Tamely Ramified Torsors and Parabolic Bundles

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(joint work with Indranil Biswas)

Mehta and Seshadri have shown that a unitary representation of the topological fundamental group of a punctured Riemann surface gives rise to a (polystable) parabolic vector bundle (of degree 0): roughly, an ordinary vector bundle, and for each cusp, a weighted filtration of the corresponding stalk (see [4]). This association is explicit, algebraic in nature, and, in fact, one to one. This gives an algebraic approach of étale fundamental groups, namely if X/k is a proper scheme over an algebraically closed field of characteristic 0, endowed with a (simple) normal crossings divisor, representations of the étale fundamental group of the complement are identified with the category of essentially finite parabolic vector bundles. This holds in positive characteristic as well if one replaces the étale fundamental group by a tamely ramified version of Nori's fundamental group scheme ([6, 2]).

More or less by construction, if one fixes a tamely ramified Galois cover, the denominators of the weights of the associated parabolic vector bundles divide the ramification indices. The other way round, if one starts from a finite set of essentially finite parabolic vector bundles, one can ask how the weights relate to the ramification indices of the minimal tamely ramified Galois cover trivializing them all. In a recent joint work with Indranil Biswas, we give an answer to this question in the abelian case.

More precisely, starting from X/k a base scheme over a field, endowed with a simple normal crossings divisor, we define tamely ramified torsors $Y \to X$ under an abelian finite group scheme G as fppf locally induced by Kummer covers. We relate the existence of such torsors with prescribed ramification data (but allowing G to vary) with the existence of essentially finite parabolic vector bundles with prescribed weights along the ramification locus.

Let us now give precise definitions and our statement. We fix a scheme X of finite type over a field k, and D a simple normal crossings divisor on X, meaning $D = \cup_{i\in I} D_i$ is the union of a finite family of irreducible, smooth divisors, crossing normally. We denote the corresponding family by $\mathbf{D} = (D_i)_{i\in I}$ and add to our data a family $\mathbf{r} = (r_i)_{i\in I}$ of positive integers. Given a closed point x of X, we set $(\mathbf{r}_x)_i$ as r_i if x belongs to D_i and 1 otherwise; this defines a local multi-index $\mathbf{r}_x$.

Definition 1. *Let G/k be a finite abelian group scheme. A* tamely ramified G-torsor *with ramification data $(\mathbf{D}, \mathbf{r})$ is the data of a scheme Y endowed with an action of G and a finite and flat G-invariant morphism $Y \to X$ such that for each closed point x of X, there exists a monomorphism $\mu_{\mathbf{r}_x} \to G$ defined over an extension k'/k such that in a* fppf *neighbourhood of x in X, the morphism $Y \to X$ is isomorphic to $Z \times^{\mu_{\mathbf{r}_x}} G$, where $Z \to Spec\, R$ is a Kummer cover locally defined by the choice of equations of $\mathbf{D}$ at x.*

Definition 2 (C.Simpson around 1990)**.** *A parabolic bundle $\mathcal{E}.$ on $(X, \mathbf{D})$ with weights in $\prod_{i\in I} \frac{1}{r_i}\mathbb{Z}$ is the data*

(1) For all $\mathbf{m} \in \prod_{i \in I} \frac{1}{r_i}\mathbb{Z}$ *of a locally free sheaf* $\mathcal{E}_{\mathbf{m}}$*, verifying* $\mathcal{E}_{\mathbf{m}'} \subset \mathcal{E}_{\mathbf{m}}$ *for* $\mathbf{m} \leq \mathbf{m}'$ *(for the component-wise partial order)*

(2) For $\mathbf{m} \in \prod_{i \in I} \frac{1}{r_i}\mathbb{Z}$*, and* $\mathbf{n} \in \mathbb{Z}^I$*, of pseudo-period isomorphisms*

$$\mathcal{E}_{\mathbf{m}+\mathbf{n}} \simeq \mathcal{E}_{\mathbf{m}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\sum_{i \in I} n_i D_i)$$

compatible between themselves, and with inclusions above.

The category $\mathrm{Par}(X, \mathbf{D})$ of parabolic vector bundles with arbitrary rational weights form an abelian tensor category, the tensor product being given by a convolution formula. Nori defines *finite* parabolic vector bundles as objects of this category satisfying a non trivial tensor relation ([6]). In order to get an abelian category in positive characteristic as well, one introduces *essentially finite* parabolic vector bundles as the kernels of morphisms between two finite parabolic vector bundles ([3]). Under the assumptions of our theorem, the corresponding full subcategory $\mathrm{EF}\,\mathrm{Par}(X, \mathbf{D})$ is even tannakian, so that it makes sense to consider the monodromy group of an essentially finite parabolic vector bundle.

Given a closed point x of D, and $\mathbf{l} \in \mathbb{Z}^I$, we will write that a parabolic vector bundle $\mathcal{E}.$ on $(X, \mathbf{D})$ *admits* $\mathbf{l}/\mathbf{r}$ *as a weight at* x if $\left(\mathcal{E}_{(\mathbf{l}+\mathbf{1})/\mathbf{r}}\right)_x \subset \left(\mathcal{E}_{\mathbf{l}/\mathbf{r}}\right)_x$ is not an equality.

Theorem 1 (I.Biswas-B., 2017). *Assume that* X/k *is proper, of finite type, geometrically {connected and reduced}. The two following statements are equivalent :*

(1) There exists a finite abelian group scheme G/k *and a tamely ramified* G*-torsor* $Y \to X$ *with ramification data* $(\mathbf{D}, \mathbf{r})$*,*

(2) for each closed point x *of* D*, and for all* $\mathbf{l} \in \mathbb{Z}^I$*, such that* $0 \leq \mathbf{l} < \mathbf{r}_x$*, there exists an object* $\mathcal{E}.$ *in* $\mathrm{EF}\,\mathrm{Par}(X, \mathbf{D})$ *with abelian monodromy and weights in* $\frac{1}{\mathbf{r}}\mathbb{Z}^I$*, such that* $\mathcal{E}.$ *admits* $\mathbf{l}/\mathbf{r}_x$ *as a weight at* x*.*

It is certainly natural to ask if the theorem holds for G/k finite but possibly non abelian, but this is still an open question.

The main difficulty is to identify tamely ramified torsors with ordinary torsors on natural orbifolds associated to X and the ramification data, the so-called stack of roots, that are fppf locally quotient stacks of Kummer covers. Some tools we use are closely related to the main topics of the conference. For instance, to avoid assuming that X/k has a rational point, or having to pick up one, we use Nori fundamental gerbe ([3]), whose rational points are the sections of the section conjecture (see reports by A. Schmidt and J. Stix). Another key ingredient is a Nori version ([1]) of Noohi's uniformization criterion for algebraic stacks ([5]), that was used by Sylvain Maugeais and Benjamin Collas in their study of the Galois action of the inertia stack of the moduli spaces of curves (see report by B. Collas and references therein).

References

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Grunwald Problems and Specialization of Galois Covers

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(joint work with François Legrand, Danny Neftin)

This talk summarized the results of the two papers [8] and [6] on the local behaviour of specializations of Galois covers. The underlying question, informally stated, is: Given a number field k and a finite group G, to which extent is it possible to impose local conditions on a Galois extension F/k with group G, with the extra requirement that F/k be a specialization of some prescribed k-regular G-extension $E/k(t)$? The topic is part of a larger ongoing project investigating in various ways the structure of the sets of specializations of a Galois cover.

Grunwald problems. The Grunwald problem (over a number field k) is a strengthening of the inverse Galois problem, asking about the existence of Galois extensions of k with prescribed Galois group which approximates finitely many prescribed local extensions. There are several variants which can be considered as Grunwald problems. The most classical is the following one:

Definition (Grunwald problem)**.** *Let k be a number field, G be a finite group and S be a finite set of primes of k. For each $p \in S$, let F_p/k_p be a Galois extension of k_p whose Galois group embeds into G. Does there exist a Galois extension of k with group G whose completion at p equals F_p/k_p for all $p \in S$?*

If we consider only sets S which are disjoint from a certain finite set S_0 of primes of k (depending on G), we speak of a weak Grunwald problem.
Weak Grunwald problems are known to have positive answers for several important classes of groups. In particular, the Grunwald-Wang theorem gives a positive answer for all abelian groups, exempting at most the primes of k extending the rational prime 2 (see [13]). Results by Harari ([4]) give positive answers for weak Grunwald problems for groups which are iterated semidirect products of abelian groups. Recent work of Harpaz and Wittenberg ([5]) gives a positive answer for all supersolvable groups (in particular, for all nilpotent groups). A different direction was exhibited by Saltman, who showed that all Grunwald problems have a positive answer if the group G has a generic Galois extension over k ([10]).