# Extension of Galois groups by solvable groups, and application to fundamental groups of curves

Niels Borne

**Abstract.** The issue of extending a given Galois group is conveniently expressed in terms of embedding problems. If the kernel is an abelian group, a natural method, due to Serre, reduces the problem to the computation of an étale cohomology group, that can in turn be carried out thanks to Grothendieck-Ogg-Shafarevich formula. After introducing these tools, we give two applications to fundamental groups of curves.

## 1. Informal introduction

In what follows, we will be mainly concerned by the description of the structure of the (étale) fundamental groups of algebraic curves. To have a glimpse of what the main issues are, let us fix $k$ be an algebraically closed field of characteristic 0. It is then well known that:

$$\pi_1^{et}(\mathbb{P}^1\backslash\{0,1,\infty\},\overline{x}) \simeq \widehat{F_2} \tag{1.1}$$

where $F_2$ is a free group on 2 generators and $\widehat{\cdot}$ stands for profinite completion. The proof, however, uses in an essential way analytic techniques. It is now an old but still open question to find a *purely algebraic proof* of the above isomorphism. This issue seems to be first mentioned in Grothendieck's masterpiece [1], where the author also explained that the only thing that was proven algebraically in the 1960's was the isomorphism between the abelianizations of the groups:

$$\pi_1^{et}(\mathbb{P}^1\backslash\{0,1,\infty\},\overline{x})^{ab} \simeq \widehat{F_2}^{ab} \tag{1.2}$$

The proof relies on class field theory, or to put it more simply, on the description of the generalized Jacobian of the curve.

.

Since then, not much progress has been done. In a recent joint work with Michel Emsalem [5], we could extend the scope of algebraic methods to give a proof of the isomorphism of the largest solvable quotients of the groups:

$$\pi_1^{et}(\mathbb{P}^1\backslash\{0,1,\infty\},\overline{x})^{solv} \simeq \widehat{F_2}^{solv} \tag{1.3}$$

These quotients are unfortunately very small: one can indeed use the classification of finite groups to show that any finite simple group can be generated by two generators, hence is a quotient of $\widehat{F_2}$, but such a group is of course not a quotient of $\widehat{F_2}^{solv}$, except if it is abelian. Thus our result is very far from giving an algebraic proof of (1.1), and moreover the isomorphism (1.3) is the best we can get from our method.

Strangely enough, our work stems from Serre's proof of Abhyankar's conjecture for solvable covers of the affine line in positive characteristic [14]. Let thus now $k$ be an algebraically closed field of characteristic $p > 0$. Abhyankar's conjecture states that, for a finite group $G$:

$$\exists\ \pi_1^{et}(\mathbb{A}^1,\overline{x}) \twoheadrightarrow G \iff G \text{ is quasi-}p \tag{1.4}$$

a group being, by definition, quasi-$p$ when it is generated by its $p$-Sylow-subgroups.

After a brief review of classical results on the étale fundamental groups of curves (section 2), we will explain Serre's device that reduces the issue of building covers with solvable Galois groups to the computation of an étale cohomology group (section 3). In characteristic 0, Ogg-Shafarevich's formula finally solves the problem, leading in section 4 to the algebraic proof of the obvious generalization (1.3) for an arbitrary affine curve. In characteristic $p$, the full Grothendieck-Ogg-Shafarevich is needed, which is explained, without a proof, in section 5. We finally go back to the origin of the subject by sketching Serre's celebrated proof of (1.4) for solvable groups.

## 2. Fundamental groups of curves over an algebraically closed field

### 2.1. Étale fundamental group

Let us start with a quick reminder of the étale fundamental group. Let $X$ be a connected scheme, endowed with a geometric point $\overline{x} : \operatorname{spec}\Omega \to X$. The étale fundamental group $\pi_1^{et}(X,\overline{x})$ is defined as the automorphism group of the functor $\overline{x}^* : \operatorname{Cov} X \to \text{Sets}$ that sends a finite étale cover $Y \to X$ to its fiber $Y(\overline{x})$. One can show (see [1]) that this group is profinite (that is, this is a topological group isomorphic to an inverse limit of finite discrete groups) and that the functor above factors through an equivalence $\overline{x}^* : \operatorname{Cov} X \to \pi_1^{et}(X,\overline{x}) - \text{Sets}$. In particular for a finite group $G$:

$$\exists\pi_1^{et}(X,\overline{x}) \twoheadrightarrow G \iff \exists Y \to X \text{ finite connected étale cover } / \operatorname{Gal}(Y/X) \simeq G$$

### 2.2. Comparison theorems

We suppose in this section that $X \to \operatorname{spec}\mathbb{C}$ is a connected scheme, locally of finite type. Let $X^{an}$ the associated complex analytic space. Then it is known (see [1], XII) that the functor

$$\operatorname{Cov} X \to \operatorname{Cov} X^{an}$$

that sends a finite cover $Y \to X$ to $Y^{an} \to X^{an}$ identifies the finite étale covers of $X$ with those of $X^{an}$. An obvious consequence is that

$$\widehat{\pi_1(X^{an}, \overline{x})} \simeq \pi_1^{et}(X, \overline{x})$$

where in the left hand side $\pi_1$ stands for the usual topological fundamental group and $\widehat{\cdot}$ for the profinite completion of a group $G$:

$$\widehat{G} = \varprojlim_{\#G/I<\infty} G/I$$

### 2.3. Invariance under algebraically closed base change

Let now $k$ be an arbitrary algebraically closed field, and $X$ a proper connected $k$-scheme. If $k'/k$ is an algebraically closed extension and $\overline{x'} : \operatorname{spec}\Omega \to X_{k'}$ is a geometric point, define $\overline{x} : \operatorname{spec}\Omega \to X$ as its image in $X$. Then there is a canonical isomorphism (see [1], X):

$$\pi_1^{et}(X_{k'}, \overline{x'}) \simeq \pi_1^{et}(X, \overline{x})$$

### 2.4. Curves in characteristic zero

From these general results, one can deduce the structure of the étale fundamental group of a smooth projective curve $X$ of genus $g$ defined over an algebraically closed field $k$ of characteristic 0. The principle is that such a $X$ is in fact defined over a subfield $k_0 \subset k$ of finite transcendence degree over the prime field $\mathbb{Q}$. This enables to embed $k_0$ into $\mathbb{C}$, and applying the results of §2.3 (twice) and §2.2, one gets that $\pi_1^{et}(X, \overline{x})$ is isomorphic to the profinite completion of the topological fundamental group of a Riemann surface of genus $g$.

More generally, similar techniques apply to work out the structure of the étale fundamental group of any smooth curve over an algebraically closed field $k$ of characteristic 0. With the above notations, let $U \subset X$ be a non-empty open subset, and $r = \#(X \backslash U)$ the number of "holes" (possibly 0). One can show along the above lines that

$$\pi_1^{et}(U, x) \simeq \widehat{\Gamma_{g,r}}$$

where $\Gamma_{g,r}$ is the quotient of the free group on $2g+r$ generators $a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_r$ by the unique relation $[a_1, b_1] \cdots [a_g, b_g] = c_1 \cdots c_r$. In particular, $\pi_1^{et}(U, x)$ is free (as a profinite group) on $2g + r - 1$ generators as soon as $r > 0$ (that is, when $U$ is affine).

### 2.5. Positive characteristic phenomenons

We follow the notations of the previous section, but we now work over an algebraically closed field $k$ of characteristic $p > 0$. For $g \geq 2$, there is not a single example of a curve $X$ of genus $g$ where the structure of the étale fundamental group $\pi_1^{et}(X, \overline{x})$ is fully understood! So we must somehow simplify the problem, and for this purpose we introduce, for a profinite group $G$, two quotients:

$$G^{p'} = \varprojlim_{\substack{I \lhd G \text{ open} \\ [G:I] \text{ prime to } p}} \frac{G}{I}$$

and

$$G^{p} = \varprojlim_{\substack{I \lhd G \text{ open} \\ [G:I] \text{ a power of } p}} \frac{G}{I}$$

**2.5.1.** $p'$ **part.** Thanks to *specialisation theory*, one can show:

$$\pi_1^{et}(U, x)^{p'} \simeq \widehat{\Gamma_{g,r}}^{p'}$$

This isomorphism was one of the early successes of Grothendieck's theory of the étale fundamental group (see [1]). So as far as $p'$-quotients are concerned, nothing new occurs in comparison with characteristic 0. The only known proof uses comparison theorems.

**2.5.2.** $p$ **part (complete curves).** But for $p$-quotients the situation is completely different. They are no longer controlled by the genus but by the *Hasse-Witt invariant*

$$h = \dim_{\mathbb{F}_p} H^1(X, \mathbb{F}_p)$$

that is, the first étale cohomology group with coefficients in the constant sheaf $\mathbb{F}_p$. One can show than $0 \leq h \leq g$, and thanks to cohomological arguments, Shafarevich proved the following:

**Theorem 2.1 (Shafarevich).** *The group* $\pi_1^{et}(X, x)^p$ *is a free pro-p group on h-generators, that is*

$$\pi_1^{et}(X, x)^p \simeq \widehat{F_h}^p$$

*Remark* 2.2. 1. Shafarevich's original proof was quite intricate and was heavily simplified with the rise of étale cohomology (see [6]). In contrast to the previous result, this is an algebraic theorem.

2. In particular, if one considers the abelianizations of the above groups, one gets, with obvious notations, for a fixed prime $l$:

$$\pi_1^{et}(X, x)^{ab,l} \simeq \begin{cases} \mathbb{Z}_l^{\oplus 2g} & \text{for } l \neq p \\ \mathbb{Z}_p^{\oplus h} & \text{for } l = p \end{cases}$$

This illustrates the general trend that (for complete curves) there are less covers in positive characteristic.

**2.5.3. Mixed covers (affine curves).** We now consider only affine curves, that is, the number $r$ of holes is greater than 1. Because of *wild ramification* strange things occur:

- The affine line $\mathbb{A}^1$ is not simply connected[1].
- Even worse, the profinite group $\pi_1^{et}(U,x)$ is not topologically of finite type.

However, the set of finite quotients of the étale fundamental group is known[2]. To state this, for a finite group $G$, we denote by $p(G)$ the group generated by its $p$-Sylow subgroups, and $n_G$ the minimal number of generators of $G$. Then the celebrated Abhyankar conjecture states:

**Theorem 2.3 (Raynaud [10],Harbater [8]).**

$$\exists \pi_1^{et}(U,\overline{x}) \twoheadrightarrow G \iff n_{G/p(G)} \leq 2g+r-1$$

The proof, unfortunately, uses a transcendental argument at some point. But a first crucial step, performed by Serre, was to prove the theorem when $X=\mathbb{A}^1$, the affine line, and $G$ is solvable, and this was done by algebraic means (see [14], and section 6).

## 3. Embedding problems

### 3.1. Definition

An embedding problem is a diagram in the category of profinite groups:

$$\begin{array}{ccccccccc} & & & & & & \pi & & \\ & & & & & & \downarrow{\scriptstyle a} & & \\ 1 & \longrightarrow & A & \longrightarrow & G & \xrightarrow{q} & H & \longrightarrow & 1 \end{array}$$

where the vertical arrow is an epimorphism and the horizontal sequence is exact. It is said to have a *weak solution* if there exists a continuous homomorphism $\beta:\pi\to G$ lifting $\alpha$, i.e. $q\circ\beta=\alpha$. There is a *strong solution* if one can choose moreover $\beta$ to be an epimorphism.

Clearly, weak solutions are in one to one correspondence with the sections of the exact sequence:

$$1\to A\to G\times_H \pi\to\pi\to 1\ .$$

[1] As the existence of Artin-Schreier covers shows.

[2] This set does not determine the group up to isomorphism, see also Proposition 4.2.

**3.2. Embedding problems with irreducible kernels**

Let $l$ be a prime number. We assume that $G$ is finite and $A$ is a $l$-elementary abelian group irreducible as $\mathbb{F}_l[H]$-module. Then a weak solution is strong if and only if it does not come from a section of the exact sequence:

$$1 \to A \to G \to H \to 1 .$$

One can use this fact to give a cohomological criterion of existence of a strong solution of the embedding problem. We distinguish between the two following situations:

**3.2.1. Case of a non-split exact sequence.** We denote (abusively) $\mathrm{cl}(G)$ the class of th extension $1 \to A \to G \to H \to 1$ in $H^2(H, A)$. Then the embedding problem has a strong solution if and only if the image of $\mathrm{cl}(G)$ by $H^2(H, A) \to H^2(\pi, A)$ is the trivial class.

**3.2.2. Case of a split exact sequence.** If the exact sequence we started from splits, and $\mathcal{S}$ denotes the set of its sections, one has the equality: $|A^H| \cdot |\mathcal{S}| = |H^1(H, A)| \cdot |A|$. Similarly if $\tilde{\mathcal{S}}$ stands for the set (possibly infinite) of sections of the exact sequence $1 \to A \to G \times_H \pi \to \pi \to 1$, then $|A^H| \cdot |\tilde{\mathcal{S}}| = |H^1(\pi, A)| \cdot |A|$ Note that $H^1(H, A) \hookrightarrow H^1(\pi, A)$. We deduce from these facts that the embedding problem has a strong solution in this case if and only if:

$$\dim_{\mathbb{F}_l} H^1(\pi, A) > \dim_{\mathbb{F}_l} H^1(H, A) .$$

**3.3. Étale sheaves**

We will be interested in such embedding problems mainly when $\pi = \pi_1^{et}(X, \overline{x})$ is the étale fundamental group of a smooth, connected algebraic curve over an algebraically closed field. In this case, the data of the epimorphism $\alpha : \pi_1^{et}(X, \overline{x}) \to H$, together with the action $\rho : H \to \mathrm{Aut}(A)$ given by conjugation, define a locally constant sheaf of $\mathbb{F}_l$-vector spaces $\underline{A}$ on the étale site $X_{et}$, by the formula $\underline{A} = (\pi_*(A_Y))^H$, where $\pi : Y \to X$ is the cover associated to $\alpha$, and $A_Y = \mathrm{Hom}_Y(\cdot, A \times Y)$ is the constant sheaf with stalk $A$ on $Y$. We will also denote this locally constant sheaf by $\pi_*^H(A_Y)$ in the sequel.

This is a well known fact from descent theory that this process defines, when $\alpha$ and $\rho$ vary, an equivalence between continuous representations of $\pi_1^{et}(X, \overline{x})$ with values in $\mathbb{F}_l$-vector spaces and locally constant sheaves of $\mathbb{F}_l$-vector spaces on the étale site $X_{et}$. In the opposite direction, one simply associates to such a sheaf its stalk $F_{\overline{x}}$ at the chosen geometric point, with the natural action.

**3.4. Comparison of cohomologies**

The reason to switch to étale sheaves is that we have both a better intuition and a better grasp of their cohomology than the one of the corresponding representations. To compare them, remember that to an $H$-Galois cover $\pi : Y \to X$ is associated the Hochschild-Serre spectral sequence:

$$E_2^{p,q} = H^p(H, H^q(Y, \pi^* F)) \Longrightarrow H^{p+q}(X, F) = E^{p+q}$$

This spectral sequence is cohomological (that is $E_2^{p,q} = 0$ for $p < 0$ or $q < 0$) hence gives rise to a five-term short exact sequence, that in the case of $F = \underline{A}$ amounts to

$$0 \to H^1(H, A) \to H^1(X, \underline{A}) \to H^1(Y, A_Y)^H \to H^2(H, A) \to H^2(X, \underline{A})$$

Going to the inductive limit over all $\alpha : \pi_1^{et}(X, \overline{x}) \to H$, we get the following facts:

- $H^1(\pi_1^{et}(X, \overline{x}), A) \simeq H^1(X, \underline{A})$
- $H^2(\pi_1^{et}(X, \overline{x}), A) \hookrightarrow H^2(X, \underline{A})$

### 3.5. $l$-cohomological dimension of a curve

We recall a general definition:

**Definition 3.1.** Let $X$ be a scheme, and $l$ be a prime number.

1. an abelian sheaf $F$ on $X_{et}$ is $l$-torsion if the natural morphism
$$\varinjlim_{n \to \infty} {}_{l^n}F \to F$$
where ${}_{l^n}F = \ker(F \stackrel{\times l^n}{\to} F)$, is an isomorphism.
2. The *$l$-cohomological dimension* of $X$ is the greatest integer $n = \mathrm{cd}_l(X)$ (possibly $\infty$) such that there exists a $l$-torsion sheaf $F$ with $H^n(X, F) \neq 0$.

The cohomology of étale torsion sheaves is controlled by the following classical result.

**Theorem 3.2 (Artin [2]).** *Let $X$ be a complete smooth algebraic curve over a separably closed field $k$ of characteristic $p$, and $l$ be a prime number distinct from $p$.*

1. $\mathrm{cd}_l X = 2$
2. *if $U \subsetneq X$ is a non empty affine open subset then* $\mathrm{cd}_l U = 1$.

*Sketch of the proof.* 1. We have to show that $H^n(X, F) = 0$ for $n > 2$ and $F$ a $l$-torsion sheaf. It is enough to show this when $F$ is constructible (for curves, this means locally constant on a dense open subset, with finite stalks, see also §5.3). Indeed, the cancellation is stable by extension, and any $l$-torsion sheaf can be filtered by constructible sheaves. Then, since a constructible sheaf is locally constant on a stratification, one can in turn reduce to the case where $F = j_! F'$ for $j : U \to X$ an open immersion, and $F'$ is locally constant. Here $j_!$ denotes the "extension by 0" operation, described on the stalks by:

$$(j_! F)_{\overline{x}} = \begin{cases} F_{\overline{x}} & \text{for } x \in U \\ 0 & \text{for } x \notin U \end{cases}$$

Using a trick called "la méthode de la trace", one reduces the problem again to the case where $F = j_! \left(\mathbb{Z}/l\right)_U$. The idea is that it is enough to control the cancellation of the cohomology after a pullback to a finite étale cover. If we denote by $i : X' = X\backslash U \to X$ the closed immersion, with the reduced structure, the exact sequence

$$0 \to j_! \left(\frac{\mathbb{Z}}{l}\right)_U \to \left(\frac{\mathbb{Z}}{l}\right)_X \to i_* \left(\frac{\mathbb{Z}}{l}\right)_{X'} \to 0$$

shows that one can suppose that $F = (\mathbb{Z}/l)_X$. Since $l \neq p$, there is a non canonical isomorphism $(\mathbb{Z}/l)_X \simeq \mu_l$. One can then use Kummer's theory, and Tsen's theorem, that asserts that $H^n(X, \mathbb{G}_m)(l) = 0$ for $n \geq 2$ (where $\cdot(l)$ stands for the $l$-primary part), to work out the following:

$$H^n(X, \mu_l) = \begin{cases} 0 & \text{for } n > 2 \\ \frac{\operatorname{Pic} X}{l \operatorname{Pic} X} & \text{for } n = 2 \end{cases}$$

which concludes the proof of the first case.

2. For the same reason, it is enough to show that $H^n(U, \mu_l) = 0$ for $n \geq 2$. But if $A = \underline{\operatorname{Pic}}^0_{X/k}$, then $A(k) \twoheadrightarrow \operatorname{Pic}(U)$ (because $U$ is affine), and $A(k)$ is $l$-divisible (because $A \xrightarrow{\times l} A$ is étale). Hence $H^2(U, \mu_l) = \frac{\operatorname{Pic} U}{l \operatorname{Pic} U} = 0$.

□

## 4. Largest pro-solvable $p'$-quotient of the fundamental group of an affine curve

The aim of this section is to prove the following theorem.

### 4.1. Statement

We fix some notations:

- $X$ a smooth projective curve over an algebraically closed field $k$ of characteristic $p \geq 0$,
- $g$ the genus of $X$,
- $U = X \setminus \{a_1, \dots, a_r\}$, with $r \geq 1$ (so that $U$ is affine),
- for a profinite group $G$, let $G^{solv,p'}$ be the inverse limit of its finite solvable quotients of order prime to $p$,
- $\widehat{F_N}$ a free group on $N$ generators.

**Theorem 4.1 (B.-Emsalem [5] for the algebraic proof).** *If $\overline{x}$ is a geometric point of $U$ then:*

$$\pi_1^{et}(U, \overline{x})^{solv,p'} \simeq \widehat{F_{2g+r-1}}^{solv,p'}$$

### 4.2. The $\mathcal{P}_G$ property

For a finite group $G$, we denote by $n_G$ the minimal number of generators of $G$. Let $\mathcal{P}_G$ be the property

$$\boxed{n_G \leq 2g + r - 1 \iff \exists\ \pi_1^{et}(U, \overline{x}) \twoheadrightarrow G}$$

Theorem 4.1 implies that $\mathcal{P}_G$ is true for $G$ solvable of order prime to $p$. But the following well known Proposition shows that the converse is also true.

**Proposition 4.2 (see for instance [7]).** *For $\pi$ a profinite group, define* $\mathrm{Im}(\pi) = \{G/H, H \lhd G, H$ *open* $\}$. *If $\pi$ and $\pi'$ are two profinite groups such that* $\mathrm{Im}(\pi) = \mathrm{Im}(\pi')$ *and $\pi$ is topologically of finite type, then $\pi \simeq \pi'$.*

*Sketch of a proof.* The main tool is the following fact: if $(E_i)_{i\in I}$ is a projective system of non empty finite sets, then $\varprojlim_{i\in I} E_i \neq \emptyset$. □

To now prove that the property $\mathcal{P}_G$ holds for $G$ solvable of order prime to $p$, we will show the slightly stronger statement:

**Proposition 4.3.** *Fix an exact sequence of finite groups:* $1 \to A \to G \to H \to 1$ . *If $A$ is solvable, $\#G$ is prime to $p$, and $\mathcal{P}_H$ holds, then $\mathcal{P}_G$ holds.*

Moreover, it is easy to see that it is enough to show the Proposition when $A$ is abelian, $l$-elementary (for a prime $l \neq p$), and irreducible as a $\mathbb{F}_l[H]$-module.

The hypothesis is that $\mathcal{P}_H$ is true. If both assertions in $\mathcal{P}_H$ are false then the same holds for $\mathcal{P}_G$, hence $\mathcal{P}_G$ is true. One can thus suppose that both assertions in $\mathcal{P}_H$ are true. In particular, one can fix an epimorphism $\pi_1^{et}(U, \overline{x}) \twoheadrightarrow H$. Let $\pi : V \to U$ the corresponding $H$-Galois cover. One can now apply the general technique explained in §3, and this leads to the following discussion.

**4.2.1. Case of a non split exact sequence.** Let us suppose that $\mathrm{cl}(G)$ is not the trivial class in $H^2(H, A)$.

Then on the one hand $H^2(\pi_1^{et}(U, \overline{x}), A) = 0$ since $U$ is affine, according to Theorem 3.2 and §3.4. The argument in §3.2.1 shows that the fixed epimorphism $\pi_1^{et}(U, \overline{x}) \twoheadrightarrow H$ always lifts to $G$.

On the other hand, the fact that the exact sequence $1 \to A \to G \to H \to 1$ does not split, and the fact that $A$ is irreducible, enable to show easily that $n_G = n_H$. Hence $n_G \leq 2g + r - 1$ holds.

So both assertions in $\mathcal{P}_G$ are true, and $\mathcal{P}_G$ holds.

**4.2.2. Case of a split exact sequence.** Let now suppose that $\mathrm{cl}(G) = 0$ in $H^2(H, A)$. Then the arguments in §3.2.2 and in §3.4 show that the fixed epimorphism $\pi_1^{et}(U, \overline{x}) \twoheadrightarrow H$ lifts to $G$ if and only if

$$\dim_{\mathbb{F}_l} H^1(U, \underline{A}) > \dim_{\mathbb{F}_l} H^1(H, A) .$$

Using Ogg-Shafarevich formula to compute the first term in the next section, we will show that this last condition is equivalent to $n_G \leq 2g + r - 1$. This will conclude the proof. Indeed then $n_G \leq 2g + r - 1 \implies \exists\ \pi_1^{et}(U, \overline{x}) \twoheadrightarrow G$ is clear. The other way round, if we assume $\exists\ \pi_1^{et}(U, \overline{x}) \twoheadrightarrow G$, then we have lifted the composite $\pi_1^{et}(U, \overline{x}) \twoheadrightarrow G \twoheadrightarrow H$ (which does not need to coincide with the one we started with), hence $n_G \leq 2g + r - 1$.

*Remark* 4.4. The proof shows in fact that if $n_G \leq 2g+r-1$ then *every* embedding problem has a strong solution. In this sense the issue is much simpler in the present situation than in Serre's original context (see §6).

### 4.3. Ogg-Shafarevich formula

We recall that $X$ is a smooth projective curve over an algebraically closed field $k$ of characteristic $p \geq 0$, $g$ denotes the genus of $X$, and $U = X \backslash \{a_1, \ldots, a_r\}$ with $r \geq 1$, is an affine open subset. Ogg-Shafarevich formula enables to compute the Euler-Poincaré characteristic $\chi(X, F) = \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{F}_l} H^i(X, F)$ of a constructible sheaf $F$ (see §5.3 for more details on this notion). Using the exact sequence of relative cohomology, it translates into the following affine version.

**Theorem 4.5 (Ogg-Shavarevich, see [9]).** *Let $F$ be a constructible sheaf of $\mathbb{F}_l$-vector spaces on $X$ that is tamely ramified at infinity and unramified on $U$. Then*

$$\chi(U, F_{|X}) = \chi(U, \mathbb{F}_l) \dim_{\mathbb{F}_l} F_{\overline{\nu}}$$

*where $\overline{\nu}$ is the generic point of $U$.*

This formula enables to conclude the proof of Proposition 4.3 (and thus of Theorem 4.1). To explain this, we fix an epimorphism $\pi_1^{et}(U, \overline{x}) \twoheadrightarrow H$. Let $\pi : V \to U$ be the associated $H$-Galois cover. By a slight abuse, we denote also by $\pi : Y \to X$ its normalisation in $X$. Let $A$ be an irreducible $\mathbb{F}_l[H]$-module, and $G = A \ltimes H$.We can now apply Theorem 4.5 to the constructible sheaf $\pi_*^H(A_Y)$ on $X$. Note that this sheaf does not need to be locally constant, but its restriction $\pi_*^H(A_Y)_{|U} = \underline{A}$ is. Using the standard fact $\chi(U, \mathbb{F}_l) = 2 - 2g - r$, we get that

$$\dim_{\mathbb{F}_l} H^1(U, \underline{A}) = (2g + r - 2) \dim_{\mathbb{F}_l} A + \dim_{\mathbb{F}_l} A^H \ .$$

So the equivalence of $\dim_{\mathbb{F}_l} H^1(U, \underline{A}) > \dim_{\mathbb{F}_l} H^1(H, A)$ with $n_G \leq 2g + r - 1$ results from the following easily shown group-theoretic Lemma, applied with $N = 2g + r - 1$:

**Lemma 4.6.** *Let $l$ be a prime, $N$ an integer. Let moreover $A$ be an $l$-elementary abelian group that is irreducible for the action of a group $H$ whose minimal number of generators $n_H$ is less than $N$. Denote by $G$ the semi-direct product $G = A \rtimes H$. Then:*

$$\dim_{\mathbb{F}_l} H^1(H, A) < (N - 1) \dim_{\mathbb{F}_l} A + \dim_{\mathbb{F}_l} A^H \iff n_G \leq N \ .$$

### 4.4. Remark on groups whose order is divisible by $p$

In the proof of Proposition 4.3, the hypothesis that $\#G$ is prime to the characteristic $p$ of $k$ is only used to ensure that the constructible sheaf $F = \pi_*^H(A_Y)$ on $X$ is tamely ramified. We can in fact weaken this hypothesis and allow $\#H$ to be divisible by $p$, if we impose instead this condition on $F$.

**Proposition 4.7.** *Fix an epimorphism $\pi_1(U, \overline{x}) \twoheadrightarrow H$ where $H$ is finite group of any order, and let $\pi : V \to U$ be the corresponding Galois $H$-cover. Suppose that $A$ is an $l$-elementary abelian group that is irreducible for the action of $H$, and consider* *the embedding problem:*

$$\begin{array}{ccccccccc} & & & & & & \pi_1(U,\overline{x}) & & \\ & & & & & & \downarrow & & \\ 1 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 1 \end{array}$$

*If the corresponding sheaf on* $F = \pi_*^H(A_Y)$ *on* $X$ *is tamely ramified, and* $n_G \leq 2g + r - 1$*, the embedding problem has a strong solution.*

*Remark* 4.8. 1. If $\pi : Y \to X$ is tamely ramified, then so is $F$.

2. Moreover by specialisation theory and analytical methods, one can show that

$$\widehat{F_{2g+r-1}} \twoheadrightarrow \pi_1(U, \underline{x})^{tame}$$

(see [1]). So in other words, the condition $n_G \leq 2g + r - 1$ on a finite group $G$ is necessary to be realised as a Galois group of a tame cover of $X$. The Proposition above says that, for some very special groups $G$, this condition is sufficient. Since the epimorphism is not an isomorphism (there are less tame covers than in characteristic zero), it is not always sufficient. It is interesting to note, however, that in this situation algebraic and analytic techniques are complementary, rather than opposed.

## 5. Grothendieck-Ogg-Shafarevich formula

There are two reasons why we now need a refined version of Ogg-Shafarevich formula, due to Grothendieck, that takes into account the wild ramification of constructible sheaves. The first reason is that the former, tame version of the formula, was originally proved but transcendental methods, using precisely the theorem describing the structure of the largest prime to $p$-quotient of the fundamental group of a curve. The second reason is this refined formula is the crux of Serre's approach of Abhyankar's conjecture.

### 5.1. Artin and Swan characters

**5.1.1. Definition.** Let

- $R$ be a complete discrete valuation ring,
- $k = R/\mathfrak{m}$ its residue field,
- $\pi$ a uniforming parameter,
- $K = \operatorname{frac} R$,
- $L/K$ a finite Galois extension with group $G$,
- $v_L$ the (normalized) valuation of $L$.

We suppose that $k$ algebraically closed of characteristic $p$. For $g$ in $G$, $g \neq 1$, put $i_G(g) = v_L(g\pi - \pi)$.

**Definition 5.1.** The *Artin character* $a_G : G \to \mathbb{Z}$ is defined by

$$g \mapsto \begin{cases} -i_G(g) & \text{if } g \neq 1 \\ \sum_{g\neq 1} i_G(g) & \text{if } g = 1 \end{cases}$$

*Remark* 5.2.
1. $a_G(1) = v_L(\mathcal{D}_{L/K})$ is the valuation of the different.
2. Define the *higher ramification groups* by

$$g \in G_i \iff i_G(g) \geq i+1 \text{ or } g = 1$$

These groups obviously form a decreasing sequence of normal groups starting from $G_0 = G$; one can moreover show that $G_i = \{1\}$ for $i \gg 0$, that $G_i$ is a $p$-group for $i \geq 1$, and that $G_0/G_1$ is cyclic of order prime to $p$. An alternative description of $a_G$ is then given by the easily proved formula:

$$a_G = \sum_{i=0}^{\infty} \frac{\#G_i}{\#G} \operatorname{Ind}_{G_i}^{G}(u_{G_i})$$

where $u_{G_i}$ is the character of the augmentation representation: $u_{G_i} = r_{G_i} - 1$, where $r_{G_i}$ stands for the character of the regular representation. In particular $a_G = 0$ if and only if $G = 1$.

**Definition 5.3.** The *Swan character* $sw_G : G \to \mathbb{Z}$ is defined by

$$sw_G = a_G - u_G$$

*Remark* 5.4.

$$sw_G = \sum_{i=1}^{\infty} \frac{\#G_i}{\#G} \operatorname{Ind}_{G_i}^{G}(u_{G_i})$$

and $sw_G = 0$ if and only if $G_1 = 1$ (that is, exactly when $L/K$ is tamely ramified).

**5.1.2. Artin and Swan representations.** The functions $a_G$ and $sw_G$ are central, that is, constant over conjugacy classes. Moreover, it was already known to Weil (in 1948, see [15]) that they come from complex representations, more precisely that for any complex character $\chi : G \to \mathbb{C}$, the scalar product $\langle a_G, \chi \rangle$ is a nonnegative integer. But a lot more can be said:

**Theorem 5.5 (Serre [12]).** *Let $l$ a prime distinct from $p$.*
1. *Artin and Swan characters can be realized over $\mathbb{Q}_l$.*
2. *There exists a* projective *$\mathbb{Z}_l[G]$-module $Sw_G$ so that $\mathbb{Q}_l \otimes_{\mathbb{Z}_l} Sw_G$ has $sw_G$ for character.*

*Remark* 5.6.
1. $Sw_G$ is unique up to isomorphism.
2. The augmentation character $u_G$ is defined (over any field) as the character of the augmentation representation $U_G = \ker(\operatorname{tr} : \mathbb{Q}_l[G] \to \mathbb{Q}_l)$, so $a_G$ is the character of the representation (called the Artin representation) $A_G = Sw_G \oplus U_G$.

### 5.2. Weil's formula

Let us now recall Weil's original motivation to introduce these representations.

Let $\pi : Y \to X$ be a Galois cover of smooth projective curves over an algebraically closed field $k$, with Galois group $G$. We denote by $g_Y$ and $g_X$ the genus of the curves. By functoriality $G$ acts on

$$H^i(Y, \mathbb{Q}_l) \simeq \begin{cases} \mathbb{Q}_l & i = 0, 2 \\ \mathbb{Q}_l^{\oplus 2g_Y} & i = 1 \\ 0 & i > 2 \end{cases}$$

Weil's formula will compute the characters of these representations.

Let $y \in |Y|_0$ be a closed point, $x = \pi(y)$. We can apply what we have just seen in §5.1 to $R = \widehat{\mathcal{O}_{X,x}}$ and $L = \operatorname{frac} \widehat{\mathcal{O}_{Y,y}}$. The Galois group is the decomposition group and is denoted by $G_y$. We will write $A_y$ for the Artin representation, this is a finite type $\mathbb{Q}_l[G_y]$-module.

Now let $x \in |X|_0$ be a closed point, and put $A_x = \operatorname{Ind}_{G_y}^G A_y$ for any lifting $y \mapsto x$. This is independent of the choice of the lifting.

Let $R_{\mathbb{Q}_l}(G)$ be the subgroup of the character group $R_{\overline{\mathbb{Q}}_l}(G)$ generated by characters of $G$ over $\mathbb{Q}_l$ (or equivalently, the Grothendieck group of the category of finite type $\mathbb{Q}_l[G]$-modules). For such a module $V$, denote by $[V]$ its class in $R_{\mathbb{Q}_l}(G)$.

**Theorem 5.7 (Weil's formula, see [13]).**

$$\sum_{i=0}^{2} (-1)^i [H^i(X, \mathbb{Q}_l)] = (2 - 2g_x)[\mathbb{Q}_l[G]] - \sum_{x \in |X|_0} [A_x]$$

*Remark* 5.8. 1. This can be seen as an equivariant version of Hurwitz formula.
2. The proof uses a Lefschetz formula in étale cohomology, see [9].

### 5.3. Constructible sheaves

Since Grothendieck-Ogg-Shafarevich formula deals with constructible sheaves, we give a more precise definition of these, valid on any scheme.

**Definition 5.9.** 1. A sheaf of abelian groups on $X_{et}$ is *locally constant* if there exists an étale covering $(X_i \to X)_{i \in I}$ and abelian groups $(G_i)_{i \in I}$ such that $F_{|X_i} \simeq \operatorname{Hom}_{X_i}(\cdot, X_i \times G_i)$.
2. $F$ is *locally constant finite* if the $G_i$'s are finite.

*Remark* 5.10. 1. Let $G$ a finite étale commutative étale group scheme over $X$. Then the sheaf $\operatorname{Hom}_X(\cdot, X \times G)$ represented by $G$ is locally constant. Besides, descent theory asserts that this functor gives an equivalence of categories from the category of finite étale commutative étale group schemes over $X$ to the category of locally constant finite abelian sheaves on $X_{et}$.
2. Locally constant finite sheaves are not stable under direct images. For if $\operatorname{supp} F = \overline{\{x \in X / F_{\overline{x}} \neq 0\}}$ and $i : X' \to X$ is a closed immersion, then for

any étale sheaf $F'$ on $X'_{et}$, by definition of the stalks $\operatorname{supp} i_* F' \subset X'$. But if $F$ is locally constant finite and non-zero, and $X$ is irreducible, then $\operatorname{supp} F = X$.

**Definition 5.11.** A sheaf of abelian groups on $X_{et}$ is *constructible* if for every irreducible closed subscheme $X'$ of $X$, there exists a non-empty open subset $U \subset X'$ such that $F_{|U}$ is locally constant finite.

*Remark* 5.12. One can show:

1. constructible sheaves form an abelian category,
2. if $f : X' \to X$ is a proper (and finitely presented) morphism and $F$ is a constructible abelian sheaf on $X'_{et}$, so is $R^q f_* F$ on $X_{et}$ for all $q \geq 0$.

**5.4. Wild conductor**

We return for a while to the local setting. Let $R$ be a discrete valuation ring, with fraction field $K$, and perfect residue field $k$. We are interested in describing constructible sheaves on the étale site of $\operatorname{spec} R$. The decomposition theorem in étale cohomology takes the following simple form. Denote the closed point by $x : \operatorname{spec} k \to \operatorname{spec} R$ and the generic point by $\nu : \operatorname{spec} K \to \operatorname{spec} R$. Let $\overline{K}$ be a separable closure of $K$, $\overline{v}$ an extension from $v$ to $\overline{K}$, and $I(\overline{v})$ the corresponding inertia group. An étale sheaf $F$ on $\operatorname{spec} R$ gives rise to

$$\begin{cases} F_{\overline{\nu}} & \text{a } G_K - \text{module} \\ F_{\overline{x}} & \text{a } G_k - \text{module} \\ F_{\overline{x}} \to (F_{\overline{\nu}})^{I(\overline{v})} & \text{a } G_k - \text{equivariant morphism} \end{cases}$$

The sheaf $F$ is constructible is these modules are finite. Moreover one can then recover $F$ from this data.

Suppose from now on that $R$ is complete, and that $k$ is algebraically closed of characteristic $p \geq 0$. Let $F$ be a constructible sheaf of $\mathbb{F}_l$-modules, with $l \neq p$, and let $L/K$ be Galois extension with group $G$ trivializing $F_{\overline{\nu}}$.

**Definition 5.13.** The (exponent of the) wild conductor of $F$ is

$$\alpha(F) = \dim_{\mathbb{F}_l} \operatorname{Hom}_G(Sw_G, F_{\overline{\nu}})$$

*Remark* 5.14.

1. $\alpha(F) = \sum_{i=1}^{\infty} \frac{\#G_i}{\#G} \dim_{\mathbb{F}_l} \frac{F_{\overline{\nu}}}{F_{\overline{\nu}}^{G_i}}$, in particular $\alpha(F) = 0$ if and only if $G_1$ acts trivially on $F$ (one says that $F$ is tamely ramified),
2. $\alpha(F)$ is additive in (short exact sequences) in $F$ (because $Sw_G$ is projective),
3. $\alpha(F)$ is independent of the choice of $L/K$.

**5.5. Conductor**

We now return to the global situation. Let $X$ be a smooth algebraic curve over an algebraically closed field $k$ of characteristic $p$, and let $F$ be a constructible sheaf of $\mathbb{F}_l$-modules, with $l \neq p$. Fix $\pi : Y \to X$ a Galois étale cover such that $\pi^* F$ is generically constant. Denote the generic point by $\nu : \operatorname{spec} K \to X$ and fix a closed point $x : \operatorname{spec} k \to X$.

Applying what we have seen in §5.4 to $R = \widehat{\mathcal{O}_{X,x}}$, and to the restriction of $F$ to $\operatorname{spec} R$, we get a local wild conductor $\alpha_x(F)$.

**Definition 5.15.** The (exponent of the) conductor of $F$ at $x$ is

$$\epsilon_x(F) = \alpha_x(F) + \dim_{\mathbb{F}_l} F_{\overline{\nu}} - \dim_{\mathbb{F}_l} F_{\overline{x}}$$

*Remark* 5.16. $\epsilon_x(F)$ is additive in (short exact sequences) in $F$.

**Lemma 5.17.** *Let $\nu : \operatorname{spec} K \to X$ be the generic point of $X$ and suppose that the natural morphism $F \to \nu_* \nu^* F$ is an isomorphism. Then for any lifting $y \mapsto x$:*

1. $\dim_{\mathbb{F}_l} F_{\overline{x}} = \dim_{\mathbb{F}_l} (F_{\overline{\nu}})^{G_y}$ ,
2. $\epsilon_x(F) = \sum_{i=0}^{\infty} \frac{\# G_{y,i}}{\# G_y} \dim_{\mathbb{F}_l} \frac{F_{\overline{\nu}}}{F_{\overline{\nu}}^{G_{y,i}}}$

### 5.6. Euler-Poincaré formula

We keep the notations of previous paragraph. As usual, for a constructible sheaf $F$, $\chi(X, F) = \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{F}_l} H^i(X, F)$, and $\chi(X) = \chi(X, \mathbb{F}_l) = 2 - 2g_X$.

**Theorem 5.18 (Grothendieck-Ogg-Shafarevich, see [11]).**

$$\chi(X, F) = \chi(X) \dim_{\mathbb{F}_l} F_{\overline{\nu}} - \sum_{x \in |X|_0} \epsilon_x(F) \ .$$

*References to a proof.* Apart from Raynaud's report on Grothendieck's proof [11], one may want to refer to similar proofs in [3] and [9], or to the more recent and completely different proof in [4]. □

**Corollary 5.19.** *Let $U \subsetneq X$ be a nonempty (affine) open subset such that $F$ is unramified on $U$. Then:*

$$\chi(U, F_{|U}) = \chi(U) \dim_{\mathbb{F}_l} F_{\overline{\nu}} - \sum_{x \in |U|_0} \alpha_x(F) \ .$$

The Corollary is clear from the sequence of relative cohomology of the pair $(X, U)$ and from the fact that $\dim_{\mathbb{F}_l} F_x = \sum_i (-1)^i \dim_{\mathbb{F}_l} H^i_x(X, F)$.

*Remark* 5.20. Note that if $r$ is the number of points of $X \setminus U$, and $g$ is the genus of $X$, then $\chi(U) = 2 - 2g - r$.

## 6. Serre's proof of solvable Abhyankar's conjecture for the affine line

### 6.1. Statement

Let $p$ a prime number. Remember that, for a finite group $G$, we denote by $p(G)$ the subgroup generated by the $p$-Sylow subgroups of $G$. We will call $\mathcal{P}_G$ the following property:

$$\boxed{G = p(G) \iff \exists\ \pi_1^{et}(\mathbb{A}^1, \overline{x}) \twoheadrightarrow G}$$

Abhyankar's conjecture for the affine line states that $\mathcal{P}_G$ is true for any finite group $G$. Serre proved this conjecture for $G$ solvable at the beginning of the 1990's. He in fact showed the following stronger statement.

**Theorem 6.1 (Serre, [14]).** *Fix an exact sequence of finite groups:* $1 \to A \to G \to H \to 1$ . *If* $A$ *is solvable, and* $\mathcal{P}_H$ *holds, then* $\mathcal{P}_G$ *holds.*

In the property $\mathcal{P}_G$, the direct sense is the difficult one, so we will mainly concentrate on this.

### 6.2. Sketch of a proof

**6.2.1. Reduction steps.** By standard *dévissages*, we can reduce to the case where $A$ is abelian, $l$-elementary ($l$ any prime, possibly $p$) and irreducible for the action of $H$.

**6.2.2. A local system.** By hypothesis, the property $\mathcal{P}_H$ is true. The case where both assertions of $\mathcal{P}_H$ are false is easy, as before. So we can assume that $H = p(H)$, and that we are given a $\phi : \pi_1^{et}(\mathbb{A}^1, \overline{x}) \twoheadrightarrow H$, and try to extend it to $G$. Let $\pi : V \to U = \mathbb{A}^1$ the étale $H$-cover corresponding to $\phi$. The data of $\phi$, together with the action of $H$ on $A$ by conjugation, defines a local system $\underline{A}_\phi$ of $\mathbb{F}_l$-vector spaces on $\mathbb{A}^1_{et}$ by the usual formula: $\underline{A}_\phi = \pi_*^H(A_V)$. The reason why we put emphasis on $\phi$ will appear later.

**6.2.3. Case of a non split exact sequence.** .

Let us assume the exact sequence $1 \to A \to G \to H \to 1$ does not split. According to Theorem 3.2, we have that $\mathrm{cd}_l\, \mathbb{A}^1 = 1$ (this is actually also true for $l = p$, albeit with a different proof). Thus according to §3.4, $H^2(\pi_1^{et}(\mathbb{A}^1, \overline{x}), A) = 0$. By the reasoning explained in §3.2.1, we get that $\phi$ always lifts to $G$. Moreover it is easy to check that $G = p(G)$, so $\mathcal{P}_G$ is true.

**6.2.4. Case of a split exact sequence.** We now suppose that the exact sequence $1 \to A \to G \to H \to 1$ splits. We will only deal with the case $l \neq p$ and show that if $G = p(G)$, then $\exists\ \pi_1^{et}(\mathbb{A}^1, \overline{x}) \twoheadrightarrow G$ (although $\phi$ does not necessary lift to $G$). According to the conclusions of §3.2.2, $\phi$ lifts to $G$ if and only if

$$\dim_{\mathbb{F}_l} H^1(H, A) < \dim_{\mathbb{F}_l} H^1(\mathbb{A}^1, \underline{A}_\phi) ,$$

(note that according to §3.4, $H^1(\pi_1^{et}(\mathbb{A}^1, \overline{x}), A) = H^1(\mathbb{A}^1, \underline{A}_\phi)$). Applying Grothendieck-Ogg-Shafarevich formula to compute the last term, we get:

**Lemma 6.2.**

$$\dim_{\mathbb{F}_l} H^1(\mathbb{A}^1, \underline{A}_\phi) = \alpha_\infty(\underline{A}_\phi) - \dim_{\mathbb{F}_l} A$$

*Proof.* Grothendieck-Ogg-Shafarevich formula gives

$$\chi(\mathbb{A}^1, \underline{A}_\phi) = \chi(\mathbb{A}^1) \dim_{\mathbb{F}_l} A - \alpha_\infty(\underline{A}_\phi) .$$

But $\chi(\mathbb{A}^1) = 2 - 2g - r = 1$, and $\chi(\mathbb{A}^1, \underline{A}_\phi) = \dim_{\mathbb{F}_l} H^0(\mathbb{A}^1, \underline{A}_\phi) - \dim_{\mathbb{F}_l} H^1(\mathbb{A}^1, \underline{A}_\phi)$. Now $H^0(\mathbb{A}^1, \underline{A}_\phi) = A^H$, and this last group must be trivial. Indeed else by irreducibility $A^H = A$, and $G \simeq A \times H$, and this contradicts the fact that $G = p(G)$, since we assume $l \neq p$. □

So to sum-up, we have

$$\dim_{\mathbb{F}_l} H^1(H, A) < \alpha_\infty(\underline{A}_\phi) - \dim_{\mathbb{F}_l} A \ ,$$

and $\phi$ lifts to $G$ if and only if the inequality is strict. However, it may happen that the inequality above is an equality. This occurs for instance for Artin-Schreier covers. Suppose that we are in this situation. It is then necessary to increase the ramification by the following trick. Fix an integer $m \geq 1$, not divisible by $p$. Denote by $V_m \to \mathbb{A}^1$ the base change of the original étale $H$-cover $V \to \mathbb{A}^1$ by the Kummer morphism $\mathbb{A}^1 \to \mathbb{A}^1$ defined by $T \mapsto T^m$. Because $H$ is quasi-$p$, and $p$ does not divide $m$, the covers $V \to \mathbb{A}^1$ and $\mathbb{A}^1 \to \mathbb{A}^1$ are linearly disjoint, so $V_m$ is irreducible, and $V_m \to \mathbb{A}^1$ defines in turn an epimorphism $\phi_m : \pi_1^{et}(\mathbb{A}^1, \overline{x}) \twoheadrightarrow H$. The next easy Lemma shows that $\alpha_\infty(\underline{A}_{\phi_m}) = m\alpha_\infty(\underline{A}_\phi)$, so $\phi_m$ lifts to $G$.

**Lemma 6.3.** *Let $f : X' \to X$ a finite separable morphism, where $X$ and $X'$ are smooth curves over an algebraically closed field of characteristic $p$. Let $F$ be a constructible sheaf of $\mathbb{F}_l$-vector spaces on $X_{et}$, with $l \neq p$, and $x' \in X'$ a closed point. Then*

$$\alpha_{x'}(f^*F) = (\deg f)_{x'} \alpha_{f(x')}(F) \ .$$

## References


[1] *Revêtements étales et groupe fondamental.* Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud, Lecture Notes in Mathematics, Vol. 224.

[2] *Théorie des topos et cohomologie étale des schémas. Tome 3.* Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.

[3] *Cohomologie $l$-adique et fonctions $L$.* Lecture Notes in Mathematics, Vol. 589. Springer-Verlag, Berlin, 1977. Séminaire de Géometrie Algébrique du Bois-Marie 1965–1966 (SGA 5), Edité par Luc Illusie.

[4] Ahmed Abbes and Takeshi Saito. The characteristic class and ramification of an $l$-adic étale sheaf. *Invent. Math.*, 168(3):567–612, 2007.

[5] Niels Borne and Michel Emsalem. Note sur la détermination algébrique du groupe fondamental pro-résoluble d'une courbe affine. *J. Algebra*, 320(6):2615–2623, 2008.

[6] Richard M. Crew. Etale $p$-covers in characteristic $p$. *Compositio Math.*, 52(1):31–45, 1984.

[7] Michael D. Fried and Moshe Jarden. *Field arithmetic*, volume 11 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, third edition, 2008. Revised by Jarden.

[8] David Harbater. Abhyankar's conjecture on Galois groups over curves. *Invent. Math.*, 117(1):1–25, 1994.

[9] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[10] M. Raynaud. Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d'Abhyankar. *Invent. Math.*, 116(1-3):425–462, 1994.

[11] Michel Raynaud. Caractéristique d'Euler-Poincaré d'un faisceau et cohomologie des variétés abéliennes. In *Séminaire Bourbaki, Vol. 9*, pages Exp. No. 286, 129–147. Soc. Math. France, Paris, 1995.

[12] Jean-Pierre Serre. Sur la rationalité des représentations d'Artin. *Ann. of Math. (2)*, 72:405–420, 1960.

[13] Jean-Pierre Serre. *Corps locaux*. Hermann, Paris, 1968. Deuxième édition, Publications de l'Université de Nancago, No. VIII.

[14] Jean-Pierre Serre. Construction de revêtements étales de la droite affine en caractéristique *p*. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(6):341–346, 1990.

[15] André Weil. *Sur les courbes algébriques et les variétés qui s'en déduisent*. Actualités Sci. Ind., no. 1041 = Publ. Inst. Math. Univ. Strasbourg **7** (1945). Hermann et Cie., Paris, 1948.

Niels Borne
Université Lille 1, Cité scientifique
U.M.R. CNRS 8524, U.F.R. de Mathématiques
59 655 Villeneuve d'Ascq Cédex
France
e-mail: `Niels.Borne@math.univ-lille1.fr`