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Cohomology of G-sheaves in positive characteristic

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Abstract

Let G be a finite group, and X a noetherian G-scheme defined on an algebraically closed field k, whose characteristic divides the order of G. We define a refinement of the equivariant K-theory of X devoted to give a better account of the information related to modular representation theory. The construction relies in an essential way on the work of M. Auslander in modular representation theory and the use of sheaves of "rings with several objects". The main applications of this "modular K-theory" are in dimension one, where we show how it allows to extend the work of S. Nakajima.

Keywords: Equivariant K-theory; Modular representation theory; Galois modules

Contents

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1. Introduction

1.1. Galois modules in positive characteristic

This work is an attempt at better understanding Galois modules in positive characteristic. We define them here as spaces of global sections of coherent G-sheaves on a proper scheme X over an algebraically closed field k of positive characteristic, endowed with the action of a finite group G. So Galois modules are representations of G, and they are known to be related to the ramification of the action (i.e. to the fixed points).

When the characteristic p of k does not divide the order of G, i.e. in the reductive case, Galois modules are quite well understood, and the situation is similar to the one in characteristic zero. But when p does divide the order of G, the situation is much more mysterious.

1.2. The role of equivariant K-theory

Indeed, the general approach to the description of Galois modules is the use of equivariant K-theory, which provides a Euler–Poincaré characteristic $\chi(G, \cdot) : K_0(G, X) \to R_k(G)$ with values in the Brauer characters group of G. One can use an equivariant

Lefschetz formula to compute it explicitly. This is satisfactory in the reductive case, since then the Brauer character of a representation characterizes its isomorphism class. But this last fact is false as soon as p divides the order of G, because then the Galois modules are modular representations, so their Brauer characters only describe their Jordan–Hölder series, and not their decomposition in indecomposables (or equivalently their isomorphism classes). This is particularly dramatic when G is a p-group, because then $R_k(G) \simeq \mathbb{Z}$ contains no equivariant information at all.

The usual way to improve the situation is to make assumptions on the ramification of the action. For example, on a smooth projective curve, one assumes that the action is tamely ramified, one can show that a coherent G-sheaf of large degree has a space of global sections which is a projective $k[G]$-module, so that its Brauer character is enough to describe its isomorphism class. This is part of the Noether-like criterion given by Nakajima [22], and this point of view was developed later on (see [4]).

For a wild action, equivariant K-theory is much less effective. In particular, it is unable to explain the modular Riemann–Roch formula given by Nakajima [22], which describes the structure of the space of global sections $H^0(X, \mathcal{L})$ of an invertible sheaf of large degree on a smooth projective curve with an action of $G = \mathbb{Z}/p$.

A first look at the properties a refined version of equivariant K-theory should satisfy suggests that important changes are needed. Indeed, the Lefschetz formula shows that if the action of G is free, the equivariant Euler–Poincaré characteristic is a multiple of $[k[G]]$ in $R_k(G)$. However, another work of Nakajima [21] shows that such a "symmetry principle" fails to hold in a refined sense if we use the standard Zariski cohomology of sheaves to define the refined Euler–Poincaré characteristic.

1.3. Modular K-theory: main properties

In this work, we present a refinement of the equivariant K-theory of a noetherian G-scheme X over k. To do so, we introduce, for each full subcategory $\mathcal{A}$ of the category $k[G]\,\mathbf{mod}$ of $k[G]$-modules of finite type, the notion of $\mathcal{A}$-sheaf on X. This is, roughly, a sheaf of modules over the Auslander algebra $\mathcal{A}_X$, itself defined as a certain sheaf of algebras with several objects over the quotient scheme $Y = X/G$, obtained by mimicking the functorial definition of the Auslander algebra of $k[G]$ (see [1]).

We embed the category Qcoh (G, X) of quasicoherent G-sheaves on X as a reflective subcategory of the category Qcoh $(\mathcal{A}, X)$ of quasicoherent $\mathcal{A}$-sheaves on X (see §5.2.2), so that each G-sheaf $\mathcal{F}$ can be seen as a $\mathcal{A}$-sheaf $\underline{\mathcal{F}}$. The category of coherent $\mathcal{A}$-sheaves on X is abelian, and its K-theory, in the sense of Quillen, is denoted by $K_i(\mathcal{A}, X)$. This construction is functorial in both variables.

Of course, the main test of validity for this new definition is the case $X = \operatorname{spec} k$. Modular K-theory is satisfying for groups with cyclic p-Sylows, where p is the characteristic of k, in the sense that for each $k[G]$-module of finite type V, its class $[\underline{V}]$ in $K_0(\mathcal{A}, \operatorname{spec} k)$ characterizes its isomorphism class (see Theorem 2.4). This seems a nice analog from the fact that the Brauer class of a *projective* $k[G]$-module determines it up to isomorphism: indeed $K_0(\mathcal{A}, \operatorname{Spec} k) = K_0\,(\mathbf{mod}\ \mathcal{A})$, where $\mathbf{mod}\ \mathcal{A} = [\mathcal{A}^{op}, k\,\mathbf{mod}\,]$ is the category of right modules (i.e. contravariant k-linear functors in k-vectors spaces) on the ring with several objects $\mathcal{A}$, and for *any* $k[G]$-module V, the representable functor $\underline{V} = \mathcal{A}(\cdot, V)$ is projective in **mod** $\mathcal{A}$. For an arbitrary finite group G, one grasps certainly more information than the Brauer character, but until now it is not clear to the author if the definition enables to get back the isomorphism class.

Using the functoriality in $\mathcal{A}$, we can compare both K-theories. Indeed if $\mathcal{A}$ and $\mathcal{A}'$, seen as rings with several objects, are Morita equivalent, then their modules and their K-theory are the same (see Proposition 6.2). In particular, since the category $\mathcal{P}$ of projective $k[G]$-modules of finite type is Morita equivalent to the category with only the free object $k[G]$, the groups $K_i(\mathcal{P}, X)$ coincide with the equivariant K-theory. So in the reductive case, since $\mathcal{P} = \mathcal{A}$, we get nothing else than equivariant K-theory. Hence something new happens only when we consider a $\mathcal{A}$ satisfying $\mathcal{P} \subsetneq \mathcal{A}$, and in this case we have a surjective homomorphism $K_0(\mathcal{A}, X) \twoheadrightarrow K_0(G, X)$.

Since $K_i(\mathcal{A}, X)$ has the usual functorial properties in X (or rather in the quotient $Y = X/G$), it makes sense to ask whether there is a localization long exact sequence for modular K-theory. We answer positively (see Theorem 6.7), but only on a certain surjectivity assumption, which is always fulfilled for equivariant K-theory, but not for modular K-theory.

The computation of modular Euler–Poincaré characteristics reduces to the computation of standard Euler–Poincaré characteristics on the quotient scheme, thanks to the formula:

$$\chi(\mathcal{A}, \underline{\mathcal{F}}) = \sum_{I \in S} \chi(\underline{\mathcal{F}}(I))[S_I] \tag{1}$$

(see Lemma 6.16). Let us explain our notations: the sum is taken over a system S of representatives of the isomorphism classes of indecomposable objects in $\mathcal{A}$. For each such object I, S_I denotes the simple right $\mathcal{A}$-module obtained by taking the quotient of the representable functor $\underline{I} = \mathcal{A}(\cdot, I)$ by its radical. $\chi(\underline{\mathcal{F}}(I))$ denotes the ordinary Euler–Poincaré characteristic of the sheaf $\pi_*^G(I^\vee \otimes_k \mathcal{F})$ on the quotient Y, where $\pi : X \to Y = X/G$ is the canonical map. To end with, the modular Euler–Poincaré characteristic here is computed with the cohomology of $\underline{\mathcal{F}}$, which is very different from the cohomology of $\mathcal{F}$ (in symbols, with the right $\mathcal{A}$-modules $H^i(X, \underline{\mathcal{F}})$, in general different from $\underline{H^i(X, \mathcal{F})}$ for $i > 0$). We give comparison results in Section 5.2.10.

1.4. Applications

Among the applications, we show that the symmetry principle holds in modular K-theory. Indeed, if the action of G on X is free, then for any G-sheaf $\mathcal{F}$ we have an equality in $K_0(\mathcal{A}, \operatorname{spec} k)$:

$$\chi(\mathcal{A}, \underline{\mathcal{F}}) = \chi(\pi_*^G \mathcal{F}) \left[\underline{k[G]}\right]$$

(see Proposition 7.1). This means that the various indecomposable $k[G]$-modules actually occur in the cohomology of $\underline{\mathcal{F}}$. A consequence of this formula is that for an acyclic $\mathcal{F}$, the space of global sections $H^0(X, \mathcal{F})$ is a free $k[G]$-module; unfortunately, this is rather easily shown by standard means. So our formula just gives the right correction terms for nonacyclic $\mathcal{F}$. We give in §7.1.1 a concrete description of the correction term $H^1(X, \underline{\Omega_X})$ in the original example of Nakajima (the sheaf of differentials on a curve).

The other applications all concern the one-dimensional case, i.e. the case when X is a projective curve over k. The reason to stick to this case is just our lack of understanding of the equivariant situation in dimension greater than one, although there is a satisfying Riemann–Roch theorem for Deligne–Mumford stacks (see [26]).

Thanks to the localization sequence, we give a description of the additive structure of $K_0(\mathcal{A}, X)$ (see Theorem 7.5) when the group acts with normal stabilizers. This is done by introducing a group of class of cycles with coefficients in the modular representations $A_0(\mathcal{A}, X)$, and enables to define a first Chern class such that the usual Riemann–Roch formula holds.

As a second concrete application we get the following (see Theorem 7.12):

Theorem 1.1. *Let X be a projective curve over an algebraically closed field k of positive characteristic p, endowed with the faithful action of finite group G. Suppose that G has cyclic p-Sylows and acts with normal stabilizers. Let $\mathcal{E}$, $\mathcal{E}'$ be two locally free G-sheaves on X, of same rank, and such that* $H^1(X, \mathcal{E}) = 0$, $H^1(X, \mathcal{E}') = 0$. *Then*:

(i) *if the ramification locus X_{ram} is not empty, there exists for each $P \in X_{\mathrm{ram}}$, a couple (V_P, V'_P) of representations of the stabilizer G_P, such that*

$H^0(X, \mathcal{E}) \oplus \bigoplus_{P \in X_{\mathrm{ram}}} \mathrm{Ind}_{G_P}^{G} V_P \simeq H^0(X, \mathcal{E}') \oplus \bigoplus_{P \in X_{\mathrm{ram}}} \mathrm{Ind}_{G_P}^{G} V'_P$.

(ii) *Let moreover $\phi : \mathcal{E} \to \mathcal{E}'$ be a morphism of G-sheaves that is an isomorphism outside the strict closed G-subset X' of X. Then the previous statement holds if one replaces X_{ram} by X'.*

This seems interesting from a practical point of view: the problem in obtaining explicit formulas lies in the fact that we have in general no explicit description of the modular representation theory (i.e. of the indecomposables) of a given group. As far as modular degree is concerned, we need only to understand the modular representation theory of the stabilizers, and it seems very likely to the author that this is already known (or at least should be easily understood) because with our assumptions there are simply semi-direct products of a cyclic p-group by a cyclic p'-group. To sum up, it would be nice to obtain a formula for the modular degree (of an invertible G-sheaf, say) involving at each ramification point P the various $k[G_P]$-indecomposables, with coefficients depending essentially on the ramification data at P (i.e. on the higher ramification groups at P).

However, we perform an actual computation only in the case of a cyclic group of order n. More precisely, we extend the result of Nakajima [22] to the case of an action of an arbitrary cyclic group, giving a recursive algorithm to compute explicitly the structure of modular representation of the space $H^0(X, \mathcal{L})$ of global sections of an invertible sheaf of large degree on the curve (see Section 7.2.4). The representation theory of G is easily described: write $n = p^v a$, where a is prime to p. For each integer $0 \leqslant j \leqslant p^v$ define $V_j = k[G]/(\sigma - 1)^j$. Then the set of modules $\{\psi \otimes V_j \ / \ \psi \in \widehat{G},\ 1 \leqslant j \leqslant p^v\}$ is a skeleton of the indecomposables in $\mathcal{A}$. The modular Euler–Poincaré characteristics in $K_0(\mathcal{A}, \operatorname{Spec} k)$ takes the explicit form

$$\chi(\mathcal{A}, \underline{\mathcal{L}}) = \sum_{\psi \in \widehat{G}} \sum_{j=1}^{p^v} \sum_{i=1}^{j} \chi(\operatorname{gr}_0 \underline{\mathcal{L}}(\psi \otimes V_i))[S_{\psi \otimes V_j}]$$

(see Lemma 7.13). As before, for an indecomposable I, S_I denotes the largest semi-simple quotient of the representable functor $\underline{I} = \mathcal{A}(\cdot, I)$. By definition, the sheaves $\operatorname{gr}_0 \underline{\mathcal{L}}(\psi \otimes V_j)$ are invertible sheaves on the quotient Y, given by $\operatorname{gr}_0 \underline{\mathcal{L}} \otimes (\psi \otimes V_j) = \operatorname{coker} (\pi_*^G((\psi \otimes V_{j-1})^\vee \otimes_k \mathcal{L}) \to \pi_*^G((\psi \otimes V_j)^\vee \otimes_k \mathcal{L}))$, where $\pi : X \to Y = X/G$ is the natural map. One can represent these sheaves by divisors thanks to the key Proposition 7.15 which describes their behaviour under extension:

$$\operatorname{gr}_0^G \underline{\mathcal{L}}(V_j) \simeq \operatorname{gr}_0^P \underline{\operatorname{gr}_0^H \underline{\mathcal{L}}(V_l)}(V_{j'})$$

and thus allows to reduce to the case of a cyclic group of order p, and then use Nakajima's original formula.

To illustrate the use of our algorithm, we give an explicit expression of the structure of $H^0(X, \mathcal{L})$ when G is a cyclic p-group: see Theorem 7.23. The expression of the coefficient of a given indecomposable in this decomposition involves the use of all the ramification jumps of the corresponding cover.

From this rather brutal computation we can deduce a Noether-like criterion, showing in a more qualitative way how ramification and Galois modules are linked (see Theorem 7.26). Remember that a $k[G]$-module is said *relatively H-projective* if it is a direct summand of a module induced from H.

Theorem 1.2. *Let $\pi : X \to Y$ be a (generically) cyclic Galois p-cover of projective curves over k of group G,* $\operatorname{ram} \pi$ *the largest ramification subgroup of π, and H a subgroup of G. Then the following assertions are equivalent*:

(i) $\operatorname{ram} \pi \subset H$.
(ii) $\forall \mathcal{L} \in \operatorname{Pic}_G X \quad \deg \mathcal{L} > 2g_X - 2 \Longrightarrow H^0(X, \mathcal{L})$ *is relatively H-projective.*
(iii) $\exists \mathcal{M} \in \operatorname{Pic} Y$ *so that* $\#G \deg \mathcal{M} > 2g_X - 2$ *and* $H^0(X, \pi^*\mathcal{M})$ *is relatively H-projective.*

Even if it is likely that a direct cohomological proof exists, the author was unable to find one.

1.5. Modular K-theory: construction

We give here some indications about the organization of this article.

Section 2 is devoted to the analysis of the zero-dimensional case. It relies in an essential way on the work of Auslander, who first realized the interest of rings with several objects for modular representation theory. The main idea is that for $\mathcal{A} = k[G]\,\mathbf{mod}$, the isomorphism classes of indecomposable objects of $\mathcal{A}$ are in one-to-one correspondence, via the Yoneda embedding, with isomorphism classes of simple objects of $\mathbf{mod}\ \mathcal{A}$, and so are well detected by K-theory there. Only the reinterpretation in terms of Grothendieck groups we give seems not to have been used before, even if it is not surprising in itself. The strongest result, obtained for groups with cyclic p-Sylows, is Theorem 2.4.

In Section 3, we sum up briefly the tools of enriched category theory needed in the sequel. We give a proof only of the facts that we could not find in the literature.

Section 4 is a preliminary to the next section on $\mathcal{A}$-sheaves. We define a ringed scheme as a scheme Y, endowed with a category enriched in the closed category $\mathbf{Qcoh}\,Y$, which is a way to express the notion of "scheme with a sheaf of algebras with several objects". The Auslander algebra $\mathcal{A}_X$ of a G-scheme X will be an example. However, it seems convenient to deal with some problems at this level of generality, especially the problem of showing the existence of an adjunction between pull-back and push-forward for sheaves of modules over ringed schemes (see Proposition 4.11).

In the next step (Section 5), we define the Auslander algebra $\mathcal{A}_X$ of a G-scheme X, which is a category enriched on $\mathbf{Qcoh}\,Y$, where $Y = X/G$, and introduce some variants. Then $\mathcal{A}$-sheaves on X are defined as sheaves for the ringed scheme $(Y, \mathcal{A}_X)$, and are studied in the rest of the section. Here, the flexibility we have in choosing the subcategory $\mathcal{A}$ of $k[G]\,\mathbf{mod}$ proves useful, since additional structures on $\mathcal{A}$ give rise to additional structures on $\mathbf{Qcoh}\,(\mathcal{A}, X)$. For instance if $\mathcal{A}$ is monoidal, then $\mathbf{Qcoh}\,(\mathcal{A}, X)$ is closed (see §5.2.9), as Day [5] noted in a more general context. Another example can be found in 5.2.11, where it is shown that if $\mathcal{A}$ is stable by radical and duality, and both operations commute, then every monomorphism preserving $\mathcal{A}$-sheaf (and in particular those coming from G-sheaves) has a canonical filtration, a helpful fact to compute modular Euler–Poincaré characteristics.

Section 6 is devoted to the definition of modular K-theory, along classical lines. After showing the localization theorem (see Theorem 6.7), we analyse the case of a free action in §6.7, and show that, as in the equivariant case, the modular K-theory coincides with the usual K-theory of the quotient Y. We end the section by proving the Lefschetz-like formula 1 (see Lemma 6.16).

Finally, the last section, Section 7, deals with applications; its content as already been discussed in §1.4.

2. Modular representation theory following Auslander

2.1. Modules over a ring with several objects

The idea that a small additive category behaves like a (not necessarily commutative) ring is due to Mitchell (see [19]). We present quickly the basic notions of the theory, following the exposition of Auslander in [1] (see also [25]). We will most of the time omit the word "small", although it is logically necessary in the definition of a ring with several objects, to avoid the discussion on universes needed to make the definitions coherent.

2.1.1. Definition
As usual **Ab** denotes the category of abelian groups. All categories, functors considered in Section 2 are additive,[1] i.e. enriched over **Ab**.

[1] We do not use the terminology "preadditive category". So an additive category does not need to have finite coproducts.

For a category $\mathcal{A}$, we will denote by $\mathbf{Mod}\,\mathcal{A}$ the category $[\mathcal{A}^{op}, \mathbf{Ab}]$ of contravariant functors from $\mathcal{A}$ to $\mathbf{Ab}$, and natural transformations between them. A (*right*) $\mathcal{A}$-*module* is by definition an object in $\mathbf{Mod}\,\mathcal{A}$. $\mathcal{A}$-modules obviously form an abelian category.

The usual Yoneda embedding $\mathcal{A} \to \mathbf{Mod}\,\mathcal{A}$ sends the object V to the contravariant representable functor $\underline{V} = \mathcal{A}(\cdot, V)$. A $\mathcal{A}$-module is said to be *of finite type* if it is a quotient of a finite direct sum of representable functors. We will call $\mathbf{mod}\,\mathcal{A}$ the full subcategory of $\mathbf{Mod}\,\mathcal{A}$ consisting of objects of finite type.

The *projective completion* of $\mathcal{A}$, denoted by $Q\mathcal{A}$, is the full subcategory of $\mathbf{Mod}\,\mathcal{A}$ whose objects are the projective $\mathcal{A}$-modules of finite type (equivalently, direct summands of finite direct sums of representable functors). Since representable functors are projective, the Yoneda embedding factorizes trough $Q\mathcal{A}$. The resulting embedding $\mathcal{A} \to Q\mathcal{A}$ is an equivalence if and only if $\mathcal{A}$ is a category with finite coproducts where idempotents split (i.e. for all $e : V \to V$ idempotent in $\mathcal{A}$, e has a kernel in $\mathcal{A}$).

2.1.2. Morita equivalence

By definition, two categories $\mathcal{A}$, $\mathcal{A}'$ are *Morita equivalent* if the corresponding categories of modules $\mathbf{Mod}\,\mathcal{A}$, $\mathbf{Mod}\,\mathcal{A}'$ are equivalent. This is known to be the case if and only if the projective completions $Q\mathcal{A}$ and $Q\mathcal{A}'$ are equivalent (see [1, Proposition 2.6]). In particular, $\mathcal{A}$ and $Q\mathcal{A}$ are Morita equivalent.

2.1.3. Change of ring

Proposition 2.1. *Let $\mathcal{A}$ be a category and $\mathcal{A}'$ be a full subcategory of $\mathcal{A}$. The restriction functor $R : \mathbf{Mod}\,\mathcal{A} \to \mathbf{Mod}\,\mathcal{A}'$ admits as a right adjoint the functor K defined on objects by*:

$$\begin{aligned} K : \mathbf{Mod}\,\mathcal{A}' &\longrightarrow \mathbf{Mod}\,\mathcal{A} \\ F &\longrightarrow (V \longrightarrow \mathbf{Mod}\,\mathcal{A}'(\mathcal{A}(\cdot, V)_{|\mathcal{A}'}, F)) \end{aligned}$$

Moreover the counit of this adjunction is an isomorphism: $RK \simeq 1$ (*equivalently*, K *is fully faithful*).

In particular, *if $\mathcal{A}$ is projectively complete*, *and every object of $\mathcal{A}$ is a direct summand of a finite direct sum of objects of $\mathcal{A}'$*, *this adjunction is an equivalence* $\mathbf{Mod}\,\mathcal{A} \simeq \mathbf{Mod}\,\mathcal{A}'$.

Proof. See [1, Propositions 3.4 and 2.3]. □

This proposition is an example of *enriched Kan extension*, a notion that we will use later in a larger context: see §3. We will be particularly interested in the following situation.

Definition 2.2. When $\mathcal{A}$ is Morita equivalent to (the full subcategory generated by) a finite set of its objects, we will say that $\mathcal{A}$ *admits a finite set of additive generators*.

2.2. The finite representation type case

In this paragraph, we give an interpretation in terms of Grothendieck groups of the classical link between rings with several objects and modular representation theory (i.e. the study of representations of a finite group G over an algebraically closed field k of characteristic dividing the order of G).

2.2.1. The ring with several objects $\mathbf{mod}\ \mathcal{A}_{\mathrm{tot}}$

More generally fix an algebraically closed field k, R a (nonnecessarily commutative) finite-dimensional k-algebra. Our main interest lies in the case $R = k[G]^{op}$, the opposite of the group algebra of a finite group G, when the characteristic p of k divides the order of the cardinal of G.

The idea is that to study the category $\mathcal{A}_{\mathrm{tot}} = \mathbf{mod}\ R$, the category of right R-modules of finite type, it is useful to consider it as a ring with several objects, and thus consider the associated module category $\mathbf{Mod}\,\mathcal{A}_{\mathrm{tot}}$.

Definition 2.3. The algebra R is said to be *of finite representation type* if the number of isomorphism classes of indecomposable objects of $\mathcal{A}_{\mathrm{tot}}$ is finite.

So it means precisely that $\mathcal{A}_{\mathrm{tot}}$ has an additive generator. We try to keep the letter $\mathcal{A}$ for an arbitrary subcategory of $\mathbf{mod}\ R$ (often supposed to possess a finite set of generators), and sometimes for an arbitrary additive category.

2.2.2. Grothendieck groups of categories of modules

For an arbitrary exact category $\mathcal{A}$ (see [23]), we will use the traditional notation $K_0(\mathcal{A})$ to denote its *Grothendieck group*: explicitly, it is the quotient of the free abelian group generated by the isomorphism classes $[V]$ of objects V of $\mathcal{A}$ by the subgroup generated by the expressions $[V] = [V'] + [V'']$ associated to exact sequences $0 \to V' \to V \to V'' \to 0$ in $\mathcal{A}$. Any abelian category $\mathcal{A}$ will be endowed with its canonical exact structure. Moreover, any additive category $\mathcal{A}$ has also a canonical exact structure consisting of the only *split* exact sequences, and we will always use the notation $\mathcal{A}^{\mathrm{split}}$ for it.

Since we wish to introduce finer invariants of the R-modules than the usual Brauer character (i.e. for a R-module V, simply its class $[V]$ in $K_0(\mathcal{A}_{\mathrm{tot}})$), we have to introduce larger groups. The immediate idea is to suppress relations in the definition of $K_0(\mathcal{A}_{\mathrm{tot}})$ and thus consider the group $K_0(\mathcal{A}_{\mathrm{tot}}^{\mathrm{split}})$. We have an obvious epimorphism $K_0(\mathcal{A}_{\mathrm{tot}}^{\mathrm{split}}) \twoheadrightarrow K_0(\mathcal{A}_{\mathrm{tot}})$, and moreover the Krull–Schmidt theorem tells us that the class $[V]$ of an object V of $\mathcal{A}_{\mathrm{tot}}$ in $K_0(\mathcal{A}_{\mathrm{tot}}^{\mathrm{split}})$ determines its isomorphism class (see Lemma 2.5).

However, if the group $K_0(\mathcal{A}_{\mathrm{tot}}^{\mathrm{split}})$ we have just built is certainly the right one, the way we have built it is wrong. Indeed, if we try to export this construction in higher dimension, by considering instead of $\mathcal{A}_{\mathrm{tot}}$ the category $\mathrm{Coh}\,(G, X)$ of coherent G-sheaves on a noetherian k-scheme X, the group $K_0(\mathrm{Coh}\,(G, X)^{\mathrm{split}})$ is much too large:

in the reductive case, i.e. when $p = \operatorname{char} k \nmid \# G$ (and in particular when $G = 1$), we do not recover the usual equivariant K-theory.

Since the interesting exact structures in the K-theory of schemes comes from abelian categories, we have to reinterpret the group $K_0(\mathcal{A}_{\text{tot}}^{\text{split}})$ in terms of the Grothendieck group of an abelian category.

For this purpose, if the ring $\mathcal{A}_{\text{tot}}$ is right noetherian (see Definition 2.7), the category $\mathbf{mod}\, \mathcal{A}_{\text{tot}}$ is a candidate, since the evaluation $F \to F(R)$ provides an exact functor $\mathbf{mod}\, \mathcal{A}_{\text{tot}} \to \mathcal{A}_{\text{tot}}$ which in turn induces an epimorphism $K_0\,(\mathbf{mod}\, \mathcal{A}_{\text{tot}}) \twoheadrightarrow K_0(\mathcal{A}_{\text{tot}})$ (this is a consequence of Proposition 2.1 applied with $\mathcal{A}'$ the category with only the object R: restriction to $\mathcal{A}'$ is evaluation $F \to F(R)$ at R, the right adjoint is the Yoneda embedding $V \to \underline{V} = \operatorname{Hom}_R(\cdot, V)$, and the natural isomorphism $\underline{V}(R) \simeq V$ proves the surjectivity). More precisely we have a commutative diagram:

$$\begin{array}{ccc} K_0(\mathcal{A}_{\text{tot}}^{\text{split}}) & & \\ & \searrow\!\!\!\!\searrow & \\ \downarrow & & K_0(\mathcal{A}_{\text{tot}}) \\ & \nearrow\!\!\!\!\nearrow & \\ K_0\,(\mathbf{mod}\, \mathcal{A}_{\text{tot}}) & & \end{array}$$

where the two diagonal arrows are the ones already described and the vertical one is induced by the Yoneda embedding.

That $K_0\,(\mathbf{mod}\, \mathcal{A}_{\text{tot}})$ contains pertinent information is shown by the following result, which is essentially a reinterpretation of a theorem of Auslander and Reiten:

Theorem 2.4. *Suppose given a (nonnecessarily commutative) algebra R over an algebraically closed field k, with R finite dimensional over k and of finite representation type, and let $\mathcal{A}_{\text{tot}} = \mathbf{mod}\, R$. Then:*

(i) *the ring $\mathcal{A}_{\text{tot}}$ is right noetherian,*
(ii) *the Yoneda embedding $\mathcal{A}_{\text{tot}} \to \mathbf{mod}\, \mathcal{A}_{\text{tot}}$ induces an isomorphism*

$$K_0(\mathcal{A}_{\text{tot}}^{\text{split}}) \simeq K_0\,(\mathbf{mod}\, \mathcal{A}_{\text{tot}}),$$

(iii) *two R-modules of finite type V, V' are isomorphic if and only if in $K_0\,(\mathbf{mod}\, \mathcal{A}_{\text{tot}})$:*

$$[\mathcal{A}_{\text{tot}}(\cdot, V)] = [\mathcal{A}_{\text{tot}}(\cdot, V')]$$

The rest of this paragraph is devoted to a proof of the theorem, which is split in the next four sections. Some of the results are stronger than strictly needed, because we intend to apply them in the more general context where the algebra R is not of finite representation type, and $\mathcal{A} \subset \mathbf{mod}\, R$ has a finite set of generators.

2.2.3. $K_0(\mathcal{A}_{\text{tot}}^{\text{split}})$ is abelian free of finite rank

Lemma 2.5. *Let $\mathcal{A}$ be a full subcategory of $\mathcal{A}_{\text{tot}}$, and V, V' be two objects of $\mathcal{A}$. Then $[V] = [V']$ in $K_0(\mathcal{A}^{\text{split}})$ if and only if $V \simeq V'$.*

Proof. Starting from $[V] = [V']$ we easily get the existence of an object W of $\mathcal{A}$ such that $V \oplus W \simeq V' \oplus W$. But then the classical Krull–Schimdt theorem (see for instance [17, Corollary 19.22]) allows to say that in fact $V \simeq V'$. $\square$

Lemma 2.5 proves that part (iii) of Theorem 2.4 is a consequence of part (ii). Moreover:

Proposition 2.6. *If $\mathcal{A}$ is a projectively complete full subcategory of $\mathcal{A}_{\text{tot}}$, then $K_0(\mathcal{A}^{\text{split}})$ is isomorphic to the free abelian group generated by isomorphism classes of indecomposables objects of $\mathcal{A}$.*

Proof. Because $\mathcal{A}$ be is projectively complete, the notion of indecomposable in $\mathcal{A}$ is the same as the one in $\mathcal{A}_{\text{tot}}$, and we thus can use the Krull–Schimdt theorem again to define a morphism from $K_0(\mathcal{A}^{\text{split}})$ into this free group. The fact that it is an isomorphism is immediate. $\square$

*2.2.4. K_0 (**mod** $\mathcal{A}_{\text{tot}}$) is abelian free*

We begin by a few definitions concerning an arbitrary additive category taken from [19]:

Definition 2.7. An additive category $\mathcal{A}$ is called *right artinian* (resp. *right noetherian*, resp. *semi-simple*) when for each object V of $\mathcal{A}$, the functor $\mathcal{A}(\cdot, V)$ is an artinian(resp. noetherian, resp semi-simple) object of **Mod** $\mathcal{A}$.

The dual (left) notion is obtained as usual by replacing $\mathcal{A}$ by $\mathcal{A}^{op}$.

Now we recall the definition of the Kelly radical (see [14]).

Definition 2.8. The *Kelly radical rad $\mathcal{A}$* of an additive category $\mathcal{A}$ is the two-sided ideal of $\mathcal{A}$ defined by

$$\operatorname{rad} \mathcal{A}(V, V') = \{f \in \mathcal{A}(V, V') / \ \forall g \in \mathcal{A}(V', V) \ \ 1_V - gf \text{ is invertible}\}$$

for all pair of objects (V, V') of $\mathcal{A}$.

In [19] it is shown that, as in the one object case, the notions of left and right semisimplicity coincide, but that however, a right artinian ring need not be right noetherian (the problem being that rad $\mathcal{A}$ is not necessarily nilpotent). In consequence Mitchell suggests the following definition.

Definition 2.9. An additive category $\mathcal{A}$ is said to be *semi-primary* if

(i) $\mathcal{A}/\operatorname{rad} \mathcal{A}$ is semi-simple,
(ii) rad $\mathcal{A}$ is nilpotent.

The advantage of this definition is that the classical Hopkins–Levitzki Theorem holds now with several objects:

Proposition 2.10. *Let $\mathcal{A}$ be a semi-primary additive category, and F a right $\mathcal{A}$-module. The following are equivalent*:

(i) *F is noetherian*,
(ii) *F is artinian*,
(iii) *F is of finite length.*

Proof. The classical proof applies unchanged: see for instance [17, §4.15]. □

Corollary 2.11. *Let $\mathcal{A}$ be an additive category. If $\mathcal{A}$ is right artinian and* $\operatorname{rad}\mathcal{A}$ *is nilpotent, then any $\mathcal{A}$-module of finite type is of finite length.*

Proof. The category $\mathcal{A}/\operatorname{rad}\mathcal{A}$ is right artinian of zero radical, hence semi-simple [19, Theorem 4.4]. So $\mathcal{A}$ is semi-primary.

Now if F is a module of finite type, F is artinian, hence of finite length by Proposition 2.10. □

Lemma 2.12. *If R is a finite-dimensional k-algebra, and $\mathcal{A}$ is a subcategory of $\mathcal{A}_{\text{tot}} =$ **mod** R with a finite set of additive generators, then*:

(i) *the category $\mathcal{A}$ is right artinian*
(ii) $\operatorname{rad}\mathcal{A}$ *is nilpotent.*

Proof. (i) First note that an arbitrary additive category $\mathcal{A}$ is right artinian if and only if its projective completion $Q\mathcal{A}$ is. By hypothesis, there exists a one object full subcategory $\mathcal{A}'$ of $\mathcal{A}_{\text{tot}} = \mathbf{mod}\ R$ such that $Q\mathcal{A}' = Q\mathcal{A}$, and $\mathcal{A}'$ is obviously right artinian, since it is a finite-dimensional k-algebra.

(ii) the restriction along $\mathcal{A} \to Q\mathcal{A}$ provides a bijection between two-sided ideals of $Q\mathcal{A}$ and two-sided ideals of $\mathcal{A}$ [25, Proposition 2]. This bijection is compatible with the product of ideals, and sends $\operatorname{rad} Q\mathcal{A}$ to $\operatorname{rad}\mathcal{A}$ [25, Proposition 10]. With the notations of the proof of (i), it is thus enough to show that $\operatorname{rad}\mathcal{A}'$ is nilpotent, but this is only the classical fact that the Jacobson radical of an artinian ring is nilpotent (see [17, Theorem 4.12]). □

Proposition 2.13. *If R is a finite-dimensional k-algebra, and $\mathcal{A}$ is subcategory of $\mathcal{A}_{\text{tot}} =$ **mod** R with a finite set of additive generators, then* **mod** $\mathcal{A}$ *is an abelian category, and every object of* **mod** $\mathcal{A}$ *is of finite length. Hence the group* $K_0\,(\mathbf{mod}\ \mathcal{A})$ *is abelian free, generated by the classes of simple objects in* **mod** $\mathcal{A}$.

Proof. Since $\mathcal{A}$ is Morita equivalent to an one object right artinian (hence right noetherian) ring, **mod** $\mathcal{A}$ is an abelian category. The second assertion results from Corollary 2.11 and Lemma 2.12. The third assertion is a consequence of the dévissage Theorem (see [23, §5 Theorem 4, Corollary 1]) for K_0, i.e. of the Jordan–Hölder Theorem. □

2.2.5. $\operatorname{rk}(K_0(\mathcal{A}_{\text{tot}}^{\text{split}})) = \operatorname{rk}(K_0\,(\mathbf{mod}\ \mathcal{A}_{\text{tot}}))$

The justification of this equality is the well-known remark which pushed Auslander to introduce functor categories in the study of representation of Artin algebras: there is a one-to-one correspondence between isomorphism classes of *indecomposables* right R-modules of finite type and isomorphism classes of *simples* right $\mathcal{A}_{\mathrm{tot}}$-modules of finite type. This is easily proved directly (see [10, §1.2]). In this paragraph we give an interpretation of this fact in a more general context.

According to Proposition 2.13, we have, if R is of finite representation type: $K_0\,(\mathbf{mod}\ \mathcal{A}_{\mathrm{tot}}) = K_0\,((\mathbf{mod}\ \mathcal{A}_{\mathrm{tot}})_{\mathrm{ss}})$, where $(\mathbf{mod}\ \mathcal{A}_{\mathrm{tot}})_{\mathrm{ss}}$ is the full subcategory of $\mathbf{mod}\ \mathcal{A}_{\mathrm{tot}}$ whose objects are the semi-simple right $\mathcal{A}_{\mathrm{tot}}$-modules. So what is left to do is to describe $(\mathbf{mod}\ \mathcal{A}_{\mathrm{tot}})_{\mathrm{ss}}$. We can in fact describe $(\mathbf{mod}\ \mathcal{A})_{\mathrm{ss}}$ for a *semi-local* category $\mathcal{A}$, in the following sense:

Definition 2.14. A category $\mathcal{A}$ is said to be *semi-local* if the category $\mathcal{A}/\mathrm{rad}\,\mathcal{A}$ is semi-simple.

This notion is linked with the notion of radical in the following way.

Definition 2.15. Let $\mathcal{A}$ be a category and F a right $\mathcal{A}$-module. We denote by $\mathrm{rad}\,F$ and call the *radical* of F the intersection of all maximal submodules of F.

For $F : \mathcal{A}^{op} \to k\,\mathbf{mod}$ a right $\mathcal{A}$-module, and $\mathcal{I}$ a right ideal of $\mathcal{A}$, we denote by $F\mathcal{I}$ the right $\mathcal{A}$-submodule of F given by, for each object V of $\mathcal{A}$:

$$F\mathcal{I}(V) = \left\{ x \in F(V) / \exists (W_i)_{1 \leqslant i \leqslant n} \in \mathrm{obj}\mathcal{A}\ \exists y_i \in F(W_i) \exists \alpha_i \in \mathcal{I}(V, W_i) / x = \sum_{i=1}^{n} y_i \cdot \alpha_i \right\}.$$

With these notations we have:

Lemma 2.16. *Let $\mathcal{A}$ be a semi-local category and F a right $\mathcal{A}$-module. Then* $\mathrm{rad}\,F = F\,\mathrm{rad}\,\mathcal{A}$.

Proof. The usual proof [17, §24.4] applies without change. □

Proposition 2.17. *Let $\mathcal{A}$ be a semi-local category. There is a natural isomorphism*:

$$Q(\mathcal{A}/\mathrm{rad}\,\mathcal{A}) \simeq (\mathbf{mod}\ \mathcal{A})_{\mathrm{ss}}.$$

Proof. Since $\mathcal{A}/\mathrm{rad}\,\mathcal{A}$ is semi-simple, the inclusion $Q(\mathcal{A}/\mathrm{rad}\,\mathcal{A}) \subset \mathbf{mod}\ (\mathcal{A}/\mathrm{rad}\,\mathcal{A})$ is an equality. So the proposition will follow from the next lemma. □

Lemma 2.18. *Let $\mathcal{A}$ be a semi-local category and $\mathcal{B} = \mathcal{A}/\mathrm{rad}\,\mathcal{A}$.*

(i) *The functor $R : \mathbf{Mod}\,\mathcal{B} \to \mathbf{Mod}\,\mathcal{A}$ induced by $\mathcal{A} \twoheadrightarrow \mathcal{B}$ is fully faithful.*
(ii) *The image of* $\mathbf{mod}\ \mathcal{B}$ *under R is* $(\mathbf{mod}\ \mathcal{A})_{\mathrm{ss}}$.

Proof. (i) According to Lemma 2.16, we can define a functor $K : \mathbf{Mod}\,\mathcal{A} \to \mathbf{Mod}\,\mathcal{B}$ by setting on objects $K(F) = F/\mathrm{rad}\,F$. This is a left adjoint of R (a left Kan extension along $\mathcal{A} \twoheadrightarrow \mathcal{B}$), and since $KR \simeq 1_{\mathcal{B}}$, R is fully faithful (see for instance [18, Chapter IV, §3, Theorem 1]).

(ii) To see first that $(\mathbf{mod}\ \mathcal{A})_{\mathrm{ss}} \subset R\,(\mathbf{mod}\ \mathcal{B})$, let F be an object of $(\mathbf{mod}\ \mathcal{A})_{\mathrm{ss}}$. Then according to Lemma 2.16 and [25, Proposition 9], we have $\mathrm{rad}\,F = F\,\mathrm{rad}\ \mathcal{A} = 0$, hence $F \simeq F/\mathrm{rad}\,F = RK(F)$.

To show the opposite inclusion $R(\mathbf{mod}\ \mathcal{B}) \subset (\mathbf{mod}\ \mathcal{A})_{\mathrm{ss}}$, fix G a semi-simple right $\mathcal{B}$-module. Since R is additive we can assume that G is in fact simple. Let $F \subset R(G)$ be a right $\mathcal{A}$ submodule of finite type. Then we have a diagram:

$$\begin{array}{ccc} F & \hookrightarrow & G \\ & \searrow \quad \nearrow & \\ & F/\mathrm{rad}\,F = K(F) & \end{array}$$

Since G is simple, and $K(F) \to G$ is a monomorphism, there are only two possibilities: either $K(F) = G$, and then $F = R(G)$, or $K(F) = 0$, which implies, by Lemma 2.16, Nakayama's Lemma (which is still valid for rings with several objects), and the fact that F is finitely generated, that $F = 0$. Hence $R(G)$ has no proper subobject in $\mathbf{mod}\ \mathcal{A}$ and by definition is in $(\mathbf{mod}\ \mathcal{A})_{\mathrm{ss}}$. $\square$

Now we can apply Proposition 2.17 to the case where $\mathcal{A}$ is a projectively complete subcategory of $\mathcal{A}_{\mathrm{tot}} = \mathbf{mod}\ R$, without finiteness assumption. To show that $\mathcal{A}$ is semi-local, choose a skeleton $\mathcal{S}$ of the category of indecomposables in $\mathcal{A}$, possibly infinite, so that $Q\mathcal{S} = \mathcal{A}$.

Lemma 2.19. *With notations as above*

$$Q(\mathcal{S}/\mathrm{rad}\,\mathcal{S}) = \mathcal{A}/\mathrm{rad}\,\mathcal{A}.$$

Proof. Thanks to [25, Proposition 10], we can identify $\mathcal{S}/\mathrm{rad}\,\mathcal{S}$ to a whole subcategory of $\mathcal{A}/\mathrm{rad}\,\mathcal{A}$. But now the equality of the Lemma is obvious, since the functor $\mathcal{A} \to \mathcal{A}/\mathrm{rad}\,\mathcal{A}$ is additive, hence preserves direct sums. $\square$

Now, since semi-simplicity is Morita invariant (a ring is semi-simple if and only if all its modules are), we are reduced to show the semi-simplicity of $\mathcal{S}/\mathrm{rad}\,\mathcal{S}$.

Recall that a *corpoid* is an additive category where all nonzeros maps are invertible. Given a field k and a set S, we have an associated corpoid, denoted by kS, defined by $kS(V, V) = k$ for any object V, and $kS(V, V') = 0$ for $V \neq V'$.

Lemma 2.20. *With notations as above, let S be the underlying set of $\mathcal{S}$. Then*

$$\mathcal{S}/\mathrm{rad}\,\mathcal{S} = kS.$$

Proof. For any object V of $\mathcal{S}$, the ring $\mathcal{S}(V, V)$ is local (see [17, Theorem 19.17]). Hence $\mathcal{S}(V, V)/\operatorname{rad}\mathcal{S}(V, V)$ is a skew field, finite dimensional over k. Since k is supposed to be algebraically closed, it must be k itself.

Now consider two different objects V, V' of $\mathcal{S}$. We have to show that $\operatorname{rad}\mathcal{S}(V, V') = \mathcal{S}(V, V')$. Suppose this is not the case, and let f be an element of $\mathcal{S}(V, V')$ be not in $\operatorname{rad}\mathcal{S}(V, V')$. According to the definition of the Kelly radical (see 2.8), there exists g in $\mathcal{S}(V', V)$ such that $1_V - gf$ is not invertible in $\mathcal{S}(V, V)$. Because $\mathcal{S}(V, V)$ is local, gf must be invertible (see [17, Theorem 19.1]). Since V' is indecomposable, this implies that $V \simeq V'$, and this contradicts the definition of $\mathcal{S}$. □

Since $\mathcal{S}/\operatorname{rad}\mathcal{S}$ has zero radical, and is immediately seen, thanks to Lemma 2.20, as right artinian, it is semi-simple [19, Theorem 4.4]. Hence $\mathcal{A}$ is semi-local, and combining Proposition 2.17 and Lemma 2.20 we get:

Proposition 2.21. *Let R be a finite-dimensional k-algebra, $\mathcal{A}$ a projectively complete subcategory of $\mathcal{A}_{\text{tot}} = \mathbf{mod}\ R$. Let also S be the set of isomorphism classes of indecomposables right R-modules of finite type contained in $\mathcal{A}$. Then there is a natural isomorphism*

$$Q(kS) \simeq (\mathbf{mod}\ \mathcal{A})_{\text{ss}}.$$

In particular, S is in one-to-one correspondence with the set of isomorphism classes of simple objects in **mod** $\mathcal{A}$.

2.2.6. The morphism $K_0^{\text{split}}(\mathcal{A}_{\text{tot}}) \to K_0\,(\mathbf{mod}\ \mathcal{A}_{\text{tot}})$ is an epimorphism

Theorem 2.22 (*Auslander–Reiten*). *Let R be a finite-dimensional k-algebra, and $\mathcal{A}_{\text{tot}} =$ **mod** R. Then every simple right $\mathcal{A}_{\text{tot}}$-module F admits a projective resolution of the form*

$$0 \to \mathcal{A}_{\text{tot}}(\cdot, V'') \to \mathcal{A}_{\text{tot}}(\cdot, V') \to \mathcal{A}_{\text{tot}}(\cdot, V) \to F \to 0.$$

Proof. See [10, §1.3]. □

Corollary 2.23. *Let R be a finite dimensional k-algebra of finite type. Then the morphism $K_0^{\text{split}}(\mathcal{A}_{\text{tot}}) \to K_0\,(\mathbf{mod}\ \mathcal{A}_{\text{tot}})$ is an epimorphism.*

Proof. This is a direct consequence of Proposition 2.13 and of Theorem 2.22. □

2.2.7. k-Categories and additive categories

Since we started with a k-algebra R, we could have worked with k-categories, i.e. categories enriched in **Mod** k, rather than with additive categories.

However, for modules categories, it does not make a significant difference. Indeed, for a k-category $\mathcal{A}$, denoted by $\mathcal{A}_0$, the underlying additive category. Then it is easy to see that restriction along the forgetful functor $\mathbf{Mod}\, k \to \mathbf{Ab}$ induces an equivalence

$$[\mathcal{A}^{op}, \mathbf{Mod}\, k]_0 \simeq [\mathcal{A}_0^{op}, \mathbf{Ab}].$$

So we can enrich our functor categories to see them as k-categories. This formulation has some advantages, in particular:

Lemma 2.24. *Let $\mathcal{A}$ be a category enriched over* $\mathbf{mod}\; k$, *admitting a finite set of additive generators. Then*

$$[\mathcal{A}^{op}, \mathbf{mod}\; k]_0 \simeq \mathbf{mod}\; (\mathcal{A}_0).$$

Proof. This is a direct consequence of the Yoneda Lemma. □

This applies in particular for $\mathcal{A}_{\text{tot}} = \mathbf{mod}\; R$, for R is a finite-dimensional k-algebra of finite representation type.

2.3. The general case

2.3.1. What is left

Let R be a finite-dimensional k-algebra. If one makes no assumption on the representation type of R, one cannot really work with the ring $\mathcal{A}_{\text{tot}}$ any longer, since it need not be the right noetherian. Instead we fix a projectively complete subcategory $\mathcal{A}$ of $\mathcal{A}_{\text{tot}}$, admitting a finite set of additive generators. For such a category $\mathcal{A}$, Propositions 2.13 and 2.21 hold, which shows that $K_0\,(\mathbf{mod}\; \mathcal{A})$ and $K_0(\mathcal{A}^{\text{split}})$ are free abelian groups of the same finite rank. However, we cannot say that the morphism induced by Yoneda between these two groups is an isomorphism.

We can be more precise: if we suppose that $\mathcal{A}$ contains the free object R, hence the category of projective right R-modules $\mathcal{P} = QR$, then we have a natural commutative diagram:

$$\begin{array}{ccc} K_0(\mathcal{A}^{\text{split}}) & \longrightarrow & K_0(\mathcal{A}_{\text{tot}}^{\text{split}}) \\ \downarrow & & \downarrow \\ K_0\,(\mathbf{mod}\; \mathcal{A}) & \twoheadrightarrow & K_0(\mathcal{A}_{\text{tot}}) \end{array}$$

2.3.2. Inverse of the devissage isomorphism

It is convenient to give an explicit description of the inverse isomorphism of the one given by devissage, so we introduce some notations.

Again we fix a projectively complete subcategory $\mathcal{A}$ of $\mathcal{A}_{\text{tot}} = \mathbf{mod}\; R$, admitting a finite set of additive generators, $\mathcal{S}$ a skeleton of the subcategory of indecomposables of $\mathcal{A}$, and S the underlying finite set.

Definition 2.25. For each object I of $\mathcal{S}$, we denote by S_I the simple object of **mod** $\mathcal{A}$ consisting of the quotient of the projective functor $\underline{I} := \mathcal{A}(\cdot, I)$ by the intersection $\operatorname{rad} I$ of its maximal subobjects.

Lemma 2.26. *Let F be an object of* **mod** $\mathcal{A}$.
We have in K_0 (**mod** $\mathcal{A}$):

$$[F] = \sum_{I \in S} \dim_k F(I)[S_I].$$

Proof. We have a canonical morphism $K_0\,(\mathbf{mod}\ \mathcal{A}) \to \operatorname{Map}(S, \mathbb{Z})$ sending $[F]$ to $I \to \dim_k F(I)$. This map sends $[S_J]$ to

$$I \to \dim_k \mathbf{mod}\ \mathcal{A}(\underline{I}, S_J) = I \to \dim_k \mathbf{mod}\ \mathcal{A}(S_I, S_J) = I \to \delta_{I,J}$$

hence it is in fact an isomorphism, and the formula follows. □

3. Enriched Kan extension

We give a brief summary of the notions of enriched category theory we will need in the sequel. We follow essentially the lines of the foundational papers [6,8,15] by Day, Eilenberg and Kelly.

3.1. Notations

As in [6,15], a *closed category* $\mathcal{S}$ will denote more precisely a symmetric closed monoidal category as defined in [8].

Briefly (we follow the presentation in [7]), a category $\mathcal{S}$ is *monoidal* if it is endowed with an associative tensor product $\otimes : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ and with an unit object I (that is, an object I of $\mathcal{S}$ and natural isomorphisms $I \otimes \cdot \simeq \mathrm{Id}_{\mathcal{S}}$, $\cdot \otimes I \simeq \mathrm{Id}_{\mathcal{S}}$), the whole data satisfying moreover the classical coherence axioms.

A monoidal category $\mathcal{S}$ is *symmetric* if, $\tau : \mathcal{S} \times \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ being the permutation functor, a natural isomorphism $\otimes \simeq \otimes \circ \tau$ is given, also verifying coherence axioms.

A symmetric monoidal category $\mathcal{S}$ is *closed* if for each object X of $\mathcal{S}$, the functor $\cdot \otimes X : \mathcal{S} \to \mathcal{S}$ admits a right adjoint $\mathcal{S}(X, \cdot) : \mathcal{S} \to \mathcal{S}$.

The definitions of $\mathcal{S}$-category, $\mathcal{S}$-functor, $\mathcal{S}$-natural transformation used are also those given in [8].

Roughly, a *$\mathcal{S}$-category* $\mathcal{B}$ is given by

(i) a class $\operatorname{Obj} \mathcal{B}$ of objects,
(ii) for any two objects B, C in $\operatorname{Obj} \mathcal{B}$, an object $\mathcal{B}(B, C)$ of $\mathcal{S}$,
(iii) for any three objects B, C, D in $\operatorname{Obj} \mathcal{B}$ a map $\circ : \mathcal{B}(B, C) \otimes \mathcal{B}(C, D) \to \mathcal{B}(B, D)$ in $\mathcal{S}$,

(iv) for any object B in $\mathrm{Obj}\,\mathcal{B}$, a map $i_B : I \to \mathcal{B}(B, B)$ in $\mathcal{S}$, such that the composition is associative and i is a unit for $\circ$.

A *$\mathcal{S}$-functor* F between two $\mathcal{S}$-categories $\mathcal{B}$, $\mathcal{B}'$ is given by

(i) a function (also denoted by F) $\mathrm{Obj}\,\mathcal{B} \to \mathrm{Obj}\,\mathcal{B}'$,
(ii) for any two objects B, C in $\mathrm{Obj}\,\mathcal{B}$, a map $\mathcal{B}(B, C) \to \mathcal{B}'(FB, FC)$ in $\mathcal{S}$, preserving the units and the composition.

A $\mathcal{S}$-category (resp. a $\mathcal{S}$-functor) defines a category (resp. a functor) in the usual sense, obtained by pushing the data along the closed functor $\mathcal{S}(I, \cdot) : \mathcal{S} \to \mathcal{E}ns$.

An *$\mathcal{S}$-natural transformation* ϕ between two $\mathcal{S}$-functors F, G: $\mathcal{B} \to \mathcal{B}'$ is given by each object B in $\mathrm{Obj}\,\mathcal{B}$ by a map $\phi_B : FB \to GB$ in the category underlying $\mathcal{B}'$ such that for any two objects B, C in $\mathrm{Obj}\,\mathcal{B}$ the following diagram commutes in $\mathcal{S}$:

$$\begin{array}{ccc} \mathcal{B}(B, C) & \xrightarrow{F} & \mathcal{B}'(FB, FC) \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle \mathcal{B}'(\cdot,\phi_C)} \\ \mathcal{B}'(GB, GC) & \xrightarrow[\mathcal{B}'(\phi_B,\cdot)]{} & \mathcal{B}'(FB, GC) \end{array}$$

We will often stress the presence of a structure of closed or enriched category by using bold letters.

We will write $\mathcal{S}-\mathrm{CAT}$ for the hypercategory (or strict 2-category) whose objects are the $\mathcal{S}$-categories, 1-arrows are the $\mathcal{S}$-functors, and 2-arrows are the $\mathcal{S}$-natural transformations. In particular we define as usual $\mathcal{C}at = \mathcal{E}ns - \mathrm{CAT}$ and $\mathcal{H}yp = \mathcal{C}at - \mathrm{CAT}$.

We will write $\mathcal{C}l$ for the 2-category of closed categories.

The notion of adjunction between $\mathcal{S}$-functors used is the one defined in [15, §2]. It implies the existence of an adjunction between the underlying functors, but is not implied by this one.

Let $\mathcal{B}$ be an $\mathcal{S}$-category, $X \in \mathrm{Obj}\,\mathcal{S}$, $B \in \mathrm{Obj}\,\mathcal{B}$.

The tensor $X \otimes B \in \mathrm{Obj}\,\mathcal{B}$ is characterized by the existence of an $\mathcal{S}$-natural isomorphism

$$\mathcal{B}(X \otimes B, C) \simeq [X, \mathcal{B}(B, C)],$$

where the brackets in the right-hand side denote the internal Hom of $\mathcal{S}$.

Dually the cotensor $[X, B] \in \mathrm{Obj}\,\mathcal{B}$ is characterized by the existence of an $\mathcal{S}$-natural isomorphism

$$\mathcal{B}(C, [X, B]) \simeq [X, \mathcal{B}(C, B)]$$

(see [15, §4]).

The definition of a complete (resp. cocomplete) $\mathcal{S}$-category $\mathcal{B}$ is the one given in [6, §2] (this should of course not be confused with the notion of projective completion of an additive category introduced above). It implies in particular the existence of cotensor (resp. tensor) objects, and the existence of small ends (resp. coends), see [6, §3.3].

Finally, if $\mathcal{B}$ and $\mathcal{B}'$ are $\mathcal{S}$-categories, with $\mathcal{B}'$ small and $\mathcal{S}$ complete, the category of $\mathcal{S}$-functors between $\mathcal{B}'$ and $\mathcal{B}$, and $\mathcal{S}$-natural transformations between these functors, can be enriched in an $\mathcal{S}$-category $[\mathcal{B}', \mathcal{B}]$ by setting, for two $\mathcal{S}$-functors F, G:

$$[\mathcal{B}', \mathcal{B}](F, G) = \int_{B'} \mathcal{B}(FB', GB')$$

(see [6, §4]).

If $\mathcal{B}$ is complete (resp. cocomplete) then $[\mathcal{B}', \mathcal{B}]$ is also complete (resp. cocomplete), and limits (resp. colimits) are formed termwise (see [16, Chapter 3, §3.3]).

3.2. Enriched left Kan extension

Proposition 3.1 (*Day–Kelly*). *Let $\mathcal{S}$ be a closed category, $i : \mathcal{B}' \to \mathcal{B}$ an $\mathcal{S}$-functor, where $\mathcal{B}$ and $\mathcal{B}'$ are small, and $\mathcal{C}$ a cocomplete $\mathcal{S}$-category. Then the $\mathcal{S}$-functor $R = [i, 1] : [\mathcal{B}, \mathcal{C}] \to [\mathcal{B}', \mathcal{C}]$ admits as a left adjoint the functor Q given on objects by, for $F \in \mathrm{Obj}[\mathcal{B}', \mathcal{C}]$:*

$$Q(F) = \int^{B'} \mathcal{B}(\cdot, iB') \otimes FB'$$

Proof. See [6, §6.1]. □

3.3. Enriched right Kan extension

Proposition 3.2. *Let $\mathcal{S}$ be a closed category, $i : \mathcal{B}' \to \mathcal{B}$ a $\mathcal{S}$-functor, where $\mathcal{B}$ and $\mathcal{B}'$ are small, and $\mathcal{C}$ a complete $\mathcal{S}$-category. Then the $\mathcal{S}$-functor $R = [i, 1] : [\mathcal{B}, \mathcal{C}] \to [\mathcal{B}', \mathcal{C}]$ admits as a right adjoint the functor K given on objects by, for $F \in \mathrm{Obj}[\mathcal{B}', \mathcal{C}]$:*

$$K(F) = \int_{B'} [\mathcal{B}(\cdot, iB'), FB'].$$

If moreover i is fully faithful, the counit of this adjunction is an isomorphism: $RK \simeq 1$ (equivalently, K is fully faithful).

Proof. The first statement is the dual of Proposition 3.1, and the second follows from the enriched Yoneda Lemma (see [6, §5]). □

Corollary 3.3. *Suppose moreover that* $\mathcal{C} = \mathcal{S}$. *Then the* $\mathcal{S}$*-functor* $R = [i, 1] : [\mathcal{B}, \mathcal{S}] \to [\mathcal{B}', \mathcal{S}]$ *admits as a right adjoint the functor*:

$$K : [\mathcal{B}', \mathcal{S}] \longrightarrow [\mathcal{B}, \mathcal{S}]$$
$$F \longrightarrow (V \longrightarrow [\mathcal{B}', \mathcal{S}](\mathcal{B}(V, i\cdot), F)).$$

Proof. By definition $K(F)(V) = \int_{B'} [\mathcal{B}(V, iB'), FB']$. Since $\mathcal{S}$ is symmetric, the cotensor $[\cdot, \cdot]$ in $\mathcal{S}$ coincide with the internal Hom of $\mathcal{S}$. The formula given follows from the definition of the $\mathcal{S}$-category $[\mathcal{B}', \mathcal{S}]$. □

Corollary 3.4. *Suppose moreover that* $\mathcal{C} = \mathcal{S}$, i *is fully faithful, and every object of* $\mathcal{B}$ *is a retract of an object of* $\mathcal{B}'$. *Then the adjunction of Corollary* 3.3 *is an equivalence.*

Proof. Consider the composite embedding $\mathcal{B}'^{op} \to \mathcal{B}^{op} \to [\mathcal{B}, \mathcal{S}]$. Since $\mathcal{B}^{op}$ is dense in $[\mathcal{B}, \mathcal{S}]$, it follows from [16, Chapter 5, Proposition 5.20], that $\mathcal{B}'^{op}$ is dense in $[\mathcal{B}, \mathcal{S}]$. But Keppy [16, Chapter 5, Theorem 5.1(ii)], and the enriched Yoneda Lemma again show that the restriction $R : [\mathcal{B}, \mathcal{S}] \to [\mathcal{B}', \mathcal{S}]$ is fully faithful. It follows now from [16, Chapter 1, §1.11] that the unit $1 \Longrightarrow RK$ of the adjunction is an isomorphism. □

In fact, the part of classical Morita theory briefly described in §2.1.2 lifts to the general enriched context (see in particular [16, Chapter 5, Proposition 5.28]). Since we will not use these results, we do not recall them.

4. Rings with several objects on a scheme

All schemes considered in the sequel are supposed noetherian.

4.1. Category of quasi-coherent sheaves on a scheme

Proposition 4.1. *Let* Y *be a scheme, and* $\operatorname{Qcoh} Y$ *the category of quasi-coherent sheaves on* Y. *There is a closed category* $\mathbf{Qcoh}\, Y$ *whose underlying category is* $\operatorname{Qcoh} Y$.

Proof. We have to give first the seven data (in fact six independent) defining a closed category as given in [8, Chapter I, §2], and we choose the natural ones. Checking axioms CC1–CC5 can be done locally, hence deduced from the corresponding facts for categories of modules over a commutative ring, or even directly.

Moreover, there is a tensor product on $\operatorname{Qcoh} Y$ defined by the existence of a natural isomorphism

$$\operatorname{Qcoh} Y(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \operatorname{Qcoh} Y(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})). \tag{2}$$

It lifts to a Qcoh Y-natural transformation

$$\mathcal{H}om(\mathcal{F}\otimes\mathcal{G},\mathcal{H})\simeq\mathcal{H}om(\mathcal{F},\mathcal{H}om(\mathcal{G},\mathcal{H})).$$

Hence we deduce from [8, Chapter II, Theorem 5.3], that the closed category defined above is in fact monoidal.

At last, using the definition 2 of the tensor product and the natural symmetry of $\mathcal{H}om$

$$\text{Qcoh}\,Y(\mathcal{F},\mathcal{H}om(\mathcal{G},\mathcal{H}))\simeq\text{Qcoh}\,Y(\mathcal{G},\mathcal{H}om(\mathcal{F},\mathcal{H}))$$

one defines a symmetry for the tensor product and check axioms MC6–MC7 of [8, §Chapter III, §1]. □

Proposition 4.2. **Qcoh** Y *is complete and cocomplete.*

Proof. According to the definition given in [6, §2], it suffices to show that Qcoh Y is complete and cocomplete. Since (co)limits must commute with localization, one-first construct them locally, which is possible, because the categories of modules over a ring are complete and cocomplete (see [24, 7.4.3, 8.4.3]). Since limits are universal, one can glue the local limits together to get global limits. □

Proposition 4.3. *Let* $f : Y' \to Y$ *be a morphism of schemes. The functor* f_* : Qcoh $Y' \to$ Qcoh Y *can be lifted in a closed functor* f_* : **Qcoh** $Y' \to$ **Qcoh** Y.

Proof. To lift f_* : Qcoh $Y' \to$ Qcoh Y one needs to specify a natural transformation $\hat{f}_* : f_*\mathcal{H}om(\mathcal{F}',\mathcal{G}') \to \mathcal{H}om(f_*\mathcal{F}', f_*\mathcal{G}')$, and a morphism $f_*^0 : \mathcal{O}_Y \to f_*\mathcal{O}_{Y'}$, and once again one chooses the obvious ones. To check axioms CF1–CF3 (resp. MF4) given in [8, Chapter I, §3] (resp. Chapter III, §1) is a long but easy task. □

Proposition 4.4. *Let* $f : Y' \to Y$ *be a morphism of schemes. The functor* f^* : Qcoh $Y \to$ Qcoh Y' *can be lifted in a closed functor* f^* : **Qcoh** $Y \to$ **Qcoh** Y'.

Proof. To lift f^* : Qcoh $Y' \to$ Qcoh Y, one needs in a similar way to specify a natural transformation $\hat{f}^* : f^*\mathcal{H}om(\mathcal{F},\mathcal{G}) \to \mathcal{H}om(f^*\mathcal{F}, f^*\mathcal{G})$, and a morphism $f^{*0} : \mathcal{O}_Y \to f^*\mathcal{O}_{Y'}$. For f^{*0} one makes the natural choice of the inverse of the isomorphism given from f_*^0 by adjunction. To construct $\hat{f}^*$, one first notices that the composition

$$f_*\mathcal{H}om(f^*\mathcal{F},\mathcal{G}') \to \mathcal{H}om(f_*f^*\mathcal{F}, f_*\mathcal{G}') \to \mathcal{H}om(\mathcal{F}, f_*\mathcal{G}')$$

given by $\hat{f}_*$ and adjunction, is in fact a natural isomorphism. One gets then immediately a natural $\hat{f}^*$, also given by adjunction. Using the Proposition 4.3, one checks axioms CF1–CF3 and MF4 again. □

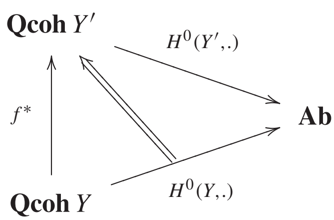
Proposition 4.5. *Let $f : Y' \to Y$ be a morphism of schemes. The usual adjunction between f^* and f_* in $\mathcal{C}at$ can be lifted in an adjunction in the 2-category of closed categories $\mathcal{C}l$.*

Proof. One checks that the unit and counit of the adjunction are in fact closed natural transformations, i.e. that they verify axioms CN1 and CN2 given in [8, Chapter I, §4].

Corollary 4.6. *The functors f^* and f_* belong to a natural $\mathcal{C}at$-enriched adjunction between* $\mathbf{Qcoh}\,Y - \mathrm{CAT}$ *and* $\mathbf{Qcoh}\,Y' - \mathrm{CAT}$.

Proof. According to [15, §2], we can just push the adjunction of Proposition 4.5 along the canonical 2-functor $\mathcal{C}l \to \mathcal{H}yp$ sending $\mathcal{S}$ to $\mathcal{S} - \mathrm{CAT}$, to get an adjunction in $\mathcal{H}yp$, i.e., by definition, a $\mathcal{C}at$-enriched adjunction. □

Corollary 4.7. *There is a natural 2-arrow in the 2-category $\mathcal{C}l$ of closed categories*:



Proof. This is obtained by pushing in the 2-category $\mathcal{C}l$ the unit of the adjunction of Proposition 4.5 along the closed functor $H^0(Y, .)$. □

4.2. Ringed schemes

Definition 4.8. Let Y be a scheme.

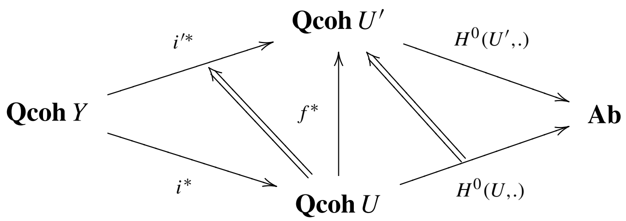
(i) A *ring* (*with several objects*) *on* Y is, by definition, a category $\mathcal{A}$ enriched on $\mathbf{Qcoh}\,Y$. The pair $(Y, \mathcal{A})$ is called a *ringed scheme*.

(ii) A morphism of ringed schemes $(Y', \mathcal{A}') \to (Y, \mathcal{A})$ is a couple $(f, f^\#)$ where $f : Y' \to Y$ is a scheme morphism and $f^\# : \mathcal{A} \to f_*\mathcal{A}'$ is a morphism of $\mathbf{Qcoh}\,Y$-categories.

Remark. There is an obvious notion of 2-arrow between two ringed schemes morphisms of same source and target, and ringed schemes thus form a 2-category.

A natural operation to consider is, for every open $i : U \to Y$, to push $\mathcal{A}$ via $H^0(U, \cdot) \circ i^* : \mathbf{Qcoh}\,Y \to \mathbf{Qcoh}\,U \to \mathbf{Ab}$, which gives an additive category $\mathcal{A}(U)$. Moreover, if $i' : U' \to Y$ is another open, every Y-inclusion $f : U' \to U$ gives an additive restriction functor $\mathcal{A}(U) \to \mathcal{A}(U')$. Indeed from the lax-functor $(Sch/Y)^{op} \to \mathcal{C}l$

sending $i : U \to Y$ to $\mathbf{Qcoh}\, U$, and Corollary 4.7, we get the diagram in $\mathcal{C}l$:



We can then push the resulting 2-arrow via the canonical 2-functor $\mathcal{C}l \to \mathcal{H}yp$ sending $\mathcal{S}$ to $\mathcal{S} - \mathrm{CAT}$, and then evaluate at $\mathcal{A}$. In this way, $\mathcal{A}$ can be seen as a sheaf of rings with several objects on Y.

4.3. Quasicoherent sheaves on ringed schemes

4.3.1. Definition

Definition 4.9. Let $(Y, \mathcal{A})$ be a ringed scheme.

(i) A quasicoherent sheaf on Y is by definition an enriched functor from $\mathcal{A}^{op}$ to $\mathbf{Qcoh}\, Y$.
(ii) A morphism of $\mathcal{A}$-sheaves is an enriched natural transformation.
(iii) We denote by $\mathbf{Qcoh}\,(Y, \mathcal{A}) = [\mathcal{A}^{op}, \mathbf{Qcoh}\, Y]$ the corresponding enriched category, and by $\mathrm{Qcoh}\,(Y, \mathcal{A})$ the underlying category.

When $Y = \mathrm{Spec}\, B$ is affine, then clearly the global sections functor induces an equivalence $\mathbf{Qcoh}\,(Y, \mathcal{A}) \simeq [\mathcal{A}^{op}(Y), B\,\mathbf{Mod}\,]$ as $\mathbf{Qcoh}\, Y \simeq B\,\mathbf{Mod}$ enriched categories.

4.3.2. Functoriality: definition

To define push-forward and pull-back for sheaves on ringed schemes we need some notations.

First consider the simplest case of a morphism $(1, i) : (Y, \mathcal{A}') \to (Y, \mathcal{A})$. Because of Proposition 4.2 we can apply the results quoted in Section 3. Thus we get left and right adjoints for the restriction functor $\mathbf{Qcoh}\,(Y, \mathcal{A}) \to \mathbf{Qcoh}\,(Y, \mathcal{A}')$. The left adjoint will be denoted as usual by $\otimes_{\mathcal{A}'} \mathcal{A}$.

Now consider a general morphism $(f, f^{\#}) : (Y', \mathcal{A}') \to (Y, \mathcal{A})$.

We denote by *adj* the $\mathcal{C}at$-natural isomorphism

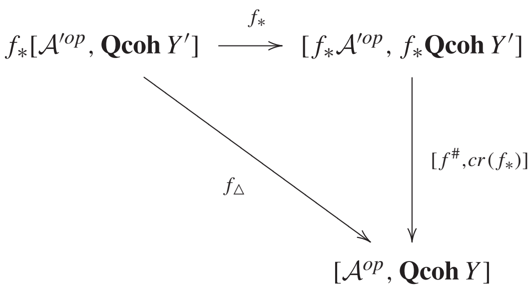
$$\mathrm{adj} : \mathbf{Qcoh}\, Y - \mathrm{CAT}(\mathcal{B}, f_*\mathcal{B}') \simeq \mathbf{Qcoh}\, Y' - \mathrm{CAT}(f^*\mathcal{B}, \mathcal{B}')$$

given by Corollary 4.6. We write ε (resp. μ) for the counit (resp. for the unit) of the adjunction between f^* and f_* in $\mathcal{C}l$.

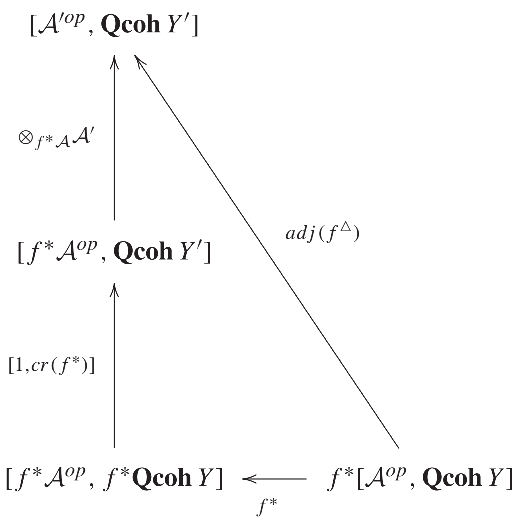
Recall that if $\Phi : \mathcal{S}' \to \mathcal{S}$ is any closed functor, there is an associated $\mathcal{S}$-functor $\Phi\mathcal{S}' \to \mathcal{S}$, and we will denote it by $cr(\Phi)$ (see [8, Chapter I, Theorem 6.6]).

Definition 4.10. Let $(f, f^{\#}) : (Y', \mathcal{A}') \to (Y, \mathcal{A})$ be a morphism of ringed schemes.

(i) We define $f_{\triangle} : f_*\mathbf{Qcoh}\,(Y', \mathcal{A}') \to \mathbf{Qcoh}\,(Y, \mathcal{A})$ as the $\mathbf{Qcoh}\,Y$-functor making the following diagram commute:



(ii) We define $f^{\triangle} : \mathbf{Qcoh}\,(Y, \mathcal{A}) \to f_*\mathbf{Qcoh}\,(Y', \mathcal{A}')$ as the $\mathbf{Qcoh}\,Y$-functor making the following diagram commute:



4.3.3. Functoriality: adjunction

Proposition 4.11. *Let $(f, f^{\#}) : (Y', \mathcal{A}') \to (Y, \mathcal{A})$ be a morphism of ringed schemes. The couple $(f^{\triangle}, f_{\triangle})$ is part of a $\mathbf{Qcoh}\,Y$-adjunction between $\mathbf{Qcoh}\,(Y, \mathcal{A})$ and $f_*\mathbf{Qcoh}\,(Y', \mathcal{A}')$.*

Proof. This is a consequence of the five following lemmas.

Lemma 4.12. *The* $\mathcal{C}at$*-natural isomorphism*

$$\mathrm{adj} : \mathbf{Qcoh}\, Y - \mathrm{CAT}(\mathcal{B}, f_*\mathcal{B}') \simeq \mathbf{Qcoh}\, Y' - \mathrm{CAT}(f^*\mathcal{B}, \mathcal{B}')$$

given by Corollary 4.6 *lifts to a* $\mathbf{Qcoh}\, Y - \mathrm{CAT}$*-natural isomorphism*

$$\widehat{\mathrm{adj}} : [\mathcal{B}, f_*\mathcal{B}'] \simeq f_*[f^*\mathcal{B}, \mathcal{B}'].$$

Proof. There are natural morphisms:

$$f_*[f^*\mathcal{B}, \mathcal{B}'] \xrightarrow{f_*} [f_*f^*\mathcal{B}, f_*\mathcal{B}'] \xrightarrow{[\mu_{\mathcal{B}},1]} [\mathcal{B}, f_*\mathcal{B}']$$

and

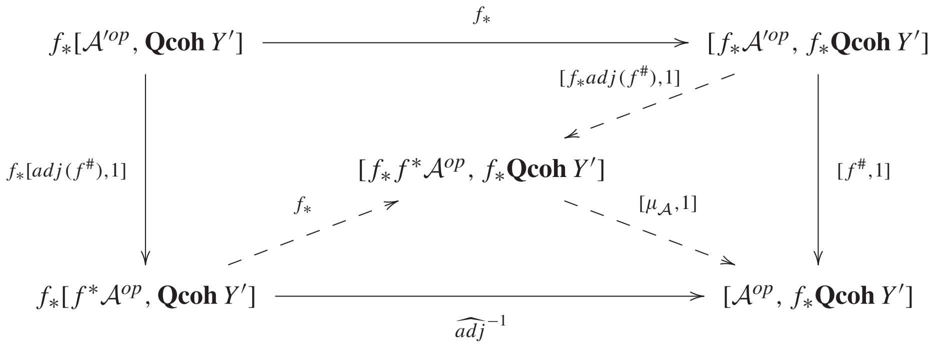
$$f^*[\mathcal{B}, f_*\mathcal{B}'] \xrightarrow{f^*} [f^*\mathcal{B}, f^*f_*\mathcal{B}'] \xrightarrow{[1,\varepsilon_{\mathcal{B}'}]} [f^*\mathcal{B}, \mathcal{B}']$$

and the morphism associated to the second one by adjunction is an inverse of the first one. □

Lemma 4.13. *The following diagram in* $\mathbf{Qcoh}\, Y - \mathrm{CAT}$ *is commutative*:

$$\begin{array}{ccc}
f_*[\mathcal{A}'^{op}, \mathbf{Qcoh}\, Y'] & \xrightarrow{f_*} & [f_*\mathcal{A}'^{op}, f_*\mathbf{Qcoh}\, Y'] \\
\Big\downarrow {\scriptstyle f_*[adj(f^{\#}),1]} & & \Big\downarrow {\scriptstyle [f^{\#},1]} \\
f_*[f^*\mathcal{A}^{op}, \mathbf{Qcoh}\, Y'] & \overset{\widehat{adj}}{\simeq} & [\mathcal{A}^{op}, f_*\mathbf{Qcoh}\, Y'] \\
\Big\uparrow {\scriptstyle f_*[1,cr(f^*)]} & & \Big\uparrow {\scriptstyle [1,adj^{-1}(cr(f^*))]} \\
f_*[f^*\mathcal{A}^{op}, f^*\mathbf{Qcoh}\, Y] & \xleftarrow[adj^{-1}(f^*)]{} & [\mathcal{A}^{op}, \mathbf{Qcoh}\, Y]
\end{array}$$

Proof. Let us consider first the following subdivision of the top square:



The quadrilateral commutes because of the $\mathbf{Qcoh}\, Y-$CAT-naturality of f_*. The bottom triangle commutes by definition of $\widehat{\text{adj}}$, and the right triangle by definition of adj.

So the top square commutes, and the bottom square is treated in a similar way. □

Lemma 4.14. *The couple* $(\otimes_{f^*\mathcal{A}}\mathcal{A}', [\text{adj}(f^\#), 1]))$ *is part of a natural adjunction.*

Proof. This is by definition of the tensor product, as given in §4.3.2. □

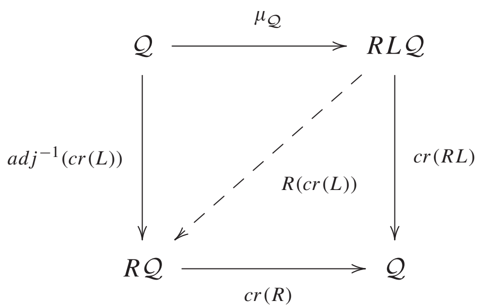
Note that we can push this adjunction along the 2-functor $f_* : \mathbf{Qcoh}\, Y' - \text{CAT} \to \mathbf{Qcoh}\, Y - \text{CAT}$, to get a first adjunction in the diagram of Lemma 4.13. The following lemma gives a second one.

Lemma 4.15. *The couple* $([1, \text{adj}^{-1}(\text{cr}(f^*))], [1, \text{cr}(f_*)])$ *is part of a natural adjunction.*

Proof. It is of course enough to show that $(\text{adj}^{-1}(\text{cr}(f^*), \text{cr}(f_*))$ is a part of a natural adjunction in $\mathbf{Qcoh}\, Y - \text{CAT}$, since we can then push it along the 2-functor $[\mathcal{A}^{op}, \cdot] : \mathbf{Qcoh}\, Y - \text{CAT} \to \mathbf{Qcoh}\, Y - \text{CAT}$. Hence the following lemma allows to conclude:

Lemma 4.16. *Let* $\mathcal{Q}$ *and* $\mathcal{Q}'$ *be two closed categories*, $L : \mathcal{Q} \to \mathcal{Q}'$ *and* $R : \mathcal{Q}' \to \mathcal{Q}$ *two closed functors part of closed adjunction* (L, R, ε, μ) (*where* ε *is the counit and* μ the unit), *and* $\text{adj} : \mathcal{Q} - \text{CAT}(\mathcal{B}, R\mathcal{B}') \simeq \mathcal{Q}' - \text{CAT}(L\mathcal{B}, \mathcal{B}')$ *the induced* $\mathcal{C}at$*-natural isomorphism. Then the pair* $(\text{adj}^{-1}(\text{cr}(L)), \text{cr}(R))$ *is part of a natural* $\mathcal{Q}$*-adjunction.*

Proof. Applying [8, Chapter I, Proposition 8.10] to the unit: $\mu : 1 \Longrightarrow RL : \mathcal{Q} \to \mathcal{Q}$, we get a 2-arrow in $\mathcal{Q} - \text{CAT}$: $\mu : 1 \Longrightarrow \text{cr}(RL)\mu_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$. But since the following diagram



commutes (as one sees thanks to the dotted arrow), we get in fact a (candidate) unit $\mu : 1 \Longrightarrow \mathrm{cr}(R)\mathrm{adj}^{-1}(\mathrm{cr}(L)) : \mathcal{Q} \to \mathcal{Q}$. The (candidate) counit is built by the same argument, using moreover the $\mathcal{C}at$-adjunction between $\mathcal{Q} - \mathrm{CAT}$ and $\mathcal{Q}' - \mathrm{CAT}$. □

We can then compose the two adjunctions we have described in the diagram of Lemma 4.13, and this shows the proposition. □

4.3.4. Stalks

Starting from a ringed scheme $(Y, \mathcal{A})$, we have for each point Q of Y a localization $\mathcal{A}_Q$ given by $\mathcal{A}_Q = i^*\mathcal{A}$, where $i : \operatorname{spec} \mathcal{O}_{Y,Q} \to Y$ is the canonical morphism. Pulling back along the ringed scheme morphism $(\operatorname{spec} \mathcal{O}_{Y,Q}, \mathcal{A}_Q) \to (Y, \mathcal{A})$ gives a stalk functor:

$$\begin{array}{ccc} \mathbf{Qcoh}\,(Y, \mathcal{A}) & \longrightarrow & \mathbf{Qcoh}\,(\operatorname{spec} \mathcal{O}_{Y,Q}, \mathcal{A}_Q) \\ \mathcal{F} & \longrightarrow & \mathcal{F}_Q \end{array}$$

4.3.5. Qcoh $(Y, \mathcal{A})$ *is abelian*

Definition 4.17. Let $(Y, \mathcal{A})$ be a ringed scheme. For each object V of $\mathcal{A}$, we denote by $\langle V \rangle$ the full subcategory of $\mathcal{A}$ containing only the object V. We will write p_V or $\cdot(V)$ for the canonical projection: $\mathbf{Qcoh}\,(Y, \mathcal{A}) \to \mathbf{Qcoh}\,(Y, \langle V \rangle)$.

For each V, the category Qcoh $(Y, \langle V \rangle)$ is the category of quasicoherent sheaves of modules on a (one object !) algebra on Y, hence it is an abelian category.

Proposition 4.18. (i) *The category* Qcoh $(Y, \mathcal{A})$ *is an abelian category.*

(ii) *A sequence $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ of $\mathcal{A}$-sheaves on Y is exact if and only if for each object V of $\mathcal{A}$ the sequence $\mathcal{F}'(V) \to \mathcal{F}(V) \to \mathcal{F}''(V)$ is exact in* Qcoh $(Y, \langle V \rangle)$.

Proof. Because of Proposition 4.2, Qcoh $(Y, \mathcal{A})$ is complete and cocomplete, and limits are formed termwise. So (ii) follows from (i). To show (i), the only thing to prove is that every monomorphism is a kernel, and the dual assertion.

Consider the functor given by the canonical projections:

$$p : \mathrm{Qcoh}\,(Y, \mathcal{A}) \to \prod_V \mathrm{Qcoh}\,(Y, \langle V \rangle).$$

It is faithful, and we know from Section 4.3.2 that each p_V admits a right adjoint, hence preserves colimits, hence p itself preserves colimits.

Dually, one sees that p also preserves limits.

Now if f is a monomorphism in $\mathrm{Qcoh}\,(Y, \mathcal{A})$, then $p(f)$ is a monomorphism in $\prod_V \mathrm{Qcoh}\,(Y, \langle V \rangle)$, which is abelian, hence $p(f) = \mathrm{Ker}(\mathrm{Coker}\, p(f)) = p(\mathrm{Ker}\,(\mathrm{Coker}\, f))$, and because p is faithful $f = \mathrm{Ker}(\mathrm{Coker}\, f)$, hence f is a kernel. The dual assertion follows similarly. $\square$

Since localization commutes with $\cdot(V)$, we get immediately from Proposition 4.18:

Proposition 4.19. *The sequence of $\mathcal{A}$-sheaves: $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is exact if and only if for each Q in Y the sequence of stalks $\mathcal{F}'_Q \to \mathcal{F}_Q \to \mathcal{F}''_Q$ is exact.*

5. Sheaves of modules for an Auslander algebra on a G-scheme

5.1. Auslander algebras associated with a G-scheme over a field

5.1.1. Definition

In the sequel, we fix an algebraically closed field k, and a scheme X over k, endowed with an *admissible* action of a finite group G, so that the quotient scheme $Y = X/G$ exists. The quotient morphism will be denoted by $\pi : X \to Y = X/G$. We call this data *a G-scheme over k*.

We recall briefly the definition of a G-sheaf:

Definition 5.1. Let $\mathcal{F}$ be a quasicoherent sheaf on the G-scheme X. A *G-linearization* of $\mathcal{F}$ is the data of a collection $(\psi_g)_{g \in G}$ of sheaf morphisms $\psi_g : g_* \mathcal{F} \to \mathcal{F}$ satisfying the following conditions:

(1) $\psi_1 = 1$
(2) $\psi_{hg} = \psi_h \circ h_*(\psi_g)$ in other words, the following diagram commutes:

$$\begin{array}{ccccc} h_* g_* \mathcal{F} & \xrightarrow{h_* \psi_g} & h_* \mathcal{F} & \xrightarrow{\psi_h} & \mathcal{F} \\ \| & & & \nearrow_{\psi_{hg}} & \\ (hg)_* \mathcal{F} & & & & \end{array}$$

A *G-sheaf on X* is by definition a quasicoherent sheaf on X endowed with a G-linearization.

Morphism of G-sheaves is morphism of sheaves commuting with the action. In this way, G-sheaves on X form an abelian category $\mathrm{Qcoh}\,(G, X)$.

If $\mathcal{G}$ is a sheaf on the quotient Y, the sheaf $\pi^*\mathcal{G}$ has a natural structure of G-sheaf, and we get a functor $\pi^* : \mathrm{Qcoh}\, Y \to \mathrm{Qcoh}\,(G, X)$. This functor admits as usual a right adjoint, denoted by π_*^G, which to a G-sheaf $\mathcal{F}$ associates the sheaf on Y given by $U \to (\mathcal{F}(\pi^{-1}U))^G$.

Lemma 5.2. *Let X be a G-scheme. The closed structure on* $\mathbf{Qcoh}\, X$ *induces on* Qcoh (G, X) *a structure of enriched category* $\mathbf{Qcoh}\,(G, X)$ *over* $\mathbf{Qcoh}\, Y$ *such that for two G-sheaves* $\mathcal{F}$, $\mathcal{F}'$:

$$\mathbf{Qcoh}\,(G, X)(\mathcal{F}, \mathcal{F}') = \pi_*^G(\mathbf{Qcoh}\, X(\mathcal{F}, \mathcal{F}')).$$

The functor $\pi_*^G : \mathrm{Qcoh}\,(G, X) \to \mathrm{Qcoh}\, Y$ *can be lifted in a natural way as a* $\mathbf{Qcoh}\, Y$*-functor* $\pi_*^G : \mathbf{Qcoh}\,(G, X) \to \mathbf{Qcoh}\, Y$.

Proof. We can first define a closed category $\mathbf{Qcoh}\,(G, X)$ whose underlying category is $\mathrm{Qcoh}\,(G, X)$ by setting $\mathbf{Qcoh}\,(G, X)(\mathcal{F}, \mathcal{F}') = \mathbf{Qcoh}\, X(\mathcal{F}, \mathcal{F}')$, seen with its usual action, and taking as "structural functor" $H^0(X, \cdot)^G$. Moreover, the functor $\pi_*^G : \mathrm{Qcoh}\,(G, X) \to \mathrm{Qcoh}\, Y$ can be lifted in a natural way to a closed functor $\pi_*^G : \mathbf{Qcoh}\,(G, X) \to \mathbf{Qcoh}\, Y$, and applying [8, Chapter I, Proposition 6.1], we get the wished structure. □

In the sequel, we will not use the closed structure of $\mathbf{Qcoh}\,(G, X)$ described in the proof above, so we will always use this notation to refer to the (poorer) structure of $\mathbf{Qcoh}\, Y$-category.

Definition 5.3. Let X be a G-scheme over k, $s_X : X \to \mathrm{Spec}\, k$ the structure morphism, and $\mathcal{A}$ a full subcategory of $k[G]\,\mathbf{mod}$.

The associated *Auslander algebra over Y* is the ring $\mathcal{A}_X$ over Y equal to the full subcategory of $\mathbf{Qcoh}\,(G, X)$ whose objects are of the form $s_X^* V$, for all objects V of $\mathcal{A}$.

In other words, the objects of $\mathcal{A}_X$ are the same as those of $\mathcal{A}$, the morphism sheaves are given by: $\mathcal{A}_X(V, V') = \mathbf{Qcoh}\,(G, X)(s_X^* V, s_X^* V')$, and the neutral and the composition are those induced by the ones of $\mathbf{Qcoh}\,(G, X)$.

5.1.2. Comparison to the constant algebra

Definition 5.4. Let X be a G-scheme over k, $s_Y : Y \to \mathrm{Spec}\, k$ the structure morphism of the quotient, and $\mathcal{A}$ a full subcategory of $k[G]\,\mathbf{mod}$.

The constant Auslander algebra over Y is the ring $\mathcal{A}_X^c = s_Y^* \mathcal{A}$.

Proposition 5.5. *There is a natural morphism of rings over Y:* $\mathcal{A}_X^c \to \mathcal{A}_X$ *which is an isomorphism if the action of G on X is trivial.*

Proof. To define the morphism, we can use adjunction, and the second assertion is clear. □

5.1.3. Comparison to the "free" algebra

Definition 5.6. Let X be a G-scheme over k, $s_X : X \to \operatorname{Spec} k$ the structure morphism, and $\mathcal{A}$ a full subcategory of $k[G]\,\mathbf{mod}$.

The "*free*" *Auslander algebra over* Y associated is the ring $\mathcal{A}_X^f$ over Y equal to the full subcategory of $\mathbf{Qcoh}\,Y$ whose objects are of the form $\pi_*^G s_X^* V$, for all objects V of $\mathcal{A}$.

In other words the objects of $\mathcal{A}_X^f$ are the same as those of $\mathcal{A}$, the morphism sheaves are given by: $\mathcal{A}_X^f(V, V') = \mathbf{Qcoh}\,Y(\pi_*^G s_X^* V, \pi_*^G s_X^* V')$, and the neutral and the composition are those induced by the ones of $\mathbf{Qcoh}\,Y$.

Proposition 5.7. *There is a natural morphism of rings over* Y: $\mathcal{A}_X \to \mathcal{A}_X^f$ *which is an isomorphism if the action of G on X is free.*

Proof. The existence of the morphism is no more than the last assertion of Lemma 5.2. When the action is free, it is a classical result in descent theory that π_*^G : $\operatorname{Qcoh}(G, X) \to \operatorname{Qcoh} Y$ is an equivalence of categories (see for instance [27, Theorem 4.23, Theorem 4.45]), and this implies the proposition. □

5.1.4. Functoriality

Let $f : X' \to X$ a morphism of G-schemes. This defines a map between quotient schemes fitting in a commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow[\tilde{f}]{} & Y \end{array} \qquad (3)$$

Lemma 5.8. (i) $f : X' \to X$ *induces a morphism of ringed schemes* $(\tilde{f}, \tilde{f}^\#)$: $(Y', \mathcal{A}_{X'}) \to (Y, \mathcal{A}_X)$.

(ii) $\operatorname{adj}(\tilde{f}^\#) : \tilde{f}^* \mathcal{A}_X \to \mathcal{A}_{X'}$ *is an isomorphism if the diagram* 3 *above is fibred*, *i.e. if* $X' = X \times_Y Y'$.

Proof. (i) To construct $\tilde{f}^\#$, we start from the isomorphism $f^*\mathbf{Qcoh}\,X(s_X{}^*V, s_X{}^*W) \to \mathbf{Qcoh}\,X'(s_{X'}{}^*V, s_{X'}{}^*W)$ given by the fact that $f^* : \mathbf{Qcoh}\,X \to \mathbf{Qcoh}\,X'$ is a closed functor. This is, in fact, a G-isomorphism and gives by adjunction a G-morphism $\mathbf{Qcoh}\,X(s_X{}^*V, s_X{}^*W) \to f_*\mathbf{Qcoh}\,X'(s_{X'}{}^*V, s_{X'}{}^*W)$. Applying π_*^G and using the fact that $\pi_*^G f_* = \tilde{f}_* \pi'^G_*$, we get the map $\mathcal{A}_X(s_X{}^*V, s_X{}^*W) \to \tilde{f}_* \mathcal{A}_{X'}(s_{X'}{}^*V, s_{X'}{}^*W)$ that we needed.

(ii) By base change, the canonical 2-arrow $\tilde{f}^*\pi_*^G \Longrightarrow \pi'^G_* f^*$ is an isomorphism, and the result follows. □

5.1.5. Change of group

To deal with the problem of change of group, we have to enlarge slightly our definition of the Auslander algebra to include the case of the basic data being a *functor* $F : \mathcal{A} \to k[G]\,\mathbf{mod}$, not only an inclusion. We still denote by $\mathcal{A}_X$ (instead of the better $\mathcal{A}_F$) the corresponding ring over $Y = X/G$ whose objects are those of $\mathcal{A}$, and whose morphisms are given by $\mathcal{A}_X(V, V') = \mathbf{Qcoh}\,(G, X)(s_X^* FV, s_X^* FV')$.

Now let $\alpha : H \to G$ be a group morphism. We define $(\alpha^*\mathcal{A})_X$ as the ring on $Z = X/H$ corresponding to the functor $\alpha^* F : \mathcal{A} \to k[G]\,\mathbf{mod} \to k[H]\,\mathbf{mod}$. Let $\tilde{\alpha} : Z \to Y$ be the canonical morphism. There is a natural morphism $\tilde{\alpha}^\# : \mathcal{A}_X \to \tilde{\alpha}_*(\alpha^*\mathcal{A})_X$ of rings over Y, in other words, we have a morphism of ringed schemes

$$(\tilde{\alpha}, \tilde{\alpha}^\#) : (Z, (\alpha^*\mathcal{A})_X) \to (Y, \mathcal{A}_X)$$

Suppose moreover that α is an inclusion, and $X = G \times^H X'$, for an H-scheme X', with quotient $Y' = X'/H$. We have a canonical H-morphism $X' \to X_{|H}$ and a corresponding morphism of ringed schemes for $\alpha^*\mathcal{A}$.

Lemma 5.9. *With notations as above, the canonical morphism of ringed schemes*

$$(Y', (\alpha^*\mathcal{A})_{X'}) \to (Z, (\alpha^*\mathcal{A})_X) \to (Y, \mathcal{A}_X)$$

is an isomorphism.

Proof. This is the fact that restriction along $X' \to X_{|H} \to X$ induces an equivalence $\mathrm{Qcoh}\,(G, X) \simeq \mathrm{Qcoh}\,(H, X')$. □

5.2. $\mathcal{A}$-sheaves

5.2.1. Definition

Definition 5.10. Let X be a G-scheme over k, $\pi : X \to Y$ the quotient, and $\mathcal{A}$ a full subcategory of $k[G]\,\mathbf{mod}$. An $\mathcal{A}$-sheaf on X is, by definition, a quasicoherent sheaf for the ringed space $(Y, \mathcal{A}_X)$. More precisely, we define $\mathbf{Qcoh}\,(\mathcal{A}, X)$ as $\mathbf{Qcoh}\,(Y, \mathcal{A}_X)$, and $\mathrm{Qcoh}\,(\mathcal{A}, X)$ as $\mathrm{Qcoh}\,(Y, \mathcal{A}_X)$.

5.2.2. From G-sheaves to $\mathcal{A}$-sheaves

For any $\mathcal{A}$, we have a $\mathbf{Qcoh}\,Y$-functor:

$$U_{\mathcal{A}} : \mathbf{Qcoh}\,(G, X) \longrightarrow \mathbf{Qcoh}\,(\mathcal{A}, X)$$

$$\mathcal{F} \longrightarrow \underline{\mathcal{F}} = (V \longrightarrow \mathbf{Qcoh}\,(G, X)(s_X^* V, \mathcal{F}))$$

obtained by composing the Yoneda embedding $\mathbf{Qcoh}\,(G,X) \to [\mathbf{Qcoh}\,(G,X)^{op}, \mathbf{Qcoh}\,Y]$ and restriction along $\mathcal{A}_X^{op} \to \mathbf{Qcoh}\,(G,X)^{op}$.

Lemma 5.11. *Let $\mathcal{A} = \langle k[G]\rangle$ be the category with only the free object $k[G]$. Then $U_{\langle k[G]\rangle}$ is an equivalence of categories.*

Proof. Notice that $\langle k[G]\rangle_X$ is defined as the sheaf of one object algebras on Y given by $\mathbf{Qcoh}\,(G,X)(s_X^*(k[G]), s_X^*(k[G]))$, but this is easily seen as isomorphic to $((\pi_*\mathcal{O}_X) * G)^{op}$, the opposite of the sheaf of twisted algebrae defined by the action of G on $\pi_*\mathcal{O}_X$. So $\langle k[G]\rangle_X^{op}$ is identified with $(\pi_*\mathcal{O}_X) * G$, and under this isomorphism $U_{\langle k[G]\rangle}$ sends the G-sheaf $\mathcal{F}$ to the $(\pi_*\mathcal{O}_X) * G$-sheaf $\pi_*\mathcal{F}$. Since π is affine, this is an equivalence. $\square$

Proposition 5.12. *Let X be a G-scheme over k, $\mathcal{A}$ a full subcategory of $k[G]\,\mathbf{mod}$ containing $k[G]$, the free object of rank 1. Then $\mathbf{Qcoh}\,(G,X)$ is a reflective subcategory of $\mathbf{Qcoh}\,(\mathcal{A},X)$. More precisely, the functor $U_{\mathcal{A}}$ admits a left adjoint R such that the counit $RU_{\mathcal{A}} \Longrightarrow 1$ is an isomorphism.*

Proof. The only thing to verify to be able to apply Corollary 3.3 is that the composite of the equivalence $U_{\langle k[G]\rangle}$ and the right Kan extension $\mathbf{Qcoh}\,(\langle k[G]\rangle, X) \to \mathbf{Qcoh}\,(\mathcal{A},X)$ coincides with $U_{\mathcal{A}}$, but this is immediate. $\square$

Corollary 5.13. *Suppose moreover that $\mathcal{A}$ contains only projective objects. Then $U_{\mathcal{A}}$ is an equivalence.*

Proof. Lemma 3.4 allows to reduce to the case where $\mathcal{A} = \langle k[G]\rangle$, which was the object of Lemma 5.11. $\square$

5.2.3. Change of ring

Proposition 5.14. *Let X be a G-scheme over k, $\mathcal{A}$ a full subcategory of $k[G]\,\mathbf{mod}$, and $\mathcal{A}'$ be a full subcategory of $\mathcal{A}$.*

(i) *The restriction functor $R : \mathbf{Qcoh}\,(\mathcal{A},X) \to \mathbf{Qcoh}\,(\mathcal{A}',X)$ admits as a right adjoint the functor K defined on objects by*

$$\begin{array}{ll} K : \mathbf{Qcoh}\,(\mathcal{A}',X) \longrightarrow \mathbf{Qcoh}\,(\mathcal{A},X) & \\ \qquad \mathcal{F} \longrightarrow (V \longrightarrow \mathbf{Qcoh}\,(\mathcal{A}',X)(\mathcal{A}_X(\cdot,V)_{|\mathcal{A}'_X}, \mathcal{F})) & \end{array}$$

Moreover $RK \simeq 1$ (equivalently, K is fully faithful).

(ii) *In particular, if $\mathcal{A}$ is projectively complete, and every object of $\mathcal{A}$ is a direct summand of a finite direct sum of objects of $\mathcal{A}'$, this adjunction is an equivalence $\mathbf{Qcoh}\,(\mathcal{A},X) \simeq \mathbf{Qcoh}\,(\mathcal{A}',X)$.*

Proof. Condition (i) follows from Corollary 3.3 and Proposition 4.2, and (ii) from Corollary 3.4. □

5.2.4. Action of **Qcoh** Y

Starting from the external $\mathcal{H}om$:

$$(\mathbf{Qcoh}\,Y)^{op} \otimes \mathbf{Qcoh}\,(\mathcal{A}, X) \longrightarrow \mathbf{Qcoh}\,(\mathcal{A}, X)$$

$$\mathcal{G} \otimes \mathcal{F} \longrightarrow \mathcal{H}om(\mathcal{G}, \mathcal{F}) = (V \longrightarrow \mathcal{H}om(\mathcal{G}, \mathcal{F}(V)))$$

we get as a left adjoint an action of **Qcoh** Y on **Qcoh** $(\mathcal{A}, X)$:

$$\mathbf{Qcoh}\,Y \otimes \mathbf{Qcoh}\,(\mathcal{A}, X) \longrightarrow \mathbf{Qcoh}\,(\mathcal{A}, X)$$

$$\mathcal{G} \otimes \mathcal{F} \longrightarrow \mathcal{G} \otimes \mathcal{F}$$

5.2.5. Functoriality

Definition 5.15. Let $f : X' \to X$ be a morphism of G-schemes, and $(\tilde{f}, \tilde{f}^{\#})$ the morphism of ringed schemes associated by Lemma 5.8. The pull-back $f^{\triangle}$ (resp. the push-forward $f_{\triangle}$) given by Definition 4.10 will be denoted by $f^{\mathcal{A}}$ (resp. $f_{\mathcal{A}}$).

In particular, for each point Q of $\mathcal{A}$, we get a stalk functor:

$$\mathbf{Qcoh}\,(\mathcal{A}, X) \longrightarrow \mathbf{Qcoh}\,(\mathcal{A}, X \times_Y \operatorname{spec} \mathcal{O}_{Y,Q})$$

$$\mathcal{F} \longrightarrow \mathcal{F}_Q$$

Indeed, with the notations of §4.3.4, Lemma 5.8(ii) implies that $(\mathcal{A}_X)_Q \simeq \mathcal{A}_{X \times_Y \operatorname{spec} \mathcal{O}_{Y,Q}}$.

5.2.6. Adjunction

Proposition 5.16. *Let $f : X' \to X$ be a morphism of G-schemes. The couple $(f^{\mathcal{A}}, f_{\mathcal{A}})$ is part of a* **Qcoh** Y*-adjunction between* **Qcoh** $(\mathcal{A}, X)$ *and* $\tilde{f}_*$**Qcoh** $(\mathcal{A}, X')$.

Proof. This follows from Proposition 4.11. □

5.2.7. Representable sheaves

Let X be a G-scheme, and $\mathcal{A}$ be a full subcategory of $k[G]$**mod**. Then the category **Qcoh** $(\mathcal{A}, X) = [\mathcal{A}_X^{op}, \mathbf{Qcoh}\,Y]$ contains for each V the corresponding representable functor, which we denote by $(\mathcal{A}_X)_V$, and call a *representable sheaf*. There is an obvious local notion of *locally representable sheaf*. Moreover, if $f : X' \to X$ is any G-morphism, then one checks that $f^{\mathcal{A}}(\mathcal{A}_X)_V \simeq (\mathcal{A}_{X'})_V$, hence both notions are preserved by arbitrary pullback.

5.2.8. Change of group

We keep the notations of §5.1.5: X is a G-scheme, $\alpha : H \to G$ a group morphism, and $F : \mathcal{A} \to k[G]\mathbf{mod}$ a functor. The associated morphism of ringed schemes

$$(\tilde{\alpha}, \tilde{\alpha}^{\#}) : (Z, (\alpha^*\mathcal{A})_X) \to (Y, \mathcal{A}_X)$$

and Definition 4.10 provides a restriction functor $\alpha^{\mathcal{A}} : \mathbf{Qcoh}\,(\mathcal{A}, X) \to \tilde{\alpha}_*\mathbf{Qcoh}\,(\alpha^*\mathcal{A}, X_{|H})$ and an induction functor $\alpha_{\mathcal{A}} : \tilde{\alpha}_*\mathbf{Qcoh}\,(\alpha^*\mathcal{A}, X_{|H}) \to \mathbf{Qcoh}\,(\mathcal{A}, X)$. Again, Proposition 4.11 implies that $(\alpha^{\mathcal{A}}, \alpha_{\mathcal{A}})$ is part of a natural adjunction between $\mathbf{Qcoh}\,(\mathcal{A}, X)$ and $\tilde{\alpha}_*\mathbf{Qcoh}\,(\alpha^*\mathcal{A}, X_{|H})$.

If, moreover, α is an inclusion, and $X = G \times^H X'$, for a H-scheme X', with quotient $Y' = X'/H$, Lemma 5.9 implies that there is a canonical equivalence

$$\mathbf{Qcoh}\,(\mathcal{A}, X) \simeq \mathbf{Qcoh}\,(\alpha^*\mathcal{A}, X').$$

5.2.9. Internal homs and tensor product

For the time being, we have just considered $\mathbf{Qcoh}\,(\mathcal{A}, X)$ as an enriched category over $\mathbf{Qcoh}\,Y$. But the work of B.Day (see [5]) implies that if $\mathcal{A}$ is a submonoidal category of $k[G]\mathbf{mod}$ (endowed with the tensor product over k), then $\mathbf{Qcoh}\,(\mathcal{A}, X)$ carries the structure of a monoidal closed symmetric category, for which the functor of evaluation at the unit $\cdot(k) : \mathbf{Qcoh}\,(\mathcal{A}, X) \to \mathbf{Qcoh}\,Y$ is closed. Since by pushing $\mathbf{Qcoh}\,(\mathcal{A}, X)$, seen as enriched over itself, along this functor, we recover $\mathbf{Qcoh}\,(\mathcal{A}, X)$ seen as enriched category over $\mathbf{Qcoh}\,Y$, we keep the same notation.

The starting fact is the following: suppose $\mathcal{A}$ is a full submonoidal category of $k[G]\mathbf{mod}$, and X is a G-scheme. Then the Auslander algebra $\mathcal{A}_X^{op}$ has a natural structure of a monoidal symmetric category *over* $\mathbf{Qcoh}\,Y$. So [5, §3,§4] shows that there is a canonical structure of monoidal closed symmetric category on $\mathbf{Qcoh}\,(\mathcal{A}, X)$, whose unit object is

$$\underline{\mathcal{O}_X} : V \to \pi_*^G(s_X^* V^\vee)$$

whose internal homs are given by

$$\mathbf{Qcoh}\,(\mathcal{A}, X)(\mathcal{F}, \mathcal{G})(V) = \int_W \mathbf{Qcoh}\,Y(\mathcal{F}(W), \mathcal{G}(W \otimes_k V))$$

and whose tensor product is given by a convolution formula

$$\mathcal{F} \otimes \mathcal{G}(V) = \int^W \mathcal{F}(W) \otimes_{\mathcal{O}_Y} \mathcal{G}(V \otimes_k W^\vee).$$

Tensor product with a representable sheaf can be made more explicit:

Lemma 5.17.

$$(\mathcal{F} \otimes (\mathcal{A}_X)_V)(W) \simeq \mathcal{F}(W \otimes_k V^\vee).$$

Proof. Since $\mathcal{A}_X$ is dense (or adequate) in **Qcoh** $(\mathcal{A}, X)$, it suffices to check this on $\mathcal{F} = (\mathcal{A}_X)_{V'}$. But then it boils down to the fact that the tensor product on **Qcoh** $(\mathcal{A}, X)$ extends the tensor product on $\mathcal{A}_X$. □

We deduce a projection formula in this context:

Proposition 5.18. *Let $f : X' \to X$ be any G-morphism, $\mathcal{F}$ a locally representable $\mathcal{A}$-sheaf on X, $\mathcal{G}$ a quasicoherent $\mathcal{A}$-sheaf on X'. Then the natural morphism*

$$\mathcal{F} \otimes f_{\mathcal{A}}\mathcal{G} \to f_{\mathcal{A}}(f^{\mathcal{A}}\mathcal{F} \otimes \mathcal{G})$$

is an isomorphism.

Proof. This is a local problem, so we can suppose $\mathcal{F}$ representable. But now we can use Lemma 5.17 to conclude. □

5.2.10. Cohomology

Proposition 5.19. *Let X be a G-scheme over k, and $\mathcal{A}$ a full subcategory of $k[G]$***mod**. *The category* Qcoh $(\mathcal{A}, X)$ *has enough injective objects.*

Proof. Using Propositions 4.19 and 5.16, we see that the classical proof (see [11]) applies without change. □

Proposition 5.16 also shows that the functor $f_{\mathcal{A}}$ is left exact, hence the following definition.

Definition 5.20. Let $f : X' \to X$ be a morphism of G-schemes. We denote by $R^i f_{\mathcal{A}}$: Qcoh $(\mathcal{A}, X') \to$ Qcoh $(\mathcal{A}, X)$ the ith derived functor of $f_{\mathcal{A}}$.

In particular, when $X = \operatorname{spec} k$, we denote the derived functors of the global sections functor by $H^i(X', \cdot)$.

In view of Corollary 5.12, it is natural to compare the usual cohomology of a G-sheaf to the one of the corresponding $\mathcal{A}$-sheaf. We give three comparison results.

Proposition 5.21. *Let X be a G-scheme over k, $\mathcal{A}$ a full subcategory of $k[G]$* **mod** *containing $k[G]$, the free object of rank 1. Suppose given a G-sheaf $\mathcal{F}$ on X. There is a canonical G-isomorphism: $H^i(X, \underline{\mathcal{F}})(k[G]) \simeq H^i(X, \mathcal{F})$.*

Proof. This is an immediate consequence of the exactness of the evaluation $\cdot(k[G])$ and of the isomorphism $\underline{\mathcal{F}}(k[G]) \simeq \pi_* \mathcal{F}$. □

Proposition 5.22. *Let X be a G-scheme over k, $\mathcal{A}$ a full subcategory of $k[G]$* **mod** *containing $k[G]$, the free object of rank* 1. *Suppose that the action of G on X is tame. Then for each G-sheaf on X we have a spectral sequence*:

$$\mathrm{Ext}^p_{k[G]}(\cdot, H^q(X, \mathcal{F})) \Rightarrow H^{p+q}(X, \underline{\mathcal{F}}).$$

Proof. The tameness of the action says that the functor π_*^G is exact, and so is the functor $\mathcal{F} \to \underline{\mathcal{F}}$. Hence the result is a direct consequence of a Theorem of Grothendieck describing the derived functors of a composite functor. □

In particular, both cohomologies coincide for a reductive action, i.e. when the characteristic of k does not divide the order of G. But note that they can differ even for a free action, as soon as $p = \mathrm{car}\, k | \#G$.

Remember that the equivariant cohomology functors $H^i(X, G, \cdot)$ are defined as the derived functors of $H^0(X, \cdot)^G$ (see [12]).

Proposition 5.23. *Let X be a G-scheme over k, $\mathcal{A}$ a full subcategory of $k[G]$* **mod** *containing $k[G]$, the free object of rank* 1, *and k, the trivial representation. Then for each G-sheaf on X we have a spectral sequence*:

$$H^p(X, R^q U_{\mathcal{A}} \mathcal{F})(k) \Rightarrow H^{p+q}(X, G, \mathcal{F}).$$

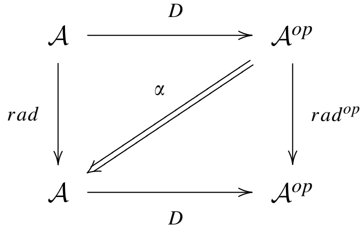
Proof. This is a consequence of the isomorphism $H^0(X, \underline{\mathcal{F}})(k) \simeq H^0(X, \mathcal{F})^G$. □

5.2.11. Canonical filtration

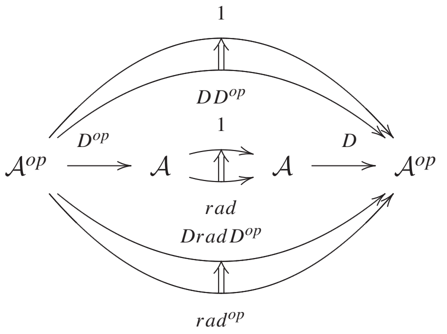
Carrying on with the idea of Nakajima [22], we will see that, if we impose some more structure on $\mathcal{A}$, then the kernel preserving $\mathcal{A}$-sheaves (and in particular those coming from G-sheaves) have a useful natural filtration of algebraic nature.

Since $k[G]$ is a semilocal ring, for each object V of $k[G]$ **mod**, we have $\mathrm{rad}\, V = (\mathrm{rad}\, k[G])V$. In particular there is a functor $\mathrm{rad} : k[G]\,\mathbf{mod} \to k[G]\,\mathbf{mod}$. Moreover we have a natural duality functor $D : k[G]\,\mathbf{mod} \to (k[G]\,\mathbf{mod})^{op}$ sending V to its dual $V^\vee$.

The data we need is the following: $\mathcal{A}$ is as usual a full subcategory of $k[G]$ **mod**, that we suppose stable under rad and D. We need also a compatibility between these two operations, in the sense that we suppose given a natural isomorphism α:



Using α, the natural transformation rad $\Longrightarrow 1$, and the canonical isomorphism β : $D\,D^{op} \Longrightarrow 1$ we get finally a natural transformation $\mathrm{rad}^{op} \Longrightarrow 1$, as sketched in the following diagram:



where the bottom natural transformation is $\alpha D^{op} \circ rad^{op} \beta^{-1}$.

This is a priori a natural transformation between additive functors from $\mathcal{A}^{op}$ to $\mathcal{A}^{op}$, but if X is a G-scheme, this extends to a natural transformation of the corresponding functors from $\mathcal{A}_X^{op}$ to $\mathcal{A}_X^{op}$, which we note the same way.

Applying now the 2-functor: $(\cdot, \mathbf{Qcoh}\, Y) : (\mathbf{Qcoh}\, Y - \mathrm{CAT})^{op} \to \mathcal{C}at$ we get a natural transformation $(\mathrm{rad}^{op}, \mathbf{Qcoh}\, Y) \Longrightarrow 1$ between endofunctors of $\mathrm{Qcoh}\,(\mathcal{A}, X)$. We define R as $(\mathrm{rad}^{op}, \mathbf{Qcoh}\, Y)$.

Let now $\mathcal{F}$ be a $\mathcal{A}$-sheaf on X such that $\mathcal{F}$, as a functor, preserves monomorphisms (this is in particular the case if $\mathcal{F} = \underline{\mathcal{G}}$ for a G-sheaf $\mathcal{G}$). Then the natural morphism $R\mathcal{F} \to \mathcal{F}$ is itself a monomorphism, and moreover $R\mathcal{F}$ preserves monomorphisms, so that $\mathcal{F}$ has a canonical filtration. We sum up the construction in the following definition:

Definition 5.24. Let $\mathcal{A}$ be a full subcategory of $k[G]$, stable under radical rad and duality D.

(i) We will say that *radical and duality commute* to say that we fix an isomorphism $\mathrm{rad}^{op} D \Longrightarrow Drad$.

(ii) Suppose that radical and duality commute, and let, moreover, X be a G-scheme, and $\mathcal{F}$ a $\mathcal{A}$-sheaf on X, whose underlying functor preserves monomorphisms. In these circumstances, we will denote the induced filtration on $\mathcal{F}$ by

$$\cdots \subset R^i\mathcal{F} \subset \cdots \subset R^2\mathcal{F} \subset R^1\mathcal{F} \subset \mathcal{F}$$

and the associated graded $\mathcal{A}$-sheaf by

$$\mathrm{gr}_i\, \mathcal{F} = R^i\mathcal{F}/R^{i+1}\mathcal{F}.$$

The existence of an isomorphism $\mathrm{rad}^{op} D \Longrightarrow Drad$ is not the general rule. Indeed, we have a natural isomorphism:

$$\mathrm{rad}(V^\vee) \simeq \left(\frac{V}{\mathrm{soc}\, V}\right)^\vee.$$

where $\operatorname{soc} V$ is the sum of all simple submodules of V, but in general $V/\operatorname{soc} V$ is *not* isomorphic to $\operatorname{rad} V$.

We finish by a positive example, to be used in the sequel: let G be a cyclic of order n, with fixed generator σ. We choose as ring with several object $\mathcal{A} = k[G]\,\mathbf{mod}$, the whole category of all $k[G]$-modules of finite type.

The representation theory of G is easily described, as follows. Write $n = p^v a$, where a is prime to p. For each integer $0 \leqslant j \leqslant p^v$ define $V_j = k[G]/(\sigma - 1)^j$. Then the set of modules $\{\psi \otimes V_j \;/\; \psi \in \widehat{G},\; 1 \leqslant j \leqslant p^v\}$ is a skeleton of the indecomposables in $\mathcal{A}$.

To construct a natural isomorphism $V/\operatorname{soc} V \simeq \operatorname{rad} V$, we can restrict to the indecomposable skeleton we have just fixed. Now the maps

$$\psi \otimes (\sigma - 1) : \psi \otimes V_j \to \psi \otimes V_j$$

give a natural transformation, $1 \Longrightarrow 1$, which canonically factorizes in a natural isomorphism $V/\operatorname{soc} V \simeq \operatorname{rad} V$.

6. Modular K-theory

6.1. Definition

Definition 6.1. Let X be a G-scheme over k and $\mathcal{A}$ a full subcategory of $k[G]\,\mathbf{mod}$. A quasicoherent $\mathcal{A}$-sheaf $\mathcal{F}$ on X is said to be *coherent* if for each G-invariant open affine $U = \operatorname{spec} R$ of X, the restriction $\mathcal{F}_{|U}$ is of finite type in $\mathbf{Qcoh}\,(\mathcal{A}, U) \simeq [\mathcal{A}_X(U)^{op}, R^G\,\mathbf{Mod}\,]$ (i.e. if, seen in $[\mathcal{A}_X(U)^{op}, R^G\,\mathbf{Mod}\,]$, it is a quotient of a finite sum of representable objects). We denote by $\operatorname{Coh}(\mathcal{A}, X)$ the full subcategory of $\operatorname{Qcoh}(\mathcal{A}, X)$ whose objects are the coherent $\mathcal{A}$-sheaves.

This notion is Morita invariant, at least in the following sense:

Proposition 6.2. *Let X be a G-scheme over k, $\mathcal{A}$ a full subcategory of $k[G]\,\mathbf{mod}$, and $\mathcal{A}'$ be a full subcategory of $\mathcal{A}$. Suppose that $\mathcal{A}$ is projectively complete, and that every object of $\mathcal{A}$ is a direct summand of a finite direct sum of objects of $\mathcal{A}'$. Then restriction along $\mathcal{A}' \to \mathcal{A}$ induces an equivalence* $\operatorname{Coh}(\mathcal{A}, X) \simeq \operatorname{Coh}(\mathcal{A}', X)$.

Proof. We know from Proposition 5.14 that restriction along $\mathcal{A}' \to \mathcal{A}$ induces an equivalence $\operatorname{Qcoh}(\mathcal{A}, X) \simeq \operatorname{Qcoh}(\mathcal{A}', X)$. Because of the hypothesis, this restriction sends $\operatorname{Coh}(\mathcal{A}, X)$ to $\operatorname{Coh}(\mathcal{A}', X)$. Moreover, the left adjoint $\otimes_{\mathcal{A}'_X}\mathcal{A}_X$ is an inverse equivalence, and since it is right exact and preserves representables, it sends $\operatorname{Coh}(\mathcal{A}', X)$ to $\operatorname{Coh}(\mathcal{A}, X)$. $\square$

The $\mathbf{Qcoh}\,Y$ enriched functor $\mathbf{Coh}Y \to \mathbf{Qcoh}\,Y$ allows to identify $\mathbf{Qcoh}\,Y{-}\mathrm{CAT}(\mathcal{A}_X^{op}, \mathbf{Coh}Y)$ with a subcategory of $\operatorname{Qcoh}(\mathcal{A}, X)$. Since $\mathcal{A}_X$ is in fact enriched in $\mathbf{Coh}Y$, $\operatorname{Coh}(\mathcal{A}, X)$ is a subcategory of $\mathbf{Qcoh}\,Y - \mathrm{CAT}(\mathcal{A}_X^{op}, \mathbf{Coh}Y)$. Moreover:

Lemma 6.3. *Suppose $\mathcal{A}$ admits a finite set of additive generators. Then* $\mathrm{Coh}\,(\mathcal{A}, X) =$ $\mathbf{Qcoh}\,Y - \mathrm{CAT}(\mathcal{A}_X^{op}, \mathbf{Coh}Y)$.

Proof. This is a local question, hence we can conclude by applying Lemma 2.24, which is of course valid with the field k replaced by any commutative noetherian ring. □

Lemma 6.4. *Suppose $\mathcal{A}$ admits a finite set of additive generators. Then* $\mathrm{Coh}\,(\mathcal{A}, X)$ *is an abelian category.*

Proof. This is clear from Proposition 6.2, which allows to reduce to the case when $\mathcal{A}$ has only one object. This follows also from Lemma 6.3, because we can follow word for word the proof of Proposition 4.18. □

Definition 6.5. Let X be a G-scheme over k and $\mathcal{A}$ a full subcategory of $k[G]\,\mathbf{mod}$ admitting a finite set of additive generators. We denote by $K_i(\mathcal{A}, X)$ the Quillen ith group of the abelian category $\mathrm{Coh}\,(\mathcal{A}, X)$.

6.2. Functoriality

6.2.1. Pullback

Let $f : X' \to X$ be a morphism of G-schemes over k such that the morphism $\tilde{f} : Y' \to Y$ between quotient schemes is flat, and $X' = X \times_Y Y'$. Then the functor $f^{\mathcal{A}} : \mathrm{Coh}\,(\mathcal{A}, X) \to \mathrm{Coh}\,(\mathcal{A}, X')$ is exact, and hence induces a map in K-theory.

6.2.2. Pushforward

Lemma 6.6. *Let $f : X' \to X$ be a morphism of G-schemes over k and $\mathcal{A}$ a full subcategory of* $k[G]\,\mathbf{mod}$, *admitting a finite set of additive generators. Suppose that the morphism $\tilde{f} : Y' \to Y$ between quotient schemes is proper, then*:

(i) *For each coherent $\mathcal{A}$-sheaf $\mathcal{F}$ on X', and each nonnegative integer i, the $\mathcal{A}$- sheaf $R^i f_{\mathcal{A}}\mathcal{F}$ is coherent.*
(ii) *There exists an integer n, such that for any integer $i > n$, and any coherent $\mathcal{A}$-sheaf $\mathcal{F}$ on X', we have $R^i f_{\mathcal{A}}\mathcal{F} = 0$.*

Proof. Because cohomology commutes with the projection p_V (i.e. for any V in $\mathrm{obj}\,\mathcal{A}$, $R^i f_{\mathcal{A}}\mathcal{F}(V) = R^i f_{\langle V\rangle}(\mathcal{F}(V)))$, Lemma 6.3 allows to reduce to the case where $\mathcal{A}$ has only one object (one can also use Proposition 6.2 to reduce to this case). But since $R^i f_{\langle V\rangle}(\mathcal{F}(V))$, seen in $\mathrm{Qcoh}\,Y$, is nothing else that $R^i \tilde{f}(\mathcal{F}(V))$, the Lemma results from [12, 3.2.1, 1.4.12]. □

Now, given such a $f : X' \to X$, we can follow the argument given in [23, §7, 2.7] to define a map

$$f_{\mathcal{A}} : K_i(\mathcal{A}, X') \to K_i(\mathcal{A}, X)$$

in the following two cases:

(i) $\tilde{f}$ is finite,
(ii) Y' admits an ample line bundle (then we have to use the action of $\operatorname{Qcoh} Y$ on $\operatorname{Qcoh}(\mathcal{A}, X)$ defined in Section 5.2.4).

Given $f : X' \to X$ and $g : X'' \to X'$, both satisfying the condition of Lemma 6.6, and one of the conditions above, then the formula $(fg)_{\mathcal{A}} = f_{\mathcal{A}} g_{\mathcal{A}}$ holds.

6.3. Localization

Theorem 6.7. *Let $i : X' \to X$ be a morphism of G-schemes over k, and $\mathcal{A}$ a full subcategory of $k[G]$* **mod**, *admitting a finite set of additive generators.*

Suppose that morphism $\tilde{i} : Y' \to Y$ between quotient schemes is a closed immersion, and that $i^{\#} : \mathcal{A}_X \to \tilde{i}_* \mathcal{A}_{X'}$ is an epimorphism.

Denote by U the pullback by $\pi : X \to Y$ of the complement of Y' in Y, and by $j : U \to X$ the canonical inclusion.

Then there is a long exact sequence:

$$\cdots \to K_i(\mathcal{A}, X') \xrightarrow{i_{\mathcal{A}}} K_i(\mathcal{A}, X) \xrightarrow{j^{\mathcal{A}}} K_i(\mathcal{A}, U) \to \cdots \to K_1(\mathcal{A}, X') \xrightarrow{i_{\mathcal{A}}} K_1(\mathcal{A}, X) \xrightarrow{j^{\mathcal{A}}} K_1(\mathcal{A}, U) \to K_0(\mathcal{A}, X') \xrightarrow{i_{\mathcal{A}}} K_0(\mathcal{A}, X) \xrightarrow{j^{\mathcal{A}}} K_0(\mathcal{A}, U) \to 0$$

Proof. The hypothesis on $\mathcal{A}$ and Proposition 6.2 allows reduce to the case where $\mathcal{A}$ has only one object, what we will do from now on.

The idea is of course to apply [23, §5, Theorem 5], but to do so we need the two following facts.

Denote by $\mathrm{Coh}\,(\mathcal{A}, X)_{Y'}$ the full subcategory of $\mathrm{Coh}\,(\mathcal{A}, X)$ consisting of sheaves with support from Y'. Being the kernel of the restriction functor $j^{\mathcal{A}}$, this is a Serre subcategory, and the first step consists of showing that $j^{\mathcal{A}}$ induces an equivalence

$$\mathrm{Coh}\,(\mathcal{A}, X)/\mathrm{Coh}\,(\mathcal{A}, X)_{Y'} \simeq \mathrm{Coh}\,(\mathcal{A}, U). \tag{4}$$

This is the object of Section 6.4. Note that the notion of quotient category used here (quotient as an example of localization) has nothing to see with the notion of quotient used in Section 2 (Quotient by a two-sided ideal).

Then we will show that we can apply the hypothesis of the dévissage Theorem [23, §5, Theorem 4] to the functor $i_{\mathcal{A}} : \mathrm{Coh}\,(\mathcal{A}, X') \to \mathrm{Coh}\,(\mathcal{A}, X)_{Y'}$: this is the aim of Section 6.5.

We sum up the notations we used in the proof in the folloing diagram:

$$\begin{array}{ccccc} X' & \xrightarrow{\;i\;} & X & \xleftarrow{\;j\;} & U \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{|U}} \\ Y' & \xrightarrow[\;\tilde{i}\;]{} & Y & \xleftarrow[\;\tilde{j}\;]{} & O \end{array} \tag{5}$$

□

6.4. Restriction of coherent $\mathcal{A}$-sheaves to an open

Because we lack a complete reference we recall the classical arguments.

Proposition 6.8. *Let $F : \mathcal{B} \to \mathcal{C}$ be an exact functor between abelian categories such that*

(i) *for any object N of $\mathcal{C}$, there is an object M of $\mathcal{B}$ and an isomorphism $FM \simeq N$,*
(ii) *for any objects M, M' of $\mathcal{B}$, and any map $x : FM \to FM'$ in $\mathcal{C}$, there exists a diagram in $\mathcal{B}$*

$$\begin{array}{ccc} & M'' & \\ {\scriptstyle u}\swarrow & & \searrow{\scriptstyle v} \\ M & & M' \end{array}$$

such that the diagram

$$\begin{array}{ccc} & FM'' & \\ {\scriptstyle Fu}\swarrow & & \searrow{\scriptstyle Fv} \\ FM & \xrightarrow{\quad x \quad} & FM' \end{array}$$

commutes in $\mathcal{C}$, and such that Fu is an isomorphism.

Then the canonical functor $\mathcal{B}/\mathrm{Ker}\,F \to \mathcal{C}$ is an equivalence of categories.

Proof. Condition (i) shows that $\mathcal{B}/\mathrm{Ker}\,F \to \mathcal{C}$ is essentially full, and (ii) that it is fully faithful. □

Hence equivalence (4) will follow from the three following lemmas.

Lemma 6.9. *Let $\mathcal{F}$ be a coherent $\mathcal{A}$-sheaf on X, and $\beta : \mathcal{G} \to \mathcal{F}_{|U}$ a monomorphism in* Coh $(\mathcal{A}, U)$. *There exists a monomorphism $\alpha : \mathcal{G}' \to \mathcal{F}$ in* Coh $(\mathcal{A}, X)$ *such that $\alpha_{|U} = \beta$ (as subobjects of $\mathcal{F}_{|U}$).*

Proof. As in [2, Proposition 1], let $\mathcal{G}'$ be the sheaf on Y associated to the presheaf $O' \to \{s \in \mathcal{F}(O')/\exists t \in \mathcal{G}(O \cap O')/\beta(t) = s_{|O\cap O'}\}$, and $\alpha : \mathcal{G}' \to \mathcal{F}$ be the canonical map. In [2] it is shown that α is a map in Coh Y such that $\alpha_{|U} = \beta$ as subobjects of $\mathcal{F}_{|U}$ in Coh O. But from its definition, one sees at once that $\mathcal{G}'$ is stable under the action of $\mathcal{A}$, hence has a unique structure of $\mathcal{A}$-sheaf such that α is an arrow in Coh $(\mathcal{A}, X)$. □

Lemma 6.10. *Let $\mathcal{G}$ be a coherent $\mathcal{A}$-sheaf on U. There exists a coherent $\mathcal{A}$-sheaf $\mathcal{F}$ on X such that $\mathcal{F}_{|U} = \mathcal{G}$.*

Proof. Follows Lemma 6.9 as in the proof of [2, Proposition 2]. □

Lemma 6.11. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two $\mathcal{A}$-sheaves on X, and $\gamma : \mathcal{F}_{|U} \to \mathcal{F}'_{|U}$ a morphism in* Coh $(\mathcal{A}, U)$. *There exists a diagram in* Coh $(\mathcal{A}, X)$

$$\begin{array}{ccc} & \mathcal{H} & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ \mathcal{F} & & \mathcal{F}' \end{array}$$

such that the diagram

$$\begin{array}{ccc} & \mathcal{H}_{|U} & \\ {\scriptstyle\alpha_{|U}}\swarrow & & \searrow{\scriptstyle\beta_{|U}} \\ \mathcal{F}_{|U} & \xrightarrow{\gamma} & \mathcal{F}'_{|U} \end{array}$$

commutes in Coh $(\mathcal{A}, U)$, *and such that $\alpha_{|U}$ is an isomorphism.*

Proof. Apply Lemma 6.9 to the graph of γ. □

6.5. Dévissage

We have to show that each object $\mathcal{F}$ in Coh $(\mathcal{A}, X)$ with support in Y' has a finite filtration whose quotients are objects in the image of $i_{\mathcal{A}} : \mathrm{Coh}\,(\mathcal{A}, X') \to \mathrm{Coh}\,(\mathcal{A}, X)$.

Let $\mathcal{J}_{X'} = \ker(i^\# : \mathcal{A}_X \to \tilde{i}_* \mathcal{A}_{X'})$. Since we have an exact sequence $0 \to \mathcal{F}\mathcal{J}_{X'} \to \mathcal{F} \to i_{\mathcal{A}} i^{\mathcal{A}} \mathcal{F} \to 0$, it is enough to show that for n large enough $\mathcal{F}\mathcal{J}_{X'}^n = 0$.

Define $\mathcal{I}_{Y'} = \ker(\mathcal{O}_Y \to \tilde{i}_* \mathcal{O}_{Y'})$, the ideal sheaf of Y'. Since the support of $\mathcal{F}$ is included in Y', the Nullstellensatz ensures that for n large enough we have $\mathcal{I}_{Y'}^n \subset \operatorname{ann} \mathcal{F}$ hence the following lemma will be enough to conclude.

Lemma 6.12. *For n large enough $\mathcal{J}_{X'}^n \subset \mathcal{I}_{Y'} \mathcal{A}_X$.*

Proof. This proof was suggested to me by Vistoli.

First remember that we can suppose that $\mathcal{A}$ has only one object, and denote by I this $k[G]$-module. Then G acts by ring homomorphisms on $\operatorname{End}_k I$, and if $s_X : X \to \operatorname{Spec} k$ denotes the structure morphism, we have that $\mathcal{A}_X = \pi_*^G s_X^* \operatorname{End}_k I$.

Define $\mathcal{I}_{X'} = \ker(\mathcal{O}_X \to i_* \mathcal{O}_{X'})$. The left exactness of π_*^G implies that $\mathcal{J}_{X'} = \pi_*^G (\mathcal{I}_{X'} s_X^* \operatorname{End}_k I)$. Hence for all n:

$$\mathcal{J}_{X'}^n = (\pi_*^G (\mathcal{I}_{X'} s_X^* \operatorname{End}_k I))^n \subset \pi_*^G ((\mathcal{I}_{X'} s_X^* \operatorname{End}_k I)^n) = \pi_*^G (\mathcal{I}_{X'}^n s_X^* \operatorname{End}_k I).$$

Fix an integer r. Applying the Nullstellensatz again, we have that for large $n : \mathcal{I}_{X'}^n \subset (\pi^* \mathcal{I}_{Y'})^r$, hence for large n:

$$\mathcal{J}_{X'}^n \subset \pi_*^G ((\pi^* \mathcal{I}_{Y'})^r s_X^* \operatorname{End}_k I) = (I_{Y'}^r \pi_* (s_X^* \operatorname{End}_k I))^G = \mathcal{I}_{Y'}^r \pi_* (s_X^* \operatorname{End}_k I) \cap \mathcal{A}_X.$$

Since Y is noetherian, we can apply the Artin–Rees Lemma to conclude that the filtration $(\mathcal{I}_{Y'}^r \pi_* (s_X^* \operatorname{End}_k I) \cap \mathcal{A}_X)_{r \geqslant 0}$ is $\mathcal{I}_{Y'}$-stable. In particular for large r we have $\mathcal{I}_{Y'}^r \pi_* (s_X^* \operatorname{End}_k I) \cap \mathcal{A}_X \subset \mathcal{I}_{Y'} \mathcal{A}_X$, and the lemma is shown. □

6.6. A criterion of surjectivity

Proposition 6.13. *Let $i : X' \to X$ be a morphism of G-schemes over k, such that*

(i) *There is a normal subgroup H of G, such that H acts trivially on X', and G/H acts freely on X',*
(ii) *The morphism $\tilde{i} : Y' \to Y$ between quotient schemes is a closed immersion.*

Then the canonical morphism $i^\# : \mathcal{A}_X \to \tilde{i}_* \mathcal{A}_{X'}$ is an epimorphism.

Proof. For the moment being, we do not use hypothesis (i), and define $\nu : X \to Z = X/H$ and $\mu : Z \to Y = Z/P$ as the quotient morphisms, and similarly for X', so that we get a P-morphism $\hat{f} : Z' \to Z$ fitting in the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle \nu'}\downarrow & & \downarrow{\scriptstyle \nu} \\ Z' & \xrightarrow{\hat{f}} & Z \\ {\scriptstyle \mu'}\downarrow & & \downarrow{\scriptstyle \mu} \\ Y' & \xrightarrow[\tilde{f}]{} & Y \end{array}$$

Define a new ring $\mathcal{A}_X^{c,H}$ on Y by setting

$$\mathcal{A}_X^{c,H}(V, W) = \mu_*^P(\mathbf{Qcoh}\, Z(s_Z^* V, s_Z^* W)^H),$$

where $s_Z : Z \to \operatorname{spec} k$ is the structure morphism.

This definition is functorial in X.

Moreover, since $\pi_*^G = \mu_*^P \nu_*^H$, there is a canonical morphism $\mathcal{A}_X^{c,H} \to \mathcal{A}_X$, also functorial in X, so that we get a commutative diagram:

$$\begin{array}{ccc} \mathcal{A}_X & \longrightarrow & \tilde{f}_* \mathcal{A}_{X'} \\ \uparrow & & \uparrow \\ \mathcal{A}_X^{c,H} & \longrightarrow & \tilde{f}_* \mathcal{A}_{X'}^{c,H} \end{array}$$

The first part of hypothesis (i) means that $\nu' = 1$, and this implies that $\mathcal{A}_{X'}^{c,H} \simeq \mathcal{A}_{X'}$.

Since the question is local, we can use the second part of hypothesis (i) to drop the ramification locus of μ and thus reduce to the case where P acts freely on Z. But then descent theory implies that $\tilde{f}^* \mathcal{A}_X^{c,H} \simeq \mathcal{A}_{X'}^{c,H}$.

So the Proposition now follows from hypothesis (ii). □

6.7. *The case of a free action*

Proposition 6.14. *Let X be a G-scheme over k and $\mathcal{A}$ a full subcategory of $k[G]$**mod** admitting a finite set of additive generators, and containing $k[G]$, the free object of rank 1. If the action of G on X is free, then the functor $U_{\mathcal{A}} : \operatorname{Qcoh}(G, X) \to \operatorname{Qcoh}(\mathcal{A}, X)$ is an equivalence of categories.*

Proof. We can first compose with the equivalence $\pi^* : \operatorname{Qcoh} Y \to \operatorname{Qcoh}(G, X)$, and by Proposition 5.14 suppose that $\mathcal{A}$ has only one object I. Define $\mathcal{E} = \pi_*^G(s_X^* I)$.

Then the composite functor is given by: $\mathcal{G} \to \mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{E}^\vee$. But we have also a functor $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{A}_X} \mathcal{E}$ in the opposite direction, and natural units and counits. To check that these are isomorphisms is a local problem, and so is deduced by classical Morita theory, because by descent theory $\mathcal{E}$ is locally free, hence is locally a progenerator. □

Corollary 6.15. *If the action of G on X is free then for each i we have a natural isomorphism $K_i(Y) \simeq K_i(\mathcal{A}, X)$.*

Proof. One checks that the equivalence of categories of Proposition 6.14 preserves coherence. □

6.8. Euler characteristics of $\mathcal{A}$-sheaves

Let X be a proper k-scheme, endowed with the action of a finite group G, and $\mathcal{A}$ be a full subcategory of $k[G]\,\mathbf{mod}$, admitting a finite set of additive generators. Denoting by $s_X : X \to \operatorname{spec} k$ the structure morphism, we get a modular Euler characteristic $\chi(\mathcal{A}, \cdot) = (s_X)_\mathcal{A} : K_0(\mathcal{A}, X) \to K_0\,(\mathbf{mod}\ \mathcal{A})$ defined by the usual formula $\chi(\mathcal{A}, \mathcal{F}) = \sum_{i \geqslant 0} (-1)^i [H^i(X, \mathcal{F})]$.

For a sheaf $\mathcal{G}$ on Y, we denote as usual by $\chi(\mathcal{G})$ its ordinary Euler characteristic.

Lemma 6.16. *Let $\mathcal{A}$ be a projectively complete full subcategory of $k[G]\,\mathbf{mod}$, admitting a finite set of additive generators. Fix $\mathcal{S}$ a skeleton of the subcategory $\mathcal{I}$ of indecomposables of $\mathcal{A}$, and let S be the underlying finite set. Let also $\mathcal{F}$ be an $\mathcal{A}$-sheaf on X. Then have in $K_0\,(\mathbf{mod}\ \mathcal{A})$:*

$$\chi(\mathcal{A}, \mathcal{F}) = \sum_{I \in S} \chi(\mathcal{F}(I))[S_I].$$

Proof. Follows directly from the fact that $(s_X)_\mathcal{A}$ commutes with restriction along $\langle I \rangle \to \mathcal{A}$ and Lemma 2.26. □

Lemma 6.17. *Suppose moreover that $\mathcal{A}$ is stable by radical and duality, that duality and radical commute, and that $\mathcal{F}$ preserves monomorphisms. Then*

$$\chi(\mathcal{A}, \mathcal{F}) = \sum_{I \in S} \sum_{i \geqslant 0} \chi(\mathrm{gr}_i\, \mathcal{F}(I))[S_I].$$

Proof. Since $\mathcal{A}$ admits a finite set of additive generators, the canonical filtration of $\mathcal{F}$ (see §5.2.11) is finite. Hence $[\mathcal{F}] = \sum_{i \geqslant 0} [\mathrm{gr}_i\, \mathbf{mod}\, F]$ in $K_0(\mathcal{A}, X)$, and the formula follows. □

7. Applications

7.1. Symmetry principle

We can prove in our context a formula given by Elllingsrud and Lønsted [9, Theorem 2.4], following the proof of these authors:

Proposition 7.1. *Let X be a proper k-scheme, endowed with a free action of a finite group G, and $\mathcal{A}$ be a full subcategory of $k[G]$* **mod**, *admitting a finite set of additive generators, and containing $k[G]$, the free object of rank 1. Then for each G-sheaf $\mathcal{F}$ on X we have the equality in K_0 (**mod** $\mathcal{A}$):*

$$\chi(\mathcal{A}, \underline{\mathcal{F}}) = \chi(\pi_*^G \mathcal{F}) \left[\underline{k[G]}\right].$$

Proof. We can, of course, suppose that $\mathcal{A}$ is projectively complete. First, using Lemma 2.26, it is easily seen that $\left[\underline{k[G]}\right] = \sum_{I \in \mathcal{S}} \dim_k I[S_I]$. Moreover Lemma 6.16 says that $\chi(\mathcal{A}, \underline{\mathcal{F}}) = \sum_{I \in \mathcal{S}} \chi(\pi_*^G(I^\vee \otimes_k \mathcal{F}))[S_I]$. But since the action is free, π is étale, and for any sheaf $\mathcal{G}$ on Y, $\chi(\pi^* \mathcal{G}) = \# G \chi(\mathcal{G})$ (see [20]), and the proposition follows. □

Note that we recover the formula of Ellingsrud and Lønsted by evaluating at $k[G]$, in other words, we have lifted the classical formula along K_0 (**mod** $\mathcal{A}$) $\to K_0(k[G]$**mod**).

7.1.1. An example of computation

We come back to Nakajima's original result on the Galois module structure of $\mathrm{H}^0(X, \Omega_X)$ on a curve X (see [21]). This section contains no improvement but just indicates how to link this result with the symmetry principle.

Let X be a smooth projective curve over an algebraically closed field k (i.e. a one-dimensional integral scheme, which is proper over $\operatorname{spec} k$ and regular), endowed with a free action of a finite group G. Moreover, let $\mathcal{A}$ be a full subcategory of $k[G]$**mod**, admitting a finite set of additive generators, and containing $k[G]$, the free object of rank 1.

We give some more details on the cohomology of the sheaf of differentials on X.

For each object V of $\mathcal{A}$, we choose a projective hull $P(V)$, and then define $\Omega(V) = \ker P(V) \twoheadrightarrow V$, and inductively for $i \geqslant 0$: $\Omega^{i+1}(V) = \Omega(\Omega^i(V))$, $\Omega^0(V) = V$ (the notation Ω is unfortunate in our context, but since seems to be used in modular representation theory, we keep it, hoping not to confuse the reader).

Proposition 7.2.

$$H^0(X, \Omega_X) \oplus P(\Omega k) \simeq \Omega^2 k \oplus k[G]^{\oplus g_Y - 1} \oplus P(k).$$

Proof. This is a special case of [13, Theorem 2]. □

Proposition 7.3. *There is equality in* $K_0\,(\mathbf{mod}\ \mathcal{A})$:

$$\left[\underline{H^1(X,\Omega_X)}\right] = \left[\underline{\Omega^2 k}\right] - \left[\underline{P(\Omega k)}\right] + \left[\underline{P(k)}\right].$$

Proof. Proposition 7.2 says that $H^0(X,\Omega_X)$ is projectively equivalent to $\Omega^2 k$, so the comparison Proposition 5.22 gives a five-terms exact sequence:

$$0 \to \mathrm{Ext}^1_{k[G]}(\cdot, \Omega^2 k) \to H^1(X, \underline{\Omega_X}) \to \underline{k} \to \mathrm{Ext}^2_{k[G]}(\cdot, \Omega^2 K) \to 0.$$

Now, the long exact sequences associated with the short exact sequences:

$$0 \to \Omega k \to P(k) \to k \to 0$$

and

$$0 \to \Omega^2 k \to P(\Omega k) \to \Omega k \to 0$$

allow to conclude quickly. □

In our opinion, the above proposition sheds some light on the appearance of the $\Omega^2 k$ term in the Nakajima's Proposition 3 of [22], and related formula (like the one of Proposition 7.2). More precisely, Proposition 7.2 implies, of course, that

$$\left[\underline{H^0(X,\Omega_X)}\right] = \left[\underline{\Omega^2 k}\right] - \left[\underline{P(\Omega k)}\right] + [P(k)] + \left[\underline{k[G]^{\oplus g_Y - 1}}\right]$$

and so $\underline{\Omega_X}$ verifies the symmetry principle in $K_0\,(\mathbf{mod}\ \mathcal{A})$, whereas Ω_X does not, in the sense that $\left[\underline{H^0(X,\Omega_X)}\right] - \left[\underline{H^1(X,\omega_X)}\right]$ is in general *not* a multiple of $\left[\underline{k[G]}\right]$ in $K_0\,(\mathbf{mod}\ \mathcal{A})$.

7.2. Galois modules on projective curves in a positive characteristic

7.2.1. Hypothesis

Following our previous paper [3], we show how to describe the group $K_0(\mathcal{A}, X)$ when X is a projective curve over an algebraically closed field k. By projective curve we mean here a one-dimensional integral scheme, which is proper over $\operatorname{spec} k$ and regular. We suppose that X/k is endowed with a faithful action of a finite group G. We will also suppose that G acts with normal stabilizers. We denote as usual by $\pi: X \to Y = X/G$ the quotient. $\mathcal{A}$ is a fixed subcategory of $k[G]\mathbf{mod}$ admitting a finite set of additive generators.

7.2.2. Additive structure of $K_0(\mathcal{A}, X)$

For each G-invariant subset U, consider its complement $X' = X - U$, endowed with the reduced structure, and denoted by $f : X' \to X$ the corresponding closed immersion. Locally X' is of the form $G \times^{G_P} P$, for a point P of stabilizer G_P. Since X' is reduced, $P \simeq \operatorname{spec} k$, hence G_P acts trivially on P. Moreover, since we make the hypothesis that G_P is normal, we can apply Proposition 6.13 to deduce that $\mathcal{A}_X \to \tilde{f}_* \mathcal{A}_{X'}$ is an epimorphism. Now from Theorem 6.7 we get an exact sequence

$$\cdots \to K_1(\mathcal{A}, U) \to K_0(\mathcal{A}, X') \to K_0(\mathcal{A}, X) \to K_0(\mathcal{A}, U) \to 0.$$

Since the category formed by all the X', when U varies between the nonempty G-invariant open subsets of X, is pseudofiltered, the sequence remains exact after taking inductive limits on X' (see [24, Theorem 14.6.6]). Moreover, the generic point ξ of X can be written as a G-scheme as

$$\xi = \lim_{\substack{\leftarrow \\ U}} U.$$

We get as in [23, §7, Proposition 2.2] that for each nonnegative integer i

$$K_i(\mathcal{A}, \xi) \simeq \lim_{\substack{\rightarrow \\ U}} K_i(\mathcal{A}, U).$$

Moreover, since the action of G on X is faithful, ξ is endowed with a free action of G, with quotient η, the generic point of Y. So according to Corollary 6.15, we have for each nonnegative integer i that $K_i(\eta) \simeq K_i(\mathcal{A}, \xi)$. As is well known, $K_0(\eta) \simeq \mathbb{Z}$, and $K_1(\eta) \simeq R(Y)^*$, where $R(Y)$ is the function field of Y.

Definition 7.4. (i) The group of $\mathcal{A}$-cycles on X, denoted by $Z_0(\mathcal{A}, X)$, is by definition

$$Z_0(\mathcal{A}, X) = \lim_{\substack{\rightarrow \\ X'}} K_0(\mathcal{A}, X').$$

where the limit is taken on all the reduced strict closed G-subschemes of X.

(ii) The group of classes of $\mathcal{A}$-cycles on X for the rational equivalence, denoted by $A_0(\mathcal{A}, X)$, is by definition the cokernel of the canonical morphism $R(Y)^* \to Z_0(\mathcal{A}, X)$ defined by the connection morphisms in the long exact sequences of K-theory.

(iii) We denoted by $\gamma : A_0(\mathcal{A}, X) \to K_0(\mathcal{A}, X)$ and $\operatorname{rk} : K_0(\mathcal{A}, X) \to \mathbb{Z}$ the canonical morphisms.

Theorem 7.5. *Let X be a projevtive curve over an algebraically closed field k, endowed with the faithful action of a finite group G, acting with normal stabilizers. Moreover, let $\mathcal{A}$ be a subcategory of $k[G]$***mod** *admitting a finite set of additive generators,* *and containing the free object* $k[G]$. *Then the following morphism*:

$$\begin{aligned} \phi : \mathbb{Z} \oplus \mathbf{mod}\,(\mathcal{A}, X) &\longrightarrow K_0(\mathcal{A}, X) \\ (r, D) &\longrightarrow r[\underline{\mathcal{O}_X}] + \gamma(D) \end{aligned}$$

is an isomorphism.

Proof. As already seen, we have an exact sequence

$$0 \to A_0(\mathcal{A}, X) \to K_0(\mathcal{A}, X) \to \mathbb{Z} \to 0,$$

where the last map is the rank map. But this one is given by $\mathcal{F} \to \mathrm{rk}(\mathcal{F}(k[G]))$, hence $r \to r[\underline{\mathcal{O}_X}]$ is clearly a section. □

Definition 7.6. We denote by $c_1 : K_0(\mathcal{A}, X) \to A_0(\mathcal{A}, X)$, and call *first Chern class*, the morphism composed of the inverse $\phi^{-1} : K_0(\mathcal{A}, X) \to \mathbb{Z} \oplus A_0(\mathcal{A}, X)$ of the isomorphism of Theorem 7.5, followed by the second projection $\mathbb{Z} \oplus A_0(\mathcal{A}, X) \to A_0(\mathcal{A}, X)$.

Lemma 7.7. *The morphism* $\deg_{\mathcal{A}} : Z_0(\mathcal{A}, X) \to K_0\,(\mathbf{mod}\ \mathcal{A})$ *corresponding to the cone* $((s_{X'})_{\mathcal{A}} : K_0(\mathcal{A}, X') \to K_0\,(\mathbf{mod}\ \mathcal{A}))_{X'}$ *is trivial on the image of* $R(Y)^* \to Z_0(\mathcal{A}, X)$. *We denote also by* $\deg_{\mathcal{A}} : A_0(\mathcal{A}, X) \to K_0\,(\mathbf{mod}\ \mathcal{A})$ *the induced morphism.*

Proof. Let $\langle k \rangle$ denote the full subcategory of $k[G]\mathbf{mod}$ containing only the trivial representation. Fix an object V of $\mathcal{A}$, corresponding to a unique k-linear functor $\langle k \rangle \to \mathcal{A}$. According to [23, §5, Theorem 5], the localization sequence is functorial for the restriction along this functor, hence we obtain a commutative diagram:

$$\begin{array}{ccccc} K_1(\mathcal{A}, \xi) & \longrightarrow & Z_0(\mathcal{A}, X) & \xrightarrow{\deg_{\mathcal{A}}} & K_0(\mathcal{A}, \mathrm{spec}\ k) \\ \downarrow & & \downarrow & & \downarrow \\ K_1(\langle k \rangle, \xi) & \longrightarrow & Z_0(\langle k \rangle, X) & \xrightarrow{\deg_{\langle k \rangle}} & K_0(\langle k \rangle, \mathrm{spec}\ k) \end{array}$$

But the bottom line is identified with $K_1(\eta) \to Z_0(Y) \to \mathbb{Z}$, and [23, §7, Lemma 5.16], implies that the first map sends a function to its divisor. Since Y is also a projective curve, the bottom line is thus a complex, and we can conclude from Lemma 2.26 (note that by Morita invariance, we can reduce to the case where $\mathcal{A}$ is projectively complete). □

Corollary 7.8. *Suppose the hypothesis of Theorem* 7.5 *are verified.*

Then for any coherent $\mathcal{A}$-sheaf $\mathcal{F}$ on X we have in $K_0(\mathbf{mod}\ \mathcal{A})$:

$$\chi(\mathcal{A}, \mathcal{F}) = \mathrm{rk}\mathcal{F}\ \chi(\mathcal{A}, \underline{\mathcal{O}_X}) + \deg_{\mathcal{A}} c_1(\mathcal{F}).$$

Proof. According to Theorem 7.5 we have $[\mathcal{F}] = \mathrm{rk}\mathcal{F}\ [\underline{\mathcal{O}_X}] + \gamma(c_1(\mathcal{F}))$ in $K_0(\mathcal{A}, X)$, and the formula is obtained by pushing along $s_X : X \to \operatorname{spec} k$. $\square$

7.2.3. Local character of modular Chern classes

It seems desirable to compute explicitly the modular first Chern class of a locally free G-sheaf. If the modular representation theory of the stabilizers is known (which seems very likely, because under our hypothesis they are semi-direct products of a cyclic p-group by a cyclic p'-group) this computation is probably possible, because, as expressed in this paragraph, it is essentially (modulo a Chern class with coefficients in free representations) of local nature.

We need two preparatory results. The first is a vanishing theorem.

Proposition 7.9. *Let $\mathcal{E}$ be a locally free G-sheaf on the curve X.*
If $\mathcal{E}$ is acyclic, so is $\underline{\mathcal{E}}$.

Proof. By definition, $\underline{\mathcal{E}}$ is the enriched functor $V \to \pi_*^G(s_X^*(V^\vee) \otimes_{\mathcal{O}_X} \mathcal{E})$ in $\mathbf{Coh}(\mathcal{A}, X) = [\mathcal{A}_X^{op}, \mathbf{Coh}Y]$. Since the evaluation at V is an exact functor, $H^i(X, \underline{\mathcal{E}})$ is the k-linear functor $V \to H^i(Y, \pi_*^G(s_X^*(V^\vee) \otimes_{\mathcal{O}_X} \mathcal{E}))$ in $\mathbf{mod}\ \mathcal{A} = [\mathcal{A}^{op}, \mathbf{mod}\ k]$, so, since $\dim Y = 1$, is zero for $i > 1$. It remains only to show that $H^1(X, \underline{\mathcal{E}}) = 0$.

Lemma 7.10. *If W is an injective $k[G]-$module so that $V \hookrightarrow W$, and $H^1(X, \underline{\mathcal{E}})(W) = 0$, then $H^1(X, \underline{\mathcal{E}})(V) = 0$.*

Proof. Let $V' = \operatorname{coker}(V \hookrightarrow W)$. Since $\mathcal{E}$ is locally free we have an exact sequence of G-sheaves on X:

$$0 \to s_X^*(V'^\vee) \otimes_{\mathcal{O}_X} \mathcal{E} \to s_X^*(W^\vee) \otimes_{\mathcal{O}_X} \mathcal{E} \to s_X^*(V^\vee) \otimes_{\mathcal{O}_X} \mathcal{E} \to 0.$$

Now, the sheaf $R^1\pi_*^G(s_X^*(W^\vee) \otimes_{\mathcal{O}_X} \mathcal{E})$ is torsion, and its stalk at the point closed Q of Y is $H^1(G_P, W^\vee \otimes_k \mathcal{E}_P)$, where $P \to Q$ is any lifting of Q in X. Since $W^\vee$ is $k[G]$-projective, this is zero, hence we have an exact sequence on Y:

$$\begin{aligned} 0 &\to \pi_*^G(s_X^*(V'^\vee) \otimes_{\mathcal{O}_X} \mathcal{E}) \to \pi_*^G(s_X^*(W^\vee) \otimes_{\mathcal{O}_X} \mathcal{E}) \\ &\to \pi_*^G(s_X^*(V^\vee) \otimes_{\mathcal{O}_X} \mathcal{E}) \to R^1\pi_*^G(s_X^*(V'^\vee) \otimes_{\mathcal{O}_X} \mathcal{E}) \to 0, \end{aligned}$$

Put $\mathcal{F} = \operatorname{coker}((\pi_*^G(s_X^*(V'^\vee) \otimes_{\mathcal{O}_X} \mathcal{E}) \to \pi_*^G(s_X^*(W^\vee) \otimes_{\mathcal{O}_X} \mathcal{E}))$. Since by hypothesis $H^1(X, \underline{\mathcal{E}})(W) = H^1(Y, \pi_*^G(s_X^*(W^\vee) \otimes_{\mathcal{O}_X} \mathcal{E})) = 0$ and $\dim Y = 1$ we must have $H^1(Y, \mathcal{F}) = 0$. But since $\mathcal{F} = \ker(\pi_*^G(s_X^*(V^\vee) \otimes_{\mathcal{O}_X} \mathcal{E}) \to R^1\pi_*^G(s_X^*(V'^\vee) \otimes_{\mathcal{O}_X} \mathcal{E}))$

and $R^1\pi_*^G(s_X^*(W^\vee)\otimes_{\mathcal{O}_X}\mathcal{E})$ is torsion, hence $H^1(Y, R^1\pi_*^G(s_X^*(W^\vee)\otimes_{\mathcal{O}_X}\mathcal{E}))=0$, this implies in turn $H^1(X,\underline{\mathcal{E}})(V)=H^1(Y,\pi_*^G(s_X^*(V^\vee)\otimes_{\mathcal{O}_X}\mathcal{E}))=0$. $\square$

To conclude the proof of the proposition, notice that we can enlarge $\mathcal{A}$, if needed, without changing the cohomology, hence we can suppose that $\mathcal{A}$ contains an injective W, with $V\hookrightarrow W$, and we even can take W free. So it remains only to show that $H^1(X,\underline{\mathcal{E}})(k[G])=0$, but this follows from Proposition 5.21. $\square$

The second preparatory result states that induction in modular K-theory (see §5.1.5, §5.2.8) is what we believe it to be.

Lemma 7.11. *Let G be a finite group with cyclic p-Sylows, H a subgroup, $\alpha : H\to G$ the inclusion, $S=\operatorname{Spec} k$, with trivial action of H, $S'=G\times^H S$, and $\mathcal{A}$ the ring with several objects $k[G]$**mod**, $\mathcal{B}$ the ring with several objects $k[H]$**mod**. Then:*

(i) *$\alpha^*\mathcal{A}$ is Morita equivalent to $\mathcal{B}$.*
(ii) *We have the following commutative diagram:*

$$
\begin{array}{ccc}
K_0(\mathcal{A},S') & \xrightarrow{(s_{S'})_{\mathcal{A}}} & K_0(\mathbf{mod}\ \mathcal{A}) \\
\uparrow{\scriptstyle c^*} & & \\
K_0(\alpha^*\mathcal{A},S) & & \\
\uparrow{\scriptstyle K_0(\cdot,S)} & & \uparrow{\scriptstyle \mathcal{Y}_{\mathcal{A}}} \\
K_0(\mathcal{B},S)=K_0(\mathbf{mod}\ \mathcal{B}) & & \\
\uparrow{\scriptstyle \mathcal{Y}_{\mathcal{B}}} & & \\
K_0(\mathcal{B}^{split}) & \xrightarrow[\operatorname{Ind}_H^G]{} & K_0(\mathcal{A}^{split})
\end{array}
$$

where the vertical rows are isomorphisms, $s_{S'} : S'\to S$ is the structural morphism, $\mathcal{Y}_\cdot$ are (induced by) the Yoneda embedding, and c^* is defined by restriction along the canonical isomorphism $\mathcal{A}_{S'}\simeq\alpha^*\mathcal{A}$.

Proof. (i) This is clear, because according to Mackey formula, for each $k[H]$-module W, W is a direct summand of $(\operatorname{Ind}_H^G W)_{|H}$.

(ii) Considering the underlying diagram of functors, this boils down to the fact that we have a natural isomorphism $\mathcal{A}(V,\operatorname{Ind}_H^G W)\simeq\mathcal{B}(V_{|H},W)$ for each $k[G]$-module V and each $k[H]$-module W, which is true, because for group rings, induction and coinduction coincide. $\square$

Theorem 7.12. *Let X be a projective curve over an algebraically closed field k of positive characteristic p, endowed with the faithful action of finite group G. Suppose* *that G has cyclic p-Sylows and acts with normal stabilizers. Let* $\mathcal{E}$, $\mathcal{E}'$ *be two locally free G-sheaves, of X of same rank, and such that* $H^1(X, \mathcal{E}) = 0$, $H^1(X, \mathcal{E}') = 0$. *Then*:

(i) *if the ramification locus* X_{ram} *is not empty, there exists for each* $P \in X_{\text{ram}}$, *a couple* (V_P, V'_P) *of representations of the stabilizer* G_P, *such that* $H^0(X, \mathcal{E}) \oplus \oplus_{P \in X_{\text{ram}}} \text{Ind}\,{}^G_{G_P} V_P \simeq H^0(X, \mathcal{E}') \oplus \oplus_{P \in X_{\text{ram}}} \text{Ind}\,{}^G_{G_P} V'_P$.

(ii) *Let moreover* $\phi : \mathcal{E} \to \mathcal{E}'$ *be a morphism of G-sheaves that is an isomorphism outside the strict closed G-subset* X' *of X. Then the previous statement holds if one replaces* X_{ram} *by* X'.

Proof. (i) By Theorem 7.8 we have $\chi(\mathcal{A}, \underline{\mathcal{E}}) = \text{rk}\,\mathcal{E}\ \chi(\mathcal{A}, \underline{\mathcal{O}_X}) + \deg_{\mathcal{A}} c_1(\underline{\mathcal{E}})$ and $\chi(\mathcal{A}, \underline{\mathcal{E}'}) = \text{rk}\,\mathcal{E}'\ \chi(\mathcal{A}, \underline{\mathcal{O}_X}) + \deg_{\mathcal{A}} c_1(\underline{\mathcal{E}'})$ in $K_0\,(\textbf{mod}\ \mathcal{A})$. But Proposition 7.9 implies also that $\chi(\mathcal{A}, \underline{\mathcal{E}}) = [\underline{H^0(X, \mathcal{E})}]$ and $\chi(\mathcal{A}, \underline{\mathcal{E}'}) = [\underline{H^0(X, \mathcal{E}')}]$, so since by hypothesis $\text{rk}\,\mathcal{E} = \text{rk}\,\mathcal{E}'$ we get that $[\underline{H^0(X, \mathcal{E})}] - [\underline{H^0(X, \mathcal{E}')}] = \deg_{\mathcal{A}} c_1(\underline{\mathcal{E}}) - \deg_{\mathcal{A}} c_1(\underline{\mathcal{E}'})$. Denote by x' the quantity $c_1(\underline{\mathcal{E}}) - c_1(\underline{\mathcal{E}'})$ and let X' be a strict closed G-subset of X so that x belongs to $K_0(\mathcal{A}, X')$, and $X' \cap X_{\text{ram}} \neq \emptyset$. By definition $[\underline{H^0(X, \mathcal{E})}] - [\underline{H^0(X, \mathcal{E}')}] = \deg_{\mathcal{A}} x' = (s_{X'})_{\mathcal{A}} x'$. In X' the ramified orbits contribute to $(s_{X'})_{\mathcal{A}} x'$, according to Lemma 7.11, by an element of the type $\underline{\text{Ind}^G_{G_P} y}$, where y belongs to $K_0^{split}(k[G_P]\,\textbf{mod}\,)$. Similarly, the unramified orbits contribute to $(s_{X'})_{\mathcal{A}} x'$ by an element of the type $N\underline{k[G]}$, where N is an integer, but this is of the previous type, with P in $X' \cap X_{\text{ram}}$ and $y = [k[G_P]]$. Finally, Theorem 2.4 allows to conclude.

(ii) Denote by U the complement of X'. The morphism ϕ is an isomorphism on U, hence $\underline{\phi}$ is an isomorphism on U, and so $[\underline{\mathcal{E}}] = [\underline{\mathcal{E}'}]$ in $K_0(\mathcal{A}, U)$. According to Theorem 6.7 we can write $[\underline{\mathcal{E}}] - [\underline{\mathcal{E}'}] = i_{\mathcal{A}} x'$ where x' belongs to $K_0(\mathcal{A}, X')$ and $i : X' \to X$ is the inclusion. We now conclude exactly in the same way as in (i), after applying $(s_X)_{\mathcal{A}}$. □

7.2.4. Explicit expression for the action of a cyclic group

To the hypothesis of §7.2.1, we add the fact that G is cyclic of order n, and we fix a generator σ. Moreover we choose as ring with several objects $\mathcal{A} = k[G]\,\textbf{mod}\,$, the whole category of all $k[G]$-modules of finite type. We follow the notations given in §5.2.11.

So by definition, for each coherent kernel-preserving $\mathcal{A}$-sheaf $\mathcal{F}$ on X, we denote by $\text{gr}_0\,\mathcal{F}$ the $\mathcal{A}$-sheaf on X defined on indecomposables by the existence of n exact sequences of sheaves on Y:

$$0 \to \mathcal{F}(\psi \otimes V_{j-1}) \to \mathcal{F}(\psi \otimes V_j) \to \text{gr}_0\, u\mathcal{F}(\psi \otimes V_j) \to 0.$$

The reader who is just concerned by the cyclic case may skip §5.2.11 and consider this as a definition of $\text{gr}_0\,\mathcal{F}$, since this is all we need here.

Lemma 7.13. *For any coherent G-sheaf $\mathcal{G}$ on X*:

$$\chi(\mathcal{A}, \underline{\mathcal{G}}) = \sum_{\psi \in \widehat{G}} \sum_{j=1}^{p^v} \sum_{i=1}^{j} \chi(gr_0\, \underline{\mathcal{G}}(\psi \otimes V_i))[S_{\psi \otimes V_j}]$$

Proof. This follows from Lemma 6.17, by noticing that in the particular case of a cyclic group G we have $\mathrm{gr}_i\, \mathcal{F} \simeq R^i \mathrm{gr}_0\, \mathcal{F}$. Alternatively, this is direct from the above exact sequences and Lemma 6.16. □

From now on, we will consider an invertible G-sheaf $\mathcal{L}$ on X, and show how to determine explicitly the structure of $k[G]$-module of its global sections $H^0(X, \mathcal{L})$ when $\deg \mathcal{L} > 2g_X - 2$.

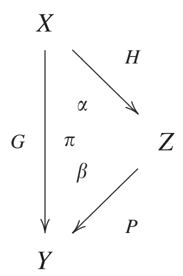
To do this, we will of course use Lemma 7.13, and show that we can describe explicitly the invertible sheaves $\mathrm{gr}_0\, \underline{\mathcal{L}}(\psi \otimes V_j)$ in terms of a G-invariant divisor D such that $\mathcal{L} \simeq \mathcal{L}_X(D)$, of the ramification data, of the genus of Y, and of some rational functions on X.

Note that the formula

$$\pi_*^G \mathcal{L}_X(D) \simeq \mathcal{L}_Y([\pi_* D/\#G])$$

(where $[\cdots]$ is the integral part of the divisor, taken coefficient by coefficient) provides an explicit description of $\pi_*^G \mathcal{L}$ (see [13, proof of Proposition 3]).

First we reduce to the case of a cyclic p-group. Remember that $n = p^v a$, with a prime to p, and let H be the subgroup of G of order a, and $P = G/H$, so that we have a tower:



(6)

Lemma 7.14. *For each ψ in the character group $\widehat{G}$, there exists a nonzero function f_ψ in the function field $R(X)$ of X such that for each invertible G-sheaf $\mathcal{L}$ on X we have*

$$\mathrm{gr}_0^G\, \underline{\mathcal{L}}(\psi \otimes V_j) \simeq \mathrm{gr}_0^P\, \underline{\alpha_*^H \mathcal{L}(f_\psi)}(V_j).$$

Proof. From the Kummer theory, we get for each ψ in $\widehat{G}$ a nonzero function f_ψ such that $\mathcal{L}_X((f_\psi)) \simeq s_X^*(\psi)$. Thus the result follows from the exactness of α_*^H. $\square$

Next, we reduce to the case of a cyclic group of order p. For this, suppose that we start from a group G of order p^v, with $v \geqslant 2$, and let H be the subgroup of order p^{v-1}, and $P = G/H$. We keep the notations of diagram 6 in this context.

Proposition 7.15. *Let $1 \leqslant j \leqslant p^v$ an integer, and write: $j = (l-1)p + j'$ with $1 \leqslant l \leqslant p^{v-1}$ and $1 \leqslant j' \leqslant p$.*

Then we have an isomorphism of invertible sheaves on Y:

$$\mathrm{gr}_0^G \underline{\mathcal{L}}(V_j) \simeq \mathrm{gr}_0^P \underline{\mathrm{gr}_0^H \underline{\mathcal{L}}(V_l)}(V_{j'}).$$

Proof. The proof requires the following lemmas.

Lemma 7.16.

$$V_{j\,|H} \simeq V_l^{\oplus j'} \oplus V_{l-1}^{\oplus p-j'}.$$

Proof. $1, \sigma - 1, \ldots, (\sigma-1)^{j'-1}$ generate Jordan blocks of size l, while $(\sigma-1)^{j'}$, $(\sigma-1)^{j'+1}, \ldots, (\sigma-1)^{p-1}$, generate Jordan blocks of size $l-1$. $\square$

Lemma 7.17. *There is an exact sequence of P-sheaves on Z:*

$$0 \to \alpha_*^H(\mathcal{L} \otimes V_{(l-1)p}) \to \alpha_*^H(\mathcal{L} \otimes V_j) \to V_{j'} \otimes \mathrm{gr}_0^H(\mathcal{L})(V_l) \to 0.$$

Proof. First ignoring the action of P, we can use Lemma 7.16 to show the existence of an exact sequence of sheaves on Z:

$0 \to \alpha_*^H(\mathcal{L} \otimes V_{(l-1)p}) \to \alpha_*^H(\mathcal{L} \otimes V_j) \to (\mathrm{gr}_0^H(\mathcal{L})(V_l))^{\oplus j'} \to 0.$

But then from the exact sequence of $k[G]$-modules

$$0 \to V_{(l-1)p} \to V_j \to V_{j'} \to 0$$

one sees that $\sigma - 1$ acts transitively on the direct summands of $(\mathrm{gr}_0^H(\mathcal{L})(V_l))^{\oplus j'}$, hence the result. $\square$

We will suppose that $j' \geqslant 2$, the case $j' = 1$ being analog, and easier. We have the following commutative diagram of P-sheaves on Z:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathrm{gr}_0{}^H(\mathcal{L})(V_l) & = & \mathrm{gr}_0{}^H(\mathcal{L})(V_l) & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \alpha_*{}^H(\mathcal{L} \otimes V_{(l-1)p}) & \longrightarrow & \alpha_*{}^H(\mathcal{L} \otimes V_j) & \longrightarrow & V_{j'} \otimes \mathrm{gr}_0^H(\mathcal{L})(V_l) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \alpha_*{}^H(\mathcal{L} \otimes V_{(l-1)p}) & \longrightarrow & \alpha_*{}^H(\mathcal{L} \otimes V_{j-1}) & \longrightarrow & V_{j'-1} \otimes \mathrm{gr}_0^H(\mathcal{L})(V_l) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

So the Proposition follows now from a diagram chase thanks to the following fact:

Lemma 7.18.

$$R^1\beta_*^P(\alpha_*^H(\mathcal{L} \otimes V_{(l-1)p})) = 0$$

Proof. One sees as in Lemma 7.17 that $\alpha_*^H(\mathcal{L} \otimes V_{(l-1)p}) \simeq V_p \otimes \alpha_*^H(\mathcal{L} \otimes V_{l-1})$, and since $V_p = k[P]$, this is enough. $\square$

So we are reduced to the case of a cyclic p-group, which was solved by Nakajima. We give a translation of [22, Theorem 1] in our context, and since our version is slightly stronger, we give a sketch of a proof.

Theorem 7.19 (*Nakajima*). *Let X be a projective curve endowed with a faithful action of $G = \mathbb{Z}/p$, and D a G-invariant divisor on X. Write $D = \pi^*\delta + \sum_{P \in X_{\mathrm{ram}}} n_P \cdot P$, where δ is a divisor on Y so that* $\operatorname{supp} \pi^*\delta \cap X_{\mathrm{ram}} = \emptyset$. *For each P in X_{ram}, let moreover N_P be the integer defined by $N_P + 1 = v_P(\sigma u_P - u_P)$, where σ is a generator of G, and u_P an uniformizer at P. Then for each integer $1 \leqslant j \leqslant p$:*

$$\mathrm{gr}_0^G \, \underline{\mathcal{L}_X(D)}(V_j) \simeq \mathcal{L}_Y\left(\delta + \sum_{P \in X_{\mathrm{ram}}} \left\lceil \frac{n_P - (j-1)N_P}{p} \right\rceil \cdot \pi_* P\right).$$

Proof. A local analysis shows (see Lemma 7.20) that the monomorphism $\pi^* \mathrm{gr}_0^G \, \underline{\mathcal{L}_X(D)}(V_j) \to \mathcal{L}_X(D)$ factorizes through $\pi^* \mathrm{gr}_0^G \, \underline{\mathcal{L}_X(D)}(V_j) \to \mathcal{L}_X(D - \sum_{P \in X_{\mathrm{ram}}} (j-1)N_P \cdot$

P), hence by applying π_*^G we get a monomorphism $\mathrm{gr}_0^G\, \underline{\mathcal{L}_X(D)}(V_j) \to \mathcal{L}_Y(\delta + \sum_{P\in X_{\mathrm{ram}}} [\frac{n_P-(j-1)N_P}{p}] \cdot \pi_* P)$. To show that this is an isomorphism is a local problem at X_{ram}, so by adding eventually to D a divisor of the form $\pi^*\gamma$ we may suppose that $\deg D > 2g_X - 2$. But then Nakajima [22, Lemma 4] shows that the two sheaves have the same space of global sections, hence they must be isomorphic. □

Lemma 7.20. *With the notations of Theorem* 7.19 *we have*

$$\pi^*\, \mathrm{gr}_0^G\, \underline{\mathcal{L}_X(D)}(V_j) \subset \mathcal{L}_X\left(D - \sum_{P\in X_{\mathrm{ram}}} (j-1)N_P \cdot P\right).$$

Proof. This is a local problem, so we can verify the inclusion on the completions of the local rings of the closed points of X. Let P be such a point, that we can suppose in X_{ram}, $Q = \pi P$, u_P an uniformizer at P, v_Q a uniformizer at Q. Localizing at P the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}_X(D)\otimes V_{j-1} & \longrightarrow & \mathcal{L}_X(D)\otimes V_j & \longrightarrow & \mathcal{L}_X(D) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \pi^*\pi_*^G(\mathcal{L}_X(D)\otimes V_{j-1}) & \longrightarrow & \pi^*\pi_*^G(\mathcal{L}_X(D)\otimes V_j) & \longrightarrow & \pi^*(\mathrm{gr}_0^G\, \underline{\mathcal{L}_X(D)}(V_j)) & \longrightarrow & 0
\end{array}$$

we get the following commutative diagram:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & V_j \otimes_k u_P^{-n_P} k[[u_P]] & \xrightarrow{\ \mu\ } & u_P^{-n_P} k[[u_P]] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle v} & & \\
\cdots & \longrightarrow & k[[u_P]] \otimes_{k[[v_Q]]} (V_j \otimes_k u_P^{-n_P} k[[u_P]])^G & \longrightarrow & k[[u_P]] \otimes_{k[[v_Q]]} (\mathrm{gr}_0^G\, \widehat{\underline{\mathcal{L}_X(D)}}(V_j)_Q) & \longrightarrow & 0
\end{array}$$

What we have to show is that the image of v is contained in the ideal $u_P^{-n_P+(j-1)N_P} k[[u_P]]$. Choose α in $V_j \otimes_k u_P^{-n_P} k[[u_P]]$. Since $V_j = k[G]/(\sigma - 1)^j$, we can write $\alpha = \sum_{i=0}^{j-1} (\sigma - 1)^i a_i$, with $a_i \in u_P^{-n_P} k[[u_P]]$. Then $\mu(\alpha) = a_0$, and if $\sigma\alpha = \alpha$, then $a_0 = (\sigma^{-1} - 1)^{j-1} a_{j-1}$. But now we can conclude, since the definition of N_P implies that for any x in $k[[u_P]]$, $v_P((\sigma^{-1} - 1)x) \geqslant v_P(x) + N_P$. □

Note, moreover, that in a relative situation like those appearing in various dévissage steps, the a priori nonequivariant isomorphisms are in fact automatically equivariant : indeed when G is a p-group acting on a projective k-scheme X, there is a structure of

G-sheaf on a given invertible sheaf. So Proposition 7.15 allows to apply Theorem 7.19 recursively, to finally have an explicit expression of $\mathrm{gr}_0^G\,\underline{\mathcal{L}_X(D)}(V_j)$, i.e. to represent this invertible sheaf on Y by a divisor.

Once these sheaves are computed, we return to the case of an arbitrary cyclic group, and show how we can make use of Lemma 7.13.

Lemma 7.21. *Suppose that G is any cyclic group. If* $\deg \mathcal{L} > 2g_X - 2$ *then* $H^1(X, \underline{\mathcal{L}}) = 0$.

Proof. This follows from Proposition 7.9. □

Now Lemma 7.13 provides, for an invertible sheaf $\mathcal{L}$ such that $\deg \mathcal{L} > 2g_X - 2$, an expression in $K_0\,(\mathbf{mod}\ \mathcal{A})$:

$$[\underline{H^0(X,\mathcal{L})}] = \sum_{\psi\in\widehat{G}} \sum_{j=1}^{p^v} \sum_{i=1}^{j} \chi(\mathrm{gr}_0\,\underline{\mathcal{L}}(\psi\otimes V_i))[S_{\psi\otimes V_j}] \tag{7}$$

where the integers $\chi(\mathrm{gr}_0\,\underline{\mathcal{G}}(\psi\otimes V_{j-i}))$ are given by the previous description and the classical Riemann–Roch formula. Theorem 2.4, (ii), implies that this characterizes fully $H^0(X,\mathcal{L})$ as a $k[G]$-module. But we can be more explicit by applying Theorem 2.4 (iii). For this we have to use the basis $\{[\underline{\psi\otimes V_j}] \mathbin{/} \psi\in\widehat{G},\ 1\leqslant j\leqslant p^v\}$ of $K_0\,(\mathbf{mod}\ \mathcal{A})$ rather than the basis $\{S_{\psi\otimes V_j} \mathbin{/} \psi\in\widehat{G},\ 1\leqslant j\leqslant p^v\}$.

Clearly, we can find the base change matrix block by block (i.e. here character by character) and thus fix a ψ in $\widehat{G}$. Let A be the Cartan matrix, i.e. the matrix whose jth column is formed by the coefficients of $[\underline{\psi\otimes V_j}]$ in the basis $\{S_{\psi\otimes V_i} \mathbin{/} 1\leqslant i\leqslant p^v\}$. Then Lemma 2.26 shows that the coefficients of A are given by $\dim_k \mathrm{Hom}_{k[G]}(V_j, V_i)$, and this is easily computed as being $\inf(i,j)$. This is independent of ψ, so all the blocks of the base change matrix are equal, and the inverse is easily computed as

$$A^{-1} = \begin{pmatrix} +2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & +2 & \ddots & \ddots & & \vdots \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & \ddots & \ddots & +2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & +1 \end{pmatrix} \tag{8}$$

7.2.5. Explicit expression for the action of a cyclic p-group

Definition 7.22. Let $\pi : X \to Y$ be a (generically) Galois cover of projective curves over k of group $G \simeq \mathbb{Z}/p$, with generator σ.

(i) For a ramified point P of X, define N_P as the integer such that $N_P + 1$ is the valuation at P of $\sigma u_P - u_P$, where u_P is an uniformizer at P.

(ii) For $0 \leqslant \alpha \leqslant p-1$ define a map $\pi_*^\alpha : Z_0(X) \to Z_0(Y)$ between 0-cycles groups by, for any divisor D on X,

$$\pi_*^\alpha D = \left[\frac{1}{p} \pi_* \left(D - \alpha \sum_{P \in X_{\mathrm{ram}}} N_P \cdot P \right) \right],$$

where $[\cdots]$ denotes the integral part of a divisor, taken coefficient by coefficient.

Theorem 7.23. *Suppose that X is a projective curve over k with a faithful action of $G \simeq \mathbb{Z}/p^v\mathbb{Z}$. For $1 \leqslant n \leqslant v$, let X_n be the quotient curve of X by the action of the subgroup of G of order p^n, and $\pi_n : X_{n-1} \to X_n$ be the canonical morphism. Moreover let D be a G-invariant divisor on X, and $H^0(X, \mathcal{L}_X(D)) \simeq \oplus_{j=1}^{p^v} V_j^{\oplus m_j}$ be the Krull–Schmidt decomposition of the global sections of $\mathcal{L}_X(D)$, where V_j is the indecomposable $k[G]$-module of dimension j. Suppose $\deg D > 2g_X - 2$. Then the integers m_j are given by*

$$\begin{cases} m_j = \deg(\pi_{v*}^{\alpha_0(j)} \cdots \pi_{1*}^{\alpha_{v-1}(j)} D) - \deg(\pi_{v*}^{\alpha_0(j+1)} \cdots \pi_{1*}^{\alpha_{v-1}(j+1)} D) \text{ if } 1 \leqslant j \leqslant p^v - 1, \\ m_{p^v} = 1 - g_{X_v} + \deg(\pi_{v*}^{p-1} \cdots \pi_{1*}^{p-1} D), \end{cases}$$

where for $1 \leqslant j \leqslant p^v$ the integers $\alpha_0(j), \ldots, \alpha_{v-1}(j)$ are the digits of the p-adic writing of $j-1$ defined by $j - 1 = \sum_{h=0}^{v-1} \alpha_h(j) p^h$ with $0 \leqslant \alpha_h(j) \leqslant p-1$.

Proof. This is a direct consequence of the following intermediate computation:

Lemma 7.24.

$$\begin{cases} m_1 = 2a_1 - a_2, \\ m_j = -a_{j-1} + 2a_j - a_{j+1} \text{ if } 2 \leqslant j \leqslant p^v - 1, \\ m_{p^v} = a_{p^v} - a_{p^v - 1} \end{cases}$$

with, for $1 \leqslant j \leqslant p^v$, $a_j = j(1 - g_{X_v}) + \sum_{i=1}^{j} \deg(\pi_{v*}^{\alpha_0(i)} \cdots \pi_{1*}^{\alpha_{v-1}(i)} D)$.

Proof. Set $\mathcal{L} = \mathcal{L}_X(D)$ and $X_v = Y$. According to equation 7 we have $[\underline{H^0(X, \mathcal{L})}] = \sum_{j=1}^{p^v} \sum_{i=1}^{j} \chi(\mathrm{gr}_0 \underline{\mathcal{L}}(V_i))[S_{V_j}]$. Set $b_j = \sum_{i=1}^{j} \chi(\mathrm{gr}_0 \underline{\mathcal{L}}(V_i))$. If we show that for each j we have equality $b_j = a_j$, we are done, according to the base change matrix given in Eq. (8). But the usual Riemann–Roch formula gives $b_j = j(1 - g_Y) + \sum_{i=1}^{j} \deg(\mathrm{gr}_0 \underline{\mathcal{L}}(V_i))$. So the only thing which remains to be shown is $\mathrm{gr}_0 \underline{\mathcal{L}}(V_i) \simeq$

$\mathcal{L}_Y(\pi_{v*}^{\alpha_0}\cdots\pi_{1*}^{\alpha_{v-1}}D)$. This is easily done by induction: the first step is Theorem 7.19, and the induction step is given by Proposition 7.15. □

7.2.6. Noether criterion with parameter

Definition 7.25. Let G be a finite group, H a subgroup, k an algebraically closed field. A $k[G]$-module of finite type V is said *relatively H-projective* if it is a direct summand of a module induced from H.

Theorem 7.26. *Let $\pi : X \to Y$ be a (generically) cyclic Galois p-cover of projective curves over k of group G,* $\operatorname{ram}\pi$ *the largest ramification subgroup of π, and H a subgroup of G. Then the following assertions are equivalent*:

(i) $\operatorname{ram}\pi \subset H$,
(ii) $\forall\mathcal{L} \in \operatorname{Pic}_G X \quad \deg\mathcal{L} > 2g_X - 2 \Longrightarrow H^0(X,\mathcal{L})$ *is relatively H-projective.*,
(iii) $\exists\mathcal{M} \in \operatorname{Pic} Y$ *so that* $\#G \deg\mathcal{M} > 2g_X - 2$ *and* $H^0(X,\pi^*\mathcal{M})$ *is relatively H-projective.*

Proof. We begin by some notations. Let p^v (resp. p^w) be the order of G (resp. H). Fixing a generator σ of G, we denote for $1 \leqslant l \leqslant p^v$ the $k[G]$-indecomposable of dimension l by $V_l^G = k[\sigma]/(\sigma-1)^l$, and similarly for H.

Lemma 7.27. *The relatively H-projective $k[G]$-modules are exactly the direct sums of the indecomposables $V^G_{lp^{v-w}}$ for $1 \leqslant l \leqslant p^w$.*

Proof. This results from the Krull–Schmidt Theorem and the fact that $\operatorname{Ind}_H^G V_l^H = V^G_{lp^{v-w}}$ for $1 \leqslant l \leqslant p^w$. □

We will also use the notations and results of §7.2.5.

We first show that (i) implies (ii). For this, note that according to Theorem 7.23, the multiplicity of V_j^G in $H^0(X,\mathcal{L})$ is given for $1 \leqslant j \leqslant p^v-1$ by:

$$\begin{aligned} m_j = {} & \deg(\pi_{v*}^{\alpha_0(j)}\cdots\pi_{w+1*}^{\alpha_{v-w-1}(j)}\pi_{w*}^{\alpha_{v-w}(j)}\cdots\pi_{1*}^{\alpha_{v-1}(j)}D) \\ & -\deg(\pi_{v*}^{\alpha_0(j+1)}\cdots\pi_{w+1*}^{\alpha_{v-w-1}(j+1)}\pi_{w*}^{\alpha_{v-w}(j+1)}\cdots\pi_{1*}^{\alpha_{v-1}(j+1)}D). \end{aligned}$$

Because of the hypothesis that $\operatorname{ram}\pi \subset H$, the coverings π_u are étale for $u > w$, hence the morphisms π_{u*}^{α} are independent of α. To show (ii), we can, according to Lemma 7.27, show that $m_j = 0$ for $p^{v-w} \nmid j$, which results from the previous remark and of:

Lemma 7.28. *Let $1 \leqslant j \leqslant p^v$ so that $p^{v-w} \nmid j$. Then $\forall u \geqslant v-w \;\; \alpha_u(j) = \alpha_u(j+1)$.*

Proof. This is clear from the definition of $\alpha_u(j)$ as the $(u+1)$th digit in the p-adic writing of $j-1$. □

Since the fact that (ii) implies (iii) is trivial, all what remains to be shown is that (iii) implies (i). For this, suppose that $\mathrm{ram}\,\pi \not\subseteq H$, that is, since the subgroups of G are totally ordered, $H \subsetneq \mathrm{ram}\,\pi$. Let $\mathcal{M}$ be an invertible sheaf on Y such that $\#G \deg \mathcal{M} > 2g_X - 2$. To show that (iii) is false, we can clearly suppose that $\mathrm{ram}\,\pi$ has order p^{w+1}. Choose $j = p^{v-w-1}$. An easy computation shows that the coefficient m_j of V_j^G in $H^0(X, \pi^*\mathcal{M})$ is

$$- \sum_{P \in X_w^{ram}} [-N_P/p]$$

and since by hypothesis $X_w^{ram} \neq \emptyset$, this is nonzero. Hence $H^0(X, \pi^*\mathcal{M})$ is nonrelatively H-projective, as was to be shown. □

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