, \bar{x}) \twoheadrightarrow \pi_1^{\text{ab}}(U_{\bar{k}}, \bar{x})$; define similarly $\pi_1^{[\text{ab}]}(X, \bar{x})$.

Proposition 31. *Let X/k be a smooth, geometrically connected, projective curve. Then the map $\pi_1(U, \bar{x}) \rightarrow \pi_1(E_D, \bar{y})$ induced by the tautological section above U yields an isomorphism*

$$\pi_1(E_D, \bar{y}) \simeq \pi_1^{[\text{ab}]}(U, \bar{x}) \times_{\pi_1^{[\text{ab}]}(X, \bar{x})} \pi_1(X, \bar{x}).$$

Proof. Recall that X_D is the curve X pinched along the étale effective divisor D . We consider a refinement of diagram (2.1) that describes $E_D \rightarrow X$ as the pullback of $\text{Pic}_{X_D}^1 \rightarrow \text{Pic}_X^1$. With a base point $\bar{y} \in E_D$ with image $\bar{x} \in X$, we have a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(E_{D, \bar{x}}, \bar{y}) & \longrightarrow & \pi_1(E_D, \bar{y}) & \longrightarrow & \pi_1(X, \bar{x}) \longrightarrow 1 \\ & & \downarrow \simeq & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1(S_{D, \bar{k}}, 1) & \longrightarrow & \pi_1(\text{Pic}_{X_D}^1, \bar{y}) & \longrightarrow & \pi_1(\text{Pic}_X^1, \bar{x}) \longrightarrow 1 \end{array}$$

with exact rows: the bottom row is exact by Corollary 23 because as a torsor under an abelian variety Pic_X^1 is an algebraic $\mathbb{K}(\pi, 1)$. And for exactness in the top row, only the injectivity of the left map needs justification which follows from the analogue injectivity in the bottom row.

We conclude that we have a commutative diagram

$$\begin{array}{ccccc} \pi_1(U, \bar{x}) & \longrightarrow & \pi_1(E_D, \bar{y}) & \longrightarrow & \pi_1(X, \bar{x}) \\ & \searrow & \downarrow & & \downarrow \\ & & \pi_1(\text{Pic}_{X_D}^1, \bar{y}) & \longrightarrow & \pi_1(\text{Pic}_X^1, \bar{x}) \end{array}$$

where the square is cartesian. Since the natural map $U \rightarrow \text{Pic}_{X_D}^1$ induces by geometric class field theory an isomorphism $\pi_1^{[\text{ab}]}(U, \bar{x}) \simeq \pi_1(\text{Pic}_{X_D}^1, \bar{y})$ and similarly $\pi_1^{[\text{ab}]}(X, \bar{x}) \simeq \pi_1(\text{Pic}_X^1, \bar{x})$ we extract the desired description. \square

4. OBSTRUCTIONS TO LIFTING GALOIS SECTIONS ALONG TORSORS UNDER TORI

In this section, we fix a smooth, proper, geometrically connected curve X of genus at least 1 over a field k of characteristic 0, and T/k a torus. Let $E \rightarrow X$ be a torsor under the torus T . Then E is also geometrically connected and therefore defines a fundamental exact sequence analogous to (1.1). The issue we want to address is: does a given section $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ of (1.1) lift to a section $\text{Gal}_k \rightarrow \pi_1(E, \bar{y})$?

4.1. Arithmetic first Chern class. We introduce in this paragraph the notion of arithmetic first Chern class of a torsor under a torus. The relevance of this notion for anabelian issues has been pointed out by Mochizuki in the case of line bundles (see [Moc99] Definition 0.3). Our definition is the straightforward generalization.

The most logical way to proceed is to use Jannsen's cohomology (see [Jan88]), that is, cohomology defined on the category of inverse systems of étale sheaves of abelian groups on the small étale site of the base scheme X . In this context, the Kummer short exact sequence takes the following form, for a couple of integers (m, n) such that $m|n$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T[n] & \longrightarrow & T & \xrightarrow{n \cdot} & T \longrightarrow 0 \\ & & \downarrow \frac{n \cdot}{m \cdot} & & \downarrow \frac{n \cdot}{m \cdot} & & \downarrow \text{id} \\ 0 & \longrightarrow & T[m] & \longrightarrow & T & \xrightarrow{m \cdot} & T \longrightarrow 0 \end{array} \quad (4.1)$$

Let us define $\mathbb{T}(T)$ (resp. $(T, \frac{n \cdot}{m \cdot})$) as the inverse system given by the left (resp. middle) column of equation (4.1). Note that the Jannsen cohomology of the right column is the usual étale cohomology of T .

Definition 32. The *first arithmetic Chern class* of a T -torsor $E \rightarrow X$ is the image of its class by the coboundary morphism

$$c_1 : H^1(X, T) \rightarrow H^2(X, \mathbb{T}(T))$$

in the Jannsen cohomology long exact sequence associated to the short exact sequence (4.1).

For an abelian group A we denote by $\text{Div}(A)$ its maximal divisible subgroup, by $\widehat{A} = \varprojlim_n A/nA$ its completion, and by $\mathbb{T}(A) = \varprojlim_n A[n]$ its Tate module.

Proposition 33. *Let X/k be a scheme of finite type. The Kummer short exact sequence induces exact sequences:*

$$0 \rightarrow \text{Div}(H^1(X, T)) \rightarrow H^1(X, T) \xrightarrow{c_1} H^2(X, \mathbb{T}(T)) \rightarrow H^2\left(X, \left(T, \frac{n \cdot}{m \cdot}\right)\right) \rightarrow \text{Div}(H^2(X, T)) \rightarrow 0$$

and

$$0 \rightarrow \widehat{H^1(X, T)} \xrightarrow{c_1} H^2(X, \mathbb{T}(T)) \rightarrow \mathbb{T}(H^2(X, T)) \rightarrow 0.$$

Moreover, the morphism

$$H^2(X, \mathbb{T}(T)) \rightarrow \varprojlim_{n \in \mathbb{N}} H^2(X, T[n])$$

is an isomorphism.

Proof. The exactness of the first sequence follows from the exactness of the Jannsen cohomology long exact sequence and the fact that for a nonnegative integer i , the image of the morphism $H^i(X, (T, \frac{n \cdot}{m \cdot})) \rightarrow H^i(X, T)$ is $\text{Div}(H^i(X, T))$. Indeed, for any inverse system of sheaves $(F_n)_{n \in \mathbb{N}}$ there is a short exact sequence (see [Jan88], (3.1)):

$$0 \rightarrow \varprojlim_{n \in \mathbb{N}}^1 H^{i-1}(X, F_n) \rightarrow H^i(X, (F_n)_{n \in \mathbb{N}}) \rightarrow \varprojlim_{n \in \mathbb{N}} H^i(X, F_n) \rightarrow 0, \quad (4.2)$$

hence the image of $H^i(X, (T, \frac{n \cdot}{m \cdot})) \rightarrow H^i(X, T)$ is the image of $\varprojlim (H^i(X, T), \frac{n \cdot}{m \cdot}) \rightarrow H^i(X, T)$, and moreover for any abelian group A the image of $\varprojlim_{n \in \mathbb{N}} (A, \frac{n \cdot}{m \cdot}) \rightarrow A$ is $\text{Div}(A)$.

We deduce the exactness of the second exact sequence from the finite level Kummer exact sequence

$$0 \rightarrow T[n] \rightarrow T \xrightarrow{n} T \rightarrow 0,$$

which leads to the exact sequence

$$0 \rightarrow H^1(X, T)/n H^1(X, T) \rightarrow H^2(X, T[n]) \rightarrow H^2(X, T)[n] \rightarrow 0.$$

Taking the limit along the system (4.1) is exact since the system $(H^1(X, T)/n H^1(X, T), \text{pr})$ with the projection maps pr is surjective, hence satisfies the Mittag–Leffler condition. We obtain the exact sequence

$$0 \rightarrow \widehat{H^1(X, T)} \xrightarrow{c_1} \varprojlim_{n \in \mathbb{N}} H^2(X, T[n]) \rightarrow \mathbb{T}(H^2(X, T)) \rightarrow 0,$$

and it remains to see that the natural map

$$H^2(X, \mathbb{T}(T)) \rightarrow \varprojlim_{n \in \mathbb{N}} H^2(X, T[n])$$

is an isomorphism (which also justifies to call the map c_1 again). Equivalently, by (4.2), we must prove

$$\varprojlim_{n \in \mathbb{N}}^1 H^1(X, T[n]) = 0.$$

The Kummer sequence provides moreover the exact sequence

$$0 \rightarrow T(X)/nT(X) \rightarrow H^1(X, T[n]) \rightarrow H^1(X, T)[n] \rightarrow 0,$$

which again by Mittag–Leffler and $\varprojlim_{n \in \mathbb{N}}^2 = 0$ leads to an exact sequence

$$0 = \varprojlim_{n \in \mathbb{N}}^1 T(X)/nT(X) \rightarrow \varprojlim_{n \in \mathbb{N}}^1 H^1(X, T[n]) \xrightarrow{\sim} \varprojlim_{n \in \mathbb{N}}^1 H^1(X, T)[n] \rightarrow 0.$$

The restriction to n -torsion of the short exact sequence of low degree terms of the Hochschild–Serre spectral sequence for $X \rightarrow \text{Spec}(k)$ and coefficients T yields exactness of

$$0 \rightarrow H^1(k, T)[n] \rightarrow H^1(X, T)[n] \rightarrow H^1(X_{\bar{k}}, T)[n]^{\text{Gal}_k}.$$

By Hilbert’s Theorem 90, the groups $H^1(k, T)[n]$ have bounded exponent and thus the system $\mathbb{T}(H^1(k, T))$ is Mittag–Leffler zero. Since $T_{\bar{k}} \cong \mathbb{G}_m^d$ for $d = \dim T$ we find non-canonically

$$H^1(X_{\bar{k}}, T) \cong \text{Pic}(X_{\bar{k}})^d$$

which has finite n -torsion. Thus the projective system $(H^1(X, T)[n])$ is an extension of a system of finite levels

$$\left(\text{im}(H^1(X, T)[n] \rightarrow H^1(X_{\bar{k}}, T)[n]^{\text{Gal}_k}) \right)$$

by a Mittag–Leffler zero system. Therefore its $\varprojlim_{n \in \mathbb{N}}^1$ vanishes and the proof is complete. \square

Remark 34. Because the morphism $H^2(X, \mathbb{T}(T)) \rightarrow \varprojlim_{n \in \mathbb{N}} H^2(X, T[n])$ is an isomorphism, it will be often sufficient to consider only Chern classes modulo n , denoted by $c_1(E)_n$, and defined as the image of $c_1(E)$ by the morphism $H^2(X, \mathbb{T}(T)) \rightarrow H^2(X, T[n])$.

Corollary 35. *For a torus T over a field k , the following sequence:*

$$0 \rightarrow H^1(k, T) \xrightarrow{c_1} H^2(k, \mathbb{T}(T)) \rightarrow \mathbb{T}(H^2(k, T)) \rightarrow 0.$$

is exact.

Proof. Since $H^1(k, T)$ is killed by the degree of any extension trivializing T , one has

$$\widehat{H^1(k, T)} = H^1(k, T),$$

and the corollary follows from Proposition 33. \square

4.2. Class of the fibration. Starting again from a T -torsor $h : E \rightarrow X$, let us consider the exact sequence

$$1 \rightarrow \pi_1(T_{\bar{k}}, 1) \rightarrow \pi_1(E, \bar{y}) \rightarrow \pi_1(X, \bar{x}) \rightarrow 1 \quad (4.3)$$

from Corollary 23 where we use the isomorphism $\pi_1(T_{\bar{k}}, 1) \simeq \pi_1(E_{\bar{x}}, \bar{y})$ of Proposition 26. This yields an abelian cohomology class

$$\pi_1(E/X) \in H^2(\pi_1(X, \bar{x}), \pi_1(T_{\bar{k}}, 1)).$$

The section s lifts to (a section of) E if and only if $s^*(\pi_1(E/X)) = 0$ in $H^2(\text{Gal}_k, \pi_1(T_{\bar{k}}, 1))$. We are therefore interested in an explicit description of the class $\pi_1(E/X)$. Before we can do that we need a digression about Chern gerbes.

Definition 36. Let E/X be a T -torsor, and $n \geq 1$. The Chern gerbe modulo n is the gerbe $\mathcal{G}_c^n(E)$ over X of liftings of the structural group of E along the morphism $\times n : T \rightarrow T$.

The class of $\mathcal{G}_c^n(E)$ in $H^2(X, T[n])$ is naturally $c_1(E)_n$, see Lemma 59 of the appendix. To avoid confusion between the two copies of T , we will adopt the following notation. We consider an exact sequence:

$$0 \rightarrow \mu \rightarrow T' \rightarrow T \rightarrow 0$$

where T, T' are geometrically connected commutative k -groups and μ is a finite étale k -group. We can associate to it the gerbe $\mathcal{G}_{E, T'}/X$ of liftings of the structural group of E to T' . The gerbe $\mathcal{G}_{E, T'}$ can be described as the quotient stack

$$E \rightarrow [E/T'] = \mathcal{G}_{E, T'} \rightarrow X.$$

Let us choose an arbitrary geometric point \bar{y} of $E_{\bar{x}}$, let \bar{z} be its image in $(\mathcal{G}_{E, T'})_{\bar{x}}$. It induces isomorphisms $(\mathcal{G}_{E, T'})_{\bar{x}} \simeq B\mu_{\bar{x}}$ and $\pi_1((\mathcal{G}_{E, T'})_{\bar{x}}, \bar{z}) \simeq \mu_{\bar{x}} = \pi_0(\mu_{\bar{k}})$ so that the diagram

$$\begin{array}{ccc} \pi_1(T_{\bar{k}}, 1) & \simeq & \pi_1(E_{\bar{x}}, \bar{y}) \\ \downarrow & & \downarrow \\ \pi_0(\mu_{\bar{k}}) & \simeq & \pi_1((\mathcal{G}_{E, T'})_{\bar{x}}, \bar{z}) \end{array}$$

is commutative, see Proposition 26 and Theorem 60 (1).

Proposition 37. *The class $\pi_1(E/X)$ is mapped under*

$$H^2(\pi_1(X, \bar{x}), \pi_1(T_{\bar{k}}, 1)) \rightarrow H^2(\pi_1(X, \bar{x}), \pi_0(\mu_{\bar{k}})) \simeq H^2(\pi_1(X, \bar{x}), \pi_1((\mathcal{G}_{E, T'})_{\bar{x}}, \bar{y}))$$

to the class $\pi_1(\mathcal{G}_{E, T'}/X)$, see Theorem 60 (2).

Proof. There is a map of short exact sequences of fundamental groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(E_{\bar{x}}, \bar{y}) & \longrightarrow & \pi_1(E, \bar{y}) & \longrightarrow & \pi_1(X, \bar{x}) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \pi_1((\mathcal{G}_{E, T'})_{\bar{x}}, \bar{z}) & \longrightarrow & \pi_1(\mathcal{G}_{E, T'}, \bar{z}) & \longrightarrow & \pi_1(X, \bar{x}) \longrightarrow 1 \end{array} \quad (4.4)$$

where the first (resp. second) row is exact by Corollary 23 (resp. Theorem 60 (2) of the appendix). Note that due to X being an algebraic $K(\pi, 1)$, all gerbes on X with locally constant band are neutralized by a finite étale cover.

The claim now follows from the definition of the push forward of extension classes. \square

Remark 38. When $T = \mathbb{G}_m$, the following Proposition 39 has been proven by Mochizuki in [Moc03] Lemma 4.4 and Lemma 4.5. We have chosen a slightly different approach, hoping that the introduction of the Chern gerbe brings further light.

Proposition 39. *Assume X is an algebraic $\mathbb{K}(\pi, 1)$, and $\text{char } k = 0$. Then the morphism $\mathbb{H}^2(\pi_1(X, \bar{x}), \pi_1(T_{\bar{k}}, 1)) \rightarrow \mathbb{H}^2(X, \mathbb{T}(T))$ is an isomorphism and sends $\pi_1(E/X)$ to $c_1(E)$.*

Proof. The exact sequence (4.2) shows that, if X is an algebraic $\mathbb{K}(\pi, 1)$, the isomorphism (3.1) also holds for inverse systems of locally constant torsion sheaves. Hence

$$\mathbb{H}^2(\pi_1(X, \bar{x}), \pi_1(T_{\bar{k}}, 1)) \rightarrow \mathbb{H}^2(X, \mathbb{T}(T))$$

is an isomorphism.

We now turn to the second statement. According to Proposition 33, the morphism

$$\mathbb{H}^2(X, \mathbb{T}(T)) \rightarrow \varprojlim_{n \in \mathbb{N}} \mathbb{H}^2(X, T[n])$$

is an isomorphism, so it is enough to show that the morphism

$$\mathbb{H}^2(\pi_1(X, \bar{x}), \pi_1(T_{\bar{k}}, 1)) \rightarrow \mathbb{H}^2(X, T[n])$$

sends $\pi_1(E/X)$ to $c_1(E)_n = [\mathcal{G}_c^n(E)]$, the Chern class modulo n . Using our unambiguous notations again as above, we claim that the morphism

$$\mathbb{H}^2(\pi_1(X, \bar{x}), \pi_1(T_{\bar{k}})) \rightarrow \mathbb{H}^2(\pi_1(X, \bar{x}), \pi_0(\mu_{\bar{k}})) \rightarrow \mathbb{H}^2(X, \mu)$$

sends $\pi_1(E/X)$ to $[\mathcal{G}_{E, T'}/X]$. By Proposition 37 it suffices to show that the class $\pi_1(\mathcal{G}_{E, T'}/X)$ is mapped to $[\mathcal{G}_{E, T'}/X]$ by (more precisely) the morphism

$$\mathbb{H}^2(\pi_1(X, \bar{x}), \pi_0(\mu_{\bar{k}})) \xleftarrow{\simeq} \mathbb{H}^2(X_{\text{fét}}, \mu) \xrightarrow{\gamma^*} \mathbb{H}^2(X, \mu).$$

The fact that X is an algebraic $\mathbb{K}(\pi, 1)$ implies that γ^* is an isomorphism and so the gerbe $\mathcal{G}_{E, T'}$ is neutralized by a finite étale cover of X . One concludes the proof by applying Theorem 60 of the appendix to the gerbe $\mathcal{G}_{E, T'}$. \square

4.3. Killing torsion obstructions. Proposition 39 enables to associate to a T -torsor $E \rightarrow X$ and a section $s : \text{Gal } k \rightarrow \pi_1(X, \bar{x})$ a class

$$s^*(c_1(E)) := s^*(\pi_1(E/X)) \in \mathbb{H}^2(k, \mathbb{T}(T)).$$

Lemma 40. *$s^*(c_1(E)) = 0$ if and only if s lifts to $\pi_1(E, \bar{y})$.*

Proof. This follows from Proposition 39. \square

We can make this obstruction more tractable thanks to the following lemma.

Lemma 41. *The Tate module $\mathbb{T}(A)$ of an abelian group A is torsion free.*

Proof. This is clear from the expression $\mathbb{T}(A) = \text{Hom}(\mathbb{Q}/\mathbb{Z}, A)$. \square

If $E \rightarrow X$ is torsion, so is $s^*(c_1(E))$, and Corollary 35 together with Lemma 41 show that the obstruction $s^*(c_1(E))$ lives in fact in $\mathbb{H}^1(k, T)$.

We add some notations, keeping those introduced at the beginning of §4. Let D be a reduced, effective divisor on the curve X . Then according to Theorem 16 and Lemma 18, the obstruction $s^*(c_1(E_D))$ to lift a section s of X to E_D belongs to $\text{Br}(D/k)$.

Let Σ be a non-empty set of primes. We will denote by $\bar{\pi}^\Sigma$ the largest pro- Σ quotient of a profinite group $\bar{\pi}$. If $1 \rightarrow \bar{\pi} \rightarrow \pi \rightarrow \text{Gal}_k \rightarrow 1$ is an extension, we will write $\pi^{[\Sigma]}$ for the pushout of π along $\bar{\pi} \rightarrow \bar{\pi}^\Sigma$.

Analogously, $\mathbb{T}^\Sigma(A)$ will be the Σ -adic Tate module of an abelian group A .

Theorem 42. *Let D be a torsion packet. Let Σ be a non-empty set of primes, co-prime to $\text{ind}(D)$. Then any section $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ lifts to a section $\text{Gal}_k \rightarrow \pi_1^{[\Sigma]}(E_D, \bar{y})$.*

Proof. The obstruction to lift s to $\pi_1^{[\Sigma]}(E_D, \bar{y})$ is

$$s^*(c_1(E_D)) \in H^2(k, \mathbb{T}^\Sigma(S_D)).$$

Since D is a torsion packet, $s^*(c_1(E_D))$ comes in fact from $H^1(k, S_D) = \text{Br}(D/k)$. The restriction operation in Galois cohomology shows that $\text{Br}(D/k)$ is killed by $\text{ind}(D)$. On the other hand, $\mathbb{T}^\Sigma(S_D)$ is a $\widehat{\mathbb{Z}}^\Sigma$ -module, and $\text{ind}_D(X)$ is invertible in $\widehat{\mathbb{Z}}^\Sigma$, thus multiplication by $\text{ind}_D(X)$ is an isomorphism of $\mathbb{T}^\Sigma(S_D)$, hence of $H^2(k, \mathbb{T}^\Sigma(S_D))$ as well. \square

Remark 43. Let ℓ be a prime and $\pi_1^{[ab, \ell]}(U, \bar{x})$ be the pushout of $\pi_1(U, \bar{x})$ along the maximal geometric abelian pro- ℓ quotient $\pi_1(U_{\bar{k}}, \bar{x}) \rightarrow \pi_1(U_{\bar{k}}, \bar{x})^{ab, \ell}$. In [BE13], Théorème 1, the authors give a set of conditions that ensures that one can lift sections along $\pi_1^{[ab, \ell]}(U, \bar{x}) \rightarrow \pi_1^{[ab, \ell]}(X, \bar{x})$. To compare this result with Theorem 42, we can set $\Sigma = \{\ell\}$. Then the morphism $\pi_1(E_D, \bar{y}) \rightarrow \pi_1^{[ab]}(U, \bar{x})$ of Proposition 31 induces a morphism $\pi_1^{[\ell]}(E_D, \bar{y}) \rightarrow \pi_1^{[ab, \ell]}(U, \bar{x})$. Thus, as far as we are concerned with lifting sections of $\pi_1(X, \bar{x})$, Théorème 1 of [BE13] is essentially a consequence of Theorem 42. The method used in [BE13] is completely different and uses a purity criterion for the ℓ -adic cohomology of U (see also Corollary 10).

5. LIFTING TO F_D OVER THE RATIONALS

In this section, k will be a field of characteristic 0 and X/k a smooth, projective, geometrically connected curve of genus ≥ 1 . Let $D \subset X$ be an effective reduced Cartier divisor and $U = X \setminus D$. We consider the associated torsors $F_D \rightarrow X$ and $E_D \rightarrow X$ from Definition 3 and study the lifting obstruction for sections $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ to the fundamental group of the respective torsors.

5.1. Vanishing of Brauer obstructions. The short exact sequence of low degree terms of the Leray spectral sequence for $X \rightarrow \text{Spec}(k)$ and \mathbb{G}_m reads

$$0 \rightarrow \text{Pic}(X) \rightarrow \text{Pic}_{X/k}(k) \xrightarrow{b} \text{Br}(k) \rightarrow \text{Br}(X). \quad (5.1)$$

By definition the map b is the Brauer obstruction map with values in the relative Brauer group $\text{Br}(X/k)$, see Definition 17, that measures the failure of a rational point of the Picard variety to describe an actual line bundle.

Proposition 44. *Let X/k be a smooth projective curve of positive genus such that $\pi_1(X/k)$ admits a section. Then the following holds.*

- (1) $b(L) = 0$ for all torsion points $L \in \text{Pic}_{X/k}(k)_{\text{tors}}$.
- (2) If k/\mathbb{Q}_p is a finite extension, then $\# \text{Br}(X/k)$ is a power of p .
- (3) If $k = \mathbb{R}$, then $\text{Br}(X/k) = 0$.

Proof. Assertion (1) is [Sti10] Proposition 12, and (2) is proven in [Sti10] Theorem 15.

Assertion (3) follows from the real section conjecture, see [Sti13b] §16.1. Indeed, any section $s : \text{Gal}_{\mathbb{R}} \rightarrow X$ comes from a point $x \in X(\mathbb{R})$, and evaluation in x yields a retraction to $\text{Br}(k) \rightarrow \text{Br}(X)$, showing $\text{Br}(X/k) = 0$. \square

Corollary 45. *Let X/\mathbb{Q} be a smooth projective curve of positive genus such that $\pi_1(X/\mathbb{Q})$ admits a section. Then the relative Brauer group $\text{Br}(X/\mathbb{Q})$ vanishes.*

Proof. We have to show that $\text{Br}(X/\mathbb{Q}) = 0$. The Hasse–Brauer–Noether theorem shows that

$$\text{Br}(X/\mathbb{Q}) \hookrightarrow \ker \left(\bigoplus_v \text{Br}(X \times_{\mathbb{Q}} \mathbb{Q}_v/\mathbb{Q}_v) \xrightarrow{\sum_v \text{inv}_v} \mathbb{Q}/\mathbb{Z} \right)$$

is injective, where v ranges over all places of \mathbb{Q} . Base change of sections implies that for all places v of \mathbb{Q} the extension $\pi_1(X \times_{\mathbb{Q}} \mathbb{Q}_v/\mathbb{Q}_v)$ splits. By Proposition 44 then $\text{Br}(X \times_{\mathbb{Q}} \mathbb{R}/\mathbb{R}) = 0$ and $\text{Br}(X \times_{\mathbb{Q}} \mathbb{Q}_p/\mathbb{Q}_p)$ is cyclic of p -power order. This forces $\text{Br}(X/\mathbb{Q}) = 0$. \square

5.2. Divisibility of line bundles. Let us recall the definition of a neighbourhood of a section.

Definition 46. A **neighbourhood** of a section $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ is a connected finite étale cover $h : X' \rightarrow X$ together with a lift

$$s' : \text{Gal}_k \rightarrow \pi_1(X', \bar{x}') \subseteq \pi_1(X, \bar{x})$$

of s . A short notation for a neighbourhood is (X', s') .

Note that neighbourhoods are always geometrically connected over k , because $\pi_1(X', \bar{x}') \rightarrow \text{Gal}_k$ is surjective by construction.

Example 47. A wealth of neighbourhoods are constructed as follows. Let $\varphi : \pi_1(X_{\bar{k}}, \bar{x}) \rightarrow G$ be a characteristic finite quotient. Then $\ker(\varphi)$ is a normal subgroup in $\pi_1(X, \bar{x})$ and

$$\pi_1(X_\varphi, \bar{x}) = \langle \ker(\varphi), s(\text{Gal}_k) \rangle = \{ \gamma s(\sigma) ; \varphi(\gamma) = 1, \sigma \in \text{Gal}_k \} \subseteq \pi_1(X, \bar{x})$$

together with the obvious lift describes a neighbourhood of s . Moreover, we have $\pi_1(X_{\varphi, \bar{k}}, \bar{x}) = \ker(\varphi)$, so that

$$\deg(X_\varphi \rightarrow X) = \#G.$$

Since $\pi_1(X_{\bar{k}}, \bar{x})$ is topologically finitely generated, the neighbourhoods X_φ form a cofinal system in the system $X_s = (X')$ of all neighbourhoods.

Proposition 48. *Let X/k be a smooth projective curve of positive genus. Let $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ be a section and let \mathcal{L} be a line bundle on X . Then the following holds.*

- (1) *For every $n \geq 1$ there is a neighbourhood (X', s') of s , such that there is a $M \in \text{Pic}_{X'}(k)$ with*

$$[\mathcal{L}|_{X'}] = M^{\otimes n}.$$

- (2) *If, moreover, for every neighbourhood (X', s') of s , the relative Brauer group $\text{Br}(X'/k)$ vanishes, then for every $n \geq 1$ there is a neighbourhood (X', s') of s , such that there is a line bundle \mathcal{M} on X' with*

$$\mathcal{L}|_{X'} \simeq \mathcal{M}^{\otimes n}.$$

Definition 49. If the conclusion (2) of Proposition 48 holds, then we say that the **line bundle \mathcal{L} is divisible locally in neighbourhoods of s** .

Proof of Proposition 48. Let $\text{Pic}_{X/k}^{n*}$ denote the subgroup of the Picard variety of line bundles of degree divisible by n . The boundary map to

$$0 \rightarrow \text{Pic}_{X/k}[n] \rightarrow \text{Pic}_{X/k} \rightarrow \text{Pic}_{X/k}^{n*} \rightarrow 0,$$

namely

$$\text{Pic}_{X/k}^{n*}(k) \rightarrow \text{H}^1(k, \text{Pic}_{X/k}[n]),$$

describes the obstruction to being divisible by n in the Picard variety for line bundles of degree divisible by n . This obstruction is natural under pullback.

We first prove assertion (1). Since $n \mid \#\pi_1(X_{\bar{k}}, \bar{x})$ we find a neighbourhood (X_1, s_1) of s with $n \mid \deg(X_1/X)$. Then $[\mathcal{L}|_{X_1}] \in \text{Pic}_{X_1/k}^{n*}(k)$ because

$$\deg(\mathcal{L}|_{X_{1, \bar{k}}}) = \deg(X_1/X) \cdot \deg(\mathcal{L}).$$

Let (X_2, s_2) be the neighbourhood of s_1 associated to the maximal abelian quotient of exponent n of $\pi_1(X_{1, \bar{k}}, \bar{x}_1)$. Then the induced map

$$\text{Pic}_{X_1/k}[n] = \text{Hom}(\pi_1(X_{1, \bar{k}}, \bar{x}_1), \mathbb{Z}/n\mathbb{Z}(1)) \rightarrow \text{Hom}(\pi_1(X_{2, \bar{k}}, \bar{x}_2), \mathbb{Z}/n\mathbb{Z}(1)) = \text{Pic}_{X_2/k}[n]$$

is the zero map. Thus the obstruction for $[\mathcal{L}|_{X_1}]$ to divisibility by n in the Picard variety vanishes after restriction to X_2 . This proves (1).

Let $M \in \text{Pic}_{X_2/k}(k)$ be an n th root of $[\mathcal{L}|_{X_2}]$. In order to prove (2) we have to investigate the Brauer obstruction for M to come from an actual line bundle. But this is the class $b(M)$ for the map b in (5.1) for X_2/k and b vanishes by assumption. This concludes the proof of (2). \square

Proposition 50. *Let X/k be a smooth projective curve of positive genus, let $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ be a Galois section, and let \mathcal{L} be a line bundle on X . Then \mathcal{L} is locally divisible in neighbourhoods of s if and only if $s^*(c_1(\mathcal{L})) = 0$.*

Proof. Let X_s be the projective limit of the pro-system of all neighbourhoods of s . Then \mathcal{L} is locally divisible in neighbourhoods of s if and only if $\mathcal{L}|_{X_s}$ is divisible in $\text{Pic}(X_s)$.

The Kummer sequence on X_s yields the exact sequence

$$\text{Pic}(X_s) \xrightarrow{n} \text{Pic}(X_s) \xrightarrow{c_1(\cdot)_n} \text{H}^2(X_s, \mathbb{Z}/n\mathbb{Z}(1)),$$

so that $\mathcal{L}|_{X_s}$ is divisible by n on X_s if and only if $c_1(\mathcal{L}|_{X_s})_n = 0$.

Naturality of the first Chern class and the isomorphism

$$\text{H}^2(X_s, \mathbb{Z}/n\mathbb{Z}(1)) = \text{H}^2(\pi_1(X_s, \bar{x}), \mathbb{Z}/n\mathbb{Z}(1)) \xrightarrow{s^*} \text{H}^2(k, \mathbb{Z}/n\mathbb{Z}(1))$$

show that $c_1(\mathcal{L}|_{X_s})_n = 0$ if and only if $s^*(c_1(\mathcal{L})_n) = 0$. By Proposition 33 we have $s^*(c_1(\mathcal{L})_n) = 0$ for all $n \geq 1$ if and only if $s^*(c_1(\mathcal{L})) = 0$. \square

Theorem 51. *Let X/k be a smooth projective curve of positive genus, and let $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ be a Galois section. Consider the following assertions.*

- (a) *All line bundles \mathcal{L} on X are locally divisible in neighbourhoods of s .*
- (b) *$s^* \circ c_1 : \text{Pic}(X) \rightarrow \text{H}^2(k, \hat{\mathbb{Z}}(1))$ vanishes.*
- (c) *The relative Brauer group $\text{Br}(X/k)$ vanishes.*
- (c') *The relative Brauer group $\text{Br}(X'/k)$ vanishes for all neighbourhoods X' of s .*

Then the following implications hold:

$$(c') \implies (a) \iff (b) \implies (c).$$

Proof. (c') \implies (a) was proven in Proposition 48 (2). The equivalence of (a) with (b) follows from Proposition 50.

For (a) \implies (c) we have to show that $b(L) = 0$ for all $L \in \text{Pic}_X(k)$. Since $b(L) \in \text{Br}(X/k)$ is torsion, there is an $n \geq 1$ such that $L^{\otimes n} = [\mathcal{M}]$ for a line bundle \mathcal{M} on X . By assumption (a), there is a neighbourhood (X', s') of s such that $\mathcal{M}|_{X'}$ admits an n th root

$$\mathcal{M}|_{X'} = \mathcal{L}'^{\otimes n}$$

with a line bundle \mathcal{L}' on X' . The difference

$$\Delta = L|_{X'} - [\mathcal{L}']$$

is an n -torsion element in $\text{Pic}_{X'}(k)$. By Proposition 44 (1) we compute

$$b(L) = b(L|_{X'}) = b(\Delta) + b([\mathcal{L}']) = 0,$$

and this proves (c). \square

5.3. Lifting to F_D . We give a criterion for a section $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ to lift to $\pi_1(F_D, \bar{y})$ when D is a torsion packet.

Proposition 52. *Let X/k be a smooth projective curve of positive genus, let $D \subset X$ be a torsion packet, and let $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ be a Galois section. Then the following are equivalent.*

- (a) *The section s lifts to a section $\text{Gal}_k \rightarrow \pi_1(F_D, \bar{y})$.*
- (b) *$s^*(c_1(\mathcal{O}_X(D))) = 0$.*

Proof. The section s lifts if and only if $s^*(c_1(F_D)) = 0$. Since T_D is the restriction of scalars of \mathbb{G}_m , Shapiro's Lemma and Hilbert's Theorem 90 imply $\text{H}^1(k, T_D) = 0$. Using Proposition 33 and Lemma 41, we find that $s^*(c_1(F_D))$ takes values in the torsion free group

$$\text{H}^2(k, \mathbb{T}(T_D)) \simeq \mathbb{T}(\text{H}^2(k, T_D)).$$

Hence, we may replace F_D by a multiple. Consider the defining sequence

$$0 \rightarrow \mathbb{G}_m \xrightarrow{i} T_D \xrightarrow{\text{pr}} S_D \rightarrow 0.$$

Then $\text{pr}_* F_D = E_D$ is torsion by D being a torsion packet and Theorem 16 (1). Thus there is $n \geq 1$ and a line bundle \mathcal{L} on X with

$$i_*[\mathbf{Isom}(\mathcal{O}, \mathcal{L})] = n \cdot [F_D].$$

Let $N : T_D \rightarrow \mathbb{G}_m$ be the norm map and $N(\cdot)$ the norm map on line bundles along the second projection $D \times_k X \rightarrow X$. Then, by Lemma 13,

$$N_* F_D \simeq \mathbf{Isom}_X(\mathcal{O}_X, \mathcal{O}_X(D))$$

and thus

$$\deg(D) \cdot [\mathcal{L}] = N_*(i_*[\mathcal{L}]) = N_*(n \cdot [F_D]) = [\mathcal{O}_X(nD)].$$

Since $H^2(k, \mathbb{T}(\mathbb{G}_m)) \simeq \mathbb{T}(H^2(k, \mathbb{G}_m))$ is torsion free (Lemma 41) and since the norm induces a retraction up to multiplication by $\deg(D)$, the map

$$i_* : H^2(k, \mathbb{T}(\mathbb{G}_m)) \hookrightarrow H^2(k, \mathbb{T}(T_D))$$

is injective. We conclude finally that $s^*(c_1(\mathcal{O}_X(D))) = 0$ if and only

$$s^*(c_1(\mathcal{O}_X(nD))) = s^*(c_1(\mathcal{L}^{\otimes \deg(D)})) = 0,$$

or equivalently if $s^*(c_1(\mathcal{L})) = 0$. By injectivity of i_* this in turn is equivalent to $s^*(c_1(n \cdot F_D)) = 0$ or, since $H^2(k, \mathbb{T}(T_D))$ is torsion free as well, $s^*(c_1(F_D)) = 0$, which completes the proof. \square

Here are some obvious dévissage properties of the lifting problem.

Lemma 53. *Let X/k be a smooth projective curve of positive genus, and let $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ be a Galois section.*

- (1) *Let $D \subset X$ be an effective reduced divisor. If s lifts to a section $\text{Gal}_k \rightarrow \pi_1(F_D, \bar{y})$, then s lifts to a section $\text{Gal}_k \rightarrow \pi_1(E_D, \bar{y})$.*
- (2) *Let $D_1 \subset D_2 \subset X$ be two divisors on X . If s lifts to a section $\text{Gal}_k \rightarrow \pi_1(F_{D_2}, \bar{y}_2)$, then s lifts to a section $\text{Gal}_k \rightarrow \pi_1(F_{D_1}, \bar{y}_1)$ (idem for E_{D_1} and E_{D_2}).*
- (3) *Consider a divisor $D \subset X$ that is the union of two effective reduced divisors D_1 and D_2 . The section $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ lifts to sections $\text{Gal}_k \rightarrow \pi_1(F_{D_i}, \bar{y}_i)$, for $i = 1, 2$, if and only if s lifts to a section $\text{Gal}_k \rightarrow \pi_1(F_D, \bar{y})$.*

Proof. Assertion (1) follows from the factorisation $\pi_1(F_D, \bar{y}) \rightarrow \pi_1(E_D, \bar{y}) \rightarrow \pi_1(X, \bar{x})$. Similarly, (2) follows from the factorisation $\pi_1(F_{D_2}, \bar{y}_2) \rightarrow \pi_1(F_{D_1}, \bar{y}_1) \rightarrow \pi_1(X, \bar{x})$ due to $F_{D_2} \rightarrow F_{D_2} \times^{T_{D_2}} T_{D_1} = F_{D_1}$ over X by Lemma 27 (resp. the factorisation $\pi_1(E_{D_2}, \bar{y}_2) \rightarrow \pi_1(E_{D_1}, \bar{y}_1) \rightarrow \pi_1(X, \bar{x})$ due to $E_{D_2} \rightarrow E_{D_2} \times^{S_{D_2}} S_{D_1} = E_{D_1}$ over X).

(3) By Proposition 7, we have $F_D \simeq F_{D_1} \times_X F_{D_2}$, so that Lemma 28 shows

$$\pi_1(F_D, \bar{y}) \simeq \pi_1(F_{D_1}, \bar{y}_1) \times_{\pi_1(X, \bar{x})} \pi_1(F_{D_2}, \bar{y}_2).$$

Assertion (3) is now obvious by the defining property of the fiber product. \square

Corollary 54. *Let X/k be a smooth projective curve of positive genus, and let $D \subset X$ be a union $D = \bigcup_{i=1}^n D_i$ of torsion packets $D_i \subset X$, and let $s : \text{Gal}_k \rightarrow \pi_1(X, \bar{x})$ be a Galois section. Then the following are equivalent.*

- (a) *The section s lifts to a section $\text{Gal}_k \rightarrow \pi_1(F_D, \bar{y})$.*
- (b) *$s^*(c_1(\mathcal{O}_X(D_i))) = 0$ for all $i = 1, \dots, n$.*

Proof. This is an immediate consequence of Proposition 52 and Lemma 53 (3). \square

5.4. Lifting to F_D over \mathbb{Q} . In this section we study the lifting problem over the field \mathbb{Q} .

Theorem 55. *Let X/\mathbb{Q} be a smooth projective curve of positive genus, and let $D \subset X$ be a union of torsion packets. Then every Galois section $s : \text{Gal}_{\mathbb{Q}} \rightarrow \pi_1(X, \bar{x})$ lifts to a section $\text{Gal}_{\mathbb{Q}} \rightarrow \pi_1(F_D, \bar{y})$.*

Proof. Let $D = \bigcup_{i=1}^n D_i$ be the decomposition into torsion packets $D_i \subset X$. By Corollary 54 we must show that $s^*(c_1(\mathcal{O}_X(D_i))) = 0$ for all $i = 1, \dots, n$. It follows from Theorem 51 that it suffices to show that for every neighbourhood (X', s') of s we have trivial relative Brauer group $\text{Br}(X'/\mathbb{Q})$, a fact by Corollary 45. \square

Corollary 56. *Let X/\mathbb{Q} be a smooth projective curve of positive genus, and let $U = X \setminus X(\mathbb{Q})$ be the complement of the set of all \mathbb{Q} -rational points. Then every Galois section $s : \text{Gal}_{\mathbb{Q}} \rightarrow \pi_1(X, \bar{x})$ lifts to a section $\text{Gal}_{\mathbb{Q}} \rightarrow \pi_1^{\text{cc}}(U, \bar{x})$.*

Proof. The divisor $D = X(\mathbb{Q}) \subset X$ is a union of torsion packets. Therefore the section s lifts to a section $\text{Gal}_{\mathbb{Q}} \rightarrow \pi_1(F_D, \bar{y})$ by Theorem 55, and since $\pi_1^{\text{cc}}(U, \bar{x}) \simeq \pi_1(F_D, \bar{y})$ by Proposition 30, this completes the proof. \square

Remark 57. Strictly speaking, we must restrict in Corollary 56 to finite sets $D \subseteq X(\mathbb{Q})$, this being a restriction only if the genus of X is 1. But a natural extension using projective limits remains true also when $X(\mathbb{Q})$ is infinite.

APPENDIX A. CLASSES OF GERBES AND GROUP EXTENSIONS

A.1. The class of a gerbe. In this paragraph we would like to make precise the correspondence between gerbes on the étale site of X and elements of $H^2(X, -)$. For a sheaf A of abelian groups on $X_{\text{ét}}$, we will denote by $H_G^1(X, A)$ and $H_G^2(X, A)$ respectively the group of classes of A -torsors and the group of classes of gerbes bound by A on X . For simplicity of notation we also define $H_G^0(X, A) = H^0(X, A)$. For any short exact sequence

$$0 \rightarrow A \rightarrow A' \rightarrow A'' \rightarrow 0$$

one has morphisms, for $i = 1, 2$,

$$\delta_{i,G} : H_G^{i-1}(X, A'') \rightarrow H_G^i(X, A)$$

which are defined in the following way:

- (1) for $a'' \in H_G^0(X, A'')$, the class of torsors $\delta_{1,G}(a'')$ is represented by the A -torsor of the lifting of a'' in a section of A' ;
- (2) for an A'' -torsor t , the class of gerbes $\delta_{2,G}(t)$ is represented by the gerbe of liftings of t to an A' -torsor.

It is a fact that $H_G^i(X, -)$, for $0 \leq i \leq 2$, endowed with $\delta_{1,G}$ and $\delta_{2,G}$ is a δ -functor in degrees at most 2 from the category of abelian sheaves on $X_{\text{ét}}$ to the category of abelian groups.

On the other hand étale cohomology $H^i(X, -)$ forms a universal δ -functor. Thus there is a unique morphism $\alpha = (\alpha_i)_{0 \leq i \leq 2}$ of δ -functors ([Gro57])

$$\alpha_i : H^i(X, -) \rightarrow H_G^i(X, -).$$

According to [Gir71], α is an isomorphism, so $H_G^i(X, -)$ is a universal δ -functor in degrees ≤ 2 .

Definition 58. For a gerbe \mathcal{G} on $X_{\text{ét}}$ bound by A , we will define the **(cohomology) class** $[\mathcal{G}]$ of \mathcal{G} in $H^2(X, A)$ as the image of the class of \mathcal{G} by the isomorphism $\alpha_2^{-1} : H_G^2(X, A) \rightarrow H^2(X, A)$.

In this correspondence the Chern class, see Definition 32, corresponds to the Chern gerbe, see Definition 36.

Lemma 59. *Let T be a torus on X and E/X a T -torsor. For n invertible on X the Chern class mod n of E/X corresponds to the Chern gerbe mod n :*

$$\alpha_2(c_1(E)_n) = \mathcal{G}_c^n(E).$$

Proof. The short exact sequence $0 \rightarrow T[n] \rightarrow T \xrightarrow{n} T \rightarrow 0$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(X, T) & \xrightarrow{\delta_2} & \mathrm{H}^2(X, T[n]) \\ \downarrow \alpha_1 & & \downarrow \alpha_2 \\ \mathrm{H}_G^1(X, T) & \xrightarrow{\delta_{2,G}} & \mathrm{H}_G^2(X, T[n]), \end{array}$$

and by definition $\alpha_2(c_1(E)_n) = \alpha_2(\delta_2([E])) = \delta_{2,G}(\alpha_1([E])) = \mathcal{G}_c^n(E)$. \square

The main result of this appendix computes $[\mathcal{G}]$ in terms of group cohomology under an assumption that is trivially verified if X is an algebraic $\mathbb{K}(\pi, 1)$.

Theorem 60. *Let X be a connected noetherian scheme with geometric point $\bar{x} \in X$, and let A be a locally constant constructible étale sheaf of abelian groups on X . Let \mathcal{G} be a gerbe on $X_{\text{ét}}$ bound by A with a geometric point \bar{z} above \bar{x} . We further assume that \mathcal{G} is neutralized by a finite étale cover of X . Then the following holds.*

- (1) *The choice of \bar{z} induces an isomorphism $\pi_1(\mathcal{G}_{\bar{x}}, \bar{z}) \simeq A_{\bar{x}}$.*
- (2) *The following sequence is exact:*

$$1 \rightarrow A_{\bar{x}} \rightarrow \pi_1(\mathcal{G}, \bar{z}) \rightarrow \pi_1(X, \bar{x}) \rightarrow 1.$$

- (3) *The natural comparison map*

$$\mathrm{H}^2(\pi_1(X, \bar{x}), A_{\bar{x}}) \rightarrow \mathrm{H}^2(X, A)$$

maps the class $\pi_1(\mathcal{G}/X)$ of the extension (2) to the class $[\mathcal{G}]$ of the gerbe \mathcal{G} .

Proof. Part (1) follows because \bar{z} induces an isomorphism $\mathcal{G}_{\bar{x}} \rightarrow \mathrm{BAut} \bar{z}$, and since $\mathcal{G}_{\bar{x}}$ is bound by $A_{\bar{x}}$, there is a given isomorphism $A_{\bar{x}} \simeq \mathrm{Aut} \bar{z}$. Via this isomorphism, $\bar{z} : \mathrm{Spec} \Omega \rightarrow \mathcal{G}_{\bar{x}}$ becomes a pointed $A_{\bar{x}}$ -torsor, and since $\mathrm{Spec} \Omega$ is simply connected, the corresponding morphism $\pi_1(\mathcal{G}_{\bar{x}}, \bar{z}) \rightarrow A_{\bar{x}}$ is an isomorphism. We will be concerned with the proof of parts (2) and (3) in Section §A.2 below. \square

A.2. Dimension shifting. If \mathcal{G} is neutralized by a finite étale cover $h : Y \rightarrow X$, then $h^*[\mathcal{G}] = 0$ and thus the class $[\mathcal{G}]$ comes from group cohomology of $\pi_1(X, \bar{x})$. This means that we can find a short exact sequence

$$0 \rightarrow A \rightarrow A' \rightarrow A'' \rightarrow 0$$

consisting of locally constant constructible étale sheaf of abelian groups on X that allows to shift $[\mathcal{G}]$ as follows. In the commutative diagram

$$\begin{array}{ccccc} & & \mathrm{H}_G^1(X, A'') & & \\ & & \alpha_1^{-1} \downarrow \simeq & \searrow \delta_G & \\ & & \mathrm{H}^1(X, A'') & & \mathrm{H}_G^2(X, A) \\ \mathrm{H}^1(\pi_1(X, \bar{x}), A''_{\bar{x}}) & \xrightarrow{\simeq} & & & \\ & \searrow \delta_{\pi_1} & & \searrow \delta & \alpha_2^{-1} \downarrow \simeq \\ & & \mathrm{H}^2(\pi_1(X, \bar{x}), A_{\bar{x}}) & \longrightarrow & \mathrm{H}^2(X, A) \end{array} \quad (\text{A.1})$$

there is an A'' -torsor $E'' \rightarrow X$ such that $\delta_G(E'') = \mathcal{G}$, i.e., \mathcal{G} is isomorphic to the gerbe of lifts of the structure group along $A' \rightarrow A''$, in other words, the quotient stack

$$\mathcal{G} = [E''/A'].$$

The class of the torsor $[E''] \in H^1(X, A'') \simeq H^1(\pi_1(X, \bar{x}), A''_{\bar{x}})$ has a description in terms of a 1-cocycle

$$a'' : \pi_1(X, \bar{x}) \rightarrow A''_{\bar{x}}$$

determined by choosing a geometric point $\bar{y} \in E''$ above \bar{x} . Alternatively, this cocycle describes a homomorphism (for example [Sti13b] §11.4)

$$s'' = s_{E''/X} : g \mapsto a''_g \cdot g$$

in the diagram

$$\begin{array}{ccc} \pi_1(X, \bar{x}) & \xrightarrow{s''} & A''_{\bar{x}} \rtimes \pi_1(X, \bar{x}) \\ & \searrow \text{id} & \downarrow \text{pr}_2 \\ & & \pi_1(X, \bar{x}). \end{array}$$

Let $[s'']$ be the cohomology class of the 1-cocycle determined by s'' . We next describe the extension class $\delta_{\pi_1}([s''])$.

Lemma 61. *The extension class $\delta_{\pi_1}([s'']) \in H^2(\pi_1(X, \bar{x}), A_{\bar{x}})$ is given by the pullback along s of the extension*

$$1 \rightarrow A_{\bar{x}} \rightarrow A'_{\bar{x}} \rtimes \pi_1(X, \bar{x}) \rightarrow A''_{\bar{x}} \rtimes \pi_1(X, \bar{x}) \rightarrow 1.$$

Proof. We lift the 1-cocycle $a'' : \pi_1(X, \bar{x}) \rightarrow A''_{\bar{x}}$ to a 1-chain

$$a' : \pi_1(X, \bar{x}) \rightarrow A'_{\bar{x}}$$

which defines a set theoretic map by $g \mapsto b'(g) = a'_g \cdot g$

$$b' : \pi_1(X, \bar{x}) \rightarrow A'_{\bar{x}} \rtimes \pi_1(X, \bar{x}).$$

A representing 2-cocycle for $\delta_{\pi_1}([s''])$ is

$$d(a') = (g, h \mapsto g(a'_h) - a'_{gh} + a'_g).$$

Here and below, as soon as the computation lives entirely in the abelian normal subgroup, then we switch to additive notation. One the other hand, the 2-cocycle describing the pullback extension is by the fiber product property

$$\begin{aligned} g, h \mapsto b'_g \cdot b'_h \cdot (b'_{gh})^{-1} &= a'_g \cdot g \cdot a'_h \cdot h \cdot (a'_{gh} \cdot gh)^{-1} \\ &= a'_g \cdot g \cdot a'_h \cdot g^{-1} \cdot a'_{gh}^{-1} = a'_g + g(a'_h) - a'_{gh} = g(a'_h) - a'_{gh} + a'_g. \end{aligned}$$

This completes the proof of the lemma. □

Let us summarize what we have done in terms of diagram (A.1):

$$\begin{array}{ccccc} & & E'' & & \\ & & \downarrow & \searrow \delta_{\mathcal{G}} & \\ [s''] & \longrightarrow & [E''] & & \mathcal{G} \\ & \searrow \delta_{\pi_1} & & \searrow \delta & \downarrow \\ & & \delta([s'']) & \longrightarrow & [\mathcal{G}]. \end{array}$$

It remains to identify $\delta_{\pi_1}([s''])$ as in Lemma 61 with $\pi_1(\mathcal{G}/X)$.

Proof of Theorem 60 part (2) and (3). Set $E' = E''$, and choose a geometric point \bar{y} of E' above \bar{z} . This induces a torsor diagram:

$$\begin{array}{ccccc} \mathrm{Spec} \Omega = E & \xrightarrow{\bar{y}} & E' & \longrightarrow & E'' \\ \bar{z} \downarrow & & \downarrow & & \downarrow \\ \mathcal{G}_{\bar{x}} & \longrightarrow & \mathcal{G} & \xrightarrow{\mathrm{pr}} & X \end{array}$$

where the top row is equivariant with respect to $A \rightarrow A' \rightarrow A''$. The translation of pointed torsors into sections, being functorial in both the base space and in the structure group, yields the commutativity of the upper squares in the map of exact sequences:

$$\begin{array}{ccccccc} \pi_1(\mathcal{G}_{\bar{x}}, \bar{z}) & \longrightarrow & \pi_1(\mathcal{G}, \bar{z}) & \longrightarrow & \pi_1(X, \bar{x}) & \longrightarrow & 1 \\ \downarrow s_{E/\mathcal{G}_{\bar{x}}} & & \downarrow s_{E'/\mathcal{G}} & & \downarrow s_{E''/X} & & \\ A_{\bar{x}} \times \pi_1(\mathcal{G}_{\bar{x}}, \bar{z}) & \longrightarrow & A'_{\bar{x}} \times \pi_1(\mathcal{G}, \bar{z}) & \longrightarrow & A''_{\bar{x}} \times \pi_1(X, \bar{x}) & \longrightarrow & 1 \\ \downarrow \mathrm{pr}_1 & & \downarrow (\mathrm{id}, \mathrm{pr}_*) & & \downarrow \mathrm{id} & & \\ 1 & \longrightarrow & A_{\bar{x}} & \longrightarrow & A'_{\bar{x}} \times \pi_1(X, \bar{x}) & \longrightarrow & A''_{\bar{x}} \times \pi_1(X, \bar{x}) \longrightarrow 1 \end{array}$$

Now the morphism $\pi_1(\mathcal{G}_{\bar{x}}, \bar{z}) \rightarrow A_{\bar{x}}$ corresponds to the $A_{\bar{x}}$ -torsor $\bar{z} : \mathrm{Spec} \Omega \rightarrow \mathcal{G}_{\bar{x}}$, hence is the isomorphism specified in Theorem 60 (1). One deduces part (2), that is, the injectivity of the morphism $\pi_1(\mathcal{G}_{\bar{x}}, \bar{z}) \rightarrow \pi_1(\mathcal{G}, \bar{z})$. The identification $\delta_{\pi_1}([s'']) = \pi_1(\mathcal{G}/X)$ follows from the definition of the pullback, hence we have also proven part (3). \square

REFERENCES

- [AM69] Michel Artin and Barry Mazur. *Etale homotopy*, volume 100 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1969.
- [BE13] Niels Borne and Michel Emsalem. Un critère d'épointage des sections l -adiques. *To appear in Bulletin de la Société Mathématique de France*, 2013.
- [BP01] Matthew Baker and Bjorn Poonen. Torsion packets on curves. *Compositio Mathematica*, 127(1):109–116, 2001.
- [CTHS05] Jean-Louis Colliot-Thélène, David Harari, and Alexei N. Skorobogatov. Compactification équivariante d'un tore (d'après Brylinski et Künnemann). *Expo. Math.*, 23(2):161–170, 2005.
- [Fer03] Daniel Ferrand. Conducteur, descente et pincement. *Bulletin de la Société Mathématique de France*, 131(4):553–585, 2003.
- [Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993.
- [Gir71] Jean Giraud. *Cohomologie non abélienne*, volume 179 of *Grundlehren math. Wiss.* Springer-Verlag, Berlin, 1971.
- [Gro57] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tohoku Mathematical Journal*, 2(9):119–121, 1957.
- [SGA1] Alexander Grothendieck. *Revêtements étales et groupe fondamental*, volume 224 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud.
- [Jan88] Uwe Jannsen. Continuous étale cohomology. *Mathematische Annalen*, 280(2):207–245, 1988.
- [Moc99] Shinichi Mochizuki. The local pro- p anabelian geometry of curves. *Inventiones Mathematicae*, 138(2):319–423, 1999.
- [Moc03] Shinichi Mochizuki. Topics surrounding the anabelian geometry of hyperbolic curves. In *Galois groups and fundamental groups*, volume 41 of *Math. Sci. Res. Inst. Publ.*, pages 119–165. Cambridge Univ. Press, Cambridge, 2003.

- [Moc07] Shinichi Mochizuki. Absolute anabelian cuspidalizations of proper hyperbolic curves. *J. Math. Kyoto Univ.*, 47(3):451–539, 2007.
- [Saï10] Mohamed Saïdi. Good sections of arithmetic fundamental groups. *Preprint*, January 2010. arXiv: 1010.1313v1[math.AG].
- [Saï12] Mohamed Saïdi. On the birational anabelian section conjecture. *Preprint*, November 2012. arXiv: 1211.7014v1[math.AG].
- [Sti02] Jakob Stix. *Projective anabelian curves in positive characteristic and descent theory for log-étale covers*, volume 354 of *Bonner Mathematische Schriften*. Mathematisches Institut Universität Bonn, 2002.
- [Sti10] Jakob Stix. On the period-index problem in light of the section conjecture. *Amer. J. Math.*, 132(1):157–180, 2010.
- [Sti12] Jakob Stix. On the birational section conjecture with local conditions. *Preprint*, March 2012. arXiv: 1203.3236v2[math.AG].
- [Sti13a] Jakob Stix. Birational p -adic Galois sections in higher dimensions. *To appear in Israel Journal of Mathematics*, 2013.
- [Sti13b] Jakob Stix. *Rational Points and Arithmetic of Fundamental Groups: Evidence for the Section Conjecture*, volume 2054 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2013.
- [Vis05] Angelo Vistoli. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 1–104. Amer. Math. Soc., Providence, RI, 2005.

NIELS BORNE, U.M.R. CNRS 8524, U.F.R. DE MATHÉMATIQUES, UNIVERSITÉ LILLE 1, 59 655 VILLENEUVE D’ASCQ CÉDEX, FRANCE

E-mail address: Niels.Borne@math.univ-lille1.fr

MICHEL EMSALEM, U.M.R. CNRS 8524, U.F.R. DE MATHÉMATIQUES, UNIVERSITÉ LILLE 1, 59 655 VILLENEUVE D’ASCQ CÉDEX, FRANCE

E-mail address: Michel.Emsalem@math.univ-lille1.fr

JAKOB STIX, MATCH - MATHEMATISCHES INSTITUT, UNIVERSITÄT HEIDELBERG, IM NEUENHEIMER FELD 288, 69120 HEIDELBERG, GERMANY

E-mail address: stix@mathi.uni-heidelberg.de