Parabolic Sheaves and Logarithmic Geometry

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1. Introduction

1.1. Motivation. The aim of this note is to give an introduction to the notion of parabolic sheaves on logarithmic schemes, as first defined in my joint work with Angelo Vistoli [BV12]. I will explain the examples we started from in order to, hopefully, enlighten the rather formal definitions given in *loc. cit.* I will conclude by a glimpse at subsequent developments.

1.2. The Context. Parabolic bundles were first introduced by V.Mehta and C.Seshadri [MS80] on a compact Riemann surface X endowed with a set D of marked points. In this initial formulation, a parabolic bundle is a vector bundle endowed with a partial flag of the residual stalk at each marked point, and also equipped with a real number in the range $[0, 1[$ called weight for each component of the flags. The celebrated Mehta-Seshadri theorem states that polystable parabolic vector bundles are in one-to-one correspondence with unitary representations of the topological fundamental group $\pi_1(X\backslash D)$ of the open Riemann surface.

C.Simpson, at the end of the 80's, reformulated and then extended the definition of a parabolic bundle to the situation where the data is a scheme X endowed with a normal crossings divisor D. In this setup, a parabolic bundle with weights in $\frac{1}{r}\mathbb{Z}$ can be seen as a decreasing sequence

$$\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_{\frac{1}{r}} \supset \cdots \supset \mathcal{E}_{\frac{r-1}{r}} \supset \mathcal{E}_1 = \mathcal{E}(-D)$$

where we assume, for simplicity, that D has only one irreducible component.

This is sufficient for certain purposes, but it is also clear that this definition is perfectible. For instance, one would like to be able to make the special locus degenerate to fill up the whole space, but with the above setup one is unable

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to restrict parabolic sheaves on (X, D) to (D, D), or more generally to restrict these along a non flat morphism $X' \to X$.

There is an even more serious issue with the definition of a parabolic sheaf. Namely, one usually works with a normal crossings divisor that contains several irreducible components. It is straightforward to generalize Simpson's definition to take this into account, by using multi-indices. However, this obvious definition gives satisfying results only in the case of a simple normal crossings divisor. When one considers instead a general normal crossings divisor, where components are allowed to self-intersect, the naive definition does not work, in the sense that the algebraic analog of Mehta-Seshadri's theorem does not hold any longer. One needs to give a more local definition in order to take into account the fact that branches of D separate étale locally, and this point was the decisive fact that led us to introduce logarithmic geometry.

1.3. Acknowledgements. I would like to express my gratitude to my co-author Angelo Vistoli, especially for his invaluable help during the preparation of this manuscript.

1.4. The Role of Logarithmic Geometry. Our definition of a parabolic sheaf works for almost any logarithmic scheme (we only impose quasi-integrality, which is a very mild assumption) and is functorial in the sense that one can pull-back parabolic sheaves along a morphism of logarithmic schemes.

This solves the first issue mentioned above, but also enables, thanks to the scope of logarithmic geometry, to consider parabolic sheaves in completely new situations, for instance when X is allowed to be singular along D. We will illustrate this fact in §3 in the special case when X is the quadratic cone.

Another application of our formalism is that, given a scheme X with a general normal crossings divisor D, the algebraic analog of Mehta-Seshadri's theorem holds, if one uses the correct definition of a parabolic bundle. In order to do so, one associates naturally to the pair (X, D) a logarithmic scheme, and one then uses the definition for logarithmic schemes.

2. Parabolic Sheaves: Old and New

There are, of course, many possible higher-dimensional generalizations of Mehta-Seshadri's definition of a parabolic vector bundle on a marked algebraic curve. But in order to be meaningful, the definition has to preserve the link with the fundamental group of the complement of the divisor $X \backslash D$. We start by recalling this connection, and then proceed to give the modern interpretation of parabolic vector bundles as ordinary vector bundles on certain orbifolds.

2.1. Parabolic Sheaves and Tamely Ramified Covers.

2.1.1. From representations of the fundamental group to parabolic bundles. In this section, we consider a scheme X over a base field k and a tamely ramified Galois cover $p : Y \to X$ with Galois group G. We assume for convenience only that the ramification locus D, which is a Cartier divisor, is irreducible, and we denote by r the ramification index : if $E = (p^*(D))_{red}$, then $p^*(D) = r \cdot E$.

In this set-up, Mehta-Seshadri's construction can be interpreted as the definition of a functor that associates to any k-linear representation $\rho : G \to \mathrm{GL}(V)$ a parabolic bundle $(\mathcal{E}_\rho)_\cdot$ on (X, D). Before defining this functor, we switch to a more convenient language.

2.1.2. From *G*-bundles to parabolic bundles. We keep the notations of previous §2.1.1.

I.Biswas realized in [Bis97] that Mehta-Seshadri's construction uses only the structure of a G-sheaf on Y of $\mathcal{F}_\rho = \mathcal{O}_Y \otimes_k V$. Thus one can associate to any G-vector bundle $\mathcal{F}$ on Y a parabolic bundle $\mathcal{E}_\cdot$ on (X, D). One obtains in this way an equivalence between G-vector bundles on Y and parabolic bundles with suitable weights on (X, D), that we will describe in full in the forthcoming §2.1.3.

2.1.3. From stacky bundles to parabolic bundles. I.Biswas rightly calls G-vector bundles on Y "orbifold bundles", they correspond namely to vector bundles on the quotient stack $[Y|G]$ by Galois descent along the étale morphism $Y \to [Y|G]$. The stack $[Y|G]$ can indeed be seen as an orbifold since the morphism $\pi : [Y|G] \to X$ is an isomorphism above $U = X \backslash D$.

My contribution [Bor07] was at this stage to show that one can replace $[Y|G]$ by a canonically isomorphic stack built out only of X and the ramification data (D, r).

Definition 2.1 ([AGV08, Cad07]). *Let X be a scheme, $\mathcal{L}$ be an invertible sheaf on X, endowed with a section s. For each positive integer $r \in \mathbb{N}^*$, one defines the stack of roots $\sqrt[r]{(\mathcal{L}, s)/X}$ as the stack classifying r-th roots of the pair $(\mathcal{L}, s)$.*

When X is instead equipped with an effective Cartier divisor D, with canonical section s_D, we will write $\sqrt[r]{D/X}$ for $\sqrt[r]{(\mathcal{O}_X(D), s_D)/X}$.

Let $\mathcal{D}\mathrm{iv}_X$ be the stack classifying pairs $(\mathcal{L}, s)$ on X. By definition of the stack of roots, the following diagram is cartesian:

$$\begin{array}{ccc} \sqrt[r]{(\mathcal{L},s)/X} & \longrightarrow & \mathcal{D}\mathrm{iv}_X \\ \downarrow & & \downarrow{\cdot\otimes r} \\ X & \xrightarrow{(\mathcal{L},s)} & \mathcal{D}\mathrm{iv}_X \end{array}$$

There is a natural morphism $\mathcal{O}_X \to \mathcal{D}\mathrm{iv}_X$ given by $f \mapsto (\mathcal{O}_X, f)$, and since this is clearly a $\mathbb{G}_m$-torsor, the morphism $\mathcal{O}_X \to \mathcal{D}\mathrm{iv}_X$ identifies canonically with $\mathbb{A}^1 \to [\mathbb{A}^1|\mathbb{G}_m]$.

Assume now that $\mathcal{L}$ is trivialized, that is we fix an isomorphism $\mathcal{L} \simeq \mathcal{O}_X$ sending s to a function f. Then the cartesian diagrams

$$\begin{array}{ccccc} \sqrt[r]{(\mathcal{L},s)/X} & \longrightarrow & \left[\mathbb{A}^1|\mu_r\right] & \longrightarrow & \left[\mathbb{A}^1|\mathbb{G}_m\right] \\ \downarrow & & \downarrow & & \downarrow{x\mapsto x^r} \\ X & \xrightarrow{f} & \mathbb{A}^1 & \longrightarrow & \left[\mathbb{A}^1|\mathbb{G}_m\right] \end{array}$$

show that $\sqrt[r]{(\mathcal{L},s)/X} \to X$ is locally isomorphic to the quotient stack

$$\left[\mathrm{spec}\left(\frac{R[x]}{x^r - f}\right)|\mu_r\right] \to \mathrm{spec}\, R \ .$$

The relevance of the stack of roots is that, with the notations of §2.1.1, the equality $p^*(D) = r \cdot E$ defines a morphism $Y \to \sqrt[r]{D/X}$. Moreover:

Lemma 2.2 (Abhyankar's lemma)**.** *With notations of §2.1.1, if D/k is smooth, then the morphism $Y \to \sqrt[r]{D/X}$ induces an isomorphism $[Y|G] \to \sqrt[r]{D/X}$.*

To explain our formulation of Biswas correspondence, we need a working definition of parabolic sheaves.

Definition 2.3. *With notations of Definition 2.1, a parabolic vector bundle on X with weights in $\frac{1}{r}\mathbb{Z}$ is a functor:*

$$\mathcal{E}_{\cdot} : \left(\frac{1}{r}\mathbb{Z}\right)^{op} \to \mathrm{Vec}(X)$$

endowed with a functorial pseudo-periodicity isomorphism

$$\mathcal{E}_{\frac{l}{r}+1} \simeq \mathcal{E}_{\frac{l}{r}} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee$$

satisfying the natural compatibility condition.

Here $\left(\frac{1}{r}\mathbb{Z}\right)^{op}$ is the opposite category of the one defined by the natural order, $\mathrm{Vec}(X)$ is the category of vector bundles on X, and the compatibility condition is that the morphism $\mathcal{E}_{\frac{l}{r}+1} \to \mathcal{E}_{\frac{l}{r}}$ coincides with the morphism $\mathrm{id} \otimes s^\vee : \mathcal{E}_{\frac{l}{r}} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \to \mathcal{E}_{\frac{l}{r}}$ via the pseudo-periodicity isomorphism.

We will denote by $\mathrm{Par}_{\frac{1}{r}}(X, (\mathcal{L}, s))$ the category of vector bundles on X with weights in $\frac{1}{r}\mathbb{Z}$ with respect to $(\mathcal{L}, s)$.

We are now ready to relate stacky vector bundles to parabolic vector bundles:

Theorem 2.4 ([Bor07]). *With notations of Definition 2.1, assume that* $s : \mathcal{O}_X \to \mathcal{L}$ *is a monomorphism, and let* $\pi : \sqrt[r]{(\mathcal{L}, s)/X} \to X$ *be the morphism to the moduli space, and* $(\mathcal{M}, t)$ *be the universal* r*-th root of* $(\mathcal{L}, s)$ *on* $\sqrt[r]{(\mathcal{L}, s)/X}$*. Then the association*

$$\mathcal{F} \mapsto \left(\frac{l}{r} \mapsto \pi_* \left(\mathcal{F} \otimes \mathcal{M}^{\otimes -l} \right) \right)$$

defines an equivalence of categories $\mathrm{Vec}(\sqrt[r]{(\mathcal{L}, s)/X}) \to \mathrm{Par}_{\frac{1}{r}}(X, (\mathcal{L}, s))$.

Following ideas of M.Nori, one can derive from Lemma 2.2 and Theorem 2.4 an algebraic version of Mehta-Seshadri's theorem for a proper, reduced scheme X over a field k, endowed with a simple normal crossings divisor D (see [Bor09]).

Unfortunately, this strategy is bound to fail with a general normal crossings divisor. The Fourier-like correspondence of Theorem 2.4 holds, but the corresponding stack of roots is not relevant any longer, since Lemma 2.2 fails badly if the divisor D is not smooth. Indeed if a local equation of D is given by hk, then the morphism $[Y|G] \to \sqrt[r]{D/X}$ identifies locally with

$$\left[\mathrm{spec} \left(\frac{R[x, y]}{x^r - h, y^r - k} \right) |\mu_r \times \mu_r \right] \to \left[\mathrm{spec} \left(\frac{R[x]}{x^r - hk} \right) |\mu_r \right]$$

which is not an isomorphism. So the stack of roots has to be changed to take into account the branches of D étale locally. Of course the definition of parabolic bundles has to be modified as well, and we will have to use sheaves as coefficients. It turns out that logarithmic geometry provides the right framework to perform these tasks.

2.2. Logarithmic Geometry.

2.2.1. Logarithmic schemes. We start by reviewing very briefly the classical theory of logarithmic schemes due to K.Kato (see [Kat89]).

Definition 2.5. *Let* X *be a scheme. A pre-log structure* (M, α) *on* X *is a sheaf* M *of commutative monoids on* X_{et} *and a morphism* $\alpha : M \to \mathcal{O}_X$*, where the last sheaf is equipped with its multiplicative law. The pre-log structure* (M, α) *is a log structure if the induced morphism* $\alpha^{-1}\mathcal{O}_X^* \to \mathcal{O}_X^*$ *is an isomorphism.*

A scheme equipped with a (pre-)log structure will be called a (pre-)log scheme. The following example is of paramount importance for us.

Example 2.6. *Let* X *be a scheme, and* $U \subset X$ *an open subset. Then one defines*

$$M_U = \{ f \in \mathcal{O}_X / f \text{ is invertible on } U \} \quad .$$

The inclusion $\alpha_U : M_U \to \mathcal{O}_X$ *defines a log structure.*

Morphisms of (pre-) log structures are defined in the natural way. To any pre-log structure (M, α), one can associate a log-structure $(M, \alpha) \to (M^a, \alpha^a)$. The sheaf M^a is obtained as the pushout of $\alpha^{-1}\mathcal{O}_X^* \to \mathcal{O}_X^*$ and $\alpha^{-1}\mathcal{O}_X^* \to M$, and the natural morphism $(M, \alpha) \to (M^a, \alpha^a)$ is universal among morphisms from (M, α) to log-structures.

Definition 2.7. *Let X be a scheme, (M, α) a log structure on X, and P an abstract monoid. A morphism $P \to M(X)$ is a Kato chart if the corresponding morphism $P_X \to M$ induces an isomorphism $P_X^a \simeq M$.*

Here is another way to formulate this notion: given an abstract monoid P, and a scheme X, the data of $P \to \mathcal{O}_X(X)$ is equivalent to the data of a morphism $f : X \to \operatorname{spec} \mathbb{Z}[P]$. Given such a data, and a log structure (M, α) on X, then the additional data of a chart $P \to M(X)$ is equivalent to an isomorphism of (M, α) with the pullback along f of the canonical log structure on $\mathbb{Z}[P]$ (that is, the log structure associated to the monoid morphism $P \to \mathbb{Z}[P]$).

Example 2.8 (The log point)**.** *Let k be a field. Then $\mathbb{A}_k^1 = \operatorname{spec} k[\mathbb{N}]$ is endowed with a natural log structure associated to the morphism $\mathbb{N} \to k[\mathbb{N}]$. Pulling back this log structure along the 0-section $0 : \operatorname{spec} k \to \mathbb{A}_k^1$, one gets a natural log structure on $\operatorname{spec} k$. One can check that it is given by the morphism $k^* \oplus \mathbb{N} \to k$ defined by :*

$$(f, n) \mapsto f \cdot 0^n = \begin{cases} f, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}$$

Definition 2.9. *Given a log-structure (M, α) on a scheme X, one defines*

$$\overline{M} = \frac{M}{\mathcal{O}_X^*} \quad .$$

This sheaf is important for several reasons. First of all, the support of $\overline{M}$ is the locus where the log structure is not trivial. Moreover, if X is a smooth scheme over a field k, endowed with a normal crossings divisor D, and $U = X \backslash D$, one can equip X with the log structure M_U (Example 2.6). Then for any geometric point $\overline{x}$ of X one has

$$\left(\overline{M}_U\right)_{\overline{x}} \simeq \mathbb{N}^{C(x)}$$

where $C(x)$ is the set of local branches of D at $\overline{x}$. This is precisely the sheaf of coefficients we need to define parabolic sheaves.

2.2.2. Deligne-Faltings logarithmic schemes. In order to be able to give the definition of parabolic sheaves on logarithmic schemes, we have to use another interpretation of logarithmic geometry. To explain the following definition, let us start from a (Kato) log-structure (M, α) on a scheme X. Passing to the quotient by the action of $\mathcal{O}_X^*$ one gets:

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & \mathcal{O}_X \\ \downarrow & & \downarrow \\ \overline{M} & \xrightarrow{L} & \mathcal{D}\mathrm{iv}_X \end{array}$$

If we assume that (M, α) is quasi-integral (that is the action of $\mathcal{O}_X^*$ on M is free), then the diagram is cartesian, hence the data of L enables to reconstruct (M, α).

Definition 2.10. *A Deligne-Faltings log structure (A, L) on a scheme X is the data of a sheaf of monoids A on X_{et} and of a morphism $L : A \to \mathcal{D}\mathrm{iv}_X$ with trivial kernel.*

Here by morphism, we mean a morphism of symmetric monoidal fibered categories over X_{et}. The condition on the kernel (weaker than injectivity for monoids!) is necessary to get back a Kato log structure (and not only a Kato pre-log structure). One shows easily that Deligne-Faltings log structures on X and (Kato) quasi-integral log structures on X are equivalent categories, through the functor defined above.

Example 2.11. *Let X be a scheme and $((\mathcal{L}_1, s_1), \cdots, (\mathcal{L}_n, s_n))$be a family of line bundles endowed with sections. This data is equivalent to the data of a monoidal functor $L_0 : (\mathbb{N}^r)_X \to \mathcal{D}\mathrm{iv}_X$, and if we put $A = (\mathbb{N}^r)_X / \ker(L_0)$, we get a Deligne-Faltings log structure $L : A \to \mathcal{D}\mathrm{iv}_X$.*

Of course, there is nothing specific to $\mathbb{N}^r$, and one could start from any morphism $L_0 : P \to \mathcal{D}\mathrm{iv}(X)$, where P is an abstract monoid. We will say that $P_X \to P_X / \ker(L_0)$ is a chart. The technical definition is as follows.

Definition 2.12. *Let (A, L) be Deligne-Faltings log structure on a scheme X. A chart for (A, L) is a morphism $P \to A(X)$ where P is an abstract monoid and the associated sheaf morphism $P_X \to A$ is a cokernel.*

It is very often useful to fix additional conditions on P, like: P is finitely generated. We will come back to this later on.

It has to be noticed that if (M, α) is a quasi-integral Kato log structure, that admits a chart, the associated Deligne-Faltings log structure also admits a chart, but the converse is not true, as will be clear when we describe the refined stack of roots in §3.3. So the notion of a chart for a Deligne-Faltings log structure is more flexible.

Despite of this, not all Deligne-Faltings log structures, even those of geometric origin, admit global charts. For instance if X is a scheme, D is a normal crossings divisor, and $U = X \backslash D$, then the Deligne-Faltings log structure $\overline{M}_U \to \mathcal{D}\mathrm{iv}_X$ admits a global chart if and only if D is a simple normal crossings divisor (and to get a Kato chart, we need the additional condition that each irreducible component of D is a principal Cartier divisor). But charts do exist étale locally.

In the sequel, we will only deal with Deligne-Faltings log structures that admits charts étale locally, and to avoid technicalities we will mostly focus on the case when a global chart exists. Our aim will be, starting from the data of a Deligne-Faltings log structure $L : A \to \mathcal{D}\mathrm{iv}_X$, with a global chart $P \to A(X)$, to define parabolic sheaves and refined stack of roots, so that the analog of Theorem 2.4 holds.

2.3. Parabolic Sheaves on the Log Point.

2.3.1. Deligne-Faltings log structure of rank 1. Let us warm up with the case $P = \mathbb{N}$. In this situation, the data of the Deligne-Faltings log structure boils down to the data of a monoidal morphism $\mathbb{N} \to \mathcal{D}\mathrm{iv}(X)$, that is, to the data of a line bundle endowed with a section $(\mathcal{L}, s)$.

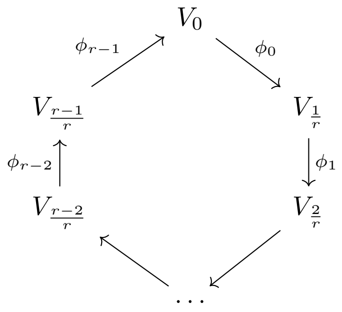
If we fix a positive integer $r \in \mathbb{N}^*$, then both the stack of roots and the parabolic sheaves have been defined in section 2.1, and there is no way (and no need !) to improve their definition further. A first question to address is: in the extreme case when $s = 0$, is Theorem 2.4 still valid ? Of course, it is natural to investigate this question when $X = \operatorname{spec} k$, that is in the case of the log point.

2.3.2. The correspondence for the log point. We start by rephrasing Example 2.6 in terms of Deligne-Faltings log structure. As it is clear from the definition of the Kato log point as a restriction, the Deligne-Faltings log point is the Deligne-Faltings log structure on $\operatorname{spec} k$ defined by the following chart $\mathbb{N} \to \mathcal{D}\mathrm{iv}(\operatorname{spec} k)$:

$$n \mapsto \begin{cases} \mathcal{O}_{\operatorname{spec} k} \xrightarrow{\mathrm{id}} \mathcal{O}_{\operatorname{spec} k}, & \text{if } n = 0 \\ \mathcal{O}_{\operatorname{spec} k} \xrightarrow{0} \mathcal{O}_{\operatorname{spec} k}, & \text{if } n \neq 0 \end{cases}$$

In others words, the Deligne-Faltings log point is defined by the pair $(\mathcal{O}_{\operatorname{spec} k}, 0)$.

If we now apply Definition 2.3 in this situation, we see that a parabolic bundle on the log point will be given by a diagram in the category of finite dimensional vector spaces:



such that the composition of r successive morphisms is 0. We have used the opposite order of the one used in 2.3 for later convenience.

On the other hand, the stack of roots is, according to the remarks following Definition 2.1:

$$\sqrt[r]{(\mathcal{O}_{\operatorname{spec} k}, 0)/\operatorname{spec} k} \simeq \left[\operatorname{spec}\left(\frac{k[x]}{x^r}\right)|\mu_r\right].$$

So a vector bundle on this stack of roots is a μ_r-equivariant vector bundle on $\operatorname{spec}\left(\frac{k[x]}{x^r}\right)$, that is, a free $\frac{k[x]}{x^r}$-module M with a $\mathbb{Z}/r$-graduation, compatible with the natural $\mathbb{Z}/r$-graduation of $\frac{k[x]}{x^r}$.

To each such module M, one can associate a parabolic bundle as above by setting $V_{\frac{l}{r}} = M_l$ and by defining $\phi_l : V_{\frac{l}{r}} \to V_{\frac{l+1}{r}}$ as the multiplication by x. Since one can reverse this procedure, one has shown that Theorem 2.4 generalizes to the log point.

3. The Correspondence Stacky-parabolic for Log Schemes

3.1. Toric Schemes. Let us fix some affine base $S = \operatorname{spec} R$. In §2.1 we started from a smooth S-scheme endowed with a normal crossings divisor. The associated Kato log structure is étale locally a pull-back of the standard log structure on $\mathbb{A}^n$ induced by:

$$\mathbb{N}^n \to \Gamma(\mathcal{O}_{\mathbb{A}^n}) = R[x_1, \cdots, x_n]$$

It we want to go beyond, we have to introduce new models. The simplest thing we can do is to replace $\mathbb{N}^n$ by a general monoid. So we put $S[P] = \mathbf{spec}\, \mathcal{O}_S[P]$ and consider the more general Kato log structure induced by:

$$P \to \Gamma(\mathcal{O}_{S[P]}) = R[P]$$

This is reasonable, of course, only if we fix some conditions on P. It makes sense to assume that P is of finite type as a monoid and integral (i.e. the cancellation law holds, equivalently, the morphism $P \to P^{gp}$ is injective). When both properties hold, we say that P is fine, and this enables to see $S[P]$ as a toric scheme. The other condition that turns out to be very useful is that P is saturated: for all $x \in P^{gp}$ and $n \in \mathbb{N}^*$, we have that $nx \in P \implies x \in P$.

As usual, we define $\widehat{P} = D_S(P^{gp})$ as the diagonalizable group scheme associated to the (finitely generated) abelian group P^{gp}. This group acts naturally on $S[P]$.

Now we will introduce toric morphisms. In §2.1 our model were tamely ramified covers, that by Abhyankar's lemma are locally induced by Kummer covers. This is the justification of the following definition.

Definition 3.1. *A morphism $P \to Q$ of monoids is Kummer if it is injective and for each $q \in Q$, there exists $n \in \mathbb{N}^*$ such that $nq \in P$.*

Under the assumptions that P and Q are fine, and $P \to Q$ is Kummer, one then shows the existence of the generalized Kummer sequence

$$0 \to \mu_{Q/P} \to \widehat{Q} \to \widehat{P} \to 0$$

where $\mu_{Q/P}$ is a finite S-group scheme.

Our generalized Kummer cover will be, of course, $S[Q] \to S[P]$. If one restricts the action of $\widehat{Q}$ on $S[Q]$ to $\mu_{Q/P}$, this morphism becomes $\mu_{Q/P}$-invariant, and it is easy to check, using the fact that Q is saturated and $P \to Q$ Kummer, that $\mathcal{O}_S[P]$ is exactly the ring of invariants of $\mathcal{O}_S[Q]$ under the action of $\mu_{Q/P}$ ([Tal14, Proposition 2.2.7]). In stacky terms, the morphism

$$\left[S[Q]|\mu_{Q/P}\right] \to S[P]$$

identifies $S[P]$ with the coarse moduli space of the stack $\left[S[Q]|\mu_{Q/P}\right]$. This stack will be the local model for the generalized stack of roots, but before defining these, we turn to a concrete example.

3.2. Sample: The Quadratic Cone.

Let us consider the following extension of monoids. We set $Q = \frac{1}{2}\mathbb{N}^2$ and

$$P = \{(\alpha, \beta) \in Q /\ \alpha + \beta \in \mathbb{N}\}$$

Since Q is free of rank 2, $S[Q] = \mathbb{A}^2_S$. We will use the exponential notation for the morphism $Q \to \Gamma(\mathcal{O}_S[Q])$ and thus write $q \mapsto x^q$, and similarly for P. For this specific example, we will write abusively $x = x^{(1/2,0)}$ and $y = x^{(0,1/2)}$ so that $\Gamma(\mathcal{O}_S[Q]) = R[x, y]$.

Since P is generated as a monoid by $(1,0)$, $(0,1)$ and $(1/2,1/2)$, we have that $\Gamma(\mathcal{O}_S[P]) = R[x^2,y^2,xy]$. By denoting $X = x^2$, $Y = y^2$ and $Z = xy$, we see that $R[x^2,y^2,xy] = R[X,Y,Z]/(XY - Z^2)$. So $S[P]$ is a closed subscheme of $\mathbb{A}^3$ that is called the quadratic cone. The point corresponding to $X = Y = Z = 0$ is clearly singular.

The Kummer sequence is also easy to compute: $Q^{gp} = \frac{1}{2}\mathbb{Z}^2$ and $P^{gp} = (\frac{1}{2},\frac{1}{2})\mathbb{Z} \oplus (\frac{1}{2},-\frac{1}{2})\mathbb{Z}$, hence there is an exact sequence

$$0 \to P^{gp} \to Q^{gp} \xrightarrow{+} \frac{\frac{1}{2}\mathbb{Z}}{\mathbb{Z}} \to 0$$

and passing to the associated diagonalizable groups

$$0 \to \mu_2 \xrightarrow{\Delta} \mathbb{G}_m^2 \to \mathbb{G}_m^2 \to 0$$

where Δ is the composite of the embedding $\mu_2 \to \mathbb{G}_m$ with the diagonal $\mathbb{G}_m \to \mathbb{G}_m^2$.

Since P, Q are fine and saturated, and $P \to Q$ is Kummer, $\mathcal{O}_S[P]$ is the ring of invariants of $\mathcal{O}_S[Q]$ under the action of $\mu_{Q/P} = \mu_2$. In other words $p : S[Q] \to S[P]$ identifies with the quotient morphism $\mathbb{A}^2 \to \mathbb{A}^2/\mu_2$ where μ_2 acts on $\mathbb{A}^2$ by $\zeta \cdot (x,y) = (\zeta x, \zeta y)$.

But $p : S[Q] \to S[P]$ is also a standard example of finite morphism which is not flat. The simplest way to see this is that smoothness descends fppf locally, so p cannot be flat, which can also be checked directly by inspecting the lengths of the fibers. So if we set $R_0 = R[x^2,y^2,xy]$, then $R[x,y]$ is not a flat R_0-module.

Similarly, the morphism $\pi : \left[S[Q]|\mu_{Q/P}\right] \to S[P]$ is finite but isn't flat (else p itself would be flat since $S[Q] \to \left[S[Q]|\mu_{Q/P}\right]$ is a μ_2-torsor, hence is flat). This fact has serious consequences, indeed the direct image of a locally free sheaf on $\left[S[Q]|\mu_{Q/P}\right]$ via π does not need to be locally free on $S[P]$ any longer (we will see a concrete example in §3.5). Hence, even if we want to analyze vector bundles on $\left[S[Q]|\mu_{Q/P}\right]$ with a correspondence of the type given Theorem 2.4, we cannot use parabolic sheaves defined in terms of vector bundles on X only, but we have to allow arbitrary (quasi-)coherent components for our parabolic sheaves instead.

3.3. Generalized Stack of Roots. As explained §1.4, we need to introduce a refined notion of stack of roots, in other words, we want to be able to define stack of roots for logarithmic schemes. Prototypes of these generalized stack of roots were first introduced by M.Olsson ([MO05]). For simplicity, we will explain their definition only in the case where the data is a Deligne-Faltings log structure on a scheme X endowed with a chart, or, which amounts to the same, a monoidal morphism $L_0 : P \to \mathcal{D}\mathrm{iv}(X)$.

Definition 3.2. *Let X be a scheme, $L_0 : P \to \mathcal{D}\mathrm{iv}(X)$ a monoidal morphism, and $P \to Q$ a Kummer morphism. The stack of roots $\sqrt[Q]{L_0/X}$ is the stack of liftings of L_0 to Q.*

We can give an alternative, more concrete definition by using a reformulation. We claim that the data of a monoidal morphism $L_0 : P \to \mathcal{D}\mathrm{iv}(X)$ is equivalent to the data of a morphism $X \to \left[S[P]|\widehat{P}\right]$. Indeed, L_0 determines a monoidal morphism $P^{gp} \to \mathcal{P}\mathrm{ic}(X)$, that is nothing else than a $\widehat{P}$-torsor T on X. If we write $L_0 : p \mapsto (\mathcal{L}_p, s_p)$, then $\mathcal{O}_T = \oplus_{p \in P^{gp}} \mathcal{L}_p$, and from this description it follows that we get in this way from the sections s_p a monoidal morphism $P_T \to \mathcal{O}_T$, or which is the same a morphism $T \to S[P]$, which is clearly $\widehat{P}$-equivariant. So we have obtained a morphism $X \to \left[S[P]|\widehat{P}\right]$, and we leave it to the reader that it is possible to go the other way round. It follows from this description that the following diagram is cartesian:

$$\begin{array}{ccc} \sqrt[Q]{L_0/X} & \longrightarrow & \left[S[Q]|\widehat{Q}\right] \\ \downarrow & & \downarrow \\ X & \xrightarrow{L_0} & \left[S[P]|\widehat{P}\right] \end{array}$$

If moreover the Deligne-Faltings log-structure admits a Kato chart, we get the following cartesian diagrams:

$$\begin{array}{ccccc} \sqrt[Q]{L_0/X} & \longrightarrow & \left[S[Q]|\mu_{Q/P}\right] & \longrightarrow & \left[S[P]|\widehat{P}\right] \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{L_0} & S[P] & \longrightarrow & \left[S[P]|\widehat{P}\right] \end{array}$$

In particular, the generalized stack of roots is a tame Artin stack in the sense of [AOV08].

3.4. Parabolic Sheaves on Log Schemes.

We now turn to parabolic sheaves. We follow the policy of the previous paragraph and we will work with the following setup: X a scheme, $P \to Q$ a Kummer morphism of monoids, $L_0 : P \to \mathcal{D}\mathrm{iv}(X)$ a monoidal morphism.

We wish to generalize Definition 2.3 to this situation. In this prototype, the Kummer morphism is the inclusion $\mathbb{N} \to \frac{1}{r}\mathbb{N}$, and the index-category of our parabolic sheaves is the category associated to the poset $\frac{1}{r}\mathbb{Z}$ (we in fact used the opposite order for historical reasons, but we will from now on deal only with covariant functors).

Definition 3.3. *Let P be a monoid. We will denote by P^{wt} the monoidal category whose objects are elements of P^{gp}, and such that a morphism $p \to p'$ is given by an element p'' in P such that $p' = p + p''$.*

We are now ready to give the definition of parabolic sheaves on log schemes:

Definition 3.4. *Let X be a scheme, $P \to Q$ a Kummer morphism of monoids, $L_0 : P \to \mathcal{D}\mathrm{iv}(X)$ a monoidal morphism. A parabolic bundle on (X, L_0) with weights in Q is a functor*

$$\mathcal{E}_{\cdot} : Q^{wt} \to \mathrm{Qcoh}(X)$$

endowed with a functorial isomorphism for each $(q,p) \in Q^{gp} \times P^{gp}$:

$$\mathcal{E}_{q+p} \simeq \mathcal{E}_q \otimes_{\mathcal{O}_X} \mathcal{L}_p$$

satisfying the natural compatibility conditions.

In this definition, we use the notation Qcoh for the category of quasi-coherent sheaves, and $L_0(p) = (\mathcal{L}_p, s_p)$.

It turns out that this definition of a parabolic sheaf is the right one for our purposes. We denote by $\mathrm{Par}_{Q/P}(X)$ the category of parabolic sheaves on (X, L_0) with weights in Q. There is a natural functor $\mathrm{Qcoh}\, X \to \mathrm{Par}_{Q/P}(X)$, $\mathcal{E} \mapsto \underline{\mathcal{E}}_{\cdot}$ left adjoint of the functor $\mathcal{E}_{\cdot} \mapsto \mathcal{E}_0$. Under the following equivalence, it corresponds to the pull-back along $\pi : \sqrt[Q]{P/X} \to X$.

Theorem 3.5 ([BV12], Theorem 6.1)**.** *Let X be a scheme, $P \to Q$ a Kummer morphism of fine and saturated monoids, $L_0 : P \to \mathcal{D}\mathrm{iv}(X)$ a monoidal morphism. We denote by $\pi : \sqrt[Q]{P/X} \to X$ the morphism from the corresponding stack of roots to its moduli space, and by $M_0 : Q \to \mathcal{D}\mathrm{iv}(\sqrt[Q]{P/X})$, $q \mapsto (\mathcal{M}_q, t_q)$ the canonical extension of L_0. Then the association:*

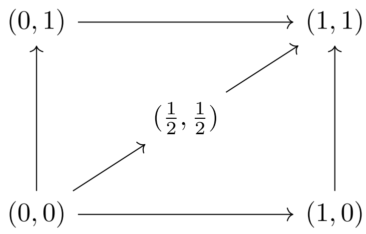
$$\mathcal{F} \mapsto (q \mapsto \pi_*(\mathcal{F} \otimes \mathcal{M}_q))$$

induces an equivalence of categories between $\mathrm{Qcoh}(\sqrt[Q]{P/X})$ *and* $\mathrm{Par}_{Q/P}(X)$.

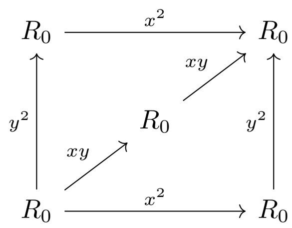
3.5. Parabolic Sheaves on the Quadratic Cone.

We illustrate now the use of Theorem 3.5 thanks to the example of the quadratic cone. We keep the notations of §3.2, in particular for convenience we write R_0 for $R[x^2, y^2, xy]$, seen as a sub-algebra of $R[x,y]$. This is the algebra of the quadratic cone $S[P]$.

We first analyze the Deligne-Faltings log structure induced by $P \to \mathcal{D}\mathrm{iv}(S[P])$ on the quadratic cone. As we have seen, this data is equivalent to the data of a monoidal functor $P^{wt} \to \mathcal{P}\mathrm{ic}(S[P])$. It is easy to check that this functor sends the fundamental domain

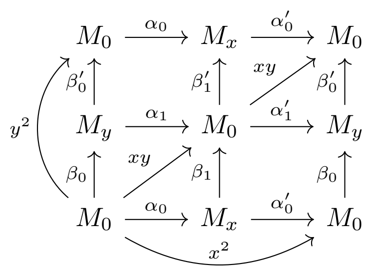


of P^{wt} to the following diagram in R_0-mod (identified with Qcoh($S[P]$)):



All invertible sheaves in this diagram are trivial because we have pulled back the canonical log structure on $[S[P]/P^{gp}]$ on the Kato chart $S[P]$.

The shape of the previous diagram and the pseudo-periodicity isomorphisms show that a general parabolic sheaf with weights in $Q = \mathbb{N}^2$ on the quadratic cone $S[P]$ is uniquely determined by the data of three R_0-modules (M_0, M_x, M_y) and eight morphisms $(\alpha_0, \alpha'_0, \alpha_1, \alpha'_1, \beta_0, \beta'_0, \beta_1, \beta'_1)$ between them so that the following diagram commutes in R_0-mod:



It may seem awkward at first sight, but it is actually pretty easy to produce natural and interesting examples of such parabolic sheaves. Let us start with the simplest example, the parabolic sheaf $\underline{\mathcal{O}_X}$ associated to the structure sheaf $\mathcal{O}_X$. A straightforward computation using Theorem 3.5 shows that this sheaf is given by the following diagram:

$$\begin{array}{ccccc}
R_0 & \xrightarrow{x} & R_1 & \xrightarrow{x} & R_0 \\
{\scriptstyle y}\uparrow & & {\scriptstyle y}\uparrow & & {\scriptstyle y}\uparrow \\
R_1 & \xrightarrow{x} & R_0 & \xrightarrow{x} & R_1 \\
{\scriptstyle y}\uparrow & & {\scriptstyle y}\uparrow & & {\scriptstyle y}\uparrow \\
R_0 & \xrightarrow{x} & R_1 & \xrightarrow{x} & R_0
\end{array}$$

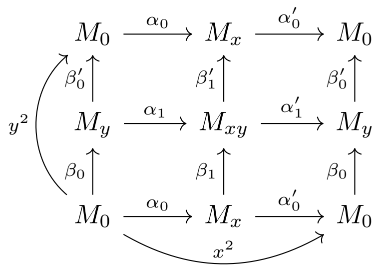
where $R_1 = xR_0 + yR_0$. Since $R[x,y] = R_0 \oplus R_1$, and $R[x,y]$ is not R_0-flat, it follows that R_1 is not R_0-flat. This illustrates the need of considering parabolic sheaves whose components are not locally free, even if we are only interested in locally free parabolic sheaves (that is parabolic sheaves with locally free counterparts on the stack of roots). Of course, for any R_0-module M, one can just apply the functor $\cdot \otimes_{R_0} M$ to the above diagram, and get an associated parabolic sheaf. This gives a concrete description of the functor $\mathrm{Qcoh}(S[P]) \to \mathrm{Par}_{Q/P}(S[P])$.

But there are more interesting examples, of course. The simplest one not of the previous type is obtained by twisting $\underline{\mathcal{O}_X}.$:

$$\begin{array}{ccccc}
R_1 & \xrightarrow{x} & R_0 & \xrightarrow{x} & R_1 \\
{\scriptstyle y}\uparrow & & {\scriptstyle y}\uparrow & & {\scriptstyle y}\uparrow \\
R_0 & \xrightarrow{x} & R_1 & \xrightarrow{x} & R_0 \\
{\scriptstyle y}\uparrow & & {\scriptstyle y}\uparrow & & {\scriptstyle y}\uparrow \\
R_1 & \xrightarrow{x} & R_0 & \xrightarrow{x} & R_1
\end{array}$$

So set $\mathcal{L}. = \underline{\mathcal{O}_X}[(\alpha,\beta)].$ for any (α,β) in $Q^{gp} = \frac{1}{2}\mathbb{Z}^2$ such that $\alpha + \beta \notin \mathbb{Z}$ (the result is clearly independent of the choice of such a couple (α,β)). Although the tensor product of parabolic sheaves is notoriously complicated, it is easy to imagine that it behaves additively with respect to the twist so that $\mathcal{L}.$ is an order 2 invertible parabolic sheaf on $S[P]$. And this is indeed what happens: one checks that $\mathcal{L}.$ corresponds to the order 2 invertible sheaf on $\sqrt[Q]{L_0/S[P]}$ given by the μ_2-torsor $S[Q] \to \sqrt[Q]{L_0/S[P]}$ (that we have already identified with $\mathbb{A}^2 \to [\mathbb{A}^2/\mu_2]$).

It is also interesting to compare parabolic sheaves on the quadratic cone with previously known cases, that is parabolic sheaves relative to a normal crossings divisor. Let us now choose for $P \to Q$ the natural embedding $\frac{1}{2}\mathbb{N}^2 \to \mathbb{N}^2$, so that $S[Q] \to S[P]$ is the standard $\mu_2 \times \mu_2$-cover $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x,y) \mapsto (x^2, y^2)$. Then a similar analysis shows that if $R_0 = R[x^2, y^2]$, a general parabolic sheaf with weights in $Q = \mathbb{N}^2$ on the plane $S[P]$ is determined by the data of four R_0-modules (M_0, M_x, M_y, M_{xy}) and eight morphisms $(\alpha_0, \alpha_0', \alpha_1, \alpha_1', \beta_0, \beta_0', \beta_1, \beta_1')$ between them so that the following diagram commutes in R_0-mod:



So parabolic sheaves are still objects of combinatorial nature, but in their new logarithmic version, they are able to take into account and reflect singularities of the base space.

3.6. Globalization. In this section, our set up was : X a scheme, $P \to Q$ a Kummer morphism of monoids, $L_0 : P \to \mathcal{D}\mathrm{iv}(X)$ a monoidal morphism. But, in general, we will meet Deligne-Faltings log structures that admit a chart only étale locally (see §2.2.2). So if we start from a Deligne-Faltings log structure $A \to \mathcal{D}\mathrm{iv}_X$, we will have to replace the Kummer morphism of monoids $P \to Q$ by a Kummer morphism of sheaves of monoids $A \to B$. There is a stack of roots $\sqrt[B]{L/X}$ obtained as a global version of Definition 3.2 (the idea of considering stacks parametrizing extension of log structures is originally due to M.Olsson, see [MO05]).

One can also extend the definition of parabolic bundles (Definition 3.4) to get a category $\mathrm{Par}_{B/A}(X)$. Of course, the correspondence with $\mathrm{Vec}(\sqrt[B]{L/X})$, that is, the analog of Theorem 3.5, still holds. All the details of these constructions can be found in our original article [BV12].

4. Survey of Recent Work

In this last section, we very briefly report on more recent work.

4.1. The Infinite Root Stack. In [TV14], M.Talpo and A.Vistoli investigate further the relationship between logarithmic geometry and stack of roots.

The interplay between logarithmic geometry and algebraic stacks has been studied in depth by M.Olsson (see [Ols03]). The new idea of Talpo-Vistoli is to introduce an infinite root stack that is, in some sense, simpler than the stacks used by M.Olsson.

The definition is, roughly, as follows. Let us start from a fine and saturated Deligne-Faltings log structure $L : A \to \mathcal{D}\mathrm{iv}_X$ on a scheme X. Then for each $n \in \mathbb{N}^*$, we can consider the natural Kummer morphism $A \to \frac{1}{n}A$, and the corresponding stack of roots $\sqrt[\frac{1}{n}A]{L/X}$. Going to the projective limit, we get the infinite root stack, denoted by $\sqrt[\infty]{L/X}$.

Even if it is not an algebraic stack, the infinite root stack turns out to be a very interesting object. It is locally the quotient of an affine scheme by a diagonalizable group. Moreover, according to the first main result of [TV14], the infinite root stack reflects faithfully the log structure it is associated with, in the sense that it enables to reconstruct (X, L).

The infinite stack of roots is also useful to give a simple definition of a quasi-coherent sheaf on a log scheme. The previous approach was to use the Kummer étale ringed topos. But it turns out to be much more convenient to define quasi-coherent sheaves on a log scheme as usual quasi-coherent sheaves on the corresponding infinite root stack. This is the point of view advocated by Talpo-Vistoli, which paves the way to the study of the K-theory of log schemes (since the K-theory of algebraic stacks is fairly well understood). Moreover, the second main result of [TV14] states that the correspondence between quasi-coherent sheaves on the infinite root stack and parabolic sheaves with arbitrary weights on the log scheme is still valid.

4.2. Moduli of Parabolic Sheaves. Let us end this survey by mentioning M.Talpo's subsequent work (see [Tal14]) on moduli spaces of parabolic sheaves on a fairly general log scheme.

In its simplest expression, when the denominators of the weights are bounded, this study relies on one hand on the above mentioned correspondence with stacky sheaves established in [BV12], and on the other hand on F. Nironi's construction of the moduli space of coherent sheaves on a Deligne-Mumford stack (see [Nir08]). When the base log scheme is projective, fine and saturated, and endowed with a global chart, the first main result of [Tal14] shows the existence of a moduli stack of parabolic sheaves that is an Artin stack (of finite type if one fixes the Hilbert polynomial).

The second main result of [Tal14] deals with the much more delicate situation when the denominators of the weights are allowed to grow arbitrarily. It is then possible, partly thanks to [TV14], to proceed to the analysis of the corresponding tower of moduli spaces.

References

[AGV08] Dan Abramovich, Tom Graber, and Angelo Vistoli. "Gromov-Witten theory of Deligne-Mumford stacks". In: *Amer. J. Math.* 130.5 (2008), pp. 1337-1398. ISSN: 0002-9327. URL: `http://dx.doi.org/10.1353/ajm.0.0017`.

[AOV08] Dan Abramovich, Martin Olsson, and Angelo Vistoli. "Tame stacks in positive characteristic". In: *Ann. Inst. Fourier (Grenoble)* 58.4

(2008), pp. 1057–1091. ISSN: 0373-0956. URL: http://aif.cedram.org/item?id=AIF_2008_58_4_1057_0.

[BV12] Niels Borne and Angelo Vistoli. "Parabolic sheaves on logarithmic schemes". In: *Adv. Math.* 231.3-4 (2012), pp. 1327–1363. ISSN: 00018708. URL: http://dx.doi.org/10.1016/j.aim.2012.06.015.

[Bis97] Indranil Biswas. "Parabolic bundles as orbifold bundles". In: *Duke Math. J.* 88.2 (1997), pp. 305–325. ISSN: 0012-7094. URL: http://dx.doi.org/10.1215/S0012-7094-97-08812-8.

[Bor07] Niels Borne. "Fibrés paraboliques et champ des racines". In: *Int. Math. Res. Not. IMRN* 16 (2007), Art. ID rnm049, 38. ISSN: 10737928. URL: http://dx.doi.org/10.1093/imrn/rnm049.

[Bor09] Niels Borne. "Sur les représentations du groupe fondamental d'une variété privée d'un diviseur à croisements normaux simples". In: *Indiana Univ. Math. J.* 58.1 (2009), pp. 137–180. ISSN: 0022-2518. URL: http://dx.doi.org/10.1512/iumj.2009.58.3734.

[Cad07] Charles Cadman. "Using stacks to impose tangency conditions on curves". In: *Amer. J. Math.* 129.2 (2007), pp. 405–427. ISSN: 00029327. URL: http://dx.doi.org/10.1353/ajm.2007.0007.

[Kat89] Kazuya Kato. "Logarithmic structures of Fontaine-Illusie". In: *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988).* Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191–224.

[MO05] Kenji Matsuki and Martin Olsson. "Kawamata-Viehweg vanishing as Kodaira vanishing for stacks". In: *Math. Res. Lett.* 12.2-3 (2005), pp. 207–217. ISSN: 1073-2780. URL: http://dx.doi.org/10.4310/MRL.2005.v12.n2.a6.

[MS80] V. B. Mehta and C. S. Seshadri. "Moduli of vector bundles on curves with parabolic structures". In: *Math. Ann.* 248.3 (1980), pp. 205–239. ISSN: 0025-5831. URL: http://dx.doi.org/10.1007/BF01420526.

[Nir08] Fabio Nironi. *Moduli Spaces of Semistable Sheaves on Projective Deligne-Mumford Stacks.* 2008. arXiv: arXiv:0811.1949 [math-AG].

[Ols03] Martin C. Olsson. "Logarithmic geometry and algebraic stacks". In: *Ann. Sci. École Norm. Sup. (4)* 36.5 (2003), pp. 747–791. ISSN: 00129593. URL: http://dx.doi.org/10.1016/j.ansens.2002.11.001.

[TV14] Mattia Talpo and Angelo Vistoli. *Infinite root stacks and quasi-coherent sheaves on logarithmic schemes.* 2014. arXiv: arXiv:1410.1164 [math-AG].

[Tal14] Mattia Talpo. *Moduli of parabolic sheaves on a polarized logarithmic scheme.* 2014. arXiv: arXiv:1410.2212 [math-AG].